PARTIAL CHOICE FUNCTIONS FOR FAMILIES OF FINITE SETS

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ABSTRACT. Let $m \geq 2$ be an integer. We show that ZF + "Every countable set of m-element sets has an infinite partial choice function" is not strong enough to prove that every countable set of m-element sets has a choice function, answering an open question from [DHHKR]. (Actually a slightly stronger result is obtained.) The independence result in the case where $m = p$ is prime is obtained by way of a permutation (Fraenkel-Mostowski) model of ZFA, in which the set of atoms (urelements) has the structure of a vector space over the finite field $\mathbb{F}_p$. The use of atoms is then eliminated by citing an embedding theorem of Pincus. In the case where m is not prime, suitable permutation models are built from the models used in the prime cases.

1. Introduction

Let $\mathrm{C}(\aleph_0, m)$ be the statement asserting that every infinite, countable set of m-element sets has a choice function. Let $\mathrm{PC}(\aleph_0, m)$ be the statement asserting that every infinite, countable set C of m-element sets has an infinite partial choice function (i.e. a choice function whose domain is an infinite subset of C), and let $\mathrm{PC}(\aleph_0, \leq m)$ denote "$\forall n \leq m\ \mathrm{PC}(\aleph_0, n)$." ($\mathrm{C}(\aleph_0, m)$ is Form 288(m), and $\mathrm{PC}(\aleph_0, m)$ is Form 373(m) in Howard and Rubin's reference [HR]. Also, $\mathrm{C}(\aleph_0, 2)$ is Form 30, and $\mathrm{PC}(\aleph_0, 2)$ is Form 18.) The main result of this paper is that for any integer $m \geq 2$, $\mathrm{PC}(\aleph_0, m)$ does not imply $\mathrm{C}(\aleph_0, m)$ in ZF. This answers questions left open in [DHHKR]. The proof of the main result will in fact show that the statement "$\forall n \in \omega$ $\mathrm{PC}(\aleph_0, \leq n)$" does not imply $\mathrm{C}(\aleph_0, m)$ in ZF.

The independence results are obtained using the technique of permutation models (also known as Fraenkel-Mostowski models). See Jech [J] for basics about permutation models and the theory ZFA (ZF modified to allow atoms). A suitable permutation model will establish the independence of $\mathrm{C}(\aleph_0, m)$ from $\mathrm{PC}(\aleph_0, m)$ in the context of ZFA. This suffices by work of Pincus in [P] (extending work of Jech and Sochor), which shows that once established under ZFA, the independence result transfers to the context

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of ZF (this is because the statement "$\forall n \in \omega$ PC($\aleph_0$, $\leq n$)" is *injectively boundable*; see [P] or Note 103 in [HR]).

The proof of the independence of C($\aleph_0$, m) from PC($\aleph_0$, m) will be broken into two sections. Section 2 is the proof of the independence result in the special case where m is prime (Theorem 2.1), and includes the deeper ideas of this paper. In Section 3, it will be shown how the general result (Theorem 3.4) follows from Theorem 2.1.

Readers with some experience with permutation models may wonder whether the model used in the proof of Theorem 2.1 is unnecessarily complicated. Section 4 explains why certain simpler models which may appear promising candidates to witness the independence of PC($\aleph_0$, 2) from C($\aleph_0$, 2) in fact fail to do so.

2. The main theorem, prime case

Theorem 2.1. *Let p be a prime integer. If ZF is consistent, then there is a model of ZF in which C($\aleph_0$, p) is false, but in which PC($\aleph_0$, $\leq n$) holds for every $n \in \omega$. (In particular, PC($\aleph_0$, p) does not imply C($\aleph_0$, p) in ZF.)*

Proof. As discussed in the Introduction, it suffices describe a permutation model in which $(\forall n \in \omega)$ PC($\aleph_0$, $\leq n$) holds and C($\aleph_0$, p) fails. Let $\mathcal{M}$ be a model of ZFAC whose set of atoms is countable and infinite; we will work in $\mathcal{M}$ unless otherwise specified. We will describe a permutation submodel of $\mathcal{M}$.

First, we set some notation for a few vector spaces over the field $\mathbb{F}_p$ with p elements. Let $W = \oplus_{i\in\omega}\mathbb{F}_p$, so each element of W is a sequence $w = (w_0, w_1, w_2, \dots)$ of elements of $\mathbb{F}_p$, with at most finitely many nonzero terms. For each $i \in \omega$, let $e_i \in W$ be the sequence such that $e_i(k) = 1$ when $k = i$ and $e_i(k) = 0$ otherwise, so $\{ e_i : i \in \omega \}$ is the canonical basis for W. Let G be the full product $\otimes_{i\in\omega}\mathbb{F}_p$ (sequences may have infinitely many nonzero elements). Finally, let $U = \mathbb{F}_p \times W$, so each element of U is a pair (a, w) with $a \in \mathbb{F}_p$, $w \in W$.

For each $w \in W$, let $U_w = \{ (a, w) : a \in \mathbb{F}_p \}$, so that $\mathcal{P} = \{ U_w : w \in W \}$ is a partition of U into sets of size p. Thinking of G as an abelian group, we define a G-action as follows, such that each $g \in G$ gives an automorphism of U, and such that the G-orbits are the elements of the partition $\mathcal{P}$ (except for $U_{\mathbf{0}}$, whose members will be fixed points). For each $(a, w) \in U$ and $g \in G$, let

$$(a, w)g = (a + \sum_{i\in\omega} w_i g_i, w)$$

(where w_i is the i^{th} entry in the sequence w, and likewise g_i; the product $w_i g_i$ is in the field $\mathbb{F}_p$, and the sum $a + \sum_i w_i g_i$ is a (finite) sum in $\mathbb{F}_p$). This action induces an isomorphism of G with a subgroup of $\mathrm{Aut}(U)$; we will henceforth identify G with this subgroup, think of the operation on G as composition instead of addition, and continue to let G act on the right.

Remark. It is clear from the given definition of G that G is abelian, and all its non-identity elements have order p. As a subgroup of $\mathrm{Aut}(U)$, G may be characterized as the group of all automorphisms of U which act on each element of the partition $\mathcal{P}$ and have order p or 1. Equivalently, G is the group of all automorphisms of U which act on each element of $\mathcal{P}$ and act trivially on $U_{\mathbf{0}}$.

Now, identify the set of atoms in $\mathcal{M}$ with the vector space U. Thus, we think of each g in G as a permutation of the set of atoms. Each permutation of U extends uniquely to an automorphism of $\mathcal{M}$, and so we will also think of G as a subgroup of $\mathrm{Aut}(\mathcal{M})$.

Let $\mathcal{I}$ be a (proper) ideal on W such that

($*1$) every infinite subset of W contains an infinite member of $\mathcal{I}$, and
($*2$) $A \in \mathcal{I} \Rightarrow \mathrm{Span}(A) \in \mathcal{I}$,

where $\mathrm{Span}(A)$ is the $\mathbb{F}_p$-vector subspace of U generated by A. For proof of the existence of such an ideal, see Lemma 2.4.

Notation and definitions regarding stabilizers and supports. For $A \subset W$ and $g \in G \subset \mathrm{Aut}(\mathcal{M})$, we say "*$g$ fixes at A*" if g fixes each atom in $\mathbb{F}_p \times A = \bigcup_{w \in A} U_w$. Let $G_{(A)}$ denote the subgroup of G consisting of elements which fix at A (i.e., $G_{(A)}$ is the pointwise stabilizer of $\bigcup_{w \in A} U_w$). If G' is a subgroup of G, then $G'_{(A)} = G' \cap G_{(A)}$. For $x \in \mathcal{M}$, we say that A *supports* x if $xg = g$ for each $g \in G$ which fixes at A, and x is *symmetric* if x has a support which is a member of $\mathcal{I}$.

Let $\mathcal{N}$ be the permutation model consisting of hereditarily symmetric elements of $\mathcal{M}$. Note that the empty set supports the partition $\mathcal{P}$ of U described above, and also supports any well-ordering of $\mathcal{P}$ in $\mathcal{M}$. So in $\mathcal{N}$, $\mathcal{P}$ is a countable partition of the set U of atoms into sets of size p. However, no choice function for $\mathcal{P}$ has a support in $\mathcal{I}$, and so $\mathcal{N} \models \neg\mathrm{C}(\aleph_0, p)$.

Remark. (1) Note, by ($*2$) above, that A supports x if and only if $\mathrm{Span}(A)$ supports x, and thus A supports x if and only if any basis for $\mathrm{Span}(A)$ supports x.

(2) Suppose A is a support for $x \in \mathcal{N}$, and suppose $g, h \in G$ are such that for all $u \in \mathbb{F}_p \times A$, $ug = uh$. Then also $xg = xh$. (This is by a typical argument about supports in permutation models.)

We now want to show that $\mathcal{N} \models (\forall n \in \omega)\, \mathrm{PC}(\aleph_0, \leq n)$. We first establish a couple of lemmas about supports of elements of $\mathcal{N}$.

Lemma 2.2. *Suppose $A \in \mathcal{I}$ and $x \in \mathcal{N}$. Either there is a finite set $B \subset W$ such that $B \cup A$ supports x, or the $G_{(A)}$-orbit of x is infinite.*

Proof. We give a forcing argument similar to one used in Shelah [S]. We set up a notion of forcing $\mathbf{Q}$ which adds a new automorphism of U like those found in $G_{(A)}$. Assume A is a subspace of W (without loss of generality, by property ($*2$) of the ideal $\mathcal{I}$). Let $A^{\perp}$ be a subspace of W complementary to A (i.e., $\mathrm{Span}(A \cup A^{\perp}) = W$ and $A \cap A^{\perp} = \{\mathbf{0}\}$), and fix a basis $\{\, w_i : i \in \omega \,\}$ for $A^{\perp}$. Conditions of $\mathbf{Q}$ shall have the following form: For any $n \in \omega$ and function $f\colon n \to \mathbb{F}_p$, let q_f be the unique automorphism of $\mathbb{F}_p \times \mathrm{Span}\{w_0, \dots w_{n-1}\} \subset U$ which fixes each U_{w_i} and maps $(0, w_i)$ to $(f(i), w_i)$. As usual, for conditions $q_1, q_2 \in \mathbf{Q}$, we let $q_1 \leq q_2$ iff $q_2 \subseteq q_1$. Thus, if $\Gamma \subset \mathbf{Q}$ is a generic filter, then $\pi = \bigcup \Gamma$ is an automorphism of $A^{\perp}$ preserving the partition $\mathcal{P}$. Easily, π extends uniquely to an automorphism of U fixing at A and preserving the partition $\mathcal{P}$, and thus we will think of such π as being an automorphism of U. Observe that $\mathbf{Q}$ is equivalent to Cohen forcing (the way we have associated each condition with a finite sequence of elements of $\mathbb{F}_p$, it is easy to think of $\mathbf{Q}$ as just adding a Cohen generic sequence in ${}^{\omega}\mathbb{F}_p$). Let $\dot{\pi}$ be a canonical name for the automorphism added by $\mathbf{Q}$. Let $(\mathbf{Q}_1, \dot{\pi}_1)$ and $(\mathbf{Q}_2, \dot{\pi}_2)$ each be copies of $(\mathbf{Q}, \dot{\pi})$.

Case 1: For some $(q_1, q_2) \in \mathbf{Q}_1 \times \mathbf{Q}_2$, $(q_1, q_2) \Vdash \check{x}\dot{\pi}_1 = \check{x}\dot{\pi}_2$.

Let $B \subset W$ be some finite support for q_1; e.g. $B = \{\, w \in W : (\exists n \in \mathbb{F}_p)\ (n, w) \in \mathrm{Dom}(q_1) \cup \mathrm{Range}(q_1) \,\}$. Let $\Gamma \subset \mathbf{Q}_1 \times \mathbf{Q}_2$ be generic over $\mathcal{M}$ with $(q_1, q_2) \in \Gamma$, and let (π_1, π_2) be the interpretation of $(\dot{\pi}_1, \dot{\pi}_2)$ in $\mathcal{M}[\Gamma]$. For any $g \in G_{(A \cup B)}$, $(g\pi_1, \pi_2)$ is another $\mathbf{Q}_1 \times \mathbf{Q}_2$-generic pair of automorphisms. Let $\Gamma_g \subset \mathbf{Q}_1 \times \mathbf{Q}_2$ such that $(g\pi_1, \pi_2)$ is the interpretation of $(\dot{\pi}_1, \dot{\pi}_2)$ in $\mathcal{M}[\Gamma_g]$.

Note that (q_1, q_2) is in both Γ and Γ_g, so $\mathcal{M}[\Gamma] \models x\pi_1 = x\pi_2$, and $\mathcal{M}[\Gamma_g] \models xg\pi_1 = x\pi_2$. Thus, $x\pi_1 = xg\pi_1$ (if desired, one can briefly reason in an extension which contains both Γ and Γ'), and it follows that $x = xg$.

We have shown that every $g \in G_{(A \cup B)}$ fixes x, which is to say that $A \cup B$ supports x, which completes the proof for Case 1.

Case 2: $\Vdash_{\mathbf{Q}_1 \times \mathbf{Q}_2} \check{x}\dot{\pi}_1 \neq \check{x}\dot{\pi}_2$.

Let $\mathcal{H}(\kappa)$ be the set consisting of sets that are hereditarily of cardinality smaller than κ, where $\kappa > 2^{\aleph_0}+|\mathrm{TC}(x)|$, and let C be a countable elementary submodel of $\mathcal{H}(\kappa)$ with $x \in C$. It is clear that there exist infinitely many elements of $G_{(A)}$ which are mutually $\mathbf{Q}$-generic over C, and in fact there is perfect set such elements by [S] (specifically, Lemma 13, applied to the equivalence relation $\mathcal{E}$ on $G_{(A)}$ defined by $\pi_1 \; \mathcal{E} \; \pi_2 \leftrightarrow x\pi_1 = x\pi_2$). More precisely, there is a perfect set $P \subset G_{(A)}$ such that for each $\pi_1, \pi_2 \in P$, (π_1, π_2) is $\mathbf{Q}_1 \times \mathbf{Q}_2$-generic over C. Thus $x\pi_1 \neq x\pi_2$ whenever $\pi_1, \pi_2 \in P$, and hence, the $G_{(A)}$-orbit of x is infinite. $\square$

Lemma 2.3. *Let $X \in \mathcal{N}$ with $|X| = n \in \omega$, and let $A \in \mathcal{I}$ be a support for X. Then for each $x \in X$, there exists some $C \subset W$ such that $|C| \leq n!$ and $A \cup C$ supports x.*

Proof. Let $\{e_0, e_1, e_2, \dots\}$ be the canonical basis for W, and for each $n \in \omega$ let $W_n = \mathrm{Span}\{e_0, \dots, e_{n-1}\}$. Let $x \in X$. Since A supports X, the $G_{(A)}$-orbit of x is contained in X, and hence is finite. By Lemma 2.2, there is a finite $B \subset W$ such that $A \cup B$ supports x. Fix N such that $B \subseteq W_N$. Let pr be the canonical projection from G to W_N, $\prod_{i\in\omega} a_i e_i \mapsto \sum_{i\in N} a_i e_i$, but restricted to the domain $G_{(A)}$. Let R be the image $\{\, \mathrm{pr}(g) \;:\; g \in G_{(A)} \,\}$, so $\mathrm{pr}\colon G_{(A)} \to R$ is a surjective map.

The action of $G_{(A)}$ on X induces a group homomorphism $\phi\colon G_{(A)} \to \mathrm{Sym}(X)$ such that if $\mathrm{pr}(g) = \mathrm{pr}(h)$, then g and h act the same way on $\mathbb{F}_p \times B$, and hence $xg = xh$. Thus the formula $\phi^*(\mathrm{pr}(g)) = \phi(g)$ gives a well-defined injective homomorphism $\phi^*\colon R \to \mathrm{Sym}(X)$. Let $K = \ker(\phi^*)$ and let C be an orthogonal complement to K in R (so that $R = K \oplus C$).

Observe $|C| = |R/K| = |\,\mathrm{Image}(\phi^*)| \leq |\,\mathrm{Sym}(X)| = n!$. It remains to check that $A \cup C$ supports x. Let $g \in G_{(A\cup C)}$. Then $\mathrm{pr}(g) = k + b$ for some $k \in K$, $b \in C$. Since g fixes at C and $C \subseteq W_N$, also $\mathrm{pr}(g)$ fixes at C and hence $b = 0$. The $\mathrm{pr}(g) \in K$, which means $\phi^*(\mathrm{pr}(g)) = \phi(g)$ is the identity element in $\mathrm{Sym}(X)$, so $xg = x$.

(*Remark:* The bound $n!$ can be improved easily, firstly by observing that C is isomorphic to an abelian subgroup of $\mathrm{Sym}(X)$, which must have cardinality quite smaller than $n!$ if $n > 2$, and secondly by replacing the subspace C with a basis for C.) $\square$

Now, to show $\mathcal{N} \models (\forall n \in \omega)\, \mathrm{PC}(\aleph_0, \leq n)$, fix $n \in \omega$, and let $Z = \{\, X_j \;:\; j \in \omega \,\}$ be a set of sets each of cardinality $\leq n$, with Z countable in $\mathcal{N}$. Let $A \in \mathcal{I}$ be a support for a well-ordering of Z, so that A is a support for each element of Z. For each $j \in \omega$, let $x_j \in X_j$ (of course, Z might not have a choice function in $\mathcal{N}$, but we are working in $\mathcal{M}$). By Lemma 2.3, since $|X_j| \leq n$, there is some $C_j \subset W$ such that $A \cup C_j$ supports x_j, and such that $|C_j| < n!$ for each j. Let $S = \bigcup_{j\in\omega} C_j$. If S is finite, then $A \cup S \in \mathcal{I}$, and $A \cup S$ is a support for the enumeration $\langle x_j \rangle_{j\in\omega}$, so in fact Z has a choice function in $\mathcal{N}$.

In case S is infinite, then we claim there exists some $D \in \mathcal{I}$ such that that $D \supset C_j$ for infinitely many j. To find this D, apply property $(*1)$ of the ideal $\mathcal{I}$, repeated $n!$ times: Let $D_1 \in \mathcal{I}$ be an infinite subset of S (which exists by $(*1)$), and let $J_1 = \{\, j \in \omega : C_j \cap D_1 \neq \emptyset \,\}$. Proceding recursively, let $D_{k+1} \in \mathcal{I}$ be an infinite subset of $\big(\bigcup_{j\in J_k} C_j\big) \smallsetminus \big(\bigcup_{k'\leq k} D_{k'}\big)$, if any exists, and $D_{k+1} = D_k$ otherwise. Let $J_{k+1} = \{\, j \in \omega : C_j \cap D_{k+1} \neq \emptyset \,\}$. Then $D = \bigcup_{k\leq n!} D_k$ has the required properties. It follows that $A \cup D$ supports an infinite subsequence of $\langle x_j \rangle_{j\in\omega}$, so Z has an infinite partial choice function in $\mathcal{N}$. □

It remains in this section to establish the existence of an ideal on $W = \oplus_{i\in\omega}\mathbb{F}_p$ having the properties needed in the proof of Theorem 2.1.

Notation and definitions.

1. For $n \in \omega \smallsetminus \{0\}$, let $\log_*(n)$ be the least $k \in \omega$ such that $(\log)^k(n) \leq 1$, where $(\log)^0(n) = n$ and $(\log)^{k+1}(n) = \log\big((\log)^k(n)\big)$.
2. Let $\{\, e_k : k \in \omega \,\}$ be the canonical basis for $W = \oplus_{i\in\omega}\mathbb{F}_p$.
3. For $w = \sum_{\ell} a_\ell e_\ell \in W$, let $\mathrm{pr}_k(w) = \sum_{\ell<k} a_\ell e_\ell$.
4. $d_k(A) = |\{\, \mathrm{pr}_k(w) : w \in A \,\}|$.
5. We say $A \subset W$ is *thin* if
$$\lim_{k\to\infty} \frac{\log_*[d_k(A)]}{\log_*(k)} = 0.$$

Lemma 2.4. *Let $\mathcal{I}$ be the set of thin subsets of W. Then*

(0) $\mathcal{I}$ is an ideal on W,

(1) every infinite subset of W contains an infinite member of $\mathcal{I}$, and

(2) $A \in \mathcal{I} \Rightarrow \mathrm{Span}(A) \in \mathcal{I}$.

Proof. (0) Clearly $\mathcal{I}$ is closed under subsets. Suppose A_1 and A_2 are thin, and let $A = A_1 \cup A_2$. Then (for any $k \in \omega$) $d_k(A) \leq d_k(A_1) + d_k(A_2)$, so
$$\frac{\log_*[d_k(A)]}{\log_*(k)} \leq \frac{\log_*[d_k(A_1) + d_k(A_2)]}{\log_*(k)} \leq \frac{1 + \max_{i=1,2}\left(\log_*[d_k(A_i)]\right)}{\log_*(k)}.$$
The limit as $k \to \infty$ must be 0, so A is thin.

(1) Let $A \subseteq W$ be an infinite set. By König's Lemma, we can find pairwise distinct $x_n \in A$ for $n \in \omega$ such that for each $i \in \omega$, $\langle x_n(i) \rangle_{n<\omega}$ is eventually constant.

Let $n_0 = 0$. For $i \in \omega$, assuming $n_0, \dots n_i$ are chosen, we can choose n_{i+1} large enough so that

$$\begin{aligned} \mathrm{pr}_{n_i}(x_{n_{i+1}}) &= \mathrm{pr}_{n_i}(x_t) \quad \text{for all } t \geq n_{i+1}, \\ \text{and} \quad \log_*(n_{i+1}) &> i+1. \end{aligned}$$

Let $A^- = \{ x_{n_i} : i \in \omega \}$. Then $d_{n_i}(A^-) \leq i+1$, and

$$\lim_{i\to\infty} \frac{\log_*(d_{n_i}(A^-))}{\log_*(n_i)} \leq \lim_{i\to\infty} \frac{\log_*(i+1)}{i} = 0.$$

Therefore A^- is an infinite, thin subset of A.

(2) For any $A \subset W$, observe that

$$d_k(\mathrm{Span}\, A) \leq p^{d_k(A)}.$$

Thus

$$\log_*(d_k(\mathrm{Span}\, A)) \;\leq\; \log_*\left(p^{d_k(A)}\right) \;\leq\; c + \log_*(d_k(A)),$$

where c is constant (e.g. $c = \log_* p$). It follows easily that if A is thin, then Span A is also thin. □

Everything needed for Theorem 2.1 has now been proven.

3. The main theorem, general case

In this section, we will show how the main theorem follows from Theorem 2.1. We first describe a general approach to making new permutation models from old ones.

Notation and definitions. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be models of ZFAC with the same pure part. For $i \in \{1,2\}$, let U_i be the set of atoms in $\mathcal{M}_i$, and assume $U_1 \cap U_2 = \emptyset$. Let G_i be a group of permutations of U_i, and let $\mathcal{I}_i$ be an ideal on U_i. Let $\mathcal{N}_i$ be the permutation submodel of $\mathcal{M}_i$ defined from G_i and the ideal of supports $\mathcal{I}_i$. (More precisely: a set $A \in \mathcal{I}_i$ *supports* $x \in \mathcal{M}_i$ if $xg = x$ whenever $g \in G_i$ and $Ag = A$. The *symmetric*$_i$ elements of $\mathcal{M}_i$ are those with supports in $\mathcal{I}_i$, and $\mathcal{N}_i$ is the class of hereditarily symmetric$_i$ members of $\mathcal{M}_i$.)

The *sum* $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ is defined as follows. Let $\mathcal{M}$ be a model of ZFAC with the same pure part as $\mathcal{N}_1$ and $\mathcal{N}_2$, and whose set of atoms is $U = U_1 \cup U_2$ (assuming U_1 and U_2 are disjoint). The group $G = G_1 \times G_2$ acts on U as follows: For $u \in U$ and $g = (g_1, g_2) \in G$, if $u \in U_i$ then $ug = ug_i$. Let $\mathcal{I}$ be the ideal on U generated by $\mathcal{I}_1 \cup \mathcal{I}_2$, and let $\mathcal{N}$ be the permutation submodel defined from G and $\mathcal{I}$.

We define two more permutation submodels of $\mathcal{M}$. The action of G_1 on $U_1 \subset U$ can be considered an action on U that happens to fix every element of U_2. Let $\tilde{\mathcal{N}}_1$ be the permutation submodel of $\mathcal{M}$ defined from G_1 and $\mathcal{I}_1$. (Observe $\mathcal{N}_1 = \tilde{\mathcal{N}}_1 \cap \mathcal{M}_1$, and that U_2 is well-orderable in $\tilde{\mathcal{N}}_1$.) Likewise, let $\tilde{\mathcal{N}}_2$ be the permutation submodel of $\mathcal{M}$ defined from G_2 and $\mathcal{I}_2$.

Lemma 3.1. *Given permutation models $\mathcal{N}_1$ and $\mathcal{N}_2$ as above, we have $\mathcal{N}_1 \oplus \mathcal{N}_2 = \tilde{\mathcal{N}}_1 \cap \tilde{\mathcal{N}}_2$.*

Proof. Let $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$. We first check that $\mathcal{N} \subseteq \tilde{\mathcal{N}}_1 \cap \tilde{\mathcal{N}}_2$ by induction on rank. Suppose $x \in \mathcal{N}$ and $x \subset \tilde{\mathcal{N}}_1 \cap \tilde{\mathcal{N}}_2$. Then x is supported by some $A = A_1 \cup A_2 \in \mathcal{I}$, with $A_1 \in \mathcal{I}_1$ and $A_2 \in \mathcal{I}_2$. But then in the action of G_1 on $\mathcal{M}$, we have $xg = x$ for every $g \in G_{1(A_1)}$, so A_1 is a support witnessing that $x \in \tilde{\mathcal{N}}_1$. Likewise, $x \in \tilde{\mathcal{N}}_2$, so $\mathcal{N} \subseteq \tilde{\mathcal{N}}_1 \cap \tilde{\mathcal{N}}_2$. The opposite inclusion is proved easily using the same ideas. $\square$

Theorem 3.2. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be permutation models with the same pure part, and let $\mathcal{N}$ be the sum $\mathcal{N}_1 \oplus \mathcal{N}_2$. If $\mathcal{N}_1$ and $\mathcal{N}_2$ both satisfy $\forall n \in \omega$ $PC(\aleph_0, \leq n)$, then so does $\mathcal{N}$.*

Proof. First, observe that the statement "$\forall n \in \omega$ PC$(\aleph_0, \leq n)$" is equivalent in ZFA to the following statement:

> $(*)$ For every $n \in \omega$, given a countable set $\{ X_j : j \in \omega \}$ of sets of cardinality at most n, there is an infinite $J \subset \omega$ such that $\bigcup_{j\in J} X_j$ is well-orderable.

Now fix $n \in \omega$ and let $Z = \{ X_j : j \in \omega \} \in \mathcal{N}$ be such that $|X_j| \leq n$ for all $j \in \omega$, and such that Z is countable in $\mathcal{N}$. By Lemma 3.1, $Z \in \tilde{\mathcal{N}}_1 \cap \tilde{\mathcal{N}}_2$. It is clear that since the statement $(*)$ holds in $\mathcal{N}_1$ and $\mathcal{N}_2$, it also holds in $\tilde{\mathcal{N}}_1$ and $\tilde{\mathcal{N}}_2$. Working in $\tilde{\mathcal{N}}_1$, by $(*)$ there is an infinite $J_1 \subset \omega$ and a support $A_1 \in \mathcal{I}_1$ for a well-ordering of the set $Z_1 = \bigcup_{j\in J_1} X_j$; that is A_1 supports every element of Z_1 (with respect to the action of G_1 on $\mathcal{M}$). But the countable family $\{ X_j : j \in J_1 \}$ is a member of $\tilde{\mathcal{N}}_2$ ($J_1 \in \tilde{\mathcal{N}}_2$ since $\tilde{\mathcal{N}}_1$ and $\tilde{\mathcal{N}}_2$ have the same subsets of ω), so working in $\tilde{\mathcal{N}}_2$, there exist an infinite $J_2 \subseteq J_1$ and an $A_2 \in \mathcal{I}_2$ such that A_2 supports every element of the set $Z_2 = \bigcup_{j\in J_2} X_j$.

Note that $Z_2 \in \mathcal{N}$. It now suffices to show that the set $A_1 \cup A_2 \in \mathcal{I}$ is a support for every element of Z_2 with respect to the action of G on $\mathcal{M}$. Let $g = (g_1, g_2) \in G_{(A_1\cup A_2)}$, and let $z \in Z_2$. Then $g_1 \in G_{1(A_1)}$ and $g_2 \in G_{2(A_2)}$, so that $zg_1 = z$ and $zg_2 = z$ (in the actions of G_1 and G_2 on $\mathcal{M}$), and it follows that $zg = z(g_1, g_2) = z$. $\square$

Theorem 3.3. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be permutation models with the same pure part, and let $\mathcal{N}$ be the sum $\mathcal{N}_1 \oplus \mathcal{N}_2$. Let $m_1, m_2 \in \omega$. If $\mathcal{N}_1 \models \neg C(\aleph_0, m_1)$ and $\mathcal{N}_2 \models \neg C(\aleph_0, m_2)$, then $\mathcal{N} \models \neg C(\aleph_0, m_1 m_2)$.*

Proof. It is straightforward to see that $\mathcal{N} \models \neg\mathrm{C}(\aleph_0,\, m_1)$ and $\neg\mathrm{C}(\aleph_0,\, m_2)$, so there are countable families $\{\, X_j \,:\, j \in \omega \,\}$ and $\{\, Y_j \,:\, j \in \omega \,\}$ in $\mathcal{N}$ with no choice functions, such that $|X_j| = m_1$ and $|Y_j| = m_2$ for each $j \in \omega$. Then $\{\, X_j \times Y_j \,:\, j \in \omega \,\} \in \mathcal{N}$ must not have a choice function in $\mathcal{N}$, so $\mathcal{N} \models \neg\mathrm{C}(\aleph_0,\, m_1 m_2)$. □

All the work is now essentially done for the proof of the main theorem, stated here:

Theorem 3.4. *Let m be an integer, $m \geq 2$. If ZF is consistent, then there is a model of ZF in which $C(\aleph_0,\, m)$ is false, but in which $PC(\aleph_0,\, \leq n)$ holds for every $n \in \omega$. (In particular, $PC(\aleph_0,\, m)$ does not imply $C(\aleph_0,\, m)$ in ZF.)*

Proof. Let $m = \prod_j p_j$ be the prime factorization of m. For each j, let $\mathcal{N}_j$ be the permutation model described in the proof of Theorem 2.1 for the prime $p = p_j$. Apply Theorems 3.2 and 3.3 to the sum $\oplus_j \mathcal{N}_j$ to obtain the desired independence result, in ZFA. The result transfers from ZFA to ZF by Pincus' embedding theorems, as described in the introduction. □

4. Simpler models not useful for the main theorem

We consider a family of permutation models, some of which may on first consideration seem to be promising candidates to witness that $\mathrm{PC}(\aleph_0,\, 2) \not\rightarrow \mathrm{C}(\aleph_0,\, 2)$. However, it will turn out that $\mathrm{PC}(\aleph_0,\, 2)$ fails in every such model.

Let $\mathcal{M}$ be a model of ZFAC whose set U of atoms is countable and infinite. Let $\mathcal{P} = \{\, U_n \,:\, n \in \omega \,\}$ be a partition of U into pairs. Let G be the group of permutations of U (equivalently, automorphisms of $\mathcal{M}$) which fix each element of $\mathcal{P}$. Let $\mathcal{I}$ be some ideal on ω. For $A \in \mathcal{I}$ and $g \in G$, we say g *fixes at* A if g fixes each element of $\bigcup_{n \in A} U_n$. Define *support* and *symmetric* by analogy with the definitions of these terms in the proof of the main theorem, and let $\mathcal{N}$ be the permutation submodel consisting of the hereditarily symmetric elements.

If $\mathcal{I}$ is the ideal of finite subsets of ω, then $\mathcal{N}$ is the "second Fraenkel model." Clearly $\mathcal{P}$ has no infinite partial choice function in the second Fraenkel model. Of course, if $\mathcal{I}$ is any larger than the finite set ideal, then $\mathcal{P}$ does have an infinite partial choice function, and it may be tempting to think that if $\mathcal{I}$ is well-chosen, then perhaps $\mathrm{PC}(\aleph_0,\, 2)$ will hold in the resulting model $\mathcal{N}$. However, we will show how to produce a set $Z = \{\, X_n \,:\, n \in \omega \,\}$ of pairs, countable in $\mathcal{N}$, with no infinite partial choice function (no matter how $\mathcal{I}$ is chosen).

Notation: For sets A and B, let $P(A, B)$ be the set of bijections from A to B. We are interested in this when A and B are both pairs, in which case $P(A, B)$ is also a pair.

Let $X_0 = A_0$. For $i \in \omega$, let $X_{i+1} = P(X_i, A_{i+1})$. The empty set supports each pair X_i, so $Z = \{ X_n : n \in \omega \}$ is a countable set in $\mathcal{N}$. Let $S \in \mathcal{I}$; we will show that S fails to support any infinite partial choice function for Z. Let $i = \min(\omega \setminus S)$, and let $g \in G$ be the permutation which swaps the elements of A_i and fixes all other atoms, so $g \in G_{(S)}$. This g fixes each element of X_n for $n < i$, but swaps the elements of X_i. By simple induction, g also swaps the elements of X_n for all $n > i$. It follows that for any $C \in \mathcal{M}$ which is an infinite partial choice function for Z, $Cg \neq C$, and thus S does not support C.

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