TRIVIAL AUTOMORPHISMS

ILIJAS FARAH AND SAHARON SHELAH

ABSTRACT. We prove that the statement 'For all Borel ideals $\mathcal{I}$ and $\mathcal{J}$ on ω, every isomorphism between Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$ and $\mathcal{P}(\omega)/\mathcal{J}$ has a continuous representation' is relatively consistent with ZFC. In this model every isomorphism between $\mathcal{P}(\omega)/\mathcal{I}$ and any other quotient $\mathcal{P}(\omega)/\mathcal{J}$ over a Borel ideal is trivial for a number of Borel ideals $\mathcal{I}$ on ω.

We can also assure that the dominating number, $\mathfrak{d}$, is equal to $\aleph_1$ and that $2^{\aleph_1} > 2^{\aleph_0}$. Therefore the Calkin algebra has outer automorphisms while all automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$ are trivial.

Proofs rely on delicate analysis of names for reals in a countable support iteration of suslin proper forcings.

1. Introduction

We start with a fairly general setting. Assume X/I and Y/J are quotient structures (such as groups, Boolean algebras, C*-algebras,...) with π_I and π_J denoting the respective quotient maps. Also assume Φ is an isomorphism between X/I and Y/J. A *representation* of Φ is a map $F\colon X \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\Phi_*"] \arrow[d, "\pi_I"'] & Y \arrow[d, "\pi_J"'] \\ X/I \arrow[r, "\Phi"'] & Y/J \end{tikzcd}$$

commutes. Since representation is not required to have any algebraic properties its existence follows from the Axiom of Choice and is therefore inconsequential to the relation of X/I, X/J and Φ.

We shall say that Φ is *trivial* if it has a representation that is itself a homomorphism between X and Y. Requiring a representation to be an isomorphism itself would be too strong since in many situations of interest there exists an isomorphism which has a representation that is a homomorphism but does not have one which is an isomorphism.

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In a number of cases of interest X and Y are structures of cardinality of the continuum and quotients X/I and Y/J are countably saturated in the model-theoretic sense (see e.g., [4]). In this situation Continuum Hypothesis, CH, makes it possible to use a diagonalization to construct nontrivial automorphisms of X/I and, if the quotients are elementarily equivalent, an isomorphism between X/I and Y/J. For example, CH implies that Boolean algebra $\mathcal{P}(\omega)/\operatorname{Fin}$ has nontrivial automorphisms ([31]) and Calkin algebra has outer automorphisms ([29] or [12, §1]). This is by no means automatic and for example the quotient group S_∞/G (where G is the subgroup consisting of finitely supported permutations) has the group of outer automorphisms isomorphic to $\mathbb{Z}$ and all of its automorphisms are trivial ([1]). Also, some quotient Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$ for Borel ideals $\mathcal{I}$ are not countably saturated and it is unclear whether nontrivial automorphisms exist (see [9]). A construction of isomorphism between quotients over two different density ideals that are not countably saturated in classical sense in [20] should be revisited using the logic of metric structures developed in [3]. As observed in [20], these two quotients have the natural structure of complete metric spaces and when considered as models of the logic of metric structures two algebras are countably saturated. This fact can be extracted from the proof in [20] or from its generalization given in [9].

We shall consider the opposite situation, but only after noting that by Woodin's Σ^2_1 absoluteness theorem ([40], [26]) Continuum Hypothesis provides the optimal context for finding nontrivial isomorphisms whenever X and Y have Polish space structure with Borel-measurable operations and I and J are Borel ideals (see [7, §2.1]).

The line of research to which the present paper belongs was started by the second author's proof that the assertion 'all automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$ are trivial' is relatively consistent with ZFC ([34]). A weak form of this conclusion was extended to some other Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$ in [18] and [17]. This line of research took a new turn when it was realized that forcing axioms imply all isomorphisms between quotients over Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$, for certain Borel ideals $\mathcal{I}$, are trivial ([32], [39], [19], [7], [10]). The first author conjectured in [11] that the Proper Forcing Axiom, PFA, implies all isomorphisms between any two quotient algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$, for a Borel ideal $\mathcal{I}$, are trivial. This conjecture naturally splits in following two rigidity conjectures.

(RC1) PFA implies every isomorphism has a continuous representation, and
(RC2) Every isomorphism with a continuous representation is trivial.

Noting that in our situation Shoenfield's Absoluteness Theorem implies that (RC2) cannot be changed by forcing and that no progress on it has been made in the last ten years, we shall concentrate on (RC1).

In the present paper we construct a forcing extension in which all isomorphisms between Borel quotients have continuous representations. This does not confirm (RC1) but it does give some positive evidence towards it.

The assumption of the existence of a measurable cardinal in the following result is used only to assure sufficient forcing-absoluteness[1] and it is very likely unnecessary.

Theorem 1. *Assume there exists a measurable cardinal. Then there is a forcing extension in which all of the following are true.*

(1) *Every automorphism of a quotient Boolean algebra $\mathcal{P}(\omega)/\mathcal{I}$ over a Borel ideal $\mathcal{I}$ has a continuous representation.*
(2) *Every isomorphism between quotient Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$ and $\mathcal{P}(\omega)/\mathcal{J}$ over Borel ideals has a continuous representation.*
(3) *Every homomorphism between quotient Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$ over Borel ideals has a locally continuous representation.*
(4) *The dominating number, $\mathfrak{d}$, is equal to $\aleph_1$.*
(5) *All of the above, and in addition we can have either $2^{\aleph_0} = 2^{\aleph_1}$ or $2^{\aleph_0} < 2^{\aleph_1}$.*

Proof of Theorem 1 will occupy most of the present paper (see §1.1 and §5 for an outline). By the above the consistency of the conclusion of the full rigidity conjecture, 'it is relatively consistent with ZFC that all isomorphisms between quotients over Borel ideals are trivial' reduces to (RC2) above.

Corollary 2. *It is relatively consistent with ZFC + 'there exists a measurable cardinal' that all automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$ are trivial while the Calkin algebra has outer automorphisms. In addition, the corona of every separable, stable C*-algebra has outer automorphisms.*

Proof. By the above, the triviality of all automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$ together with $\mathfrak{d} = \aleph_1$ and Luzin's weak Continuum Hypothesis, is relatively consistent with ZFC + 'there exists a measurable cardinal'. By [7, §1], the two latter assumptions imply the existence of an outer automorphism of the Calkin algebra. An analogous result for coronas of some other C*-algebras, including separable stable algebras, is proved in [5]. □

If α is an indecomposable countable ordinal, the *ordinal ideal* $\mathcal{I}_\alpha$ is the ideal on α consisting of all subsets of α of strictly smaller order type. If α is multiplicatively indecomposable, then the *Weiss ideal* $\mathcal{W}_\alpha$ is the ideal of all subsets of α that don't include a closed copy of α in the ordinal topology. See [7] for more on these ideals and the definition of nonpatholigical analytic p-ideals.

Corollary 3. *It is relatively consistent with ZFC + 'there exists a measurable cardinal' that every isomorphism between $\mathcal{P}(\omega)/\mathcal{I}$ and $\mathcal{P}(\omega)/\mathcal{J}$ is trivial whenever $\mathcal{I}$ is Borel and $\mathcal{J}$ is in any of the following classes of ideals is trivial:*

(1) *Nonpathological analytic p-ideals,*

[1] More precisely, we need to know that in all forcing extensions by a small proper forcing all $\mathbf{\Sigma}^1_2$ sets have the property of Baire, $\mathbf{\Pi}^1_2$-unformization and that all $\mathbf{\Pi}^1_2$ sets have the Property of Baire. By Martin–Solovay ([28]) it suffices to assume that $H(\mathfrak{c}^+)^\#$ exists

(2) *Ordinal ideals,*
(3) *Weiss ideals,*

In particular, quotient over an ideal of this sort and any other Borel ideal can be isomorphic if and only if the ideals are isomorphic.

Proof. If an isomorphism $\Phi\colon \mathcal{P}(\omega)/\mathcal{I} \to \mathcal{P}(\omega)/\mathcal{J}$ has a continuous representation and $\mathcal{J}$ is in one of the above classes, then Φ is trivial. This was proved in [7], [23] and [22]. □

In the presence of sufficient large cardinals and forcing absoluteness, the forcing notion used in the proof Theorem 1 gives a stronger consistency result. Universally Baire sets of reals were defined in [15] and well-studied since. A reader not familiar with the theory of universally Baire sets may safely skip all references to them.

Theorem 4. *Assume there are class many Woodin cardinals. Then all conclusions of Theorem 1 hold simultaneously for arbitrary universally Baire ideals in place of Borel ideals.*

The proof of Theorem 4 will be sketched in §5.

1.1. **The plan.** We now roughly outline the proof of Theorem 1. Starting from a model of CH force with a countable support iteration of creature forcings $\mathbb{Q}_{\mathbf{x}}$ (§3) and standard posets for adding a Cohen real, $\mathcal{R}$. The iteration has length $\aleph_2$ and each of these forcings occurs on a stationary set of ordinals of uncountable cofinality specified in the ground model. Forcing $\mathbb{Q}_{\mathbf{x}}$ adds a real which destroys homomorphisms between Borel quotients that are not locally topologically trivial (§2).

Now consider an isomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ between quotients over Borel ideals $\mathcal{I}$ and $\mathcal{J}$. By the standard reflection arguments (§4.3) and using the above property of $\mathbb{Q}_{\mathbf{x}}$ we show that Φ is locally topologically trivial. Finally, a locally topologically trivial automorphism that survives adding random reals has a continuous representation (Lemma 4.13).

In order to make all this work, we need to assure that the forcing iteration is sufficiently definable. In particular, we have the continuous reading of names (§4.1). A simplified version of the forcing notion with additional applications appears in [16].

The forcing notion used to prove Theorem 4 is identical to the one used in Theorem 1. With an additional absoluteness assumptions the main result of this paper can be extended to a class of ideals larger than Borel. We shall need the fact that, assuming the existence of class many Woodin cardinals, all projective sets of reals are universally Baire and, more generally, that every set projective in a universally Baire set is universally Baire. Proofs of these results use Woodin's stationary tower forcing and they can be found in [26].

1.2. **Notation and conventions.** Following [36] we denote the theory obtained from ZFC by removing the power set axiom and adding '$\beth_\omega$ exists' by ZFC*.

We frequently simplify and abuse the notation and write $\Phi \restriction a$ instead of the correct $\Phi \restriction \mathcal{P}(a)/(a \cap \mathcal{I})$ when $\Phi\colon \mathcal{P}(\omega)/\mathcal{I} \to \mathcal{P}(\omega)/\mathcal{J}$ and $a \subseteq \omega$.

If $\mathcal{I}$ is an ideal on $\mathbb{N}$ then $=^{\mathcal{I}}$ denotes the equality modulo $\mathcal{I}$ on $\mathcal{P}(\mathbb{N})$.

As customary in set theory, interpretation of the symbol $\mathbb{R}$ ('the reals') depends from the context. It may denote $\mathcal{P}(\omega)$, ω^ω, or any other recursively presented Polish space. Set-theoretic terminology and notation are standard, as in [25], [35] or [21].

2. Local triviality

We start by gathering a couple of soft results about representations of homomorphisms. A homomorphism $\Phi\colon \mathcal{P}(\omega)/\mathcal{I} \to \mathcal{P}(\omega)/\mathcal{J}$ is $\mathbf{\Delta}^1_2$ if the set

$$\{(a,b) : \Phi([a]_{\mathcal{I}}) = [b]_{\mathcal{J}}\}$$

includes a $\mathbf{\Delta}^1_2$ set $\mathcal{X}$ such that for every a there exists b for which $(a,b) \in \mathcal{X}$. We similarly define when Φ is Borel, $\mathbf{\Pi}^1_2$, or in any other pointclass.

For a homomorphism $\Phi\colon \mathcal{P}(\omega)/\mathcal{I} \to \mathcal{P}(\omega)/\mathcal{J}$ consider the ideals

$$\mathrm{Triv}^0_\Phi = \{a \subseteq \omega : \Phi \restriction a \text{ is trivial}\},$$

$$\mathrm{Triv}^1_\Phi = \{a \subseteq \omega : \Phi \restriction a \text{ has a continuous representation}\},$$

and

$$\mathrm{Triv}^2_\Phi = \{a \subseteq \omega : \Phi \restriction a \text{ is } \mathbf{\Delta}^1_2\}.$$

We say that Φ is *locally trivial* if Triv^0_Φ is nonmeager, that it is *locally topologically trivial* if Triv^1_Φ is nonmeager and hat it is *locally* $\mathbf{\Delta}^1_2$ if Triv^2_Φ is nonmeager.

By [7, Theorem 3.3.5] a fairly weak consequence of PFA implies every homomorphism between quotients over Borel p-ideals is locally continuous (and a bit more). See [7] for additional definitions.

By a well-known result of Jalali–Naini and Talagrand (for a proof see [2] or [7, Theorem 3.10.1]) for each meager ideal $\mathcal{I}$ that includes Fin there is a partition $\bar{I} = (I_n : n \in \omega)$ of ω into finite intervals such that for every infinite $c \subseteq \omega$ the set $\bar{I}_c = \bigsqcup_{n \in C} I_n$ is positive. In other words, the ideal $\mathcal{I}$ is meager if and only if for some partition $\bar{I}$ of ω into finite intervals $\mathcal{I}$ is included in the hereditary F_σ set

$$\mathcal{H}(\bar{I}) = \{a \subseteq \omega : (\forall^\infty n) I_n \not\subseteq a\}.$$

We say that $\bar{I}$ *witnesses* $\mathcal{I}$ is meager. If $\mathcal{I}$ is an ideal that has the property of Baire and includes Fin, then it is necessarily meager.

The following is a well-known consequence of the above.

Lemma 2.1. *Assume $\mathcal{I}$ is a Borel ideal and $\mathcal{K}$ is a nonmeager ideal. Then for every $c \in \mathcal{I}^+$ there is $d \in \mathcal{K}$ such that $c \cap d \in \mathcal{I}^+$.*

Proof. Since the ideal $\mathcal{I}\cap\mathcal{P}(c)$ is a proper Borel ideal on c, it is meager and we can find a partition of c into intervals $c=\bigsqcup_n I_n$ such that $\bigsqcup_{n\in y} I_n\notin\mathcal{I}$ for every infinite $y\subseteq\omega$. Let $\omega=\bigsqcup_n J_n$ be a partition such that $J_n\cap c=I_n$ for all n. Since $\mathcal{K}$ is nonmeager, there is an infinite y such that $d=\bigcup_{n\in y} J_n$ belongs to $\mathcal{K}$. Then $d\cap c=\bigsqcup_{n\in y} I_n$ is not in $\mathcal{I}$ and therefore d is as required. □

The assumption of the following lemma follows from the assumption that there exists a measurable cardinal by [28].

Lemma 2.2. *Assume that all $\mathbf{\Sigma}^1_2$ sets of reals have the property of Baire. If a homomorphism $\Phi\colon\mathcal{P}(\omega)/\mathcal{I}\to\mathcal{P}(\omega)/\mathcal{J}$ is $\mathbf{\Delta}^1_2$ then it has a continuous representation.*

Proof. By the Novikov–Kondo–Addison uniformization theorem, Φ has a $\mathbf{\Sigma}^1_2$ representation. Since this map is Baire-measurable, by a well-known fact (e.g., [7, Lemma 1.3.2]) Φ has a continuous representation. □

Given a partition $I=(I_n:n\in\omega)$ of ω into finite intervals we say that a forcing notion $\mathbb{P}$ *captures* I if there is a $\mathbb{P}$-name $\dot r$ for a subset of ω such that for every $p\in\mathbb{P}$ there is an infinite $c\subseteq\omega$ with the following property:

(1) For every $d\subseteq\bigcup_{n\in c} I_n$ there is $q_d\le p$ such that q_d forces
$$\dot r\cap\bigcup_{n\in c}\check I_n=\check d.$$

By $[a]_{\mathcal{I}}$ we denote the equivalence class of set a modulo the ideal $\mathcal{I}$. When the ideal is clear from the context we may write $[a]$ instead of $[a]_{\mathcal{I}}$.

3. Creatures

Two suslin proper forcing notions are used in the proof of Theorem 1. One is the Lebesgue measure algebra, $\mathcal{R}$. The other shall be described in the present section. It is a creature forcing (for background see [30]).

Fix a partition $I=(I_n:n\in\omega)$ of ω into consecutive finite intervals. Also fix another fast partition $J=(J_n:n\in\omega)$ into consecutive finite intervals. For $s\subseteq\omega$ write
$$I_s=\bigcup_{j\in s} I_j \text{ and } I_{<n}=\bigcup_{j<n} I_j.$$
Let $\mathbf{x}$ denote the pair (I,J), called 'relevant parameter.' Define $(\mathrm{CR}_{\mathbf{x}},\Sigma_{\mathbf{x}})$ as follows (in terms of [30], this will be a 'creating pair').

Let $\mathfrak{c}\in\mathrm{CR}_{\mathbf{x}}$ if
$$\mathfrak{c}=(n_{\mathfrak{c}},u_{\mathfrak{c}},\eta_{\mathfrak{c}},\mathcal{F}_{\mathfrak{c}},m_{\mathfrak{c}},k_{\mathfrak{c}})$$
(we omit the subscript $\mathfrak{c}$ whenever it is clear from the context) provided the following conditions hold

(1) $u\subseteq J_n$,
(2) $\eta: I_u\to\{0,1\}$,
(3) $\mathcal{F}\subseteq\{0,1\}^{I_{J_n}}$ and each $\mu\in\mathcal{F}$ extends η,
(4) $k\le|J_n|-|u|$,

(5) if $v \subseteq J_n \setminus u$, $|v| \leq k$, and $\nu\colon I_v \to \{0,1\}$ then some $\mu \in \mathcal{F}$ extends $\eta \cup \nu$,
(6) $m < 3^{-|I_{<n}|} \log_2 k$.

For $\mathfrak{c}$ and $\mathfrak{d}$ in $\mathrm{CR}_{\mathbf{x}}$ let $\mathfrak{d} \in \Sigma_{\mathbf{x}}(\mathfrak{c})$ if the following conditions hold

(7) $n_{\mathfrak{d}} = n_{\mathfrak{c}}$,
(8) $\eta_{\mathfrak{c}} \subseteq \eta_{\mathfrak{d}}$,
(9) $k_{\mathfrak{c}} \geq k_{\mathfrak{d}}$,
(10) $\mathcal{F}_{\mathfrak{c}} \supseteq \mathcal{F}_{\mathfrak{d}}$,
(11) $m_{\mathfrak{c}} \leq m_{\mathfrak{d}}$.

For $\mathfrak{c} \in \mathrm{CR}_{\mathbf{x}}$ we define the following

(12) $\mathrm{nor}_0(\mathfrak{c}) = \lfloor 3^{-|I_{<n}|} \log_2 k \rfloor$
(13) $\mathrm{nor}(\mathfrak{c}) = \mathrm{nor}_0(\mathfrak{c}) - m$,
(14) $\mathrm{pos}(\mathfrak{c}) = \mathcal{F}$.

Therefore $\mathfrak{c}$ is a finite 'forcing notion' that 'adds' a function from I_{J_n} into $\{0,1\}$. Its 'working part' (or, the already decided part of the 'generic' function) is $\eta_{\mathfrak{c}}$ and $\mathcal{F}_{\mathfrak{c}}$ is the set of 'possibilities' for the generic function (thus the redundant notation (14) included here for the purpose of compatibility with [30]). The 'norm' $\mathrm{nor}(\mathfrak{c})$ provides a lower bound on the amount of freedom allowed by $\mathfrak{c}$ in determining the generic function.

For a relevant parameter $\mathbf{x}$ we now define the creature forcing $\mathbb{Q} = \mathbb{Q}_{\mathbf{x}}$. Let $\mathbf{H}(n) = 2^k$, where $k = I_{J_n}$. This is the number of 'generics' for $\mathfrak{c} \in \Sigma_{\mathbf{x}}$ with $n_{\mathfrak{c}} = n$. Also let

$$\phi_{\mathbf{H}}(j) = |\textstyle\prod_{i<j} \mathbf{H}(i)|.$$

Fix a function $f\colon \omega \times \omega \to \omega$ which satisfies the following conditions for all k and l in ω:

(15) $f(k,l) \leq f(k,l+1)$,
(16) $f(k,l) < f(k+1,l)$,
(17) $\phi_{\mathbf{H}}(l)(f(k,l) + \phi_{\mathbf{H}}(l) + 2) < f(k+1,l)$.

We say such f is $\mathbf{H}$-*fast* (cf. [30, Definition 1.1.12]).

We now let $\mathbb{Q}_{\mathbf{x}}$ be $\mathbb{Q}_f(\mathrm{CR}_{\mathbf{x}}, \Sigma_{\mathbf{x}})$, as in [30, Definition 1.1.10 (f)]. This means that a typical condition in $\mathbb{Q}$ is a triple

$$p = (f_p, i(p), \bar{\mathfrak{c}}(p))$$

such that (we drop subscript p when convenient)

(18) $f\colon I_{<i(p)} \to \{0,1\}$ for some $i(p) \in \omega$,
(19) $\bar{\mathfrak{c}}(p) = \langle \mathfrak{c}(p,j) : j \geq i(p) \rangle$,
(20) Each $\mathfrak{c}(p,j)$ is in $\mathrm{CR}_{\mathbf{x}}$ and satisfies $n_{\mathfrak{c}(p,j)} = j$,
(21) with $m_j = \min(I_{\min(J_j)})$ (cf. [30, Definition 1.1.10(f)]) we have

$$(\forall k)(\forall^\infty j)(\mathrm{nor}(\mathfrak{c}(p,j)) > f(k, m_j).$$

We let $q \leq p$ (where q is a condition stronger than p) if the following conditions are satisfied

(22) $f_p \subseteq f_q$,

(23) $\mathfrak{c}(q,j) \in \Sigma_{\mathbf{x}}(\mathfrak{c}(p,j))$ for $j \geq i(q)$,
(24) $f_q \restriction I_j \in \mathrm{pos}(\mathfrak{c}(p,j))$ for $j \in [i(p), i(q))$.

The idea is that $\mathbb{Q}_{\mathbf{x}}$ adds a function $\dot{f}$ from ω into $\{0,1\}$. A condition $p = (f_p, i(p), \bar{\mathfrak{c}}(p))$ decides that $\dot{f}$ extends f_p as well as $f_{\mathfrak{c}(p,j)}$ for all $j \geq i(p)$. Also, $\mathrm{pos}(\mathfrak{c}(p,j))$ is the set of possibilities for the restriction of $\dot{f}$ to I_{J_j}. The 'norms on possibilities' condition (21) affects the 'rate' at which decisions are being made.

Experts may want to take note that with our creating pair $(\mathrm{CR}_{\mathbf{x}}, \Sigma_{\mathbf{x}})$ there is no difference between $\mathbb{Q}_f$ and $\mathbb{Q}^*_f$ (cf. [30, Definition 1.2.6]) since the intervals J_n form a partition of ω. This should be noted since the results from [30] quoted below apply to $\mathbb{Q}^*_f$ and not $\mathbb{Q}_f$ in general.

3.1. **Properties of $\mathbb{Q}_{\mathbf{x}}$.** We shall need several results from [30] where the class of forcings to which $\mathbb{Q}_{\mathbf{x}}$ belongs was introduced and studied.

Lemma 3.1. *The forcing notion $\mathbb{Q}_{\mathbf{x}}$ is nonempty and nonatomic.*

Given $h\colon \omega \to \omega$ (typically increasing), we say that the creating pair $(\mathrm{CR}_{\mathbf{x}}, \Sigma_{\mathbf{x}})$ is *h-big* ([30, Definition 2.2.1]) if for each $\mathfrak{c} \in \mathrm{CR}_{\mathbf{x}}$ such that $\mathrm{nor}(\mathfrak{c}) > 1$ and $\chi\colon \mathrm{pos}(\mathfrak{c}) \to h(n(\mathfrak{c}))$ there is $\mathfrak{d} \in \Sigma_{\mathbf{x}}(\mathfrak{c})$ such that $\mathrm{nor}(\mathfrak{d}) \geq \mathrm{nor}(\mathfrak{c}) - 1$ and $\chi \restriction \mathrm{pos}(\mathfrak{d})$ is constant. We need only h-bigness in the case when $h(n) = 2$ for all n.

Lemma 3.2. *If $h(n) = 3^{|I_{<n}|}$ then the pair $(\mathrm{CR}_{\mathbf{x}}, \Sigma_{\mathbf{x}})$ is h-big.*

Proof. Fix $\mathfrak{c} = (n, u, \eta, \mathcal{F}, m, k) \in \mathrm{CR}_{\mathbf{x}}$ such that $\mathrm{nor}(\mathfrak{c}) = \lfloor 2^{-|I_{<n}|} \log_2 k \rfloor - m > 0$ and a partition $\mathcal{F} = \bigcup_{j<r} \mathcal{F}$, with $r = 3^{|I_{<n}|}$. We need to find $\mathfrak{d} \in \Sigma_{\mathbf{x}}(\mathfrak{c})$ such that $\mathrm{nor}(\mathfrak{d}) \geq \mathrm{nor}(\mathfrak{c}) - 1$ and $\mathcal{F}_{\mathfrak{d}} \subseteq \mathcal{F}_j$ for some j.

Since $\mathrm{nor}(\mathfrak{c}) = \lfloor r \log_2 k \rfloor - m > 0$, we have that $\log_2 k \geq r$ and therefore $k' = \lceil rk \rceil > 0$.

We shall find $\mathfrak{d}$ of the form $(n, v, \zeta, \mathcal{F}_j, m, k')$ for appropriate v, ζ and $j < r$. Note that $\mathrm{nor}(\mathfrak{d}) = \lfloor r \log_2 k' \rfloor - m = \mathrm{nor}(\mathfrak{c}) - 1$. We shall try to find u_j and $\eta_j\colon u_j \to \{0,1\}$ for $j < r$ as follows. If $\mathfrak{d}_0 = (n, u, \eta, \mathcal{F}_0, m, k') \in \Sigma_{\mathbf{x}}(\mathfrak{c})$, we let $\mathfrak{d} = \mathfrak{d}_0$ and stop. Otherwise, there are $v_0 \subseteq J_n \setminus u$ and $\zeta_0\colon v_0 \to \{0,1\}$ such that $\eta \cup \zeta_0$ has no extension in $\mathcal{F}_0$. Let $u_1 = u \cup v_0$ and $\eta_1 = \eta \cup \zeta_0$. If $\mathfrak{d}_1 = (n, u_1, \eta_1, \mathcal{F}_1, m, k') \in \Sigma_{\mathbf{x}}(\mathfrak{c})$, we let $\mathfrak{d} = \mathfrak{d}_1$ and stop. Otherwise, there are $v_1 \subseteq J_n \setminus u_1$ and $\zeta_1\colon v_1 \to \{0,1\}$ such that $\eta_1 \cup \zeta_1$ has no extension in $\mathcal{F}_1$. Let $u_2 = u_1 \cup v_1$ and $\eta_2 = \eta \cup \zeta_1$. Proceeding in this way, for $j < r$ we construct v_j, u_j, ζ_j and η_j such that η_j has no extension in $\mathcal{F}_j$ or we find $\mathfrak{d}_j$ witnessing r-bigness of $\mathfrak{c}$. If u_j and η_j are constructed for $j < r-1$, then $v = \bigcup_{j<r} v_j$ has cardinality $rk' = k$ and $\nu = \bigcup_{j<r} \zeta_j$ has no extension in $\mathcal{F}$. But this contradicts the assumption (4) on $\mathfrak{c}$. Therefore one of $\mathfrak{d}_j$ is as required. $\square$

A creating pair $(\mathrm{CR}_{\mathbf{x}}, \Sigma_{\mathbf{x}})$ has the *halving property* ([30, Definition 2.2.7]) if for each $\mathfrak{c} \in \mathrm{CR}_{\mathbf{x}}$ such that $\mathrm{nor}(\mathfrak{c}) > 0$ there is $\mathfrak{d} \in \Sigma_{\mathbf{x}}(\mathfrak{c})$ (usually denoted $\mathrm{half}(\mathfrak{c})$) such that

(1) $\operatorname{nor}(\mathfrak{d}) \geq \frac{1}{2}\operatorname{nor}(\mathfrak{c})$,
(2) If in addition $\operatorname{nor}(\mathfrak{c}) \geq 2$ then for each $\mathfrak{d}_1 \in \Sigma_{\mathbf{x}}(\mathfrak{d})$ such that $\operatorname{nor}(\mathfrak{d}_1) > 0$ there is $\mathfrak{c}_1 \in \Sigma_{\mathbf{x}}(\mathfrak{c})$ such that $\operatorname{nor}(\mathfrak{c}_1) \geq \frac{1}{2}\operatorname{nor}(\mathfrak{c})$ and $\operatorname{pos}(\mathfrak{c}_1) \subseteq \operatorname{pos}(\mathfrak{d}_1)$.

Lemma 3.3. *The pair* $(\mathrm{CR}_{\mathbf{x}}, \Sigma_{\mathbf{x}})$ *has the halving property.*

Proof. $\mathfrak{c} = (n, u, \eta, \mathcal{F}, m, k) \in \mathrm{CR}_{\mathbf{x}}$ such that $\operatorname{nor}(\mathfrak{c}) = 2^{-|I_{<n}|} - m > 0$. Write $r = 3^{-|I_{<n}|}$. Since $m < rk$ by (6) we have that $m_{\mathfrak{d}} = \frac{1}{2}(rk + m)$ satisfies $m' < rk$ and therefore $\mathfrak{d} = (n, u, \eta, \mathcal{F}, m_{\mathfrak{d}}, k)$ is in $\Sigma_{\mathbf{x}}(\mathfrak{c})$.

Now let us assume $\operatorname{nor}(\mathfrak{c}) \geq 2$ since otherwise there is nothing left to do. Assume $\mathfrak{d}_1 = (n, u_1, \eta_1, \mathcal{F}_1, k_1, m_1) \in \Sigma_{\mathbf{x}}(\mathfrak{d})$ is such that $\operatorname{nor}(\mathfrak{e}) > 0$. Note that $\operatorname{nor}(\mathfrak{d}_1) = r\log_2(k_1) - m_1$, $m_1 \geq m_{\mathfrak{d}}$ and $k_1 \leq k_{\mathfrak{d}} = k$.

Let $\mathfrak{c}_1 = (n, u_1, \eta_1, \mathcal{F}_1, k_1, m)$. Then

$$\operatorname{nor}(\mathfrak{c}_1) = \lfloor r \log_2 k_1 \rfloor - m = \operatorname{nor}(\mathfrak{d}_1) - m + m_1 \geq \frac{1}{2}\operatorname{nor}(\mathfrak{c}_1)$$

as required. $\square$

Recall that a forcing notion $\mathbb{P}$ is *ω^ω-bounding* if for every name $\dot{f}$ for an element of ω^ω and every $p \in \mathbb{P}$ there are $q \leq p$ and $g \in \omega^\omega$ such that $q \Vdash \dot{f}(n) \leq \check{g}(n)$ for all n.

Proposition 3.4. *Forcing notion $\mathbb{Q}_{\mathbf{x}}$ is proper, ω^ω-bounding, and both the ordering and the incomparability relation on $\mathbb{Q}_{\mathbf{x}}$ are Borel.*

Proof. In addition to bigness and halving properties of $\mathbb{Q}_{\mathbf{x}}$ proved in two lemmas above, we note that this forcing is finitary (i.e,, each $\mathrm{CR}_{\mathbf{x}}$ is finite) and simple (i.e., $\Sigma_{\mathbf{x}}(S)$ is not defined for $S \subseteq \mathrm{CR}_{\mathbf{x}}$ that contains more than one element). By [30, Corollary 2.2.12 and Corollary 3.1.2], or rather by [30, Theorem 2.2.11], it is proper and ω^ω-bounding.

It is clear that $\leq_{\mathbb{Q}_{\mathbf{x}}}$ is Borel. We check the remaining fact, that the relation $\perp_{\mathbb{Q}_{\mathbf{x}}}$ is Borel. Function $g\colon (\mathbb{Q}_{\mathbf{x}})^2 \to \omega^\omega$ defined by

$$g(p,q)(n) = \max\{\operatorname{nor}(\mathfrak{d}) : \mathfrak{d} \in \Sigma_{\mathbf{x}}(\mathfrak{c}(p,n)) \cap \Sigma_{\mathbf{x}}(\mathfrak{c}(q,n))\}$$

(with $\max \emptyset = 0$) is continuous. Since p and q are compatible if and only if $g(p,q)$ satisfies the largeness requirement (21), the incompatibility relation is Borel. $\square$

4. Forcing Iteration

In this long section we analyze properties of forcings used in our proof.

4.1. **Fusions and continuous reading of names in the iteration.**

A crucial property of the forcing iteration used in our proof is that it has the continuous reading of names (by $\mathbb{R}$ we will usually mean $\mathcal{P}(\omega)$).

Definition 4.1. Consider a countable support forcing iteration $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$ such that each $\dot{\mathbb{Q}}_\eta$ is a ground-model Suslin forcing notion which adds a generic real $\dot{g}_\xi$. Such an iteration has *continuous reading of* *names* if for every $\mathbb{P}_\kappa$-name $\dot{x}$ for a new real the set of conditions p such that there exists countable $S \subseteq \kappa$, compact $F \subseteq \mathbb{R}^S$, and continuous $h\colon F \to \mathbb{R}$ such that

$$p \Vdash \text{“}\langle \dot{g}_\xi : \xi \in S\rangle \in F \text{ and } \dot{x} = h(\langle \dot{g}_\xi : \xi \in S\rangle)\text{”}$$

is dense.

For iterations of proper forcing notions of the form P_I where I is a $\mathbf{\Sigma}^1_1$ on $\mathbf{\Pi}^1_1$ σ-ideal of Borel sets (see [43]) continuous reading of names follows from posets being ω^ω-bounding. This is a beautiful result of Zapletal ([43, Theorem 3.10.19 and Theorem 6.3.16]). While many proper forcings adding a real are equivalent to ones of the form P_I (see [42]. and [43]), this unfortunately does not necessarily apply to creature forcings as used in our proof (see [24, §3]).

Nevertheless, continuous reading of names in our iteration is a special case of the results in [36]. For convenience of the reader we shall include a proof of this fact in Proposition 4.2 below.

We shall define a sequence of finer orderings on $\mathbb{Q}_{\mathbf{x}}$ (see [30, Definition 1.2.11 (5)]). For $p \in \mathbb{Q}_{\mathbf{x}}$ and $j \in \mathbb{N}$ let

$$\chi(p, j) = \{r \in \mathbb{Q}_{\mathbf{x}} : r \leq p, f_r = f_q, \text{ and } \mathfrak{c}(r, i) = \mathfrak{c}(p, i) \text{ for all } i \leq j\}.$$

For p and q in $\mathbb{Q}_{\mathbf{x}}$ and $n \geq 1$ write

(1) $p \leq_0 q$ if $p \leq q$ and $f_p = f_q$,
(2) $p \leq_n q$ if
  (a) $p \leq_0 q$, and with $m_i = \min(I_{\min J_i})$ and
  $$k = \min\{i : \mathrm{nor}(\mathfrak{c}(q, i)) > f(n, m_i)\}$$
  we have
  (b) $q \in \chi(p, k)$, and
  (c) $\mathrm{nor}(\mathfrak{c}(p, i)) \geq f(n, m_i)$ for all i such that $\mathfrak{c}(p, i) \neq \mathfrak{c}(q, i)$.

We say that $p \in \mathbb{Q}_{\mathbf{x}}$ *essentially decides* a name for an ordinal $\dot{m}$ if there exists j such that every $q \in \chi(p, j)$ decides $\dot{m}$.

By Theorem 2.2.11, if $\dot{m}$ is a name for an ordinal and $p \in \mathbb{Q}_{\mathbf{x}}$ then for every $n \in \omega$ there exists $q \leq_n p$ which essentially decides $\dot{m}$ (of course this is behind the proof of Proposition 3.4, modulo standard fusion arguments).

Let us now consider $\mathcal{R}$, the standard poset for adding a random real. Conditions are compact subsets of $\mathcal{P}(\omega)$ of positive Haar measure μ and the ordering is reverse inclusion. For $n \in \omega$ define a finer ordering on $\mathcal{R}$ by $q \leq_n p$ if $q \leq p$ and $\mu(q) \geq (1 - 2^{-n-1})\mu(p)$. We say that $q \in \mathcal{R}$ essentially decides $\dot{m}$ if there exists j such that $q \cap [s]$ decides $\dot{m}$ for every $s \in 2^j$ such that $q \cap [s] \in \mathcal{R}$. The inner regularity of μ implies that for every name $\dot{m}$ for an ordinal, every $p \in \mathcal{R}$ and every n there exists $q \leq_n p$ which essentially decides $\dot{m}$.

In the following proposition we assume $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$ is a countable support iteration such that each $\dot{\mathbb{Q}}_\eta$ is either some $\mathbb{Q}_{\mathbf{x}}$ or $\mathcal{R}$, and that in addition the maximal condition of $\mathbb{P}_\eta$ decides whether $\dot{\mathbb{Q}}_\eta$ is $\mathcal{R}$ or $\mathbb{Q}_{\mathbf{x}}$, and in the latter case it also decides x, for all η.

Proposition 4.2. *An iteration $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$ as in the above paragraph has the continuous reading of names.*

Proof. Since $\mathbb{P}_\kappa$ is a countable support iteration of proper, ω^ω-bounding forcing notions, by [35] the iteration is proper and ω^ω-bounding.

Let $\dot{g}$ be a name for an element of ω^ω. By the above and by working below a condition, we may assume that there exists $h \in \omega^\omega$ such that $\Vdash_{\mathbb{P}} \dot{g} \leq \check{h}$. Choose a countable elementary submodel M of $H_{(2^\kappa)^+}$ containing everything relevant and let F_j, for $j \in \omega$, be an increasing sequence of finite subsets of $M \cap \kappa$ with union equal to $M \cap \kappa$. By using order $\leq_n$ in $\mathbb{Q}_{\mathbf{x}}$ and in $\mathcal{R}$ introduced above, we can construct a fusion sequence p_n such that for every n and every $\eta \in F_n$ we have

(1) $p_n \in M$,
(2) $p_{n+1} \restriction \eta \Vdash p_{n+1}(\eta) \leq_n p_n$,
(3) p_n decides the first n digits of $\dot{g}$ (we can do this since $\dot{g} \leq h$ implies there are only finitely many possibilities).
(4) $p_n \restriction \eta$ decides $\mathfrak{c}(p_n(\eta), j)$ for $j \leq n$ if $\dot{\mathbb{Q}}_\eta = \mathbb{Q}_{\mathbf{x}}$ for some x or decides $\{s \in 2^n : p_n(\eta) \cap [s] \neq \emptyset\}$ if $\dot{\mathbb{Q}}_\eta = \mathcal{R}$.

Then for every $\eta \in M \cap \kappa$ and n large enough the condition $p_{n+1} \restriction \eta$ for $n \in \mathbb{N}$ forces that $p_{n+1}(\eta) \leq_n p_n(\eta)$. Therefore we can define a fusion p of sequence p_n. Since $p_n \in M$ for all n we have that the support of p is included in $S = M \cap \kappa$. Let F be the closed subset of $\mathcal{P}(\mathbb{N})^S$ whose complement is the union of all basic open $U \subseteq \mathcal{P}(\mathbb{N})^S$ such that $p \Vdash \dot{x} \notin U$. By (3) there is a continuous function $h\colon F \to \omega^\omega$ such that $p \Vdash h(\langle \dot{g}_\xi : \xi \in S\rangle) = \dot{x}$. $\square$

4.2. **Subiterations and complexity estimates.** Assume $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$ is an iteration as in Proposition 4.2. Then for every subset $S \subseteq \kappa$ we have a well-defined subiteration

$$\mathbb{P}_S = (\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \in S, \eta \in S).$$

We shall write $\Vdash_S$ instead of $\Vdash_{\mathbb{P}_S}$ and $\Vdash$ instead of $\Vdash_{\mathbb{P}_\kappa}$. In some specific situations we have that $p \Vdash \phi$ is equivalent to $p \Vdash_S \phi$, where S is the support of p.

The following result is a key to our proof. In the context of [43] much more can be said, but Zapletal's theory does not apply to the context of creature forcings (cf. paragraph after Definition 4.1).

Lemma 4.3. *Assume $\mathbb{P}_\kappa$ is a countable support iteration of ground model ω^ω-bounding Suslin forcings. Assume B is a $\mathbf{\Pi}^1_1$ set, $p \in \mathbb{P}_\kappa$, $\dot{x}$ is a name for an element of $\mathcal{P}(\mathbb{N})$, $p \Vdash \dot{x} \in B$ and $\dot{x}$ is a $\mathbb{P}_{\mathrm{supp}(p)}$-name. Then $p \Vdash_{\mathrm{supp}(p)} \dot{x} \in B$.*

Proof. Let $S = \mathrm{supp}(p)$. Assume the contrary and find $q \leq p$ such that $q \Vdash_S \dot{x} \notin B$. Let T be a tree whose projection is the complement of B and let $\dot{y}$ be a name such that q forces (in $\mathbb{P}_S$) that $(\dot{y}, \dot{x})$ is a branch through T. Since $\mathbb{P}_S$ is an iteration of ω^ω-bounding forcings it is ω^ω-bounding ([35]) we can assume (by extending q if necessary) that $q \Vdash_S \dot{y} \leq \check{h}$ for $h \in \omega^\omega$.

Now choose a countable $M \prec H_\theta$ for a large enough θ so that M contains $\mathbb{P}_\kappa$, q, $\dot{x}$, T, h and everything relevant. Let $G \subseteq \mathbb{P}_\kappa \cap M$ be an M-generic filter containing q. Let $x = \mathrm{int}_{\mathrm{G}}(\dot{x})$. The tree $T_x = \{s : (s, x \restriction n) \in T$ for some $n\}$ is finitely branching (being included in $\{s : s(i) \leq h(i)$ for all $i < |s|\}$) and infinite. It therefore has an infinite branch by König's Lemma. This implies that $x \notin B$, contradicting the fact that $p \Vdash \dot{x} \in B$. $\square$

Recall that a forcing notion is *Suslin proper* if its underlying set is an analytic set of reals and both $\leq$ and $\perp$ are analytic relations. The following lemma is well-known.

Lemma 4.4. *Assume $\mathbb{P}$ is Suslin proper, $\dot{x}$ is a $\mathbb{P}$-name for a real, and $A \subseteq \mathbb{R}^2$ is Borel. Then for a dense set of conditions $p \in \mathbb{P}$ the set*

$$\{a : p \Vdash (\check{a}, \dot{x}) \in A\}$$

is $\mathbf{\Delta}^1_2$.

Proof. Since $\mathbb{P}$ is proper the set of all $p \in \mathbb{P}$ such that all antichains in $\dot{x}$ are countable below p is dense. For $a \subseteq \omega$ we now have that $p \Vdash (\check{a}, \dot{x}) \in A$ if there exists a countable well-founded model M of ZFC* containing everything relevant such that for every M-generic $G \subseteq M \cap \mathbb{P}$ with $p \in G$ we have that $F(a, \mathrm{int}_{\mathrm{G}}(\dot{x})) \in A$. This is a $\mathbf{\Sigma}^1_2$ statement with A as a parameter.

Alternatively, $p \Vdash (\check{a}, \dot{x}) \in A$ if for every countable well-founded model M of ZFC* and every M-generic $G \subseteq M \cap \mathbb{P}$ with $p \in G$ we have that $F(a, \mathrm{int}_{\mathrm{G}}(\dot{x})) \in A$. This is a $\mathbf{\Pi}^1_2$ statement with A as a parameter. $\square$

Lemma 4.5. *Assume $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$ is as in Proposition 4.2. Assume $\dot{x}$ is a $\mathbb{P}$-name for a real, $A \subseteq \mathbb{R}$ is Borel and $g \colon \mathbb{R}^2 \to \mathbb{R}$ is a Borel function. If $p \in \mathbb{P}$ is such that the name $\dot{x}$ is continuously read below p then the set*

$$\{a : p \Vdash g(\check{a}, \dot{x}) \in A\}$$

is $\mathbf{\Delta}^1_2$.

Proof. By Proposition 4.2, with $S = \mathrm{supp}(p)$ we have a compact $F \subseteq \mathcal{P}(\mathbb{N})^S$ and a continuous $h \colon F \to \mathcal{P}(\omega)$ such that $p \Vdash h(\langle \dot{g}_\xi : \xi \in S\rangle) = \dot{x}$.

Lemma 4.3 implies that $p \Vdash g(\check{a}, \dot{x}) \in A$ if and only if $p \Vdash_S g(\check{a}, \dot{x}) \in A$. Since S is countable, by Lemma 4.4 the latter set is $\mathbf{\Delta}^1_2$. $\square$

4.3. **Reflection.** Throughout this section we assume $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$ is a forcing iteration of proper forcings of cardinality $< \kappa$ in some model M of a large enough fragment of ZFC. We also assume $G_\kappa \subseteq \mathbb{P}_\kappa$ is an M-generic filter and let $G \restriction \xi$ denote $G \cap \mathbb{P}_\xi$. If $\dot{A}$ is a $\mathbb{P}_\kappa$-name for a set of reals we can consider it as a collection of nice names for reals. Furthermore, since $\mathbb{P}_\kappa$ is proper then we can identify $\dot{A}$ with a collection of pairs $(p, \dot{x})$ where $p \in \mathbb{P}_\kappa$ and $\dot{x}$ is a name that involves only countable antichains below p.

The intention is that p forces $\dot{x}$ is in A. With this convention we let $\dot{A} \restriction \xi$ denote the subcollection of $\dot{A}$ consisting only of those pairs $(p, \dot{x})$ such that $p \in \mathbb{P}_\xi$ and $\dot{x}$ is a $\mathbb{P}_\xi$ name,

The following 'key triviality' will be used repeatedly in proof of the main theorem. It ought to be well-known but it does not seem to appear explicitly in the literature.

Proposition 4.6. *Assume $\kappa > \mathfrak{c}$ is a regular cardinal and*

$$(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$$

is a countable support iteration of proper forcings of cardinality $< \kappa$. Assume $\dot{A}$ is a $\mathbb{P}_\kappa$ name for a set of reals. Then the set of ordinals $\xi < \kappa$ such that

$$(H(\aleph_1), \mathrm{int}_{G\restriction\xi}(\dot{A} \restriction \xi))^{V[G\restriction\xi]} \prec (H(\aleph_1), \mathrm{int_G}(\dot{A}))^{V[G]}$$

includes a club relative to $\{\xi < \kappa : \mathrm{cf}(\xi) \geq \omega_1\}$.

Proof. Since each $\mathbb{P}_\xi$ is proper ([35]), no reals are added at stages of uncountable cofinality. Therefore if $\mathrm{cf}(\eta)$ is uncountable then $H(\aleph_1)^{V[G\restriction\eta]}$ is the direct limit of $H(\aleph_1)^{V[G\restriction\xi]}$ for $\xi < \eta$. The assertion is now reduced to a basic fact from model theory: club many substructures of $(H(\aleph_1), \mathrm{int_G}(\dot{A}))^{V[G]}$ of cardinality $< \kappa$ are elementary submodels. $\square$

Definition 4.7. Using notation as in the beginning of §4.3 we say that a formula $\phi(x, Y)$ (with parameters $x \in \mathbb{R}$ and $Y \subseteq \mathbb{R}$) *reflects* (with respect to $\mathbb{P}_\kappa$) if for every name $\dot{a}$ for a real and every name $\dot{B}$ for a set of reals the following are equivalent.

(1) $V[G] \models \phi(\dot{a}, \dot{B})$, and
(2) There is a club $\mathbf{C} \subseteq \kappa$ such that for all $\xi \in \mathbf{C}$ with $\mathrm{cf}(\xi) \geq \omega_1$ we have $V[G \restriction \xi] \models \phi(\dot{a}, \dot{B} \restriction \xi)$.

Corollary 4.8. *Let $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$ be a countable support iteration of proper forcings of cardinality $< \kappa$. Assume $\dot{\mathcal{I}}$ and $\dot{\mathcal{J}}$ are $\mathbb{P}_\kappa$-names for Borel ideals on ω and $\dot{\Phi}$ is a $\mathbb{P}_\kappa$-name for an isomorphism between their quotients.*

(1) *For every name $\dot{a}$ for a real the statement $\dot{a} \in \mathrm{Triv}^1_{\dot{\Phi}}$ reflects.*
(2) *For $0 \leq j \leq 2$ the statement "$\mathrm{Triv}^j_{\dot{\Phi}}$ is meager" reflects.*
(3) *For every $\mathbb{P}_\kappa$-name $\dot{I}$ for a partition of ω into finite sets the statement $\dot{\mathcal{I}} \subseteq \mathcal{H}(\dot{I})$ reflects.*

Proof. Since the pertinent statements are projective with the interpretation of $\dot{\Phi}$ as a parameter, each of the assertions is a consequence of Proposition 4.6. $\square$

4.4. **Random reals.** We identify $\mathcal{P}(\omega)$ with 2^ω and with $(\mathbb{Z}/2\mathbb{Z})^\omega$ and equip it with the corresponding Haar measure. The following lemma will be instrumental in the proof of one of our key lemmas, Lemma 4.13.

Lemma 4.9. *Assume $\mathcal{J}$ is a Borel ideal and f and g are continuous functions such that each one of them is a representation of a homomorphism from $\mathcal{P}(\omega)$ into $\mathcal{P}(\omega)/\mathcal{J}$. If the set*

$$\Delta_{f,g,\mathcal{J}} = \{c \subseteq \omega : f(c) \neq^{\mathcal{J}} g(c)\}$$

is null then it is empty.

Proof. By the inner regularity of Haar measure we can find a compact set K disjoint from $\Delta_{f,g,\mathcal{J}}$ of measure $> 1/2$. Fix any $c \subseteq \omega$. The sets K and $K\underline{\Delta}c = \{b\Delta c : b \in K\}$ both have measure $> 1/2$ and therefore we can find $b \in K$ such that $b\Delta c \in K$. But then

$$f(c) =^{\mathcal{J}} f(c\Delta b)\Delta f(b) =^{\mathcal{J}} g(c\Delta b)\Delta g(b) =^{\mathcal{J}} g(c)$$

completing the proof. □

In the following $\mathcal{R}$ denotes the forcing for adding a random real and $\dot{x}$ is the canonical $\mathcal{R}$-name for the random real.

Corollary 4.10. *Assume $\mathcal{J}$ is a Borel ideal and f and g are continuous functions such that each is a representation of a homomorphism from $\mathcal{P}(\omega)$ into $\mathcal{P}(\omega)/\mathcal{J}$. Furthermore assume $\mathcal{R}$ forces $f(\dot{x}) =^{\mathcal{J}} g(\dot{x})$. Then $f(c) =^{\mathcal{J}} g(c)$ for all $c \subseteq \omega$.*

Proof. It will suffice to show that the assumptions of Lemma 4.9 are satisfied. This is a standard fact but we include the details. Since the set $\Delta_{f,g,\mathcal{J}}$ is Borel, if it is not null then there exists a compact set $K \subseteq \Delta_{f,g,\mathcal{J}}$ of positive measure. If M is a countable transitive model of a large enough fragment of ZFC containing codes for K, f, g, and $\mathcal{J}$ and $x \in K$ is a random real over M, then $M[x] \models f(x) =^{\mathcal{J}} g(x)$ by the assumption on f and g. However, this is a Δ^1_1 statement and is therefore true in V. But $x \in \Delta_{f,g,\mathcal{J}}$ and therefore $f(x) \neq^{\mathcal{J}} g(x)$, a contradiction. □

4.5. **Trivializing automorphisms locally and globally.** Ever since the second author's proof that all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial in an oracle-cc forcing extensions ([34]), every proof that automorphisms of a similar quotient structure proceeds in (at least) two stages. In the first stage one proves that the automorphism is 'locally trivial' and in the second stage local trivialities are pieced together into a single continuous representation (see e.g., [7, §3]). The present proof is no exception.

Throughout this subsection we assume

$$(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \mathfrak{c}^+, \eta < \mathfrak{c}^+)$$

is as in Proposition 4.2. Therefore it is a countable support iteration such that each $\dot{\mathbb{Q}}_\eta$ is either some $\mathbb{Q}_{\mathbf{x}}$ or $\mathcal{R}$, and that in addition the maximal condition of $\mathbb{P}_\eta$ decides whether $\dot{\mathbb{Q}}_\eta$ is $\mathcal{R}$ or $\mathbb{Q}_{\mathbf{x}}$, and in the latter case it also decides x, for all η. We shall write $p \Vdash_\xi \phi$ instead of $p \Vdash_{\mathbb{P}_\xi} \phi$.

Lemma 4.11. *With $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$ as above, assume that for every partition I of ω into finite intervals the set*

$$\{\xi < \mathfrak{c}^+ \Vdash_\xi \text{“}\mathbb{Q}_\xi \textit{ captures } I \textit{ and } \mathrm{cf}(\xi) \textit{ is uncountable”}\}$$

is stationary. Then every homomorphism between quotients over Borel ideals is locally $\mathbf{\Delta}^1_2$ (see §2).

Proof. Fix a name $\dot{\Phi}$ for a homomorphism between quotients over Borel ideals $\mathcal{I}$ and $\mathcal{J}$. By moving to an intermediate forcing extension containing relevant Borel codes, we may assume the ideals $\mathcal{I}$ and $\mathcal{J}$ are in the ground model. Let $G \subseteq \mathbb{P}_{\mathfrak{c}^+}$ be a generic filter.

Assume $\mathrm{Triv}^2_{\mathrm{int}_\mathrm{G}(\dot{\Phi})}$ is meager in $V[G]$ with a witnessing partition $\mathrm{int}_\mathrm{G}(\dot{I})$ (cf. the discussion before Corollary 4.8). By Corollary 4.8 the set of $\xi < \mathfrak{c}^+$ of uncountable cofinality such that $\mathrm{int}_{G\restriction\xi}(\dot{I})$ witnesses $\mathrm{Triv}^2_{\mathrm{int}_{G\restriction\xi}(\dot{\Phi}\restriction\xi)}$ is meager in $V[G \restriction \xi]$ includes a relative club.

Since the iteration of proper ω^ω-bounding forcings is proper and ω^ω-bounding ([35]) the forcing is ω^ω-bounding, we may assume $\mathrm{int}_\mathrm{G}(\dot{I})$ is a ground-model partition, $\bar{I} = (I_n : n \in \omega)$ By our assumption, there is a stationary set $\mathbf{S}$ of ordinals of uncountable cofinality such that for all $\eta \in \mathbf{S}$ we have

(1) $\Vdash_\eta$“$\dot{\mathbb{Q}}_\xi$ adds a real $\dot{x}$ that captures $\bar{I}$”.

Fix $\eta \in \mathbf{S}$ for a moment. By going to the intermediate extension we may assume $\eta = 0$. Let $\dot{y}$ be a name for a subset of ω such that

$$[\dot{y}]_\mathcal{J} = \Phi([\dot{x}]_\mathcal{I}).$$

By the continuous reading of names (Proposition 4.2) we can find condition p with support S containing 0, compact $F \subseteq \mathcal{P}(\mathbb{N})^S$ and continuous $h \colon F \to \mathcal{P}(\omega)$ such that $p \Vdash h(\langle \dot{g}_\xi : \xi \in S\rangle) = \dot{y}$. Note that $\dot{x}$ is equal to $\dot{g}_0$ hence it is “continuously read.”

Since $\mathbb{Q}_0$ captures I we can find an infinite d such that with $a = I_d$ for every $b \subseteq a$ condition $p_b \leq p$ forces $\dot{x} \cap a = b$. Also, $\mathrm{supp}(p_b) = \mathrm{supp}(p)$ and (by the definition of $\mathbb{Q}_\mathbf{x}$) the map $b \mapsto p_b$ is continuous.

By the choice of $\dot{y}$, with $c = \Phi_*(a)$ by Lemma 4.3 we have that

$$p_b \Vdash_S \dot{y} \cap c =^\mathcal{J} \Phi_*(b).$$

By Lemma 4.4 the set

$$\{(b, e) : b \subseteq a, e \subseteq c, e =^\mathcal{J} \Phi_*(b)\}$$

is $\mathbf{\Delta}^1_2$.

Therefore a and $\dot{\mathbb{Q}}_\xi$ satisfy the assumptions of Lemma 4.5 and in $V[G \restriction \xi]$ the restriction of $\mathrm{int}_{G\restriction\xi}(\dot{\Phi} \restriction \xi)$ to $\mathcal{P}(a)/\mathcal{I}$ is $\mathbf{\Delta}^1_2$, contradicting our assumption. Since assuming $\mathrm{Triv}^2_{\mathrm{int}_\mathrm{G}(\dot{\Phi})}$ was meager lead to a contradiction, this concludes the proof. □

Definition 4.12. Assume $\mathbb{P}$ is a forcing notion and $\dot{\Phi}$ is a $\mathbb{P}$-name for an isomorphism between quotients over Borel ideals $\mathcal{I}$ and $\mathcal{J}$ which extends ground-model isomorphism Φ between these quotients. We say that $\dot{\Phi}$ is $\mathbb{P}$-*absolutely locally topologically trivial* if the following apply (in order to avoid futile discussion we assume $\mathbb{P}$ is ω^ω-bounding):

(1) Φ is locally topologically trivial,
(2) $\mathbb{P}$ forces that the continuous witnesses of local topological triviality of Φ witness local topological triviality of $\dot{\Phi}$.

In order to justify this definition we note that this is not a consequence of the assumption that Φ is locally topologically trivial and $\dot{\Phi}$ is forced to be locally topologically trivial. By a result of Steprāns, there is a σ-linked forcing notion such that a trivial automorphism of $\mathcal{P}(\omega)/\operatorname{Fin}$ extends to a trivial automorphism, but the triviality is not implemented by the same function ([38]). Steprāns used this to show that there is a forcing iteration $\mathbb{P}_\kappa$ that forces Martin's Axiom and the existence of a nontrivial automorphism Φ of $\mathcal{P}(\omega)/\operatorname{Fin}$ that is trivial in $V[G \restriction \xi]$ for cofinally many ξ.

The following key lemma shows that in our forcing extension local topological triviality is always witnessed by a $\mathbf{\Pi}^1_2$ set.

Lemma 4.13. *Assume $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \kappa, \eta < \kappa)$ is as in the beginning §4.5 such that $\mathbb{Q}_0$ is $\mathcal{R}$. Also assume $\dot{\Phi}$ is a $\mathbb{P}_\kappa$-name for a $\mathbb{P}_\kappa$-absolutely locally topologically trivial isomorphism between quotients over Borel ideals $\mathcal{I}$ and $\mathcal{J}$. Then the set*

$$\{(c,d) : \Phi_*(c) =^{\mathcal{J}} d\}$$

is $\mathbf{\Pi}^1_2$.

Proof. We have $\Phi\colon \mathcal{P}(\omega)/\mathcal{I} \to \mathcal{P}(\omega)/\mathcal{J}$. Let $\dot{x}$ be the canonical $\mathbb{Q}_0$-name for the random real and let $\dot{y}$ be a $\mathbb{P}_\kappa$-name for the image of $\dot{x}$ by the extension of Φ. By the continuous reading of names (Proposition 4.2) we can find condition p with countable support S containing 0, compact $F \subseteq \mathcal{P}(\mathbb{N})^S$ and continuous $h\colon F \to \omega^\omega$ such that $p \Vdash h(\langle \dot{g}_\xi : \xi \in S\rangle) = \dot{y}$. Again $\dot{x}$ is equal to $\dot{g}_0$ hence it is "continuously read."

Consider the set $\mathcal{Z}$ of all (a,b,f,g) such that

(1) a and b are subsets of ω.
(2) $f\colon \mathcal{P}(a) \to \mathcal{P}(b)$ and $g\colon \mathcal{P}(b) \to \mathcal{P}(a)$ are continuous maps,
(3) f is a representation of a homomorphism from $\mathcal{P}(a)/\mathcal{I}$ into $\mathcal{P}(b)/\mathcal{J}$,
(4) g is a representation of a homomorphism from $\mathcal{P}(b)/\mathcal{J}$ into $\mathcal{P}(a)/\mathcal{I}$,
(5) $f(c) \in \mathcal{J}$ if and only if $c \in \mathcal{I}$,
(6) $f(g(c)) =^{\mathcal{J}} c$ for all $c \subseteq b$, and $g(f(c)) =^{\mathcal{I}} c$ for all $c \subseteq a$,
(7) p forces that $f(\dot{x} \cap \check{a}) =^{\mathcal{J}} \dot{y} \cap \check{b}$.
(8) p forces that $g(\dot{y} \cap \check{b}) =^{\mathcal{I}} \dot{x} \cap \check{a}$.

Conditions (1) and (2) state that $\mathcal{Z}$ is a subset of the compact metric space $\mathcal{P}(\omega)^2 \times C(\mathcal{P}(\omega), \mathcal{P}(\omega))^2$, where $C(X,Y)$ denotes the compact metric space of continuous functions between compact metric spaces X and Y. Since (3) states that

$$(\forall x \subseteq a)(\forall y \subseteq a) f(x \cup y) =^{\mathcal{J}} f(x) \cap f(y)$$
$$(\forall x \subseteq a) f(a) \setminus f(x) =^{\mathcal{J}} f(a \setminus x)$$

this is a $\mathbf{\Pi}^1_1$ condition, and similarly for (4). Similarly (5) and (6) are $\mathbf{\Pi}^1_1$. Lemma 4.5 implies that the remaining condition, (7), is $\mathbf{\Delta}^1_2$ (recall that $\mathcal{J}$ was assumed to be Borel). Therefore the set $\mathcal{Z}$ is $\mathbf{\Delta}^1_2$. The set

$$\mathcal{K} = \{a : (a, b, f, g) \in \mathcal{Z} \text{ for some } (b, f, g)\}$$

is easily seen to be an ideal that includes Triv^1_Φ. Since Φ is locally topologically trivial it is nonmeager.

We shall now prove a few facts about the elements of $\mathcal{Z}$.

An $(a, b, f, g) \in \mathcal{Z}$ can be re-interpreted in the forcing extension, and in particular we identify function f with the corresponding continuous function. Properties (1)–(6) are $\mathbf{\Pi}^1_1$ and therefore still hold in the extension. In particular f is forced to be a representation of an isomorphism.

For $a \in \mathcal{K}$ let f_a and g_a denote functions such that $(a, b, f_a, g_a) \in \mathcal{Z}$ for some b. For $a \in \mathrm{Triv}^1_\Phi$ let $h_a : \mathcal{P}(a) \to \mathcal{P}(\omega)$ be a continuous representation of $\Phi \restriction a$. Let Φ_* denote a representation of the extension of Φ in the forcing extension.

(9) If $a \in \mathcal{K}$ then $f_a(c) =^{\mathcal{J}} \Phi_*(c) \cap b$ for all $c \subseteq a$.

This is a consequence of Corollary 4.10, since (7) states that p forces

$$f_a(\dot{x} \cap \check{a}) =^{\mathcal{J}} \Phi_*(\dot{x}) \cap \check{b}.$$

If Φ_*^{-1} denotes a representation of Φ^{-1} then by the same argument and (8) we have

$$g_a(d) =^{\mathcal{I}} \Phi_*^{-1}(d) \cap a$$

for all $d \subseteq b$.

(10) If $a \in \mathcal{K}$ then $f_a(c) =^{\mathcal{J}} \Phi_*(c)$ for all $c \subseteq a$.

Let $d = \Phi_*(a) \setminus b$ and $c = \Phi_*^{-1}(d)$. Then $c \setminus a$ belongs to $\mathcal{I}$. Also, with $c' = c \cap a$ we have $f_a(c') =^{\mathcal{J}} \Phi_*(c) \cap b = d \cap b = \emptyset$. However, (6) and (4) together with this imply

$$c =^{\mathcal{I}} c' =^{\mathcal{I}} g_a(f_a(c')) =^{\mathcal{I}} g_a(\emptyset) =^{\mathcal{I}} \emptyset.$$

Unraveling the definitions, we have that Φ_*^{-1} sends $\Phi_*(a) \setminus b$ to $\emptyset$ modulo $\mathcal{I}$ and therefore that $\Phi_*(a) =^{\mathcal{J}} b = f_a(a)$. By applying (9) and Corollary 4.10, (10) follows.

(11) If $a \in \mathrm{Triv}^1_\Phi$ then $a \in \mathcal{K}$ and $h_a(c) =^{\mathcal{J}} \Phi_*(c) =^{\mathcal{J}} f_a(c)$ for all $c \subseteq a$.

That $a \in \mathcal{K}$ is immediate from the definitions of $\mathcal{Z}$ and $\mathcal{K}$, and $h_a(c) =^{\mathcal{J}} \Phi_*(c)$ is immediate from $a \in \mathrm{Triv}^1_\Phi$ and the definition of h_a. The last equality, $\Phi_*(c) =^{\mathcal{J}} f_a(c)$ for all $c \subseteq a$, was proved in (10).

Putting together (10) and (11) we obtain that $\mathcal{K} = \mathrm{Triv}^1_\Phi$ and that f_a witnesses $a \in \mathrm{Triv}^1_\Phi$ for every $a \in \mathcal{K}$.

(12) We have

$$\{(c,d) : \Phi_*(c) =^{\mathcal{J}} d\} = \{(c,d) : (\forall (a,b,f,g) \in \mathcal{Z}) f(c \cap a) =^{\mathcal{J}} b \cap d\}$$

Take (c,d) such that $\Phi_*(c) =^{\mathcal{J}} d$. Then for every $(a,b,f,g) \in \mathcal{Z}$ we have $\Phi_*(c \cap a) =^{\mathcal{J}} f(c \cap a)$ by (11) and (10), and therefore (c,d) belongs to the right-hand side set.

Now take (c,d) such that $\Phi_*(c)\Delta d$ is not in $\mathcal{J}$.

Assume for a moment that $e = \Phi_*(c) \setminus d \notin \mathcal{J}$. Since Φ is an isomorphism, we can find a such that $\Phi_*(a) =^{\mathcal{J}} e$. We have that a is $\mathcal{I}$ positive. Since $\mathcal{K}$ is nonmeager, by Lemma 2.1 we can find $a' \subseteq a$ such that $a' \in \mathcal{K} \setminus \mathcal{I}$. Then $f_{a'}(c \cap a')$ is $\mathcal{J}$-positive, included (modulo $\mathcal{J}$) in e, and disjoint (modulo $\mathcal{J}$) from d. Therefore $(a', f_{a'})$ witness that (c,d) does not belong to the right-hand side of (12).

We must therefore have $e = d \setminus \Phi_*(c) \notin \mathcal{J}$ (there is no harm in denoting this set by e, since the existence of the set denoted by e earlier lead us to a contradiction). Applying the above argument we can find $a' \in \mathcal{K}$ such that $c \cap a'$ is $\mathcal{I}$-positive, but its image under $f_{a'}$ is included (modulo $\mathcal{J}$) in d and disjoint (modulo $\mathcal{J}$) from $\Phi_*(c)$, which is again a contradiction.

By (12) we have the required $\mathbf{\Pi}^1_2$ definition of Φ. □

5. Proofs

Proof of Theorem 1. By §3 for every partition I of ω into finite intervals there is a forcing notion of the form $\mathbb{Q}_{\mathbf{x}}$ that adds a real which captures I. Each of these forcings is proper, real, has continuous reading of names and is ω^ω-bounding. Starting from a model of CH partition $\{\xi < \aleph_2 : \mathrm{cf}(\xi) = \aleph_1\}$ into $\aleph_1$ stationary sets. Consider a countable support iteration $(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \omega_2, \eta < \omega_2)$ of forcings of the form $\mathbb{Q}_{\mathbf{x}}$ and random reals such that for every $\dot{I}$ the set $\{\xi : \mathrm{cf}(\xi) = \omega_1$ and $\dot{\mathbb{Q}}_\xi$ is $\mathbb{Q}_{\mathbf{x}}\}$ is stationary and also $\{\xi : \mathrm{cf}(\xi) = \omega_1$ and $\dot{\mathbb{Q}}_\xi$ is the poset for adding a random real$\}$ is stationary.

Since this forcing is a countable support iteration of proper ω^ω-bounding forcings it is proper and ω^ω-bounding (by [35, §VI.2.8(D)]) and therefore $\mathfrak{d} = \aleph_1$ in the extension.

Now fix names $\dot{\mathcal{I}}$ an $\dot{\mathcal{J}}$ for Borel ideals and a name $\dot{\Phi}$ for an automorphism between Borel quotients $\mathcal{P}(\omega)/\dot{\mathcal{I}}$ and $\mathcal{P}(\omega)/\dot{\mathcal{J}}$. By Lemma 4.11, $\dot{\Phi}$ is forced to be locally $\mathbf{\Delta}^1_2$ and by Corollary 4.8 there is a stationary set $\mathbf{S}$ of ξ such that $\mathrm{cf}(\xi) = \omega_1$ such that $\dot{\Phi} \restriction \xi$ is a $\mathbb{P}_\xi$ name for a a locally $\mathbf{\Delta}^1_2$-isomorphism, and $\dot{\mathbb{Q}}_\xi$ is the standard poset for adding a random real.

By our assumption that all $\mathbf{\Sigma}^1_2$ sets have the property of Baire and Lemma 2.2, $\dot{\Phi}$ is forced to be locally topologically trivial. By Lemma 4.13, if $\xi \in \mathbf{S}$ then $\dot{\Phi}$ is $\mathbf{\Pi}^1_2$ in $V[G \restriction \xi]$. Therefore $\dot{\Phi}$ is $\mathbf{\Pi}^1_2$ in $V[G]$.

Since our assumption that there exists a measurable cardinal implies that we have $\mathbf{\Pi}^1_2$-uniformization of this graph, $f \colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ and all $\mathbf{\Pi}^1_2$ sets have the Property of Baire, Φ has a Baire-measurable representation. By a well-known fact (e.g., [7, Lemma 1.3.2]) Φ has a continuous representation.

In order to add $2^{\aleph_0} < 2^{\aleph_1}$ to the conclusions, start from a model of CH and add $\kappa \geq \aleph_3$ of the so-called Cohen subsets of $\aleph_1$ to increase $2^{\aleph_1}$ to κ while preserving CH. More precisely, we force with the poset of all countable partial functions $p\colon \aleph_3 \times \aleph_1 \to \{0,1\}$ ordered by the extension. Follow this by the iteration $\mathbb{P}_{\aleph_2}$ of $\mathbb{Q}_{\mathbf{x}}$ and $\mathcal{R}$ defined above. The above argument was not sensitive to the value of $2^{\aleph_1}$ therefore all isomorphisms still have continuous representations. Finally, the iteration does not collapse $2^{\aleph_1}$ because a simple Δ-system argument shows that it has $\aleph_2$-cc.

Proof of Theorem 4. Not much more remains to be said about this proof. Assume there exist class many Woodin cardinals, consider the very same forcing iteration as in the proof of Theorem 1 and fix names for universally Baire ideals $\dot{\mathcal{I}}$ and $\dot{\mathcal{J}}$ as well as for an isomorphism $\dot{\Phi}$ between their quotients. Proofs of lemmas from §2 show that the graph of $\dot{\Phi}$ is forced to be projective in $\dot{\mathcal{I}}$ and $\dot{\mathcal{J}}$ and therefore universally Baire itself (see [26]). It can therefore be uniformized on a dense G_δ set by a continuous function, and therefore $\dot{\Phi}$ is forced to have a continuous representation.

6. Concluding remarks

As pointed out earlier, some of the ideas used here were present in the last section of [33]. However, in the latter only automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$ were considered and, more importantly, the model constructed there does have nontrivial automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$. This follows from the very last paragraph of [37].

Question 6.1. *Are large cardinals necessary for the conclusion of Theorem 1?*

The answer is likely to be negative (as suggested by the anonymous referee) but it would be nice to have a proof.

We note that [7, Question 3.14.2] mentioned in the original version of the present note was solved by Alan Dow ([6]). Questions of whether isomorphisms with continuous representations are necessarily trivial are as interesting as ever, but since we have no new information on these questions we shall move on. Problem 6.2 reiterates one of the conjectures from [11], and a positive answer to (1) below may require an extension of results about freezing gaps in Borel quotients from [10].

Problem 6.2.
(1) Prove that PFA implies that all isomorphisms between quotients over Borel ideals have continuous representations.
(2) Prove that all isomorphisms between quotients over Borel ideals have continuous representations in standard $\mathbb{P}_{\max}$ extension ([41], [27]).

We end with two fairly ambitious questions. A positive answer to the following would be naturally conditioned on a large cardinal assumption (see [13]).

Question 6.3. *Is there a metatheorem analogous to Woodin's Σ^2_1 absoluteness theorem ([26], [40]) and the Π_2-maximality of $\mathbb{P}_{\max}$ extension ([41], [27]), that provides a positive answer to Problem 6.2 (1) or (2) automatically from Theorem 1?*

Let us temporarily abandon Boolean algebras and briefly return to the general situation described in the introduction. Attempts to generalize these rigidity results to other categories were made, with limited success, in [8]. For example, quotient group $\prod_\omega \mathbb{Z}/2\mathbb{Z}/\bigoplus_\omega \mathbb{Z}/2\mathbb{Z}$ clearly has nontrivial automorphisms in ZFC. One should also mention the case of semilattices, when isomorphisms are locally trivial but not necessarily trivial ([8]). On the other hand, PFA implies that all automorphisms of the Calkin algebra are trivial ([12]). Note that 'trivial' as defined here is equivalent to 'inner' for automorphisms of the Calkin algebra, but this is not true for arbitrary corona algebras since in some cases the relevant multiplier algebra has outer automorphisms, unlike $\mathcal{B}(H)$ (see [5], [14]).

Problem 6.4. In what categories can one prove consistency of the assertion that all isomorphisms between quotient structures based on standard Borel spaces are trivial?

6.1. **Groupwise Silver forcing.** A simpler forcing notion that can be used in our proof in place of $\mathbb{Q}_{\mathbf{x}}$ defined above ([16]). The 'relevant parameter' is $\bar{I} = (I_n : n \in \omega)$, a partition of ω into finite intervals. Forcing $\mathbb{S}_{\bar{I}}$ consists of partial functions f from a subset of ω into $\{0, 1\}$ such that the domain of f is disjoint from infinitely many of the I_n. Every condition f can be identified with the compact subset p_f of $\mathcal{P}(\omega)$ consisting of all functions extending f. Special cases of $\mathbb{S}_{\bar{I}}$ are Silver forcing (the case when $I_n = \{n\}$ for all n) and 'infinitely equal,' or EE, forcing (the case when $|I_n| = n$ for all n, see [2, §7.4.C]).

This is a suslin forcing and a fusion argument shows that it is proper, ω^ω-bounding and has continuous reading of names. Also, the proof that this forcing is ω^ω-bounding and proper are analogous to proofs of the corresponding facts for EE, [2, Lemma 7.4.14] and [2, Lemma 7.4.12], respectively). Since this forcing is of the form P_I, Zapletal's results ([43]) make its analysis a bit more convenient. Proofs of these facts and applications of $\mathbb{S}_{\bar{I}}$ to the rigidity of quotients appear in [16].

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Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario, Canada, M3J 1P3, and Matematicki Institut, Kneza Mihaila 35, Belgrade, Serbia

URL: http://www.math.yorku.ca/∼ifarah

Email address: ifarah@mathstat.yorku.ca

The Hebrew University of Jerusalem, Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, Jerusalem 91904, Israel, and, Department of Mathematics, Hill Center-Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA

Email address: shelah@math.huji.ac.il

URL: http://shelah.logic.at/