MAD SATURATED FAMILIES AND SANE PLAYER
SH935

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Abstract. We throw some light on the question: is there a MAD family (= a maximal family of infinite subsets of $\mathbb{N}$, the intersection of any two is finite) which is saturated (= completely separable i.e. any $X \subseteq \mathbb{N}$ is included in a finite union of members of the family or includes a member (and even continuum many members) of the family). We prove that it is hard to prove the consistency of the negation:

(a) if $2^{\aleph_0} < \aleph_\omega$, then there is such a family

(b) if there is no such family then some situation related to pcf holds whose consistency is large; and if $\mathfrak{a}_* > \aleph_1$ even unknown

(c) if, e.g. there is no inner model with measurables then there is such a family.

0. Introduction

We try to throw some light on the following problem:

Problem 0.1. Is there, provably in ZFC, a completely separable MAD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$, see Definition 0.3(1),(4).

Erdös-Shelah [ES72] investigates the ZFC-existence of families $\mathcal{A} \subseteq \mathcal{P}(\omega)$ with separability properties, continuing Hechler [Hec71] which mostly uses MA; now 0.1 is Problem A of [ES72], pg.209, see earlier Miller [Mil37], and see later Goldstern-Judah-Shelah [GJS91] on existence for larger cardinals. It seemed natural to prove the consistency of a negative answer by CS iteration making the continuum $\aleph_2$ but this had not worked out; the results here show this is impossible.

The celebrated matrix-tree theorem of Balcar-Pelant-Simon [BPS80], Balcar-Simon [BS89] is related to our starting point. In Gruenhut-Shelah [GS] we try to generalize it, hoping eventually to get applications, e.g. "there is a subgroup of ${}^\omega\mathbb{Z}$ which is reflexive (i.e. canonically isomorphic to the dual of its dual)" and "less", see Problem D7 of [EM02], no success so far. We then had tried to use such constructions to answer 0.1 positively, but this does not work. Simon [CK96] have proved (in ZFC), that there is an infinite almost disjoint $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ such that $B \subseteq \omega$ and $(\exists^\infty A \in \mathcal{A})[B \cap A \text{ infinite}] \Rightarrow (\exists A \in \mathcal{A})(A \subseteq B)$. Shelah-Steprans [SS11] try to continue it with dealing with Hilbert spaces.

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Here $\mathfrak{s}$ and ideals (formally $J \in \mathrm{OB}$) are central. Originally we have a unified proof using games between the MAD and the SANE players but with some parameters for the properties (why SANE? it makes sense that the opponent of MAD is SANE). As on the one hand it was claimed this is unreadable and on the other hand we have a direct proof, which was presented (for $\mathfrak{s} < \mathfrak{a}_*$), in the Hebrew University and Rutgers, we use the later one. A minor price is that the proof in §2 are saying - repeat the earlier one with the following changes. The major price is that some information is lost: using smaller more complicated cardinal invariants as well as some points in the proof which we hope will serve other proofs (including covering all cases) so we shall return to the main problem and relatives in [S$^+$] which continue this work.

A related problem of Balcar and Simon is: given a MAD family $\mathcal{B}$ we look for such $\mathcal{A}$ refining it, i.e. $(\forall B \in \mathrm{id}^+_{\mathcal{A}})(\exists A \in \mathcal{A})(A \subseteq^* B)$. At present there is no difference between the two problems (i.e. in 1.1, 2.1, 2.6 we cover this too)

Anyhow

Conclusion 0.2. 1) If $2^{\aleph_0} < \aleph_\omega$ <u>then</u> there is a saturated MAD family.
2) Moreover in (1) for any dense $J_* \subseteq [\omega]^{\aleph_0}$ we can find such a family $\subseteq J_*$.

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Definition 0.3. 1) We say $\mathcal{A}$ is an AD (family) for B when $\mathcal{A} \subseteq [B]^{\aleph_0}$ is infinite, almost disjoint (i.e. $A_1 \neq A_2 \in \mathcal{A} \Rightarrow A_1 \cap A_2$ finite). We say $\mathcal{A}$ is MAD for B <u>when</u> $\mathcal{A}$ is AD for B and is $\subseteq$-maximal among such $\mathcal{A}$'s.
2) If $B = \omega$ we may omit it.
3) For $\mathcal{A} \subseteq [\omega]^{\aleph_0}$, $\mathrm{id}_{\mathcal{A}}$ is the ideal generated by $\mathcal{A} \cup [\omega]^{<\aleph_0}$.
4) A MAD family $\mathcal{A}$ is saturated when: if $B \in \mathrm{id}^+_{\mathcal{A}}$ (see 0.7(3)) then B almost contains some member of $\mathcal{A}$ (equivalently: if $B \in \mathrm{id}^+_{\mathcal{A}}$ <u>then</u> B almost contains continuum many members of A because if $B \in \mathrm{id}^+_{\mathcal{A}}$ then there is an AD family $\mathcal{B} \subseteq [B]^{\aleph_0} \cap \mathrm{id}^+_{\mathcal{A}}$ of cardinality $2^{\aleph_0}$).

Definition 0.4.
(1) Let $\mathfrak{a}$ be the minimal cardinality of a MAD family
(2) Let $\mathfrak{a}_*$ be the minimal κ such that there is a sequence $\langle A_\alpha : \alpha < \kappa + \omega \rangle$ of pairwise almost disjoint (=with finite intersection) infinite subsets of ω satisfying:
there is no infinite set $B \subseteq \omega$ almost disjoint to A_α for $\alpha < \kappa$ but $B \cap A_{\kappa+n}$ is infinite for infinitely many n-s.

Observation 0.5. We have $\mathfrak{b} \leq \mathfrak{a}_* \leq \mathfrak{a}$.

Remark 0.6. 1) Note that if there is a MAD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ such that $B \in \mathrm{id}^+_{\mathcal{A}} \Rightarrow (\exists^{2^{\aleph_0}} A \in \mathcal{A})$ ($B \cap A$ is infinite), <u>then</u> there is a MAD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ such that $B \in \mathrm{id}^+_{\mathcal{A}} \Rightarrow (\exists^{2^{\aleph_0}} A \in \mathcal{A})(A \subseteq B)$ equivalently $B \in \mathrm{id}^+_{\mathcal{A}} \Rightarrow (\exists A \in \mathcal{A})(A \subseteq B)$; just list our tasks and fulfil them by dividing each member of $\mathcal{A}$ to two infinite sets to fulfil one task.
2) So the four variants of "there is $\mathcal{A} \ldots$" in 0.3(4), 0.6(1) are equivalent.

Notation 0.7. 1) For $A \subseteq \omega$ let $A^{[\ell]}$ be A if $\ell = 1$ and $\omega \backslash A$ if $\ell = 0$ we may replace ω by B when clear from the context.
2) For $J \subseteq [\omega]^{\aleph_0}$ let $J^\perp = \{B : B \in [\omega]^{\aleph_0}$ and $[A \in J \Rightarrow A \cap B$ finite$]\}$ and also for $\bar{A} = \langle A_s : s \in S \rangle$ let $\bar{A}^\perp = \{A_s : s \in S\}^\perp$.
3) For $\mathcal{A} \subseteq [B]^{\aleph_0}$

(a) $\mathrm{id}_{\mathcal{A}}(B)$ is the ideal of $\mathcal{P}(B)$ generated by $(\mathcal{A}{\restriction}B) \cup [B]^{<\aleph_0}$

(b) $\mathrm{id}^+_{\mathcal{A}}(B) = [B]^{\aleph_0} \backslash \mathrm{id}_{\mathcal{A}}(B)$, for $\mathcal{A}{\restriction}B$ see 7) below;

(c) if $B = \omega$ we may omit it.

4) Let $A \subseteq^* B$ means that $A \backslash B$ is finite.
5) If $\mathcal{C} \subseteq \mathcal{P}(\omega), B \subseteq \omega$ and $\eta \in {}^{\mathcal{C}}2$ then $I_{\mathcal{C},\eta}(B)$ is $\{C \subseteq B : C \subseteq^* A^{[\eta(A)]}$ for every $A \in \mathcal{C}\}$; if $B = \omega$ we may omit it.
6) In part 5), if ν is a function extending η then let $I_{\mathcal{C},\nu} = I_{\mathcal{C},\eta}$.
7) For $\mathcal{A} \subseteq \mathcal{P}(B_2)$ and $B_1 \subseteq B_2$ let $\mathcal{A}{\restriction}B_1 = \{A \cap B_1 : A \in \mathcal{A}$ satisfies $A \cap B_1$ is infinite$\}$.

Definition 0.8. 1) Let $\mathrm{OB} = \{I \subseteq [\omega]^{\aleph_0} : I \cup [\omega]^{<\aleph_0}$ is an ideal of $\mathcal{P}(\omega)\}$.
2) For $A \subseteq \omega$ let $\mathrm{ob}(A) = \{B : B \in [\omega]^{\aleph_0}$ and $B \subseteq^* A\}$ so $\mathrm{ob}(\omega) = [\omega]^{\aleph_0}$.
3) $\eta \perp \nu$ means $\neg(\eta \trianglelefteq \nu) \wedge \neg(\nu \trianglelefteq \eta)$.
4) We say $\mathcal{A}$ is AD in $J \subseteq [\omega]^{\aleph_0}$ when $\mathcal{A}$ is AD and $\mathcal{A} \subseteq J$.
5) We say $\mathcal{A}$ is MAD in $J \subseteq [\omega]^{\aleph_0}$ when $\mathcal{A}$ is AD in J and is $\subseteq$-maximal among such $\mathcal{A}$'s.
6) $J \subseteq [\omega]^{\aleph_0}$ is hereditary when $A \in [\omega]^{\aleph_0} \wedge A \subseteq^* B \in J \Rightarrow A \in J$.
7) $J \subseteq [\omega]^{\aleph_0}$ is dense when $(\forall B \in [\omega]^{\aleph_0})(\exists A \in J)[A \subseteq B]$.

1. The simple case: $\mathfrak{s} < \mathfrak{a}_*$

We here give a proof for the case $\mathfrak{s} < \mathfrak{a}_*$.

Theorem 1.1. *1) If $\mathfrak{s} < \mathfrak{a}_*$ then there is a saturated MAD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$.*
2) Moreover, given a dense $J_* \subseteq [\omega]^{\aleph_0}$ we can demand $\mathcal{A} \subseteq J_*$.

Proof. Stage A: Let $\kappa = \mathfrak{s}$, so $\mathrm{cf}(\kappa) > \aleph_0$. For part (1) let $J_* \subseteq [\omega]^{\aleph_0}$ be a dense (and even hereditary) subset of $[\omega]^{\aleph_0}$, i.e. as in part (2) and in both cases without loss of generality every finite union of members of J_* is co-infinite, i.e. $\omega \notin \mathrm{id}_{J_*}$.

Choose a sequence $\langle C^*_\alpha : \alpha < \kappa\rangle$ of subsets of ω exemplifying $\mathfrak{s} = \kappa$, i.e. $\neg(\exists B \in [\omega]^{\aleph_0}) \wedge \bigwedge\limits_\alpha (B \subseteq^* C^*_\alpha \vee B \subseteq^* \omega \backslash C^*_\alpha)$. For $i < \kappa$ and $\eta \in {}^i 2$ let $C^*_\eta = C^*_i$, the aim of this notation is to simplify later proofs where we say "repeat the present proof but ...".

Stage B: For $\alpha \leq 2^{\aleph_0}$ let AP_α, the set of α-approximations, be the set of t consisting of the following objects satisfying the following conditions:

$\boxplus_1$ (a) $\mathcal{T} = \mathcal{T}_t$ is a subtree of ${}^{\kappa >}2$, i.e. closed under initial segments

(b) let $\mathrm{suc}(\mathcal{T}) = \{\eta \in \mathcal{T} : \ell g(\eta)$ is a successor ordinal$\}$ and[1] $c\ell(\mathcal{T}) = \{\eta \in {}^{\kappa \geq}2$: if $i < \ell g(\eta)$ then $\eta \restriction i \in \mathcal{T}\}$

(c) $1 \leq |\mathcal{T}| \leq \aleph_0 + |\alpha|$

(d) $\bar{I} = \bar{I}_t = \langle I_\eta : \eta \in c\ell(\mathcal{T})\rangle = \langle I^t_\eta : \eta \in c\ell(\mathcal{T}_t)\rangle$

(e) $\bar{A} = \bar{A}_t = \langle A_\eta : \eta \in \mathrm{suc}(\mathcal{T})\rangle = \langle A^t_\eta : \eta \in \mathrm{suc}(\mathcal{T}_t)\rangle$

such that

(f) $A_\eta \in I_\eta \cap J_*$ or[2] $A_\eta = \emptyset$ and $\mathscr{S}_t = \{\eta \in \mathrm{suc}(\mathcal{T}_t) : A_\eta \neq \emptyset\}$

(g) $I_\eta = \{A \in [\omega]^{\aleph_0}$: if $i < \ell g(\eta)$ then $A \subseteq^* (C^*_{\eta\restriction i})^{[\eta(i)]}$ and if $i+1 < \ell g(\eta)$ then $A \cap A_{\eta\restriction(i+1)}$ is finite$\}$, so I_η is well defined also when $\eta \in c\ell(\mathcal{T})$.

We let

(h) $C^t_\eta = C^*_\eta$ (for generalizations)

$\boxplus_2$ $\mathrm{AP} = \cup\{\mathrm{AP}_\alpha : \alpha \leq 2^{\aleph_0}\}$

$\boxplus_3$ $s \leq_{\mathrm{AP}} t$ iff (both are from AP and)

(a) $\mathcal{T}_s \subseteq \mathcal{T}_t$

(b) $\bar{I}_s = \bar{I}_t \restriction c\ell(\mathcal{T}_s)$

(c) $\bar{A}_s = \bar{A}_t \restriction \mathrm{suc}(\mathcal{T}_s)$.

Stage C: We assert various properties of AP; of course s, t denote members of AP:

$\boxplus_4$ (a) $\leq_{\mathrm{AP}}$ partially orders AP

(b) $\eta \triangleleft \nu \in c\ell(\mathcal{T}_t) \Rightarrow I^t_\nu \subseteq I^t_\eta$

(c) if $\eta \in c\ell(\mathcal{T}_t)$ then $I^t_\eta \in$ OB, i.e. $I^t_\eta \cup [\omega]^{<\aleph_0}$ is an ideal of $\mathcal{P}(\omega)$

[1] so $c\ell(\{<>\}) = \{<>, <0>, <1>\}$
[2] the case "$A_\eta = \emptyset$" is not needed in the present proof

(d) $\langle A^t_\eta : \eta \in \mathscr{S}_t\rangle$ is almost disjoint (so $A^t_\eta \in \mathrm{ob}(\omega)$ and $\eta \ne \nu \in \mathscr{S}_t \Rightarrow A^t_\eta \cap A^t_\nu$ finite; recall that here we can assume $\mathscr{S}_t = \mathrm{suc}(\mathcal{T}_t)$)

(e) if $\eta \in c\ell(\mathcal{T}_t)$ and $\ell g(\eta) = \kappa$ then $I^t_\eta = \emptyset$

(f) if $s \le_{\mathrm{AP}} t$ <u>then</u> $c\ell(\mathcal{T}_s) \subseteq c\ell(\mathcal{T}_t)$ and $\eta \in c\ell(\mathcal{T}_s) \Rightarrow I^s_\eta = I^t_\eta$ (and clause (b) of $\boxplus_3$ follow from clauses (a),(c))

(g) • if $\nu \in c\ell(\mathcal{T}_s)\backslash\mathcal{T}_s$ and $\eta \in \mathscr{S}_s$ and $B \in I^s_\nu$ <u>then</u> $B \cap A_\eta$ is finite

• if $\nu \in \mathcal{T}_s$ and $\eta \in \mathscr{S}_s$ but $\neg(\nu \trianglelefteq \eta)$ and $B \in I^s_\nu$ <u>then</u> $B \cap A_\eta$ is finite.

[Why clause (d)? Let $\eta_0 \ne \eta_1 \in \mathscr{S}_t$, if $\eta_0 \perp \eta_1$ let $\rho = \eta_0 \cap \eta_1$ hence for some $\ell \in \{0,1\}$ we have $\rho\hat{\ }\langle\ell\rangle \trianglelefteq \eta_0, \rho\hat{\ }\langle 1-\ell\rangle \trianglelefteq \eta_1$ so $A_{\eta_k} \in I^t_{\eta_k} \subseteq I^t_{\rho\hat{\ }<k>} \subseteq \mathrm{ob}((C^t_\rho)^{[k]})$ for $k = 0,1$ hence $A_{\eta_0} \cap A_{\eta_1} \subseteq^* \mathrm{ob}((C^t_\rho)^{[\ell]}) \cap \mathrm{ob}((C^t_\rho)^{[1-\ell]}) = \emptyset$. If $\eta_0 \triangleleft \eta_1$ note that $A^t_{\eta_1} \in I^t_{\eta_1} \subseteq \mathrm{ob}(\omega\backslash A^t_{\eta_0})$ by clause $\boxplus_1(g)$. Also if $\eta_1 \triangleleft \eta_0$ similarly so clause (d) holds indeed.

Why Clause (e)? Recall the choice of $\langle C^*_\alpha : \alpha < \kappa\rangle$ and $\langle C^t_\eta : \eta \in {}^{\kappa >}2\rangle$ hence $\alpha < \kappa \Rightarrow C^t_{\eta\restriction\alpha} = C^*_\alpha$. So if $B \in I^t_\eta$, then $B \in I_{\eta\restriction(\alpha+1)}$ hence $(B \subseteq^* C^*_\alpha \vee B \subseteq^* \omega\backslash C^*_\alpha)$ for every $\alpha < \kappa$, a contradiction to the choice of $\langle C^*_\alpha : \alpha < \kappa\rangle$.]

$\boxplus_5$ (a) $\alpha < \beta \le 2^{\aleph_0} \Rightarrow \mathrm{AP}_\alpha \subseteq \mathrm{AP}_\beta$

(b) $\mathrm{AP}_0 \ne \emptyset$ (e.g. use t with $\mathcal{T}_t = \{<>\}$)

(c) if $\langle t_i : i < \delta\rangle$ is $\le_{\mathrm{AP}}$-increasing, $t_i \in \mathrm{AP}_{\alpha_i}$ for $i < \delta, \langle\alpha_i : i < \delta\rangle$ is increasing, δ a limit ordinal and $\alpha_\delta = \cup\{\alpha_i : i < \delta\}$ <u>then</u> $t_\delta = \cup\{t_i : i < \delta\}$ naturally defined belongs to $\mathrm{AP}_{\alpha_\delta}$ and $i < \delta \Rightarrow t_i \le_{\mathrm{AP}} t_\delta$

$\boxplus_6$ let J_t be the ideal on $\mathcal{P}(\omega)$ generated by $\{A^t_\eta : \eta \in \mathscr{S}_t\} \cup [\omega]^{<\aleph_0}$.

For $s \in \mathrm{AP}$ and $B \in \mathrm{ob}(\omega)$ we define:

$(*)_1$ $S_B = S^s_B := S^1_B \cup S^2_B$ where

(a) $S^1_B = S^{s,1}_B := \{\eta \in c\ell(\mathcal{T}_s) : [B\backslash A]^{\aleph_0} \cap I^s_\eta \ne \emptyset$ for every $A \in J_s\}$

(b) if $\iota = 2$ then $S^2_B = S^{s,2}_B := \{\eta \in c\ell(\mathcal{T}_s)$: for infinitely many $\nu, \eta \trianglelefteq \nu \in \mathscr{S}_s$ and the set $B \cap A_\nu$ is infinite$\}$

(c) $S^3_B = S^{3,s}_B := S_B$

$(*)_2$ $\mathrm{SP}^\iota_B = \mathrm{SP}^{s,\iota}_B := \{\eta \in \mathcal{T}_s : \eta\hat{\ }\langle 0\rangle \in S^{s,\iota}_B$ and $\eta\hat{\ }\langle 1\rangle \in S^{s,\iota}_B\}$ for $\iota = 1,2,3$.

Note

$(*)_3$ for $\iota = 1,2,3$

(a) S^ι_B is a subtree of $c\ell(\mathcal{T}_s)$

(b) $\langle\rangle \in S_B \Leftrightarrow B \in J^+_s \Leftrightarrow \langle\rangle \in S^1_B$

(c) $\mathrm{SP}^\iota_B \subseteq \mathcal{T}_s$

(d) if $B \subseteq A$ are from $[\omega]^{\aleph_0}$ then $S^\iota_B \subseteq S^\iota_A$, $\mathrm{SP}^\iota_B \subseteq \mathrm{SP}^\iota_A$.

[Why? For $\iota = 1$, the first statement holds by recalling $\boxplus_4(b)$, the second, $\langle\rangle \in S^\iota_B \Leftrightarrow B \in J^+_s$, holds as $I^s_{<>} = \mathrm{ob}(\omega)$, the third, $\mathrm{SP}^\iota_B \subseteq \mathcal{T}_s$ as by the definition of $c\ell(\mathcal{T}_s)$ we have $\eta\hat{\ }\langle\ell\rangle \in c\ell(\mathcal{T}_s) \Rightarrow \eta \in \mathcal{T}_s$. Also the fourth is obvious. For $\iota = 2$ this is even easier and for $\iota = 3$ it follows, except for clause (b), but then]

$(*)_4$ if $\eta \in S_B$ and $\nu_0 \triangleleft \nu_1 \triangleleft \ldots \triangleleft \nu_{n-1}$ list $\{\nu \triangleleft \eta : \nu \in \mathrm{SP}_B\}$ so this set is finite and we let $C_s(\eta, B) := \cap\{(C^s_{\nu_\ell})^{[\eta(\ell g(\nu_\ell))]} : \ell < n\}$, then $S_{B \cap C_s(\eta,B)} = \{\nu \in S_B : \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu\}$.

[Why? Clearly $(\forall A \in I^s_\eta)(A \subseteq^* C_s(\eta, B))$ by the definition of I^s_η, see $\boxplus_1(g)$ but $(\exists A \subseteq I^s_\eta)(|A \cap B| = \aleph_0)$ hence $B \cap C_s(\eta, B) \in \mathrm{ob}(\omega)$.

As $B \cap C_s(\eta, B) \subseteq B$ clearly $S_{B\cap C_s(\eta,B)} \subseteq S_B$. Also as $\eta \in S_B$ and as $(\forall A \in I^s_\eta)(A \subseteq^* C_s(\eta, B))$ clearly $\eta \in S_{B\cap C_s(\eta,B)}$ and moreover $\{\nu \in S_{B\cap C_s(\eta,B)} : \eta \trianglelefteq \nu\} = \{\nu \in S_B : \eta \trianglelefteq \nu\}$ by $\boxplus_4(b)$.

Also as S_B and $S_{B\cap C_s(\eta,B)}$ are subtrees clearly $\{\nu : \nu \trianglelefteq \eta\} \subseteq S_B \cap S_{B\cap C_s(\eta,B)}$ and recalling $(\forall A \in I^s_\eta)[A \subseteq^* C_s(\eta, B)]$ clearly $\eta \trianglelefteq \nu \in c\ell(\mathcal{T}_s) \Rightarrow [\nu \in S_B \Leftrightarrow \nu \in S_{B\cap C_s(\eta,B)}]$.

So to prove the equality it suffices to assume $\alpha < \ell g(\eta), \nu \in S_B, \ell g(\eta \cap \nu) = \alpha, \ell g(\nu) > \alpha$ and $\nu \in S_{B\cap C_s(\eta,B)}$ and get a contradiction. If $\ell < n$ and $\alpha = \ell g(\nu_\ell)$ then $(\forall A \in I^s_\nu)[A \subseteq^* (C^s_{\eta\restriction\alpha})^{[1-\eta(\alpha)]}]$, so an easy contradiction. If $\alpha \notin \{\ell g(\nu_\ell) : \ell < n\}$ we can get contradiction to $\eta\restriction\alpha \notin \mathrm{SP}_B$. So we are done proving $(*)_4$.]

$(*)_5$ Assume $t \in \mathrm{AP}$

- (a) for every $\eta \in c\ell(\mathcal{T}_t)$ the set $\{B \in \mathrm{ob}(\omega) : \eta \notin S_B\}$ belongs to OB
- (b) if $\iota = 1, 2, 3$ and $B = B_0 \cup \ldots \cup B_n \subseteq \omega$ then $S^\iota_B = S^\iota_{B_0} \cup \ldots \cup S^\iota_{B_n}$
- (c) if $A \in J_t, B_1 \in \mathrm{ob}(\omega)$ and $B_2 = B_1 \backslash A$ then $S^\iota_{B_2} = S^\iota_{B_1}, \mathrm{SP}^\iota_{B_2} = \mathrm{SP}^\iota_{B_1}$ for $\iota = 1, 2, 3$
- (d) $S^2_B \subseteq \mathcal{T}_t$ for $B \in \mathrm{ob}(\omega)$
- (e) if $\eta \triangleleft \nu \in c\ell(\mathcal{T}_t)$ then $\eta \in \mathcal{T}_t$
- (f) if $B \in \mathrm{ob}(\omega)$ and $s \leq_{\mathrm{AP}} t$ then
  - $\bullet_1$ $S^{s,1}_B \supseteq S^{t,1}_B \cap c\ell(\mathcal{T}_s)$
  - $\bullet_2$ $S^{s,2}_B \subseteq S^{t,2}_B \cap \mathcal{T}_s$ (inclusion in different direction? yes!)
  - $\bullet_3$ $S^{s,3}_B \supseteq S^{t,3}_B \cap c\ell(\mathcal{T}_s)$, in fact $(S^{t,2}_B \cap c\ell(\mathcal{T}_s)\backslash S^{s,2}_B \subseteq S^{s,1}_B$
  - $\bullet_4$ if $\eta \in S^s_B\backslash\mathcal{T}_t$ then $\eta \in S^{t,1}_B$
  (clause (f) is not used in the present proof).

[Why? Easy, e.g. for clause $(f), \bullet_3$, assume $\eta \in S^{t,2}_B \cap c\ell(\mathcal{T}_s)$ and $\eta \notin S^{s,2}_B$ then by the former we can find pairwise distinct $\eta_n \in \mathscr{S}_t$ such that $A^t_{\eta_n} \cap B$ is infinite for $n < \omega$. As $\eta \notin S^{s,2}_B$ necessarily $\{n : \eta_n \in \mathcal{T}_s\}$ is finite, so let n be such that $\eta_n \notin \mathcal{T}_s$; then $A^t_{\eta_n} \cap B$ witness $\eta \in S^{s,1}_B$ as promised.]

$(*)_6$ for $\iota = 1, 2, 3$ we have

- (a) if $B \subseteq \omega, \ell < 2$ and $\nu \in S^\iota_B \cap \mathcal{T}_t$ and $B \subseteq^* (C^t_\nu)^{[\ell]}$ then $\nu\hat{\ }\langle\ell\rangle \in S^\iota_B$ but $\nu\hat{\ }\langle 1-\ell\rangle \notin S^\iota_B$
- (b) if $B \subseteq \omega$ and $\nu \in S^\iota_B \cap \mathcal{T}_t$ then for some $\ell < 2$ we have $\nu\hat{\ }\langle\ell\rangle \in S^\iota_B$
- (c) $\omega \notin J_t$.

[Why? Read the definitions recalling $(*)_5(c)$. For clause (c) recall that $\nu \in \mathscr{S}_s \Rightarrow A_s \in J_s$ and by $\boxplus_1(f)$ we have $J_s \subseteq J_*$ and by Stage A, $\omega \notin \mathrm{ob}(J_*)$.]

Stage D:

$\boxplus_7$ if $\alpha < 2^{\aleph_0}, s \in \mathrm{AP}_\alpha$ and $B \in \mathrm{ob}(\omega)\backslash J_s$ then we can find $t \in \mathrm{AP}_{\alpha+1}$ such that $s \le_{\mathrm{AP}} t$ and B contains A_η for some $\eta \in \mathscr{S}_t \backslash \mathcal{T}_s$.

This is a major point and we shall prove it in Stage F below.

Stage E: We prove the theorem.

Let $\langle B_\alpha : \alpha < 2^{\aleph_0}\rangle$ list $\mathcal{P}(\omega)$ each appear $2^{\aleph_0}$ times. By induction on $\alpha \le 2^{\aleph_0}$ we choose t_α such that

$\circledast$ (a) $t_\alpha \in \mathrm{AP}_\alpha$

(b) $\beta < \alpha \Rightarrow t_\beta \le_{\mathrm{AP}} t_\alpha$

(c) if $\alpha = \beta + 1$ then either $B_\alpha \in J_{t_\beta}$ or B_α contains A_η, for some $\eta \in \mathscr{S}_{t_\alpha} \backslash \mathcal{T}_{t_\beta}$.

For $\alpha = 0$ use $\boxplus_5(b)$.

For α limit use $\boxplus_5(c)$.

For $\alpha = \beta + 1$ use $\boxplus_7$.

Now let $t \in \mathrm{AP}$ be $\cup\{t_\alpha : \alpha < 2^{\aleph_0}\}$ and recalling $(*)_6(c)$ it is easy to check that by $\boxplus_1(f)$ we have $A^t_\eta \in \mathrm{ob}(\omega)$ and by $\boxplus_4(a)$ the sequence $\bar{A}_t = \langle A_\eta : \eta \in \mathcal{S}_t\rangle$ is an AD family and is with no repetitions; moreover by $\circledast(c)$ the sequence $\bar{A}_t$ is a saturated MAD family, enough for 1.1(1) and recalling that by $\boxplus_1(f)$ it is $\subseteq J_*$ also enough for 1.1(2).

Stage F: The rest of the proof is dedicated to the proof of $\boxplus_7$ so α, s and B are given.

The proof is now split into cases.

Case 1: Some $\nu \in S_B$ is such that $\nu \in c\ell(\mathcal{T}_s)\backslash\mathcal{T}_s$.

By $(*)_5(d)$ we have $\nu \in S^1_B$. Clearly as $\nu \in S_B$ there is $B_1 \in [B]^{\aleph_0} \cap I^s_\nu$. Note that $\ell g(\nu) > 0$ as $\langle\rangle \in \mathcal{T}_s$ by clause (c) of $\boxplus_1$.

Note that $A \in I^s_\nu \wedge \eta \in \mathscr{S}_s \Rightarrow A \cap A^s_\eta$ is finite, e.g. by the proof of $\boxplus_4(d)$ or better by $\boxplus_4(g)$.

Subcase 1A: Assume $\ell g(\nu)$ is a successor ordinal.

Let $B_2 \subseteq B_1$ be such that $B_2 \in J_*$ and $B_1\backslash B_2$ are infinite. Now define t as follows: $\mathcal{T}_t = \mathcal{T}_s \cup \{\nu\}, A^t_\rho$ is A^s_ρ if $\rho \in \mathrm{suc}(\mathcal{T}_s)$ and is B_2 if $\rho = \nu$, lastly define I^t_ρ for $\rho \in \mathcal{T}_t$ as in clause (g) of $\boxplus_1$. Easy to check that t is as required; actually "$B_1\backslash B_2$ is infinite" is not used, similarly below.

Subcase 1B: Assume $\ell g(\nu)$ is a limit ordinal.

Clearly $\ell g(\nu) < \kappa$ by $\boxplus_4(e)$, as $I^s_\nu \ne \emptyset$ because $B_1 \in I^s_\nu$, clearly there is $\ell \in \{0,1\}$ such that $B'_1 := (C^s_\nu)^{[\ell]} \cap B_1$ is infinite, let $B_2 \subseteq B'_1$ be such that $B_2, B'_1\backslash B_2$ are infinite and $B_2 \in J_*$. We define t by $\mathcal{T}_t = \mathcal{T}_s \cup \{\nu, \nu\hat{}\langle\ell\rangle\}, A^t_\rho$ is A^s_ρ if $\rho \in \mathrm{suc}(\mathcal{T}_s)$ and is B_2 if $\rho = \nu\hat{}\langle\ell\rangle$ and I^t_ρ for $\rho \in \mathcal{T}_t$ is defined as in clause (g) of $\boxplus_1$.

Easy to check that t is as required.

Case 2: $\mathrm{SP}_B = \emptyset$ but not case 1.

Let $\nu^*_B := \cup\{\eta : \eta \in S^1_B\}$.

Subcase 2A: $\nu^*_B \in S^1_B$.

As $S_B \subseteq c\ell(\mathcal{T}_s)$ by the definition of S_B, as we are assuming "not case 1" necessarily $S_B \subseteq \mathcal{T}_s$ hence $\nu^*_B \in \mathcal{T}_s$ so $\ell g(\nu^*_B) < \kappa$.

We define B^*_2 as $B \cap A_{\nu^*_B}$ if $A_{\nu^*_B}$ is well defined and $B^*_2 = \emptyset$ otherwise; and for $\ell = 0,1$, let $B^*_\ell := B \cap (C^*_{\ell g(\nu^*_b)})^{[\ell]} \backslash B^*_2$.

So

$(*)_7$ $\langle B^*_0, B^*_1, B^*_2 \rangle$ is a partition of B

hence by $(*)_5(b)$ for some $\ell = 0,1,2$

$(*)_8$ $\nu^*_B \in S^1_{B^*_\ell}$

easily

$(*)_9$ $\ell \neq 2$,

and

$(*)_{10}$ $\rho := \nu^*_B{}^\frown\langle \ell \rangle \in S_{B^*_\ell}$.

[Why? By the definitions noting $\rho \in c\ell(\mathcal{T}_s)$.]

Also as $B^*_\ell \subseteq B$ clearly

$(*)_{11}$ $S_{B^*_\ell} \subseteq S_B$.

But $(*)_{10} + (*)_{11}$ contradicts the choice of ν^*_B.

<u>Subcase 2B</u>: $\nu^*_B \notin S^1_B$.

By $(*)_3(b) + (*)_6(c)$ and the assumption of $\boxplus_7$ we have $\langle\rangle \in S^1_B$ and by $(*)_6(b)$ clearly $\langle 0 \rangle \in S^1_B$ or $\langle 1 \rangle \in S_B$ hence $\nu^*_B \neq \langle\rangle$. If $\nu^*_B = \nu{}^\frown\langle \ell \rangle$ by the definition of ν^*_B we have $\nu^*_B \in S^1_B$, contradiction to the subcase assumption. Hence necessarily $\ell g(\nu^*_B)$ is a limit ordinal $\leq \kappa$, call it δ. So $\alpha < \delta \Rightarrow \nu^*_B{\restriction}\alpha \in S_B$ but $\rho \triangleleft \varrho \in c\ell(\mathcal{T}_s) \Rightarrow \rho \in \mathcal{T}_s$ hence $\alpha < \delta \Rightarrow \nu^*_B{\restriction}\alpha \in \mathcal{T}_s$. Now for every $\alpha < \delta$ let $\nu^*_{B,\alpha} := (\nu^*_B{\restriction}\alpha){}^\frown\langle 1 - \nu^*_B(\alpha)\rangle$, so clearly $\nu^*_{B,\alpha} \in c\ell(\mathcal{T}_s)\backslash S_B$ hence as $\nu^*_{B,\alpha} \notin S^2_B$, by the definition of $S^2_B \subseteq S_B$ the set $\mathcal{A}_\alpha = \{A^s_\rho : \rho \in \mathcal{S}_s, \nu^*_{B,\alpha} \trianglelefteq \rho$ and $B \cap A_\rho$ is infinite$\}$ is finite, so we can find $n = n(\alpha) < \omega$ and $A^*_{\alpha,0}, \ldots, A^*_{\alpha,n(\alpha)-1}$ enumerating $\mathcal{A}_\alpha$ but also $\nu^*_{B,\alpha} \notin S^1_B \subseteq S_B$, hence $\mathrm{ob}(B \backslash \cup \mathcal{A}_\alpha) = \mathrm{ob}(B \backslash \cup \{A^*_{\alpha,\ell} : \ell < n(\alpha)\}$ is disjoint to $I_{\nu^*_{B,\alpha}} \cap J^+_s$ and by the choice of $\mathcal{A}_\alpha$ and $(*)_4$, $\mathrm{ob}(B \backslash \cup \mathcal{A}_\alpha) = [B \backslash (A^*_{\alpha,0} \cup \ldots \cup A^*_{\alpha,n(\alpha)-1})]^{\aleph_0}$ is disjoint to $I^s_{\nu^*_{B,\alpha}}$. Let $A^*_{\alpha,n(\alpha)}$ be $A_{\nu^*_B{\restriction}\alpha}$ when defined and $\emptyset$ otherwise. By the definitions of $I^s_{\nu^*_{B,\alpha}}, I^s_{\nu^*_B{\restriction}\alpha}$ we have (for $\alpha < \delta$ of course):

$\odot_1$ (a) $[B \cap (C^s_{\nu^*_B{\restriction}\alpha})^{[1-\nu^*_B(\alpha)]} \backslash (A^*_{\alpha,0} \cup \ldots \cup A^*_{\alpha,n(\alpha)})]^{\aleph_0}$ is disjoint to $I^s_{\nu^*_B{\restriction}\alpha}$

(b) $A^*_{\alpha,\ell} \in \{A_\rho : \rho \in \mathscr{S}_t$ and $\nu^*_{B,\alpha} \trianglelefteq \rho$ (hence $A_{\alpha,\ell} \subseteq (C^s_{\nu^*_B{\restriction}\alpha})^{[1-\nu^*_B(\alpha)]}$)$\}$ for $\ell < n(\alpha)$ (not needed presently)

[Why clause (b)? By the choice of $\mathcal{A}_\alpha$.]

Let $\mathcal{A}^* = \{B \cap A : A = A^*_{\alpha,k}$ for some $\alpha < \delta, k \leq n(\alpha)$ and $B \cap A$ is infinite$\}$.

So $\mathcal{A}^*$ is a family of pairwise almost disjoint infinite subsets of B and if $\mathcal{A}^*$ is finite, still $B \backslash \cup \{A : A \in \mathcal{A}^*\}$ is infinite because $\mathcal{A}^* \subseteq J_s$ and we are assuming $B \notin J_s$.

Let $\Lambda := \{\nu \in \mathscr{S}_s : \nu^*_B \trianglelefteq \nu$ and $|A_\nu \cap B| = \aleph_0\}$, so

- the set $\{A_\nu : \nu \in \mathcal{S}_s$ and $|A_\nu \cap B| = \aleph_0\}$ is equal to $\mathcal{A}^* \cup \{A_\nu : \nu \in \Lambda\}$.

Now

$\odot_2$ there is a set B_1 such that:
- (a) $B_1 \subseteq B$ is infinite
- (b) B_1 is almost disjoint to any $A \in \mathcal{A}^*$
- (c) if Λ is finite then $\nu \in \Lambda \Rightarrow |B_1 \cap A_\nu| < \aleph_0$
- (d) if Λ is infinite then for infinitely many $\nu \in \Lambda$ we have $|B_1 \cap A_\nu| = \aleph_0$

[Why? First assume Λ is finite, so without loss of generality it is empty. If $\mathcal{A}^*$ is finite use the paragraph above on $\mathcal{A}^*$. Otherwise as $|\mathcal{A}^*| \leq |\delta| + \aleph_0 \leq \kappa = \mathfrak{s}$ and by the theorem's assumption $\mathfrak{s} < \mathfrak{a}_* \leq \mathfrak{a}$ and by the definition of $\mathfrak{a}$ it follows that $\odot_2$ holds.

Second, assume that Λ is infinite, and choose pairwise distinct $\nu_n \in \Lambda$ for $n < \omega$. Now we recall that we are assuming $\mathfrak{s} < \mathfrak{a}_*$ and apply the definition 0.4 of $\mathfrak{a}_*$ to $\mathcal{A}^*$ and $\langle A_{\nu_n} : n < \omega\rangle$ and get an infinite $B_1 \subseteq B$ as required.]

$\odot_3$ (a) $B_1 \in J_s^\perp$
- (b) if $\neg(\nu_B^* \trianglelefteq \eta)$ and $\eta \in \mathscr{S}_s$ then $B_1 \cap A_\eta$ is finite.

[Why? For clause (a), note that first $B_1 \subseteq B \subseteq \omega$, second B_1 is infinite by clause (a) of $\odot_2$, third $B_s \notin J_s$ is proved by dividing to two cases. If Λ is finite use clause (b) of $\odot_3$ proved below and clause (b) of $\odot_2$; and if Λ is infinite use $\odot_2(d)$). So let us turn to proving clause (b); we should prove that $[\eta \in \mathrm{suc}(\mathcal{T}_s) \wedge \neg(\nu_B^* \trianglelefteq \eta) \Rightarrow$ $B_1 \cap A_\eta$ finite.

If $A_\eta \in \{A^*_{\alpha,n} : \alpha < \delta, n \leq n(\alpha)\}$ then either $A_\eta \cap B$ is finite hence $A_\eta \cap B_1 \subseteq A_\eta \cap B$ is finite or $A_\eta \cap B$ is infinite hence $A_\eta \cap B \in \mathcal{A}^*$ hence by $\odot_2(b)$ the set $B_1 \cap (A_\eta \cap B)$ is finite by the choice of B_1 but $B_1 \subseteq B$ hence $B_1 \cap A_\eta$ is finite. So assume $A_\eta \notin \{A^*_{\alpha,m} : \alpha < \delta, n \leq n(\alpha)\}$, so by the choice of $A_{\alpha,n(\alpha)}$ for $\alpha < \delta$ necessarily $\neg(\eta \triangleleft \nu_B^*)$. Recall that we are assuming that $\neg(\nu_B^* \trianglelefteq \eta)$. Together for some $\alpha < \delta$ we have $\alpha = \ell g(\nu_B^* \cap \eta) < \delta$ and $\nu_B^* {\restriction} \alpha \triangleleft \eta$ and we get contradiction by the choice of $\mathcal{A}_\alpha = \{A^*_{\alpha,\ell} : \ell < n(\alpha)\}$ and $A^*_{\alpha,n(\alpha)}$.]

We shall now prove by induction on $\alpha \leq \delta$ that $B_1 \in I^s_{\nu_B^* {\restriction} \alpha}$. For $\alpha = 0$ recall $I^s_{\nu_B^* {\restriction} \alpha} = [\omega]^{\aleph_0}$, for α limit $I^s_{\nu_B^* {\restriction} \alpha} = \cap\{I^s_{\nu_B^* {\restriction} \beta} : \beta < \alpha\}$ and use the induction hypothesis. For $\alpha = \beta + 1$ first note that by $\odot_2(b)$ the set B_1 is almost disjoint to $A_{\nu_B^* {\restriction} \beta}$ if $\nu_B^* {\restriction} \beta \in \mathscr{S}_s \subseteq \mathrm{suc}(\mathcal{T}_s)$ by $\odot_2(b)$ and, second, B_1 is almost disjoint to $(C^s_{\nu_B^* {\restriction} \beta})^{[1-\nu_B^*(\beta)]}$ otherwise recalling $\odot_3(b)$ we get contradiction to the present case assumption $\mathrm{SP}_B = \emptyset$ by $(*)_6(a)+$ the induction hypothesis; together by the definition of $I^s_{\nu_B^* {\restriction} \beta}, I^s_{\nu_B^* {\restriction} \alpha}$ we have $B_1 \in I^s_{\nu_B^* {\restriction} \alpha}$. Having carried the induction, in particular $B_1 \in I^s_{\nu_B^* {\restriction} \delta} = I_{\nu_B^*}$; now recalling first $B_1 \notin J_s$ by $\odot_3(a)$, second $B_1 \subseteq B$ by $\odot_2(a)$ and third, the choice of $\mathcal{A}^*, J_s$, together they contradict the subcase assumption $\nu_B^* \notin S_B^1$.

<u>Case 3</u>: None of the above.

Without loss of generality

$\oplus_1$ if $B_1 \subseteq B$ but $B_1 \notin J_s$ then none of the two cases above holds.

We try to choose $\bar{\eta}^n = \langle \eta_\rho : \rho \in {}^n 2\rangle$ by induction on n such that:

(a) $\eta_\rho \in \mathrm{SP}_B$

(b) if $\rho = \varrho\hat{}\langle \ell \rangle$ then $\eta_\varrho\hat{}\langle \ell \rangle \trianglelefteq \eta_\rho$

(c) $\{\nu : \nu \triangleleft \eta_\rho$ and $\nu \in \mathrm{SP}_B\} = \{\eta_{\rho\restriction k} : k < \ell g(\rho)\}$.

For $n = 0$, note that $\mathrm{SP}_B \neq \emptyset$ as not case 2 (and not case 1) so we can choose $\eta_\rho \in SP_B$ with minimal length so clause (c) obviously holds. If $n = m + 1$ and $\rho \in {}^m 2$ by the induction hypothesis $\eta_\rho \in \mathrm{SP}_B$, hence $\eta_\rho \in \mathcal{T}_s$ and by the definition of SP_B for $\ell = 0, 1$ the sequence $\eta_\rho\hat{}\langle \ell \rangle$ belongs to S_B.

First assume $\{\nu \in \mathrm{SP}_B : \eta_\rho\hat{}\langle \ell \rangle \trianglelefteq \nu\} = \emptyset$. So $B_1 := B \cap C_s(\eta_\rho\hat{}\langle \ell \rangle, B) \notin J_s$ noting $C_s(\eta_\rho\hat{}\langle \ell \rangle, B) = \cap\{C^{[\rho(k)]}_{\eta_\rho \restriction k} : k \leq m\}$, recalling it is defined in $(*)_4$ from Stage C using clause (c) and $\eta_\rho\hat{}\langle \ell \rangle \in S_B$; hence $\eta_\rho\hat{}\langle \ell \rangle \in S_{B_1}$.

Now by $(*)_4$ we know $S_{B_1} = \{\nu \in S_B : \nu \trianglelefteq \eta_\rho\hat{}\langle \ell \rangle$ or $\eta_\rho\hat{}\langle \ell \rangle \trianglelefteq \nu\}$ so case 2 or case 1 holds for B_1, contradiction to $\oplus_1$.

Second, assume we have $(\exists \eta)(\eta_\rho\hat{}\langle \ell \rangle \trianglelefteq \eta \in \mathrm{SP}_B)$ so choose such $\eta_{\rho\hat{}\langle \ell \rangle}$ of minimal length.

Hence we have carried the inductive choice of $\langle \bar{\eta}^n : n < \omega \rangle$.

For each $\rho \in {}^\omega 2$ let $\eta_\rho = \cup\{\eta_{\rho\restriction n} : n < \omega\}$, clearly $\eta_\rho \in c\ell(\mathcal{T}_s)$. Also $\langle \eta_\rho : \rho \in {}^\omega 2 \rangle$ is without repetitions and each η_ρ belongs to $c\ell(\mathcal{T}_s)$, so as $|\mathcal{T}_s| < 2^{\aleph_0}$ there is $\rho \in {}^\omega 2$ such that $\eta_\rho \notin \mathcal{T}_s$. By clause (c) above we have $\{\varrho : \varrho \triangleleft \eta_\rho$ and $\varrho \in SP_B\} = \{\eta_{\rho \restriction n} : n < \omega\}$.

Note that

$\oplus_2$ $\langle C_s(\eta_{\rho\restriction k}, B) : k < \omega \rangle$ is $\subseteq$-decreasing.

Let $\mathcal{W} = \{\alpha < \ell g(\eta_\rho)$: for some $\nu \in \mathscr{S}_s$ we have $\ell g(\nu \cap \eta_\rho) = \alpha$ and $A_\nu \cap B$ is infinite$\}$.

<u>Subcase 3A</u>: Assume $\mathcal{W}$ is an unbounded subset of $\ell g(\eta_\rho)$.

In this case choose $\alpha_n \in \mathcal{W}$ such that $\alpha_{n+1} > \alpha_n \geq \ell g(\eta_{\rho\restriction n})$ for $n < \omega$ and we choose $\nu_n \in \mathscr{S}_s$ such that $\ell g(\nu_n \cap \eta_\rho) = \alpha_n$ and $A_{\nu_n} \cap B$ is infinite. So can choose an infinite $B_0 \subseteq B$ such that $n < \omega$ implies $B_0 \backslash \cup\{A_{\nu_{n\restriction k}} : k < n\} \subseteq^* C_s(\eta_\rho \restriction \alpha_n, B)$ and $(B_0 \cap A_{\nu_n} \in \mathrm{ob}(\omega))$.

So

$\oplus_3$ $B_0 \subseteq B, B_0 \notin J_s$;

$\oplus_4$ the set SP_{B_0} is empty.

[Why? By $(*)_4$ for each $n < \omega$ we have $S_{B_0} \subseteq \{\nu : \nu \trianglelefteq \eta_{\rho\restriction n} \vee \eta_{\rho\restriction n} \trianglelefteq \nu\}$, hence $S_{B_0} \cap \mathcal{T}_s \subseteq \{\nu : \nu \triangleleft \eta_\rho$ or $\eta_\rho \trianglelefteq \nu\}$ but $\eta_\rho \notin \mathcal{T}_s$ so $\mathrm{SP}_{B_0} = \emptyset$.]

$\oplus_5$ S_{B_0} is not empty.

[Why? By $(*)_3(b)$.]

By $\oplus_4 + \oplus_5$ for the set B_0, case 2 or case 1 hold, so we get contradiction to $\oplus_1$.

<u>Subcase 3B</u>: Assume $\sup(\mathcal{W}) < \ell g(\eta_\rho)$, so we can choose $n(*) < \omega$ such that $\sup(\mathcal{W}) < \ell g(\eta_{\rho\restriction n(*)})$.

Now $[\nu \in \mathscr{S}_s \wedge \eta_{\rho\restriction n(*)} \trianglelefteq \nu \Rightarrow B \cap A^s_\nu$ is finite$]$ as otherwise recalling $\eta_\rho \in c\ell(\mathcal{T}_s) \backslash \mathcal{T}_s$ necessarily $\alpha = \ell g(\eta_\rho \cap \nu) < \ell g(\eta_\rho)$ and of course $\alpha \geq \ell g(\eta_{\rho\restriction n(*)})$, but see the choice of $n(*)$; so $\eta_{\rho\restriction n(*)} \notin S^2_\beta$ hence $\eta_{\rho\restriction n(*)} \in S^1_B$, so we can choose an infinite $B_1 \subseteq B$ such that $B_1 \in I^s_{\eta_{\rho\restriction n(*)}}$. So checking by cases, $B_1 \in \mathrm{ob}(\omega)$ is almost disjoint to any $A_\nu, \nu \in \mathscr{S}_s$. Obviously $B_1 \in I^s_{\eta_\rho}$, so for it case 1 holds as exemplified by η_ρ again contradiction to $\oplus_1$. $\square_{1.1}$

2. The other cases

Theorem 2.1. *1) If $\kappa = \mathfrak{s} = \mathfrak{a}_*$ and $\mathrm{cf}([\mathfrak{s}]^{\aleph_0}, \subseteq) = \mathfrak{s}$ then there is a saturated MAD family.*
2) If $\kappa = \mathfrak{s} = \mathfrak{a}_*$ and $\mathbf{U}(\kappa) = \kappa$, see Definition 2.2 below and $J_* \subseteq [\omega]^{\aleph_0}$ is dense then there is a saturated MAD family $\subseteq J_*$.

Recall

Definition 2.2. 1) For cardinals $\partial \le \sigma < \lambda, \partial \le \theta \le \lambda$ (here usually $\sigma \le \theta$) let $\mathbf{U}_{\theta,\sigma,\partial}(\lambda) = \mathrm{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\le \sigma}$ such that for every $X \in [\lambda]^\theta$ for some $u \in \mathcal{P}$ we have $|X \cap u| \ge \partial\}$. If $\partial = \sigma$ we may omit ∂; if $\sigma = \partial = \aleph_0$ we may omit them both, and if $\sigma = \partial = \aleph_0 \wedge \theta = \lambda$ we may omit θ, σ, ∂. In the case of our Theorem, it means: $\mathbf{U}(\kappa) = \mathrm{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\kappa]^{\le \aleph_0}$ and $(\forall X \in [\kappa]^\kappa)(\exists u \in \mathcal{P})(|X \cap u| \ge \aleph_0)\}$.
2) If in addition J is an ideal on θ then $\mathbf{U}_{\theta,\sigma,J}(\lambda) = \mathrm{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\le\sigma}$ such that for every function $f : \theta \to \lambda$ for some $u \in \mathcal{P}$ the set $\{i < \theta : f(i) \in u\}$ does not belong to $J\}$.
3) Let $\mathrm{Pr}(\kappa, \theta, \sigma, \partial)$ mean: $\kappa \ge \theta \ge \sigma \ge \partial, \theta > \partial$ and we can find $(E, \bar{\mathcal{P}})$ witnessing it (if $\partial = \sigma$ we may omit ∂, if $\sigma = \partial = \aleph_0$ we may omit them both, if $\sigma = \partial = \aleph_0 \wedge \theta = \kappa$ we may omit θ, σ, ∂), which means:

- (a) $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha \in E \rangle$
- (b) E is a club of κ and $\gamma \in E \Rightarrow |\gamma|$ divide γ
- (c) if $u \in \mathcal{P}_\alpha$ then $u \in [\alpha]^{\le\sigma}$ has no last member
- (d) $\bullet_1$ $\bar{\mathcal{P}}$ is $\subseteq$-increasing
  - $\bullet_2$ $|\mathcal{P}_\alpha| < \kappa$
- (e) if $w \subseteq \kappa$ is bounded and $\mathrm{otp}(w) = \theta$ and $\sup(w) \in \mathrm{acc}(E)$ then for some u, j we have (recalling $\theta > \partial$):
  - $\bullet_1$ $|u \cap w| \ge \partial$
  - $\bullet_2$ $j \in \mathrm{acc}(E)$
  - $\bullet_3$ $u \in \mathcal{P}_j$
  - $\bullet_4$ $|w \cap j| < \theta$,i.e. $j < \sup(w)$
- (f) if $i \in \{0\} \cup E$ and $j = \min(E \backslash (i+1)), w \subseteq [i,j), \mathrm{otp}(w) = \theta$ then for some set u
  - $\bullet_1$ $u \in \mathcal{P}_j$ and $u \subseteq (i,j)$
  - $\bullet_2$ $|u \cap w| \ge \partial$.

Explanation 2.3. The proof of 2.1 is based on the proof of 1.1. The difference is that in the proof of $\odot_2$ of subcase 2B of stage F, if $\ell g(\nu^*_B) = \kappa$ it does not follow that we have $|\mathcal{A}^*| < \mathfrak{a}_*$, so we have to do something else when $|\mathcal{A}^*| = \mathfrak{a}_* = \mathfrak{s}$. By the assumption $\mathbf{U}(\kappa) = \kappa$ there is a sequence $\langle u_\alpha : \omega \le \alpha < \kappa \rangle$ of members of $[\kappa]^{\aleph_0}$ such that $u_\alpha \subseteq \alpha$ and for every $X \in [\kappa]^\kappa$ for some $\alpha, u_\alpha \cap X$ is infinite. Now if e.g. $\ell g(\nu) = \alpha \ge \omega$ we can use u_α and apply 2.5 below to appropriate $\bar{B}_\nu$ and get $\mathcal{P}_\nu$ and add it to the family $\{C^*_\alpha : \alpha < \kappa\}$ witnessing $\mathfrak{s} = \kappa$ the family $\mathcal{P}_\nu$ as in 2.5. So now we really need to use C^s_ν rather than C^*_α.

Observation 2.4. If $\mathrm{Pr}(\kappa, \theta, \sigma, \partial)$ is satisfied by $(E, \bar{\mathcal{P}})$ then we can find $(E', \bar{\mathcal{P}}')$ as in 2.2(3) but

$(d)'$ $\bullet_2$ if $j > \sup(j \cap E')$ then $|\mathcal{P}'_j| \le j$

(e) as above but instead $\sup(w) \in \mathrm{acc}(E)$ require just $\sup(w) \in E \cup \{\kappa\}$.

Proof. Use any club $E' \subseteq \mathrm{acc}(E)$ of κ such that $\delta \in E' \Rightarrow |\mathcal{P}_\delta| \le |\min(E' \backslash (\delta+1))|$ and $\delta \in \mathrm{nacc}(E') \Rightarrow \mathrm{cf}(\delta) \ne \mathrm{cf}(\theta)$ and let $\mathcal{P}'_\gamma$ be $\mathcal{P}_\gamma$ if $\gamma \in \mathrm{acc}(E')$ and be $\cup\{\mathcal{P}_\beta : \beta \in E \cap \gamma\}$ if $\gamma \in \mathrm{nacc}(E')$. $\square_{2.4}$

Claim 2.5. *Assume $\bar{B}^* = \langle B^*_n : n < \omega\rangle$ satisfies $B^*_n \in [\omega]^{\aleph_0}, B^*_{n+1} \subseteq B^*_n$ and $|B^*_n \backslash B^*_{n+1}| = \aleph_0$ for infinitely many n's. Then we can find $\mathcal{P}$ such that:*

$(*)$ (a) *$\mathcal{P} \subseteq [\omega]^{\aleph_0}$ is of cardinality $\mathfrak{b}$*

(b) *if $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ is an AD family, $B \subseteq \omega$ and $(\exists^\infty n)(B \cap B^*_n \backslash B^*_{n+1} \notin \mathrm{id}_{\mathcal{A}})$ or just there is a sequence $\langle (n_i, A_i) : i < \omega\rangle$ such that $n_i < n_{i+1}, A_i \in \mathcal{A}\backslash\{A_j : j < i\}$ and $A_i \cap B_{n_i} \backslash B_{n_i+1}$ is infinite for every i then for some countable (infinite) $\mathcal{P}' \subseteq \mathcal{P}$ for $2^{\aleph_0}$ function $\eta \in {}^{\mathcal{P}'}2$ we have: for some $\mathrm{id}_{\mathcal{A}}$-positive set $A \subseteq^* B$ we have: $A \subseteq^* C^{[\eta(C)]}$ for every $C \in \mathcal{P}'$ and $A \subseteq^* B_n$ for every n.*

Proof. Let $\mathcal{B} = \{\bar{B} : \bar{B} = \langle B_n : n < \omega\rangle$ where $B_n \subseteq \omega$ is infinite, $B_n \supseteq B_{n+1}$ and $B_n \backslash B_{n+1}$ is infinite for infinitely many $n < \omega\}$, i.e. the set of $\bar{B}$ satisfying the demands on $\bar{B}^*$.

For $\bar{B} \in \mathcal{B}$ and $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ let $\mathrm{pos}(\bar{B}, \mathcal{A}) = \{B \subseteq \omega : B \cap B_n \backslash B_{n+1} \notin \mathrm{id}_{\mathcal{A}}$ for infinitely many n's or there is a sequence $\langle (n_i, A_i) : i < \omega\rangle$ as in $(*)(b)\}$.

We shall use freely

$\odot$ assume $\bar{B} \in \mathcal{B}$ and $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ is AD

(a) if $(\exists^\infty n)(B \cap B_n \backslash B_{n+1} \in \mathrm{id}^+_{\mathcal{A}})$ then there is a sequence $\langle (n_i, A_i) : i < \omega\rangle$ as in $(*)(b)$, that is: $n_i < n_{i+1}, (B_{n_i} \backslash B_{n_i+1}) \cap A_i$ is infinite and $A_i \in \mathcal{A}\backslash\{A_j : j < i\}$ for every i

(b) if $B \in \mathrm{pos}(\bar{B}, \mathcal{A})$ then $B \in \mathrm{id}^+_{\mathcal{A}}$

(c) for $B \subseteq \omega, (\forall n)(\exists n)[n < m \wedge (B_n \backslash B_m) \cap B \in \mathrm{id}^+_{\mathcal{A}}]$ iff $\forall(\exists^\infty n)[(B_n \cap B_{n+1}) \cap B \in \mathrm{id}^+_{\mathcal{A}}]$.

So Observation 2.5 says that for every $\bar{B} \in \mathcal{B}$ there is $\mathcal{P} \subseteq [\omega]^{\aleph_0}$ of cardinality $\mathfrak{b}$ such that if $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ is an AD family and $B \in \mathrm{pos}(\bar{B}, \mathcal{A})$ then there is a countable infinite $\mathcal{P}' \subseteq \mathcal{P}$ as required in $(*)(b)$ of 2.5.

Consider the statement:

$\boxplus$ if $\bar{B} \in \mathcal{B}$ then we can find $\mathbf{B}$ such that

(a) $\mathbf{B} = \langle \bar{B}_\delta : \delta \in S^{\mathfrak{b}}_{\aleph_0}\rangle$ recalling $S^{\mathfrak{b}}_{\aleph_0} = \{\delta < \mathfrak{b} : \mathrm{cf}(\delta) = \aleph_0\}$

(b) $\delta \in S^{\mathfrak{b}}_{\aleph_0} \Rightarrow \bar{B}_\delta \in \mathcal{B}$

(c) if $\mathcal{A}$ is an AD family and $B \in \mathrm{pos}(\bar{B}, \mathcal{A})$, then for some club E of $\mathfrak{b}$, for every $\delta \in E \cap S^{\mathfrak{b}}_{\aleph_0}$ we have $(\exists^\infty n)[B \cap (B_{\delta,n} \backslash B_{\delta,n+1}) \in \mathrm{id}^+_{\mathcal{A}}]$

(d) if $\delta_1 < \delta_2$ are from $S^{\mathfrak{b}}_{\aleph_0}$ then for some $n < \omega$ the set $B_{\delta_1,n} \cap B_{\delta_2,n}$ is finite.

Why is this statement enough? By it we can find a subset $\mathcal{B}'$ of $\mathcal{B}$ of cardinality $\mathfrak{b}$ such that $\bar{B}^* \in \mathcal{B}'$ and for every $\bar{B} \in \mathcal{B}'$ for some $\mathbf{B} = \langle \bar{B}_\delta : \delta \in S^{\mathfrak{b}}_{\aleph_0}\rangle$ as in $\boxplus$ we have $\delta \in S^{\mathfrak{b}}_{\aleph_0} \Rightarrow \bar{B}_\delta \in \mathcal{B}'$. Now $\mathcal{P}$, the closure by Boolean operations of $\{B_n : \bar{B} \in \mathcal{B}'$ and $n < \omega\}$ is as required.

To show this let $\bar{B} \in \mathcal{B}'$ (e.g. $\bar{B}^*$) and an AD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ and assume $B \in \operatorname{pos}(\bar{B}, \mathcal{A})$ be given.

We choose by induction on $n < \omega$ a sequence $\langle \bar{B}_\eta, m_\eta : \eta \in {}^n 2\rangle$ such that

- $\bar{B}_\eta \in \mathcal{B}'$ moreover $(\exists^\infty n)(B_{\eta,n} \backslash B_{\eta,n+1} \in \operatorname{id}^+_{\mathcal{A}})$ for $\eta \in {}^n 2$
- $\bar{B}_\eta = \bar{B}$ if $\eta = \langle\rangle$ so $n = 0$
- $B \in \operatorname{pos}(\bar{B}_\eta, \mathcal{A}), m_\eta < \omega$ and moreover $\cap\{B_{\eta \restriction \ell, m_{\eta\restriction \ell}} : \ell \leq n\} \cap B \in \operatorname{pos}(\bar{B}_\eta, \mathcal{A}) \subseteq \operatorname{id}^+_{\mathcal{A}}$ if $\eta \in {}^n 2$
- if $\nu\hat{\ }\langle 0\rangle, \nu\hat{\ }\langle 1\rangle \in {}^n 2$ then the set $B_{\nu\hat{\ }\langle 0\rangle, m_{\nu\hat{\ }\langle 0\rangle}} \cap B_{\nu\hat{\ }\langle 1\rangle, m_{\nu\hat{\ }\langle 1\rangle}}$ is finite.

For $n = 0$ this is trivial and for $n = m+1$ we use $\boxplus(c)$, i.e. the construction of $\mathcal{B}'$. For every $n < \omega, \varrho \in {}^n 2$ let $B_\varrho = \cap\{B_{\eta\restriction k, m_{\eta\restriction k}} : k \leq n\}$. So $B_\varrho \in \operatorname{id}^+_{\mathcal{A}}$ and $m < \ell g(\varrho) \Rightarrow B_\varrho \subseteq B_{\varrho\restriction m}$ and if $\varrho_1 \neq \varrho_2 \in {}^n 2$ then the set $B_{\varrho_1} \cap B_{\varrho_2}$ is finite. Obviously $[\varrho \in {}^\omega 2 \Rightarrow (\forall n < \omega)(\exists k < \omega)(B_{\varrho\restriction n} \backslash B_{\varrho \restriction k} \in \operatorname{id}^+_{\mathcal{A}})]$ in fact $k = n+1$ is enough hence for each $\varrho \in {}^\omega 2$ there is $C_\varrho \in \operatorname{id}^+_{\mathcal{A}}$ such that $C_\varrho \subseteq^* B_{\varrho\restriction n}$ for $n < \omega$.

[Why? We try by induction on $k < \omega$ to choose $A_{\varrho,k}, A'_{\varrho,k} \in \operatorname{ob}(\omega)$ such that $A'_{\varrho,k} \in \mathcal{A}, A_{\varrho,k} \subseteq A'_{\varrho,k}$ and $m < k \Rightarrow A'_{\varrho,k} \neq A'_{\varrho,m}$ and $A_{\varrho,k} \subseteq^* B_{\varrho\restriction k}$. Now first, if we succeed then we can find $C \in \operatorname{ob}(\omega)$ such that for every $n < \omega$ we have $C \cap A_{\varrho,n}$ is infinite and $C \setminus \cup\{A'_{\varrho,m} : m < n\} \subseteq B_{\varrho\restriction k_n}$. If there is an infinite $C' \subseteq C$ almost disjoint to every member of $\mathcal{A}$, then $C_\varrho = C'$ is as required. If there is no such C' then we can find pairwise distinct $A''_n \in \mathcal{A} \setminus \{A'_{\varrho,m} : m < \omega\}$ such that $C \cap A''_n$ is infinite for every $n < \omega$. Clearly $A''_n \cap C \subseteq^* B_{\varrho\restriction m}$ for every $n, m < \omega$ and there is an infinite $C_\varrho \subseteq C$ such that $C_\varrho \subseteq^* B_{\varrho\restriction m}$ and $C_\varrho \cap A''_n$ is infinite for every $n, m < \omega$, so C_ϱ is as required.

Second, if $k < \omega$ and we cannot choose $A_{\varrho,k}$ then we can choose $C_\varrho \in \operatorname{ob}(\omega)$ such that $n < \omega \Rightarrow C_\varrho \subseteq^* B_{\varrho\restriction n}$ and $C_\varrho \cap A_{\varrho,m} = \emptyset$ for $m < k$, and C_ϱ is as required, so we are done.]

So $\mathcal{P}' = \{B_{\eta\restriction k,m} : k, m < \omega\}$ is as required.

So proving $\boxplus$ is enough.

Why does this statement hold?

Let $\bar{f} = \langle f_\alpha : \alpha < \mathfrak{b}\rangle$ be a sequence of members of ${}^\omega\omega$ witnessing $\mathfrak{b}$ and without loss of generality $f_\alpha \in {}^\omega\omega$ is increasing and $\alpha < \beta < \mathfrak{b} \Rightarrow f_\alpha <_{J^{\rm bd}_\omega} f_\beta$.

For $\alpha < \mathfrak{b}$ let $C_\alpha := \cup\{B_n \cap [0, f_\alpha(n)) : n < \omega\}$ so clearly

$$(*)_1 \quad (a) \quad \alpha < \beta \Rightarrow C_\alpha \subseteq^* C_\beta$$
$$\qquad (b) \quad \alpha < \mathfrak{b} \wedge n < \omega \Rightarrow C_\alpha \subseteq^* B_n.$$

We choose $\alpha_\varepsilon = \alpha(\varepsilon) < \mathfrak{b}$ by induction on $\varepsilon < \mathfrak{b}$, increasing with ε as follows: for $\varepsilon = 0$ let $\alpha_\varepsilon = \min\{\alpha < \mathfrak{b} : C_\alpha \text{ is infinite}\}$, for $\varepsilon = \zeta+1$ let $\alpha_\varepsilon = \min\{\alpha < \mathfrak{b} : \alpha > \alpha_\zeta$ and $C_\alpha \backslash C_{\alpha(\zeta)}$ is infinite$\}$ and for ε limit let $\alpha_\varepsilon = \cup\{\alpha_\zeta : \zeta < \varepsilon\}$. By the choice of $\bar{f}$ every α_ε is well defined, see the proof of $\oplus_\alpha$ below.

So $\langle \alpha_\varepsilon : \varepsilon < \mathfrak{b}\rangle$ is increasing continuous with limit $\mathfrak{b}$. For each $\delta \in S^{\mathfrak{b}}_{\aleph_0}$ let $\langle \varepsilon(\delta, n) : n < \omega\rangle$ be increasing with limit δ and, lastly, let $\bar{B}_\delta = \langle C_{\alpha(\delta)} \backslash \bigcup\limits_{m \leq n} C_{\alpha(\varepsilon(\delta,m))} : n < \omega\rangle$ so $B_{\delta,n} = C_{\alpha(\delta)} \backslash \bigcup\limits_{m\leq n} C_{\alpha(\varepsilon(\delta,m))}$ hence $B_{\delta,n+1} \subseteq B_{\delta,n}$ and $B_{\delta,n} \backslash B_{\delta,n+1}$ is infinite by the choice of $\alpha_{\varepsilon(\delta,n)+1}$. Clearly $\bar{B}_\delta \in \mathcal{B}$ (also follows from the proof below).

Why is $\langle \bar{B}_\delta : \delta \in S^{\mathfrak{b}}_{\aleph_0}\rangle$ as required in $\boxplus$? Clauses (a) + (b) are obvious and clause (d) is easy (as if $\delta_1 < \delta_2$ then for some n we have $\delta_1 < \alpha(\varepsilon(\delta_2, n))$ hence $B_{\delta_1,n} \cap B_{\delta_2,n} \subseteq B_{\alpha(\varepsilon(\delta_1,n))} \cap (B_{\alpha(\delta_2)} \backslash C_{\alpha(\varepsilon(\delta_2,n))}) \subseteq^* B_{\alpha(\delta_1)} \cap (B_{\alpha(\delta_2)} \backslash B_{\alpha(\delta_1)}) = \emptyset$).

Lastly, to check clause (c) of $\boxplus$ let $\mathcal{A}$ be an AD family and $B \subseteq \omega$ be such that

$(*)_2$ $u = u_B := \{n < \omega : B \cap B_n \backslash B_{n+1} \notin \text{ id}_{\mathcal{A}}\}$ is infinite, or just $\langle (n_i, A_i) : i < \omega\rangle$ is as in $\odot(a)$ for B and $u = u_B = \{n_i : i < \omega\}$.

It is enough to prove that for every $\alpha < \mathfrak{b}$

$\oplus_\alpha$ there is $\beta \in (\alpha, \mathfrak{b})$ such that $B \cap C_\beta \backslash C_\alpha \in \text{ id}^+_{\mathcal{A}}$.

[Why is it enough? As then for some club E of $\mathfrak{b}$, for every $\delta \in E \cap S^{\mathfrak{b}}_{\aleph_0}$ we have $(\forall \varepsilon < \delta)(\alpha_\varepsilon < \delta)$ and $(\forall \alpha < \delta)(\exists \beta)(\alpha < \beta < \delta \wedge C_\beta \backslash C_\alpha \in \text{ id}^+_{\mathcal{A}})$ hence $(\exists^\infty n)((C_{\alpha(\varepsilon(\delta,n+1))} \backslash C_{\alpha(\varepsilon(\delta,n))}) \in \text{ id}^+_{\mathcal{A}})$ which means $(\exists^\infty n)(B_{\delta,n} \backslash B_{\delta,n+1}) \in \text{ id}^+_{\mathcal{A}})$ as required.]

So let us prove $\oplus_\alpha$.

If $\oplus_\alpha$ fails, for every $\beta \in (\alpha, \mathfrak{b})$ there are $n = n(\beta)$ and $A_{\beta,0}, \ldots, A_{n(\beta)-1} \in \mathcal{A}$ such that $B \cap C_\beta \backslash C_\alpha \subseteq^* A_{\beta,0} \cup \ldots \cup A_{\beta,n(\beta)-1}$. Without loss of generality $n(\beta)$ is minimal hence by $(*)_1$ the sequence $\langle n(\beta) : \beta \in [\alpha, \mathfrak{b})\rangle$ is non-decreasing, but $\mathfrak{b} = \text{ cf}(\mathfrak{b}) > \aleph_0$, hence, for some $\alpha_* \in [\alpha, \mathfrak{b})$, the sequence $\langle n(\beta) : \beta \in [\alpha_*, \mathfrak{b})\rangle$ is constant and let $n(\alpha_*) = n_*$.

As $\mathcal{A}$ is AD and $B \cap C_{\alpha_*} \backslash C_\alpha \subseteq^* A_{\alpha_*,0} \cup \ldots \cup A_{\alpha_*,n_*-1}$ and $\beta \in (\alpha_*, \mathfrak{b}) \Rightarrow B \cap C_{\alpha_*} \backslash C_\alpha \subseteq B \cap C_\beta \backslash C_\alpha \subseteq A_{\beta,0} \cup \ldots \cup A_{\beta,n_*-1}$, using "$\mathcal{A}$ is almost disjoint" and the minimality of $n_{\alpha_*} = n_*$ it follows that $\{A_{\alpha_*,\ell} : \ell < n_*\} \subseteq \{A_{\beta,\ell} : \ell < n_*\}$ hence they are equal.

So

$\odot$ $\beta \in (\alpha, \mathfrak{b}) \Rightarrow B \cap C_\beta \backslash C_\alpha \subseteq^* A_{\alpha_*,0} \cup \ldots \cup A_{\alpha_*,n_*-1}$.

Let $w = \{n_i : i < \omega\}$ and $A_{n_i} \notin \{A_{\alpha_*,\ell} : \ell < n_*\}$, where n_i, A_i are from $(*)_2$ (recall $\odot(a)$) now for each $n \in w$ we have $B \cap (B_n \backslash B_{n+1})$ is infinite; but by the choice of C_α the set $(B_n \cap B_{n+1}) \cap C_\alpha$ is finite hence $B \cap B_n \backslash B_{n+1} \backslash C_\alpha \in$ is infinite so by the choice of w clearly there is $k_n \in (B \cap B_n \backslash B_{n+1} \backslash C_\alpha) \backslash A_{\alpha_*,0} \backslash \ldots \backslash A_{\alpha_*,n_*-1} \backslash \{k_0, \ldots, k_{n-1}\}$. Define $g : \omega \to \omega$ by $g(n) = k_{\min(w \backslash n)}$. By the choice of $\bar{f}$ there is $\beta \in (\alpha_*, \mathfrak{b})$ such that $u_1 := \{n < \omega : g(n) < f_\beta(n)\}$ is infinite. As f_β is increasing, clearly $n \in u_1 \Rightarrow k_{\min(w\backslash n)} = g(n) < f_\beta(n) \leq f_\beta(\min(w \backslash n)) \Rightarrow k_{\min(w\backslash n)} \in C_\beta \backslash C_\alpha$. So[3] $\{k_{\min(w\backslash n} : n \in u_1\} \in [\omega]^{\aleph_0}$ is infinite and is a subset of $B \cap C_\beta \backslash C_\alpha \backslash A_{\alpha_*,0}, \ldots, A_{\alpha_*,n_*-1}$, contradiction so $\oplus_\alpha$ indeed holds, so we are done. $\square_{2.5}$

Proof. Proof of 2.1 We prove part (2), and part (1) follows from it. We immitate the proof of 1.1.

Stage A:

Let $\kappa = \mathfrak{s}$. Let $\mathcal{P} \subseteq [\kappa]^{\aleph_0}$ witness $\mathbf{U}(\kappa) = \kappa$, for transparency we assume $\omega \in \mathcal{P}$ and $u \in \mathcal{P} \Rightarrow \text{ otp}(u) = \omega$, this holds without loss of generality as $\mathfrak{b} \leq \mathfrak{a}_* = \mathfrak{s} = \kappa$. [Why? It is enough to show that for every countable $u \subseteq \kappa$ there is a family $\mathcal{P}_u$ of cardinality $\leq \mathfrak{b}$ of subsets of u each of order type ω such that every infinite subset of u has an infinite intersection with some member of $\mathcal{P}$. Without loss of generality u is a countable ordinal α and we prove this by induction on α. For α successor

[3] we can use k_n for $n \in u_1 \cap w$ which is infinite

ordinal or not divisible by ω^2 this is trivial so let $\langle \alpha_n : n < \omega \rangle$ be an increasing sequence of limit ordinals with limit α but $\alpha_0 = 0$. Let $\langle \beta_{n,k} : k < \omega \rangle$ list $[\alpha_n, \alpha_{n+1})$ with no repetitions and let $\langle f_\epsilon \in {}^\omega\omega : \epsilon < \mathfrak{b} \rangle$ exemplifies $\mathfrak{b}$, each f_ϵ increasing and let $\mathcal{P}_\alpha = \cup\{\mathcal{P}_\beta : \beta < \alpha\} \cup \{\{\beta_{n,k} : n < \omega, k < f_\epsilon(n)\} : \varepsilon < \mathfrak{b}\}$. Clearly $\mathcal{P}_\alpha$ has the right form and cardinality.

Lastly, assume $v \subseteq u$ is infinite, if for some $\gamma < \alpha, u \cap \gamma$ is infinite use the choice of $\mathcal{P}_\gamma$. Otherwise let $f \in {}^\omega\omega$ be defined by $f(n) = \min\{k : (\exists m)[n \leq m \wedge \beta_{m,k} \in v]\}$, and use $\epsilon < \mathfrak{b}$ large enough.]

Let $\langle u_\alpha : \alpha < \kappa \rangle$ list $\mathcal{P}$ possibly with repetitions, without loss of generality $n \leq \omega \Rightarrow u_n = \omega$ and $\alpha > \omega \Rightarrow u_\alpha \subseteq \alpha$. For $\alpha < \kappa$ let $\langle \gamma(\alpha, k) : k < \omega \rangle$ list u_α in increasing order and $\gamma_{\alpha,k} = \gamma(\alpha, k)$.

Let $\langle \mathcal{U}_\alpha : \alpha < \kappa \rangle$ be a partition of κ, to sets each of cardinality κ such that $\min(\mathcal{U}_{1+\alpha}) \geq \sup(u_\alpha) + 1$ and $\omega \subseteq \mathcal{U}_0$. Let $\langle C^*_\alpha : \alpha \in \mathcal{U}_0 \rangle$ list a subset of $\mathcal{P}(\omega)$ witnessing $\mathfrak{s} = \kappa$ and as in Stage A of the proof of 1.1, the set $J_* \subseteq \text{ ob}(\omega)$ is dense and $\omega \notin \text{ id}_{J_*}$.

If $\bar{B}$ is as in the assumption of 2.5 and $\alpha \in (0, \kappa)$ let $\mathcal{P}_{\bar{B}}$ be as in the conclusion of 2.5 and for $\alpha < \kappa$ let $\langle C^*_{\bar{B},\alpha,i} : i \in \mathcal{U}_\alpha \rangle$ list $\mathcal{P}_{\bar{B}}$.

Stage B: As in the proof of 1.1 but we use $C^s_\rho (\rho \in \mathcal{T}_s)$ which may really depend on s and where $C^t_\rho, \bar{B}^t_{\nu,\beta}$ are defined in clauses $\boxplus_1(e), (g), (h), (i), (j)$ below (so the $\boxplus(e), (g), (h)$ from 1.1 are replaced) and depend just on $\mathcal{T}_t, \bar{A}_t$ and $\bar{I}_t$, too[4] where

$\boxplus_1$ (a)-(d) and (f) as in 1.1 of course and

- (e) $\bullet_1$ as before, i.e. $\bar{A} = \langle A^t_\eta : \eta \in \text{ suc}(\mathcal{T}_t) \rangle$,
  - $\bullet_2$ $\bar{C} = \bar{C}_t = \langle C^t_\eta : \eta \in \Lambda_t \rangle$ where $\Lambda_t = \{\eta : \eta \in {}^i 2$ and $i \in \mathcal{U}_0$ or $\alpha > 0, i \in \mathcal{U}_\alpha$ and $\eta {\restriction} (\sup(u_\alpha)) \in \mathcal{T}_t$ or (for 2.6) $\eta \in \mathcal{T}_t\}$
  - $\bullet_3$ we stipulate $A_\eta = \emptyset$ if $\eta \in \mathcal{T} \setminus \text{ suc}(\mathcal{T})$
- (g) as in 1.1 but replacing $C^*_{\eta \restriction i}$ by $C^t_{\eta \restriction i}$
- (h) if $i \in \mathcal{U}_0$ and $\nu \in \mathcal{T}_t \cap {}^i 2$ then $C^s_\nu = C^*_i$
- (i) if $\beta \in (0, \kappa)$ and $\nu \in {}^{\sup(u_\beta)}2$ and both $\langle C^t_{\nu \restriction i} : i \in u_\beta \rangle$ and $\langle A^t_{\nu \restriction i} : i \in u_\beta \rangle$ are well defined then we let $\bar{B}^t_{\nu,\beta} = \langle B^t_{\nu,\beta,n} : n < \omega \rangle$ be defined by $B^t_{\nu,\beta,n} = \cap\{(C^t_{\nu \restriction \gamma(\beta,k)})^{[\nu(\gamma(\beta,k))]} \setminus A^t_{\nu \restriction \gamma(\beta,k)} : k < n\}$
- (j) if $\beta \in (0, \kappa), i \in \mathcal{U}_\beta$ hence $i \geq \sup(u_\beta)$ and $\rho \in {}^i 2$ and $\bar{B}^t_{\rho \restriction \sup(u_\beta),\beta}$ is well defined then, recalling stage A, $C^t_\rho = C^*_{\bar{B}^t_{\rho \restriction \sup(u_\beta),\beta},\beta,i}$.

Note that $\mathcal{T}_t, \bar{A}_t$ determine t, i.e. $\bar{I}_t, \Lambda_t, \bar{C}_t$ and $\langle \bar{B}^t_{\nu,\beta} : \nu, \beta$ as above$\rangle$.

Stage C:

As in 1.1 we just add:

$\boxplus_4$
- (h) if $s \leq_{\text{AP}} t$ and $\bar{B}^s_{\nu,\beta}$ is well defined then $\bar{B}^t_{\nu,\beta}$ is well defined and equal to it
- (i) if $s \leq_{\text{AP}} t$ and C^s_ν is well defined then C^t_ν is well defined and equal to it, so $\Lambda_s \subseteq \Lambda_t$
- (j) C^s_ν is well defined when $\nu \in c\ell(\mathcal{T}_s) \cap {}^{\kappa >}2$.

[4] also here we require $\eta \in \text{ suc}(\mathcal{T}_s) \Rightarrow A_\eta \neq \emptyset$

In the proof of $\boxplus_4(e)$ use the choice of $\langle C^s_\nu : \nu \in {}^i 2, i \in \mathcal{U}_0\rangle$, i.e. of $\langle C^*_\alpha : \alpha \in \mathcal{U}_0\rangle$ in Stage A.

Stages D,E: As in 1.1.

Stage F: The only difference is in the proof of $\odot_2$ in subcase(2B). Recall

Case 2: $\mathrm{SP}_B = \emptyset$ but not Case 1, i.e. $S_B \subseteq \mathcal{T}_s$, recall $B \subseteq \omega, B \notin J_s$.

Subcase 2B: $\nu^*_B \notin S_B$ where $\nu^*_B = \cup\{\eta : \eta \in S_B\}$

Recall $\Lambda, \mathcal{A}^*$ are defined there.

- $\odot_2$ there is a set B_1 such that
  - (a) $B_1 \subseteq B$ is infinite
  - (b) B_1 is almost disjoint to any $A \in \mathcal{A}^*$
  - (c) if Λ is finite then $\nu \in \Lambda \Rightarrow |B_1 \cap A_\nu| < \aleph_0$
  - (d) if Λ is infinite then for infinitely many $\nu \in \Lambda$ we have $|B_1 \cap A_\nu| = \aleph_0$.

So the rest of the proof is dedicated to proving $\odot_2$. If $|\mathcal{A}^*| < \kappa$ then $\mathcal{A}^*$ has cardinality $< \kappa = \mathfrak{s}$ hence by the theorem's assumption $|\mathcal{A}^*| < \mathfrak{s} = \mathfrak{a}_*$, so $\odot_2$ follows as in the proof of 1.1. So we can assume $|\mathcal{A}^*| = \kappa$ but $|\mathcal{A}^*| \le \aleph_0 + |\ell g(\nu^*_B)|$ hence necessarily $\ell g(\nu^*_B) = \kappa$ follows and let

$$\begin{aligned}\mathcal{W} := \{\alpha < \kappa : \quad & \text{for some } \ell \le n(\alpha) \text{ we have } A^*_{\alpha,\ell} \cap B \in \mathcal{A}^* \\ & \text{(equivalently } A^*_{\alpha,\ell} \cap B \text{ is infinite) but} \\ & A^*_{\alpha,\ell} \notin \{A^*_{\alpha_1,\ell_1} : \alpha_1 < \alpha \text{ and } \ell_1 \le n(\alpha_1)\}\}.\end{aligned}$$

For $\alpha \in \mathcal{W}$ choose $\ell(\alpha) \le n(\alpha)$ such that $B \cap A^*_{\alpha,\ell(\alpha)}$ is infinite and $A^*_{\alpha,\ell(\alpha)} \notin \{A^*_{\alpha_1,\ell_1} : \alpha_1 < \alpha$ and $\ell_1 \le n(\alpha_1)\}$, in fact by $\odot_1(b)$ the last condition follows. As $n(\alpha) < \omega$ for $\alpha < \kappa$, clearly $|\mathcal{W}| = \kappa$ because $|\mathcal{A}^*| = \kappa$ and $\mathrm{cf}(\kappa) = \mathrm{cf}(\mathfrak{s}) > \aleph_0$, hence by the choice of $\mathcal{P}$ there is $u_* \in \mathcal{P}$ such that $|\mathcal{W} \cap u_*|$ is infinite; let $\alpha(*) \in [\omega, \kappa)$ be such that $u_{\alpha(*)} = u_*$ and let $\nu = \nu^*_B \restriction \sup(u_*)$; recall that $\mathrm{otp}(u_*) = \omega$; note that

- $\odot_{2.1}$ $k < \omega \Rightarrow B^s_{\nu,\alpha(*),k+1} \subseteq B^s_{\nu,\alpha(*),k} \subseteq \omega$.

[Why? By their choice in $\boxplus_1(i)$.]

Recall also that $\langle \gamma_{\alpha(*),k} : k < \omega\rangle$ list u_* in increasing order and so

- $\odot_{2.2}$ $v := \{k < \omega : \gamma_{\alpha(*),k} \in \mathcal{W}\}$ is infinite;
- $\odot_{2.3}$ $[B^s_{\nu,\alpha(*),k}]^{\aleph_0} \supseteq I^s_{\nu \restriction \gamma(\alpha(*),k)}$ for $k < \omega$.

[Why? As for $k(1) < k$, $(C^s_{\nu \restriction \gamma(\alpha(*),k(1))})^{[\nu(\gamma(\alpha)(*),k(1))]}$ and $\omega \backslash A^*_{\nu \restriction \gamma(\alpha(*),k(1))}$ belongs to $\{X \subseteq \omega : [X]^{\aleph_0} \supseteq I^s_{\nu \restriction \gamma(\alpha(*),k)}\}$ hence by the definition of $B^s_{\nu,\alpha(*),k}$ in $\boxplus_1(i)$ it satisfies $\odot_{2.3}$.]

- $\odot_{2.4}$ $k \in v \Rightarrow B \cap B^s_{\nu,\alpha(*),k} \backslash B^s_{\nu,\alpha(*),k+1}$ is infinite.

[Why? For $k \in v$ let $\beta = \gamma(\alpha(*), k), n = n(\beta)$ and $\ell = \ell(\beta)$. On the one hand

$$[B \cap A^*_{\beta,\ell}]^{\aleph_0} \subseteq [A^*_{\beta,\ell}]^{\aleph_0} \subseteq I_{\nu^*_B \restriction \beta}.$$

On the other hand $A^*_{\beta,\ell}$ is disjoint to $(C^s_{\nu^*_B \restriction \beta})^{[\nu^*_B(\beta)]} \backslash A^s_{\nu^*_B \restriction \beta}$ if $A^*_{\beta,\ell} = A^*_{\beta,n}$ trivially and is almost disjoint to $(C^s_{\nu^*_B \restriction \beta})^{[\nu(\beta)]}$ otherwise, i.e. as

$$[(C^s_{\nu^*_B \restriction \beta})^{[1-\nu(\beta)]}]^{\aleph_0} \supseteq I_{(\nu^*_\beta \restriction \beta)\hat{\ }\langle 1-\nu(\beta)\rangle} \supseteq \{A^*_{\alpha,\ell}\}.$$

Hence $(C^s_{\nu^*_B \restriction \beta})^{[\nu^*_B(\beta)]} \backslash A^*_{\beta,\ell}$ is almost disjoint to $B \cap A^*_{\beta,\ell}$, an infinite set from $I_{\nu^*_B \restriction \beta}$ hence by $\odot_{2.3}$ from $[B^s_{\nu,\alpha(*),k}]^{\aleph_0}$.

So

$$B \cap B^s_{\nu,\alpha(*),k} \backslash B^s_{\nu,\alpha(*),k+1} = B \cap B^s_{\nu,\alpha(*),k} \backslash (B^s_{\nu,\alpha(*),k} \cap ((C^s_{\nu^*_B \restriction \beta})^{[\nu^*_B(\beta)]} \backslash A^*_{\beta,k})$$

almost contains this infinite set hence is infinite as promised.]

So by the choice of $\mathcal{P}_{\bar{B}^s_\nu,\alpha(*)}$, i.e. Observation 2.5 and clauses (i),(j) of $\boxplus_1$ for some $\beta \in \mathcal{U}_{\alpha(*)}$ so $\beta \geq \alpha(*) \geq \ell g(\nu)$ we have $B \backslash (C^s_{\nu^*_B \restriction \beta})^{[\ell]} \notin J_s$ for $\ell = 0, 1$ hence $B_1 := B \backslash (C^s_{\nu^*_B})^{[\nu^*_B(\beta)]} \notin J_s$ recalling that for $\beta \in \mathcal{U}_\alpha, \alpha \neq 0$ and $\rho \in {}^\beta 2$ the set C^s_ρ depends just on $\ell g(\rho)$ and $\rho \restriction \sup(u_\alpha)$ (and our s).

Now consider B_1 instead of B, clearly S_{B_1} is a subset of S_B and $\nu^*_B \restriction (\beta + 1)$ is not in it, but $B_1 \in$ ob(B) hence $S_{B_1} \subseteq S_B$ hence S_{B_1} is $\subseteq \{\nu^*_B \restriction \gamma : \gamma \leq \beta\}$ and B_1 fall under subcase (2A) as $\beta < \kappa = \ell g(\nu^*_B)$. $\square_{2.1}$

Theorem 2.6. *There is a saturated MAD family $\mathcal{A} \subseteq J_*$ when $\mathfrak{a}_* < \kappa = \mathfrak{s}, J_* \subseteq$ ob(ω) is dense and* $\Pr(\kappa, \mathfrak{a}_*)$, *see 2.2(3).*

Proof. Proof of 2.6 We immitate the proofs of 1.1, 2.1. Note that $\mathfrak{b} \leq \mathfrak{a}_* < \mathfrak{s}$.

Stage A:

Similarly to stage A of the proof of 2.1; let $(E, \bar{\mathcal{P}}^*)$ be as in Definition 2.2(3) and Observation 2.4, as $\mathfrak{b} < \kappa$, without loss of generality $u \in \mathcal{P}^*_\alpha \Rightarrow$ otp$(u) = \omega$ for $\alpha \in [\omega, \kappa)$. As we can replace E by any appropriate club E' of κ contained in acc(E) see Observation 2.4, without loss of generality otp$(E) =$ cf(κ), min$(E) \geq \omega$ and $\gamma \in E \Rightarrow \gamma + 1 + \mathfrak{b} <$ min$(E \backslash (\gamma + 1))$. Let $\langle \gamma^*_i : i <$ cf$(\kappa)\rangle$ list E in increasing order.

Let $\langle u_\gamma : \gamma < \kappa \rangle$ be such that $\langle u_\gamma : \gamma^*_i \leq \gamma < \gamma^*_{i+1} \rangle$ list $\mathcal{P}_{\gamma^*_{i+1}}$ (which includes $\mathcal{P}_{\gamma^*_i}$) and $u_j = \omega$ for $j < \gamma^*_0$.

Let $\langle \mathcal{U}_\alpha : \alpha < \kappa \rangle$ be a partition of $\{2i + 1 : i < \kappa\}$ such that min$(\mathcal{U}_{1+\alpha}) \geq \alpha + \omega, |\mathcal{U}_{1+\alpha}| = \mathfrak{b}, |\mathcal{U}_0| = \kappa, 1 \leq \alpha < \gamma^*_i \Rightarrow \mathcal{U}_\alpha \subseteq \gamma^*_i$.

Let $\langle C^*_i : i \in \mathcal{U}_0 \rangle$ list a family of subsets of ω witnessing $\mathfrak{s} = \kappa$ also J_* is as in the proof of 1.1.

Let $\mathcal{P}_{\bar{B}}, \langle C^*_{\bar{B},\alpha,i} : i \in \mathcal{U}_\alpha \rangle$ be as in 2.1, Stage A.

Stage B:

As in 2.1, i.e. the case $\mathfrak{s} = \mathfrak{a}_*$, but we change $\boxplus_1(f)$

$\boxplus_1$ (f) • $A_\eta \in I_\eta \cap J_*$ or $A_\eta = \emptyset$ and

- $\mathscr{S}_t := \{\eta \in \mathcal{T}_t : A_\eta \neq \emptyset\} \subseteq \{\eta \in \mathcal{T}_t : \ell g(\eta) = \gamma^*_i + 1$ for some $i < \kappa\}$.

Stage C:
As in the proof of 2.1.

Stage D:
Here there is a minor change: we replace $\boxplus_7$ in 1.1, 2.1 by $\boxplus_7, \boxplus_8, \boxplus_9$ below

$\boxplus_7$ if $\alpha < 2^{\aleph_0}, s \in \mathrm{AP}_\alpha$ and $B \in J^+_s$ then there are a limit ordinal $\xi \in \kappa \backslash E$ and $t \in \mathrm{AP}_{\alpha+1}$ such that $s \leq_{\mathrm{AP}} t$ and $|S^t_B \cap {}^\xi 2| = 2^{\aleph_0}$; we may add $\mathscr{S}_t = \mathscr{S}_s$.

This is proved in Stage F.
To clarify why this is O.K. recall $\boxplus_1(f)$ and note that

$(*)$ if $s \leq_{\mathrm{AP}} t, B \in \mathrm{ob}(\omega), \eta \in S^s_B \backslash \mathcal{T}_s$ and $\eta \notin \mathcal{T}_t$ then $\eta \in S^t_B$.

Now we need

$\boxplus_8$ if $\xi \in \kappa \backslash E$ is a limit ordinal, $\alpha < 2^{\aleph_0}, t \in \mathrm{AP}_\alpha, B \in \mathrm{ob}(\omega)$ and $|S^t_B \cap {}^\xi 2| = 2^{\aleph_0}$ and $\zeta = \min(E \backslash \xi)$ then for every t_1 and $\alpha + \zeta \leq \beta < 2^{\aleph_0}$ such that $t \leq_{\mathrm{AP}} t_1 \in \mathrm{AP}_\beta$ there is t_2 satisfying $t_1 \leq_{\mathrm{AP}} t_2 \in \mathrm{AP}_{\beta+1}$ and $(\exists \eta \in \mathrm{suc}(\mathcal{T}_{t_2}))[\eta \notin \mathcal{T}_{t_1} \wedge A^{t_2}_\eta \in \mathrm{ob}(\omega) \wedge A^{t_2}_\eta \subseteq B]$.

The proof of $\boxplus_8$ is like the proof of Case 1 in Stage F in the proof of 1.1 but we elaborate; we are given β, ξ, ζ and t_1 such that $t \leq_{\mathrm{AP}} t_1 \in \mathrm{AP}_\beta$; now we choose $\rho \in S^t_B \cap {}^\xi 2 \backslash \mathcal{T}_{t_1}$ exists as $|S^t_B \cap {}^\xi 2| = 2^{\aleph_0} > |\mathcal{T}_{t_1}|$ so recalling $(*)_5(e)$ necessarily $\rho \in S^{t_1,1}_B$. Choose B_1 such that $B_1 \subseteq B, B_1 \in I^t_\rho$.

Note that for every $\varepsilon \in [\xi, \zeta + 1)$ either $C^{t_1}_\varrho$ is well defined for every $\varrho \in {}^\varepsilon 2$ such that $\rho \trianglelefteq \varrho$ and its value is the same for all such ϱ (when ε is odd) or $C^{t_1}_\varrho$ for $\rho \trianglelefteq \varrho \in {}^\varepsilon 2$ is not well defined (when ε is even). So $\mathcal{B} = \{C^{t_1}_\varrho : \rho \triangleleft \varrho \in {}^{\zeta+1 \geq} 2$ and $C^{t_1}_\varrho$ is well defined$\}$ is a family of $\leq |\zeta| < \kappa = \mathfrak{s}$ subsets of B_1 hence there is an infinite $B_2 \subseteq B_1$ such that $\rho \trianglelefteq \varrho \in {}^{\zeta >} 2 \wedge (C^t_\varrho$ well defined$) \Rightarrow B_2 \subseteq^* C^t_\varrho \vee B_2 \subseteq^* \omega \backslash C^t_\varrho$ and without loss of generality $B_2 \in J_*$.

We choose η such that $\rho \triangleleft \eta \in {}^{\zeta+1} 2$ and

$[\ell g(\rho) \leq \gamma < \zeta + 1 \wedge (C^t_{\eta \restriction \gamma}$ is well defined$) \wedge B_2 \subseteq^* (C^t_{\eta \restriction \gamma})^{[\ell]} \wedge \rho \in \{0,1\} \Rightarrow \eta(\gamma) = \ell]$. Let us define $t_2 \in \mathrm{AP}_{\beta+\zeta+2} := \mathrm{AP}_{\beta+1}$ (as $\alpha + \zeta + 1 \leq \beta$ and $|\alpha_1| = |\alpha_2| \Rightarrow \mathrm{AP}_{\alpha_1} = \mathrm{AP}_{\alpha_2}$) as follows:

(a) $\mathcal{T}_{t_2} := \mathcal{T}_{t_1} \cup \{\varrho : \varrho \trianglelefteq \eta\}$

(b) $A^{t_2}_\varrho$ is $A^{t_1}_\varrho$ if well defined, is B_2 if $\varrho = \eta$ and is $\emptyset$ if $\eta \in \mathrm{suc}(\mathcal{T}_{t_2})$ but $A^{t_2}_\varrho$ is not already defined

(c) $C^{t_2}_\varrho$ is $C^{t_1}_\varrho$ if $\varrho \in \mathcal{T}_{t_1}$ and we choose $C^{t_2}_{\eta \restriction \varepsilon}$ by induction on $\varepsilon \in [\xi, \zeta + 2]$ as follows: if it is determined by $\boxplus_1$ we have no choice otherwise let it be $\omega^{[\eta(\varepsilon)]}$.

The other objects of t_2 are determined by those we have chosen. So $\boxplus_8$ holds indeed.

$\boxplus_9$ if $s \in \mathrm{AP}_\alpha$ and $\rho \in c\ell(\mathcal{T}_s)$ then for some $t, s \le_{\mathrm{AP}} t \in \mathrm{AP}_{\alpha+3}$ and $\mathcal{T}_s \subseteq \mathcal{T}_t \subseteq \mathcal{T}_s \cup \{\rho, \rho\hat{}\langle 0\rangle, \rho\hat{}\langle 1\rangle\}$ and $I^s_\rho \neq \emptyset \Rightarrow \rho \in \mathcal{T}_t$ and $I^s_\rho \neq \emptyset \wedge \ell < 2 \wedge I^t_{\rho\hat{}<\ell>} \neq \emptyset \Rightarrow \rho\hat{}\langle \ell\rangle \in \mathcal{T}_t$.

[Why? Easier than $\boxplus_8$.]

Stage E:
Similar to 1.1 with the changes necessitated by the change in Stage D.

Stage F:
We prove $\boxplus_7$, the proof splits to cases.

Case 1: Some $\nu \in S_B$ is such that $\nu \in c\ell(\mathcal{T}_s)\backslash\mathcal{T}_s$.

Let $B_1 \in \mathrm{ob}(B) \cap I^s_\nu$, there is such B_1 as $\nu \in S_B$ but $\nu \notin S^{s,2}_B$ as $\nu \notin \mathcal{T}_s$.

Let $C_{\nu,n} \in \mathrm{ob}(\omega)$ for $n < \omega$ be such that $\cap\{C^{[\varrho(n)]}_{\nu,n} : n < \ell g(\varrho)\} \cap B_1$ is infinite for every $\varrho \in {}^{\omega>}2$.

We choose $\mathcal{T}_t = T_s \cup \{\nu\hat{}\rho : \rho \in {}^{\omega>}2\}$. For $\rho \in {}^{\omega>}2$, we choose $C^t_{\nu\hat{}\rho}$ by induction on $\ell g(\rho)$: if $\ell g(\nu\hat{}\rho) = \ell g(\nu) + n$ is even and $n \in \{2m, 2m+1\}$ then $C^t_{\nu\hat{}\rho} = C_{\nu,m}$, if $\ell g(\nu\hat{}\rho) = \ell g(\nu) + n, n$ odd we act as in the proof of $\boxplus_8$, i.e. see stage A.

Lastly, let $A^t_{\nu\hat{}\rho} = \emptyset$ when $\rho \in {}^{\omega>}2$ is not covered by the previous cases (if there is one).

Easily

$(*)$ if $s \le_{\mathrm{AP}} s_1, |\mathcal{T}_{s_1}| < 2^{\aleph_0}$ then $|S^{s_1}_B \cap {}^{\ell g(\nu)+\omega}2| = 2^{\aleph_0}$.

So we are done with Case 1.

Case 2: $SP^s_B = \emptyset$ but not Case 1 and let $\nu^*_B = \cup\{\eta : \eta \in S_B\}$.

Subcase 2A: $\nu^*_B \in S_B$
As in the proof of 1.1, ending in contradiction.

Subcase 2B: $\nu^*_B \notin S_B$
With the exception of $\odot_2$ which we elaborate this is as in the proofs of 1.1, 2.1 but in the end replace "Subcase 1B" by "Case 1". Recall from Stage 2B as in the proof of 1.1, $\delta := \ell g(\nu^*_B)$ is a limit ordinal and $\nu^*_{B,\alpha} = (\nu^*_B{\restriction}\alpha)\hat{}\langle 1 - \nu^*_B(\alpha)\rangle$ for $\alpha < \delta$ and $\langle\langle A^*_{\alpha,n} : n \le n(\alpha)\rangle : \alpha < \delta\rangle$ and $\mathcal{A}^* := \{B \cap A : A = A^*_{\alpha,k}$ for some $\alpha < \delta, k \le n(\alpha)$ and $B \cap A$ is infinite$\}$, $\Lambda = \{\nu \in \mathscr{S}_s : \nu^*_B \trianglelefteq \nu$ and $A_\nu \cap B$ is infinite$\}$ are as in Stage (2B) of the proof of 1.1. We have to prove:

$\odot_2$ there is a set B_1 such that:
- (a) $B_1 \subseteq B$ is infinite
- (b) B_1 is almost disjoint to any $A \in \mathcal{A}^*$
- (c) if Λ is finite then $\nu \in \Lambda \Rightarrow |B_1 \cap A_\nu| < \aleph_0$
- (d) if Λ is infinite then for infinitely many $\nu \in \Lambda$ we have $|B_1 \cap A_\nu| = \aleph_0$.

Why $\odot_2$ holds? If $|\mathcal{A}^*| < \mathfrak{a}_*$ then as in the proof of 1.1 the statement of $\odot_2$ follows. So we can assume $|\mathcal{A}^*| \ge \mathfrak{a}_*$ and let

$$\mathcal{W} := \{\alpha < \kappa : \quad \begin{array}{l}\text{for some } \ell \le n(\alpha) \text{ we have } A^*_{\alpha,\ell} \cap B \in \mathcal{A}^* \\ (\text{equivalently } A^*_{\alpha,\ell} \cap B \text{ is infinite})\}.\end{array}$$

and let

$$\mathcal{W}' = \{\alpha \in W : |\alpha \cap \mathcal{W}| < \mathfrak{a}_*\}.$$

Subcase 2B(α): $\sup(\mathcal{W}') \in \operatorname{acc}(E)$

For $\alpha \in \mathcal{W}$ choose $\ell(\alpha) \le n(\alpha)$ such that $B \cap A^*_{\alpha,\ell(\alpha)}$ is infinite hence $A^*_{\alpha,\ell} \notin \{A^*_{\alpha_1,\ell_1} : \alpha_1 < \alpha, \ell_1 \le n(\alpha_1)\}$. As $n(\alpha) < \omega$ for $\alpha < \kappa$, clearly $|\mathcal{W}| = |\mathcal{A}^*| \ge \mathfrak{a}_*$ hence $\operatorname{otp}(\mathcal{W}') = \mathfrak{a}_*$.

So by Definition 2.2(3), i.e. the choice of $(E, \bar{\mathcal{P}})$, there is a pair (u_*, γ^*_j) as in clause (e) there. So $u_* \in \mathcal{P}^*_{\gamma^*_j}$ hence $u_* = u_{\alpha(*)}$ for some $\alpha(*) \in [\gamma^*_j, \gamma^*_{j+1}), \gamma^*_{j+1} < \sup(\mathcal{W}')$ and $\mathcal{W} \cap \sup(E \cap \gamma^*_j) = \mathcal{W}' \cap \sup(E \cap \gamma^*_j)$ has cardinality $< \mathfrak{a}_*$ and let $\nu = \nu^*_B \restriction \sup(u_*)$; recall $\operatorname{otp}(u_*) = \omega$.

Recall also that $\langle \gamma_{\alpha(*),n} : n < \omega \rangle$ list u_* in increasing order and so $v := \{n < \omega : \gamma_{\alpha(*),n} \in \mathcal{W}\}$ is infinite and clearly $n \in v \Rightarrow B^s_{\nu,\alpha(*),n} \backslash B^s_{\nu,\alpha(*),n+1}$ is infinite as in the proof of 2.1. So by the choice of $\mathcal{P}_{\bar{B}^s_\nu,\alpha(*)}$, i.e. 2.5 and clauses (i),(j) of $\boxplus_1$ for some $\beta \in \mathcal{U}_{\alpha(*)}$ so $\beta \ge \ell g(\nu)$ we have $B \backslash (C^s_{\nu^*_B \restriction \beta})^{[\ell]} \notin J_s$ for $\ell = 0, 1$ hence $B_1 := B \backslash (C^s_{\nu^*_B})^{[\nu^*_B(\beta)]} \notin J_s$ recalling that for $\beta \in \mathcal{U}_\alpha, \alpha \ne 0$ and $\rho \in {}^\beta 2$ the set C^s_ρ depends just on $\ell g(\rho)$ and $\rho \restriction \sup(u_\alpha)$ (and our s).

We finish as in the proof of 2.1.

Subcase 2B(β): $\sup(\mathcal{W}') \in (\gamma^*_i, \gamma^*_j]$ where $j = i + 1$ so $\gamma^*_i, \gamma^*_j \in E$ and let $\gamma^* = \sup(\mathcal{W}')$.

Apply Definition 2.2(3), clause (f) to $\gamma^*_i, \gamma^*_j, \mathcal{W}' \backslash \gamma^*_j$ we get $u = u_* \in \mathcal{P}_{\gamma^*_j}$ so $u = u_{\alpha(*)} \subseteq [\gamma^*_i, \gamma^*_j)$ for some $\alpha(*) \in (\gamma^*_i, \gamma^*_j)$.

Let $\beta = \sup(u_*), \nu = \nu^*_\beta \restriction \beta$ so by $\odot_{2.1} + \odot_{2.4}$ in stage 2B in the proof of 2.1, and by the choice of $\mathcal{P}_{\bar{B}_{\nu,\alpha(*)}}$ there is a $\mathcal{Q} \subseteq \mathcal{P}_{\bar{B}^s_{\nu,\alpha(*)}}$ of cardinality $\aleph_0$ and $\Lambda \subseteq {}^{\mathcal{Q}}2$ of cardinality $2^{\aleph_0}$ such that for every $\rho \in \Lambda$ there is $B_\rho \in \operatorname{ob}(B) \cap J^+_s$ such that $A \in \mathcal{Q} \Rightarrow B_\rho \subseteq^* A^{[\rho(A)]}$ and $n < \omega \Rightarrow B_{\eta,n} := B_n \backslash \cup \{C^{[\nu^*_B(\gamma_{\alpha(*),k})]}_{\nu^*_B \restriction \gamma_{\alpha(*),k}} : k < n\} \in I^s_{\nu^*_B \restriction \gamma_{\alpha(*),n}}$.

Clearly for some $v \subseteq \mathcal{U}_{\alpha(*)} \subseteq (\gamma^*_i, \gamma^*_j)$ of cardinality $\aleph_0, \nu \triangleleft \rho \in \mathcal{T}_s \wedge \ell g(\rho) \ge \sup(v) \Rightarrow \{C^s_{\rho \restriction \varepsilon} : \varepsilon \in v\} = \mathcal{Q}$.

For $\eta \in \Lambda$ analyzing S_{B_η} and recalling $\gamma^*_{j+1} < \mathfrak{s}$ clearly $S_{B_\eta} \cap \{\nu \in \mathcal{T}_s : \ell g(\nu) \le \sup(u_*)\}$ is $\{\nu^*_B \restriction \gamma : \gamma < \sup(u_*)\}$ and $\sup(u_*) > \gamma^* + 1$, so there is no ρ such that A^s_ρ is non-empty, $\rho \in \mathcal{T}_s$ and $\sup(u_*) \le \ell g(\rho) < \gamma^*_j$, so $S_{B_\eta} \cap \{\nu \in \mathcal{T}_s : \ell g(s) < \gamma^*_j\}$ does not depend on $\langle A^s_\eta : \eta \in \operatorname{suc}(\mathcal{T}_s), \ell g(\eta) \ge \gamma^*_j \rangle$ so we can finish easily as in case 1.

Case 3: As in the proof of 2.1. $\square_{2.6}$

3. Further Discussion

The cardinal invariant $\mathfrak{s}$ plays here a major role, so the claims depend on how $\mathfrak{s}$ and $\mathfrak{a}_*$ are compared; when $\mathfrak{s} = \mathfrak{a}_*$ it is not clear whether the further assumption of 2.1(2) may fail. If $\mathfrak{s} > \mathfrak{a}_* > \aleph_1$, it is not clear if the assumption of 2.6 may fail. Recall 1.1, dealing with $\mathfrak{s} < \mathfrak{a}_*$, the first case is proved ZFC, but the others need pcf assumptions.

All this does not exclude the case $\mathfrak{s} = \aleph_{\omega+1}, \mathfrak{a}_* = \aleph_1$ hence $\mathfrak{b} = \aleph_1$, as in [She04]. Fulfilling the promise from §0 and the abstract.

Claim 3.1. *1) If there is no inner model with a measurable cardinal (and even the non-existence of much stronger statements) <u>then</u> there is a saturated MAD family, $\mathcal{A}$.*
2) Also if $\mathfrak{s} < \aleph_\omega$ there is one.
3) Moreover, if $J_* \subseteq \mathrm{ob}(\omega)$ is dense <u>then</u> we can demand $\mathcal{A} \subseteq J_*$.

Proof. As Theorems 1.1, 2.1, 2.6 cover them (using well known results). $\square_{3.1}$

We now remark on some further possibilities.

Definition 3.2. 1) We say $\mathscr{S} \subseteq \mathrm{ob}(\omega)$ is $\mathfrak{s}$-free <u>when</u>:

- (a) for every $A \in \mathrm{ob}(\omega)$ there is $B \in \mathrm{ob}(A)$ such that B induces an ultrafilter on $\mathscr{S}$; i.e. $C \in \mathscr{S} \Rightarrow A \subseteq^* C \vee A \subseteq^* (\omega \backslash C)$.

1A) We say $\mathscr{S} \subseteq \mathrm{ob}(\omega)$ is $\mathfrak{s}$-free in I <u>when</u> $I \in \mathrm{OB}$ and for every $A \in I$ there is $B \in \mathrm{ob}(A)$ which induces an ultrafilter on $\mathscr{S}$.
2) We say $\mathscr{S} \subseteq \mathrm{ob}(\omega)$ is $\mathfrak{s}$-richly free <u>when</u> clause (a) and

- (b) if $A \in \mathrm{ob}(\omega)$ and the set $\{D \cap \mathscr{S} : D$ an ultrafilter on ω containing $\mathrm{ob}(A)\}$ is infinite, <u>then</u> it has cardinality continuum.

3) We say $\mathscr{S} \subseteq \mathrm{ob}(\omega)$ is $\mathfrak{s}$-anti-free if no $B \in \mathrm{ob}(\omega)$ induces an ultrafilter on $\mathscr{S}$.
4) Let $\mathfrak{S}$ be $\{\kappa$: there is a $\subseteq$-increasing sequence $\langle \mathscr{S}_i : i < \kappa\rangle$ of $\mathfrak{s}$-richly-free families such that $\cup\{\mathscr{S}_i : i < \kappa\}$ is not $\mathfrak{s}$-free.
5) Recall $\mathfrak{s} = \min\{|\mathscr{S}| : \mathscr{S} \subseteq \mathrm{ob}(\omega)$ and no $B \in \mathrm{ob}(\omega)$ induces an ultrafilter on $\mathscr{S}\}$.
5A) Let $\mathfrak{S}_{\mathrm{cmp}} = \{\kappa$: there is a sequence $\langle \mathcal{S}_i : i < \kappa\rangle$ such that: if $u \in [\kappa]^{<\kappa} \Rightarrow \cup\{\mathcal{S}_i : i \in u\}$ is $\mathfrak{s}$-richly free but $\cup\{\mathcal{S}_i : i < \kappa\}$ is not $\mathfrak{s}$-free$\}$.
6) We say $\mathrm{ch}_{\mathrm{dim}}(\mathcal{B}) < \kappa$ when $(\exists \eta \in {}^{\mathcal{B}}2)(I = I_{\mathcal{B},\eta}) \Rightarrow \mathrm{cf}(I, \supseteq) < \kappa$ recalling 0.7(5).

Observation 3.3. 1) If $\mathscr{S}$ is $\mathfrak{s}$-free and $\mathscr{S}' \subseteq \mathscr{S}$ <u>then</u> $\mathscr{S}'$ is $\mathfrak{s}$-free.
2) If $\mathscr{S} \subseteq \mathrm{ob}(\omega)$ and $|\mathscr{S}| < \mathfrak{s}$ <u>then</u> $\mathscr{S}$ is $\mathfrak{s}$-free.
3) If $\mathscr{S}_n \subseteq \mathrm{ob}(\omega)$ is $\mathfrak{s}$-free for $n < \omega$ <u>then</u> $\cup\{\mathscr{S}_n : n < \omega\}$ is $\mathfrak{s}$-free.
3A) If $\beta < \mathfrak{t}$ and $\mathcal{S}_\alpha \in \mathrm{ob}(\omega)$ is $\mathfrak{s}$-free for $\alpha < \beta$ <u>then</u> $\cup\{\mathscr{S}_\alpha : \alpha < \beta\}$ is $\mathfrak{s}$-free.
4) $\mathfrak{s} \in \mathfrak{S}$.
5) $\kappa \in \mathfrak{S}$ iff $\mathrm{cf}(\kappa) \in \mathfrak{S}$.
6) $\kappa \in \mathfrak{S} \Rightarrow \aleph_1 \leq \kappa \leq 2^{\aleph_0}$.
7) In Definition of $\mathfrak{S}$ we can add "$\cup\{\mathscr{S}_i : i < \kappa\}$ is $\mathfrak{s}$-anti-free".
8) $\mathrm{cf}(\mathfrak{s}) > \aleph_0$, in fact $\kappa \in \mathfrak{S} \Rightarrow \mathrm{cf}(\kappa) > \aleph_0$.

Definition 3.4. 1) We say $A \in \mathrm{ob}(\omega)$ obeys $f \in {}^\omega\omega$ <u>when</u>: for every $n_1 < n_2$ from A we have $f(n_1) < n_2$.

2) Let $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle$ be a sequence of members of ${}^\omega\omega$. We say $\bar{A} = \langle A_\alpha : \alpha \in u \rangle$ obeys $\bar{f}$ <u>when</u> $u \subseteq \delta$ and A obeys f_α for $\alpha \in A$.
3) $\mathfrak{a}_{\bar{f}} =$ Min$\{|u|$: there are $B \in$ ob(ω) and $\bar{A} = \langle A_\alpha : \alpha \in u \rangle$ obeying $\bar{f}$ such that $\{A_\alpha \cap B : \alpha \in u\}$ is a MAD of $B\}$.

Remark 3.5. 1) Also note that in 1.1, 2.1, 2.6 we can replace $\mathfrak{s}$ by a smaller (or equal) cardinal invariant $\mathfrak{s}_{\rm tree}$, the tree splitting number; where.
2) Let $\mathfrak{s}_{\rm tree}$ be the minimal κ such that there is a sequence $\bar{C} = \langle C_\eta : \eta \in {}^{\kappa >}2 \rangle$ such that $C_\eta \in {\rm ob}(\omega)$ for $\eta \in {}^{\kappa >}2$ and there is no $\eta \in {}^\kappa 2$ and $A \in {\rm ob}(\omega)$ such that $\epsilon < k \Rightarrow A \subseteq^* C_\eta^{[\eta(\epsilon]}$. Note that the minimal κ for which there is such sequence $\langle C_\eta : \eta \in {}^{\kappa >}2 \rangle$ has uncountable cofinality.
3) Also in 1.1 we may weaken $\mathfrak{s} < \mathfrak{a}_*$ to $\mathfrak{s} < \mathfrak{a} \wedge \mathfrak{s} \leq \mathfrak{a}_*$.

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