## 1. Introduction

For a ring $R$ and modules $H$ and $\{M_i : i \in I\}$ over $R$, where $I$ is countable, there is a natural isomorphism between $\mathrm{Hom}(\bigoplus_{i\in I} M_i, H)$ and $\prod_{i\in I} \mathrm{Hom}(M_i, H)$. There usually isn't a canonical isomorphism when we reverse the order, between $\mathrm{Hom}(\prod_{i\in I}, H)$ and $\bigoplus_{i\in I} \mathrm{Hom}(M_i, H)$. The modules $H$ over $R$ for which such an isomorphism exists are exactly the slender modules, where a module is slender if every homomorphism $\varphi : R^\omega \to H$ is determined by a finite number of coordinates. Over $\mathbb{Z}$, the ring of the integers, the modules are simply commutative groups. Slender groups (and modules) were introduced by Łos, and first appeared in [Fuc73]. In the list of open questions in their book [EM02] Eklof and Mekler pose the following question (section C question 7), Is it provable in ZFC that there exists a group other than the ${}^n\mathbb{Z}$ for $n \in \omega$, such that both $\mathbb{Z}^*$ and $\mathbb{Z}^{**}$ are slender. The motive behind this work was the hope to construct such a group, this goal hasn't been achieved yet. From what follows it seems that it can be done under the assumptions that $\mathfrak{h} = \aleph_1$ and that there exists a saturated MAD family, on this topic see F900 [1]. ***

The original plan was to construct such a group using a a construction which is very similar to the Balcar and Simon 'Base Tree' theorem. The plan also being that this construction would be at the base of more constructions, for example, of Boolean algebras with certain properties [FILL].

Balcar and Simon's theorem for Boolean algebras appears in [BS89], For the particular case where the Algebra is $\mathbf{B} = P(\omega)/\mathrm{fin}$ and the cardinals $\kappa$ and $\lambda$ which appear there are chosen to be $\aleph_1$ and 2, respectively, then we get a 'Base Tree', i.e there is a dense tree $D \subseteq \mathbf{B}$, such that for every $b \in \mathbf{B}$ we have $|\{d \in D : |b \cap d| = \aleph_0\}| = 2^{\aleph_0}$. In this work we will prove a similar theorem for when $\lambda = 2$ in an abstract setting. This theorem stands on its own, but as mentioned above, one of the objectives in proving this theorem is that it will be able to serve in many constructions, amongst them the construction of a group $G$ that both its dual and dual's dual are slender.

In this summary we bring the abstract construction.

[1] Saharon, is this the right reference?

## 2. The Abstract Case

**Definition 2.1.** Let $\mathbb{P}$ be a quasi-order. We define $\mathfrak{h}(\mathbb{P})$ to be the minimal $\theta$ for which exists a sequence $\langle D_\alpha : \alpha < \theta \rangle$ of open dense sets in $\mathbb{P}$ such that $\bigcap\limits_{\alpha<\theta} D_\alpha$ is not open dense.

**Observation 2.2.** *Assume $\mathbb{P}$ is a quasi order:*

(1) *If $\mathfrak{h}(\mathbb{P})$ exists then it is a regular cardinal.*
(2) *If $\mathbb{P}$ is not atomic (i.e its atoms aren't dense in $\mathbb{P}$) then $\mathfrak{h}(\mathbb{P})$ exists.*
(3) *If for every $p \in \mathbb{P}$ we have $\mathfrak{h}(\mathbb{P}) = \mathfrak{h}(\mathbb{P}_{\geq p})$ then there is a sequence $\langle D_\alpha : \alpha < \mathfrak{h}(\mathbb{P}) \rangle$ of open dense sets such that their intersection is empty, moreover, we can choose the sequence to be decreasing.*

While reading this section it is good to keep in mind the special case where $\mathbb{Q} = \mathbb{P}$, $\perp_0, \perp_1$ are interpreted as incompatibility, and the relation $R$ simply as $\leq_{\mathbb{P}}$ (see below)

**Definition 2.3.** (1) We say $\mathbf{x} = (\mathbb{P}, C, \kappa)$ is a precandidate when:

(a) $\kappa \geq \aleph_0$ and for every $\theta < \kappa$ we have $2^\theta < 2^\kappa$
(b) $\mathbb{P}$ is a not atomic quasi order, $C \subseteq \mathbb{P}$ is dense with cardinality $2^\kappa$
(c) $\mathbb{P}$ is $\kappa^+$- complete (i.e. every increasing chain of length $\leq \kappa$ has an upper bound).
(d) $\mathfrak{h}(\mathbb{P}_{\geq p}) = \mathfrak{h}(\mathbb{P})$ for every $p \in \mathbb{P}$.

(2) We say $(\mathbb{P}, C, \kappa, \mathbb{Q}, \perp_0, \perp_1, \mathscr{A})$ is a candidate when in addition:

(e) $\mathbb{Q}$ is a set of cardinality $\leq 2^\kappa$.
(f) $\perp_0 \subseteq \mathbb{P} \times \mathbb{P}$ is a symmetric relation, and $\not\perp_0 \subseteq \not\perp$ (where $\perp$ is incompatibility in $\mathbb{P}$)
(g) $\perp_1 \subseteq \mathbb{P} \times \mathbb{Q}$.
(h) $\mathscr{A} = \{A_p : p \in \mathbb{P}\}$. $A_p \subseteq \mathbb{P}_{\geq p}$ for every $p \in \mathbb{P}$, and this is a $\perp_0$ in pairs set of cardinality $2^\kappa$.
(i) If $p \not\perp_1 q$ then there are $p_1, p_2$ such that $p_1 \perp_0 p_2$, $p_i \not\perp_1 q$, and $p_i$ are above an element of $A_p$ $(i = 1, 2)$.
(j) If $D$ is a dense set in $\mathbb{P}$ and $r \not\perp_\ell q$ then there is $d \in D$ such that $d \geq r$ and $d \not\perp_\ell q$ $(\ell = 0, 1)$.
(k) If $p \in \mathbb{P}, q \in \mathbb{Q}$ and $B \subseteq \mathbb{P}$ is a set of pairwise $\perp_0$ elements such that $|B| < 2^\kappa$, and for every $r \in B$ both $r \perp_1 q$ and $p \not\perp_1 q$ <u>then</u>

there are $p_1, p_2 \in \mathbb{P}$ above elements in $A_p$ such that $p_1 \perp_0 p_2$, for every $r \in B$ we have $p_i \perp_0 r$, and $p_i \not\perp_1 q$. [2] ***

(l) If $p_1 \perp_\ell x$ and $p_2 \geq p_1$ then also $p_2 \perp_\ell x$ ($\ell = 0, 1$), we call this property smoothness (actually for $\ell = 0$ the smoothness is a consequence of ($e$) above).

(3) We say $(\mathbb{P}, C, \kappa, \mathbb{Q}, \perp_0, \perp_1, \mathscr{A}, R)$ is a good candidate when in addition:

(m) $R \subseteq \not\perp_1$.

(n) Given a sequence $\bar{p} = \langle p_\varepsilon : \varepsilon < \delta \rangle$ and $q \in \mathbb{Q}$, if the following conditions hold, there is $p \in \mathbb{P}$ such that $p \not\perp_1 q$ and $p$ is an upper bound of $\bar{p}$:

(i) $\delta \leq \mathfrak{h}(\mathbb{P})$

(ii) $\langle p_\varepsilon : \varepsilon < \delta \rangle$ is $\leq_{\mathbb{P}}$-increasing.

(iii) $p_\varepsilon \not\perp_1 q$ for $\varepsilon < \delta$.

(iv) When $\mathrm{cf}(\delta) > \kappa$, for each $\varepsilon < \delta$ there is a maximal subset $\mathcal{I}_\varepsilon$ of $\{r : \bigwedge\limits_{\zeta < \varepsilon} (p_\zeta \leq_{\mathbb{P}} r)\}$ of pairwise $\perp_0$-members of $\mathbb{P}$ such that $p_\varepsilon \in \mathcal{I}$, and $r \in \mathcal{I}_\varepsilon \backslash \{p_\varepsilon\} \Rightarrow r \perp_1 q$.

As mentioned above it is a good idea to keep in mind the case where $\mathbb{Q} = \mathbb{P}$, $\perp_0, \perp_1$ are interpreted as the usual incompatibility, and $R$ is interpreted as $\leq_{\mathbb{P}}$. In this special case almost all the properties of a good candidate are immediate, since in this case if $p, q \in \mathbb{P}$ are compatible then $\mathbb{P}_{\geq p} \cap \mathbb{P}_{\geq q} \neq \emptyset$. We also note that in this case it is not necessary to choose ahead of time $\mathscr{A}$, and in fact it is enough to demand that above every $p$ there are 2 incompatible elements, and from the $\kappa^+$-completeness one can deduce the existence of $2^\kappa$ many such elements. We return to this case later on.

**Definition 2.4.** We say that the good candidate $\mathbf{x}$ is $\theta$-solvable when there is a $\theta$-solution $\bar{p}$ which means:

(a) $\bar{p}$ has the form $\langle p_\eta : \eta \in \mathrm{suc}(\mathcal{T}) \rangle$, so

(b) $\mathcal{T}$ is a subtree of ${}^{\theta>}(2^\kappa)$, i.e. closed under initial segments and

$$\mathcal{T}_\varepsilon = \{\eta \in \mathcal{T} : \lg(\eta) = \varepsilon\}, \ \mathrm{suc}(\mathcal{T}) = \cup\{\mathcal{T}_{\varepsilon+1} : \varepsilon < \theta\}$$

(b)$^+$ And if $\eta \in \mathcal{T}_{\varepsilon+1}$ then $\{\alpha : \eta^\frown\langle\alpha\rangle \in \mathcal{T}\}$ has cardinality $2^\kappa$.

[2] this seems like a very cumbersome definition, try to find a simpler more natural one. It is used in one place, in adding to the tree

(c) If $\eta \triangleleft \nu$ are from $\mathrm{suc}(\mathcal{T})$ then $p_\eta \leq_{\mathbb{P}} p_\nu$.

(d) If $\varepsilon < \theta, \eta \in \mathcal{T}_\varepsilon$ then $\{p_{\eta\frown\langle\alpha\rangle} : \eta\frown\langle\alpha\rangle \in \mathcal{T}\}$ are pairwise $\perp_0$.

(d)$^+$ Moreover $\langle p_{\eta\frown\langle\alpha\rangle} : \eta\frown\langle\alpha\rangle \in \mathcal{T}\rangle$ is maximal under (c)+(d)

(e) $\{p_\eta : \eta \in \mathrm{suc}(\mathcal{T})\}$ is a $\mathbf{x}$-dense subset of $\mathbb{P}$ i.e. $(\forall q \in \mathbb{Q})(\exists \eta \in \mathrm{suc}(\mathcal{T}))(p_\eta R q)$

The following is the main theorem in this section.

**Theorem 2.5.** *Assume $\mathbf{x} = (\mathbb{P}, \mathbf{C}, \kappa, \mathbb{Q}, \perp_{\mathbf{0}}, \perp_{\mathbf{1}}, \mathscr{A}, \mathbf{R})$ is a good candidate, and let $\langle F_\alpha : \alpha < 2^\kappa\rangle$ be a sequence of functions such that if $q, \langle p_\varepsilon : \varepsilon < \delta\rangle$ with $\mathrm{cf}(\delta) = \kappa$ satisfy clause (3n) in 2.3 then for every $\alpha < 2^\kappa$ $f_\alpha = F_\alpha(q, \langle p_\varepsilon : \varepsilon < \delta\rangle)$ is an upper bound for $\langle p_\varepsilon : \varepsilon < \delta\rangle$ and $f_\alpha R q$.*
*Then we can find $\bar{p}$ such that*

(a) $\bar{p} = \langle p_\eta : \eta \in \mathrm{suc}(\mathcal{T})\rangle$ *is an $\mathfrak{h}(\mathbb{P})$-solution*

(b) $\mathcal{T}$ *is the union of the $\subseteq$- increasing sequence $\langle \mathcal{T}^\alpha : \alpha < 2^\kappa\rangle$ such that $|\mathcal{T}^\alpha| \leq \kappa + |\alpha|$ and $\mathcal{T}^\alpha$ is $\triangleleft$-downward closed*

(c) *For every $q \in \mathbb{P}$ there is $\langle \eta_\rho : \rho \in {}^{\kappa>}2\rangle$ with the following properties:*

($\alpha$) $\rho_1 \triangleleft \rho_2 \Rightarrow \eta_{\rho_1} \triangleleft \eta_{\rho_2}$

($\beta$) $\eta_{\rho\frown\langle 0\rangle}, \eta_{\rho\frown\langle 1\rangle}$ *are $\triangleleft$- incomparable for $\rho \in {}^{\kappa>}2$*

($\gamma$) $\langle p_{\eta_\rho\restriction(\varepsilon+1)} : \varepsilon < \lg(\eta_\rho)\rangle$, q *satisfy the demands in clause 2.3(3n).*

($\delta$) *for $\varepsilon < \kappa$ limit and $\rho \in {}^\varepsilon 2$ we have $\eta_\rho = \cup\{\eta_{\rho\restriction\zeta} : \zeta < \varepsilon\}$.*

($\varepsilon$) *for $\rho \in {}^\kappa 2$ we let $\eta_\rho = \cup\{\eta_\rho\restriction\varepsilon : \varepsilon < \kappa\}$.*

($\zeta$) *for $2^\kappa$ ordinals $\alpha$, there is a set $\Lambda_\alpha = \{\rho^\alpha_\beta : \beta < |\alpha| + \kappa\}$ such that for every $\beta < \alpha$ the following hold:*
*$\rho^\alpha_\beta \in {}^\kappa 2$, $\rho^\alpha_\beta \notin \mathcal{T}^\alpha$, $\eta_{\rho^\alpha_\beta\frown\langle 0\rangle} \in \mathcal{T}^{\alpha+1}$ and*

$$p_{\eta_{\rho^\alpha_\beta\frown\langle 0\rangle}} = F_\beta(q, \langle p_{\eta_{\rho^\alpha_\beta}\restriction(\varepsilon+1)} : \varepsilon < \lg(\eta^\alpha_\beta)\rangle)$$

*Proof.* Choose $\bar{\mathcal{D}}$ such that $\bar{\mathcal{D}} = \langle \mathcal{D}_\varepsilon : \varepsilon < \mathfrak{h}(\mathbb{P})\rangle$ is a decreasing sequence of dense open subsets of $\mathbb{P}$ such that $\cap\{\lceil_\varepsilon : \varepsilon < \mathfrak{h}(\mathbb{P})\} = \emptyset$ (see Observation2.2(3)).
Before we proceed with the proof we need a few more definitions (and lemmas).

**Definition 2.6.** For every $\mu \leq 2^\kappa$ we define $K_\mu$ and $\leq_\mu$.
$K_\mu$ is the set of $\bar{p} = \langle p_\eta : \eta \in \mathrm{suc}(\mathcal{T})\rangle$ (we write $\mathcal{T} = \mathcal{T}^{\bar{p}} = \mathcal{T}[\bar{p}]$ ) satisfying clauses (a),(b),(c),(d) of Definition 2.4 for $\theta = \mathfrak{h}(\mathbb{P})$, such that

(1) $|\mathcal{T}| \leq \mu$.
(2) if $\eta \in \mathcal{T}_{\varepsilon+2}, \eta = \nu^\frown\langle\alpha\rangle$ for some $\alpha$ and $\nu$, then $p_\eta \in \mathcal{D}_\varepsilon \cap \mathbb{P}_{\geq q}$ for some $q \in \mathcal{A}(p_\nu)$.
(3) $\beta_\eta = \{\alpha : \eta^\frown\langle\alpha\rangle \in \mathcal{T}\}$ is an ordinal for every $\eta \in \mathcal{T}$, and we denote $C_\eta = \{p_{\eta^\frown\langle\alpha\rangle} : \alpha \in \beta_\eta\}$.

$\leq_\mu$ is the natural order on $K_\mu$ i.e $\bar{p} \leq_\mu \bar{q}$ iff

(1) $\bar{p}, \bar{q} \in K_\mu$.
(2) $\mathcal{T}[\bar{p}] \subseteq \mathcal{T}[\bar{q}]$.
(3) $\bar{p} = \bar{q} \restriction \mathcal{T}[\bar{p}]$.

Notice that $K_\mu \neq \emptyset$ as for $\mathcal{T} = \{\langle\rangle\}$, we have $\bar{p} = \langle\rangle \in K_\mu$.

**Lemma 2.7.** *If the sequence $\langle\bar{p}^\alpha : \alpha < \delta\rangle$ is $\leq_\mu$-increasing (in $K_\mu$), $\delta \leq \mu \leq 2^\kappa$*
*then the sequence has a $\leq_\mu -\ell ub$*

*Proof.* Let $\langle\bar{p}^\alpha : \alpha < \delta\rangle$ be a $\leq_\mu$-increasing sequence where $\bar{p}^\alpha = \langle p^\alpha_\eta : \eta \in \mathrm{suc}(\mathcal{T}^\alpha)\rangle$. Set $\mathcal{T} = \cup\{\mathcal{T}^\alpha : \alpha < \delta\}$ and let $p_\eta = p^\alpha_\eta$ for $\eta \in \mathrm{suc}(\mathcal{T})$, and $\alpha$ large enough. $\square$

**Observation 2.8.** *For $\mu_1 \leq \mu_2 \leq 2^\kappa$ we have $K_{\mu_1} \subseteq K_{\mu_2}$ and $\leq_{\mu_1} = \leq_{\mu_2} \restriction K_{\mu_1}$.*

**Definition 2.9.** We say that $\bar{p} \in K_\mu$ is full with respect to $q \in \mathbb{P}$ when for every $\eta \in \mathcal{T}[\bar{p}]$:
<u>if</u> there exists only one$\alpha$ such that

(1) $\eta^\frown\langle\alpha\rangle \in \mathcal{T}[\bar{p}]$
(2) $p_{\eta^\frown\langle\alpha\rangle} \not\leq_1 q$

<u>then</u> there is no $r \in \mathbb{P}$ such that

(a) for every $\varepsilon < \lg(\eta)$ we have $p_{\eta\restriction(\varepsilon+1)} \leq r$.
(b) $r \not\leq_1 q$.
(c) for every $\gamma$ if $\eta^\frown\langle\gamma\rangle \in \mathcal{T}[\bar{p}]$ then $p_{\eta^\frown\langle\gamma\rangle} \perp_0 r$

**Definition 2.10.** (definition+proof)
For every $q \in \mathbb{P}$ and $\bar{p} = \langle p_\eta : \eta \in \mathrm{suc}(\mathcal{T})\rangle \in K_\mu$ ($\mu \leq 2^\kappa$) full with respect to $q$, we define by induction on $\varepsilon < \kappa$ sets $S_{\bar{p},q,\varepsilon} \subseteq {}^\varepsilon 2$ and $\bar{\eta}_{\bar{p},q} = \langle\eta_{\bar{p},q,\rho} : \rho \in S_{\bar{p},q,\varepsilon}\rangle$ such that:

(1) if $\zeta < \varepsilon$ and $\rho \in S_{\bar{p},q,\varepsilon}$ then $\rho\restriction\zeta \in S_{\bar{p},q,\zeta}$ and $\eta_{\bar{p},q,\rho\restriction\zeta} \triangleleft \eta_{\bar{p},q,\rho}$

(2) if $\rho \in S_{\bar{p},q,\varepsilon}$ then $p_{\eta_{\bar{p},q,\rho}} \not\perp_1 q$

<u>Case 1</u>: $\varepsilon = 0$

$S_{\bar{p},q,\varepsilon} = \langle\rangle$ and $\eta_{\langle\rangle} = \langle\rangle$

<u>Case 2</u>: $\varepsilon$ is a limit ordinal

$S^0_{\bar{p},q,\varepsilon} = \{\rho \in {}^{\varepsilon}2 : \zeta < \varepsilon \Rightarrow \rho{\restriction}\zeta \in S_{\bar{p},q,\zeta}\}$

For every $\rho \in S^0_{\bar{p},q,\varepsilon}$ define $\eta_{\bar{p},q,\rho} = \bigcup\{\eta_{\bar{p},q,\rho\restriction\zeta} : \zeta < \varepsilon\}$

We define $S_{\bar{p},q,\varepsilon} = \{\rho \in S^0_{\bar{p},q,\varepsilon} : (\exists\gamma)(\eta_{\bar{p},q,\rho}{}^\frown\langle\gamma\rangle \in \mathcal{T}[\bar{p}])\}$

<u>Case 3</u>: $\varepsilon = \zeta + 1$

For $\rho \in S_{\bar{p},q,\zeta}$ we try to define $\eta_{\bar{p},q,\rho,i}$ by induction on $i \leq \mathfrak{h}(\mathbb{P})$ such that:

(a) $\eta_{\bar{p},q,\rho,i} \in \mathcal{T}_{\lg(\eta_{\bar{p},q,\rho})+i}$

(b) $\eta_{\bar{p},q,\rho,0} = \eta_{\bar{p},q,\rho}$

(c) if $i = j+1$ and there is a unique $\alpha$ such that $p_{\eta_{\bar{p},q,\rho,j}{}^\frown\langle\alpha\rangle} \not\perp_1 q$ then

$$\eta_{\bar{p},q,\rho,i} = \eta_{\bar{p},q,\rho,j}{}^\frown\langle\alpha\rangle \text{ for this } \alpha.$$

(d) for $i$ limit we let $\eta = \cup\{\eta_{\bar{p},q,\rho,j} : j < i\}$, and if there is $\gamma$ such that $\eta^\frown\langle\gamma\rangle \in \mathcal{T}[\bar{p}]$ then we define $\eta_{\bar{p},q,\rho,i} = \eta$.

Let $\mathbf{i}(\bar{p},q,\rho)$ be the minimal $i \leq \mathfrak{h}(\mathbb{P})$ such that $\eta_{\bar{p},q,\rho,i}$ is not defined, i.e this is the minimal $i$ for which we can't find a successor which is $\not\perp_1$ with $q$, conversely there are at least two such successors.

Now this is a crucial point:

$$\mathbf{i}(\bar{p},q,\rho) < \mathfrak{h}(\mathbb{P}).$$

Otherwise, as $\bar{p}$ is full with respect to $q$, $\{p_{\eta_{\bar{p},q,\rho,i}} : i \leq \mathfrak{h}(\mathbb{P})\} \cup \{q\}$ satisfy $(m)$ of Definition 2.3, so there is an upper bound $r$ for $\langle p_{\eta_{\bar{p},q,\rho,i}} : i < \mathfrak{h}(\mathbb{P})\rangle$, so $r \in \cap\{\mathcal{D}_\varepsilon : \varepsilon \leq \mathfrak{h}(\mathbb{P})\}$, which is a contradiction.

Let $S^*_{\bar{p},q,\varepsilon}$ be the set of all $\rho \in S_{\bar{p},q,\zeta}$ such that $\mathbf{i}(\bar{p},q,\rho)$ is a successor ordinal, and the failure of clause (c) above is in having at least two such $\alpha$-s. Now let

$$S_{\bar{p},q,\varepsilon} = \{\rho^\frown\langle 0\rangle, \rho^\frown\langle 1\rangle : \rho \in S^*_{\bar{p},q,\varepsilon}\}.$$

Lastly if $\rho^\frown\langle 0\rangle, \rho^\frown\langle 1\rangle \in S_{\bar{p},q,\varepsilon}$ let $\alpha_0 < \alpha_1$ be the first two ordinals $\alpha$ such that $p_{\eta_{\bar{p},q,\rho,\mathbf{i}(\bar{p},q,\rho)-1}{}^\frown\langle\alpha\rangle} \not\perp_1 q$ and let

$$\eta_{\bar{p},q,\rho^\frown\langle\ell\rangle} = \eta_{\bar{p},q,\rho,\mathbf{i}(\bar{p},q,\rho)-1}{}^\frown\langle\alpha_\ell\rangle \text{ for } \ell = 0,1.$$

We now define $S_{\bar{p},q} = \cup\{S_{\bar{p},q,\varepsilon} : \varepsilon < \kappa\}$ and $\bar{\eta}_{\bar{p},q} = \langle\eta_{\bar{p},q,\rho} : \rho \in S_{\bar{p},q}\rangle$.

*Remark* 2.11. For $\bar{p} \in K_\mu$, $q \in \mathbb{P}$, for every $\eta = \eta_\rho \in \bar{\eta}_{\bar{p},q}$ we have $\eta = \cup\{\eta_{\bar{p},q,\rho\restriction\varepsilon} : \varepsilon < \lg(\rho)\}$. As $\lg(\rho) < \kappa$ so $\mathrm{cf}(\lg(\eta)) < \kappa$, and $\langle q, \langle p_{\eta\restriction(\varepsilon+1)} : \varepsilon < \lg(\rho)\rangle\rangle$ satisfy clause $(m)$ in 2.3(3).

**Lemma 2.12.**
(1) *For every $\bar{p} \in K_\mu$ $(\mu \leq 2^\kappa)$, $q \in \mathbb{Q}$ there is $\bar{q}$ such that $\bar{p} \leq_\mu \bar{q}$ and $\bar{q}$ is full with respect to $q$.*
(2) *For $q \in \mathbb{Q}$ and $\bar{p}_1 \leq_\mu \bar{p}_2$ both full with respect to $q$ we have $S_{\bar{p}_1,q} \subseteq S_{\bar{p}_2,q}$ and $\bar{\eta}_{\bar{p}_1,q} = \bar{\eta}_{\bar{p}_2,q}\restriction S_{\bar{p}_1,q}$.*

*Proof.* Let $A = A_{\bar{p},q}$ be the set of all $\eta$ such that there is a unique $\alpha = \alpha_\eta$ as in parts $1, 2$ of definition 2.9 and exists an $r$ as in (a)-(c) of the definition. From clauses (i,k) in Definition 2.3(1) there exists such $r$ in $C$, $r_\eta$. define $\mathcal{T}[\bar{q}] = \mathcal{T}[\bar{p}] \cup \{\eta^\frown\langle\beta_\eta\rangle : \eta \in A\}$ (see Definition 2.6 for the definition of $\beta_\eta$, and recall that it is an ordinal), all that is left is to define $q_{\eta^\frown\langle\beta_\eta\rangle}$. If $\lg(\eta)$ is a limit ordinal we can choose $q_{\eta^\frown\langle\beta_\eta\rangle} = r_\eta$, if $\lg(\eta) = \varepsilon + 1$ by clause $(i)$ of the definition of a candidate there is $r \in \mathcal{D}_\epsilon \cap C$ such that $r \geq r_\eta$ and $r \not\leq_1 q$. Using the smoothness (clause $(k)$ of the above definition) we conclude that $r \perp_0 p_{\eta^\frown\langle\gamma\rangle}$ for every $\gamma \in \beta_\eta$. Clearly $\bar{q}$ is full with respect to $q$, and $\bar{p} \leq_\mu \bar{q}$ as required.
(2) follows directly from the definition of being full with respect to a condition. $\square$

We now have the tools to prove Theorem **??**.
Let $\bar{y} = \langle(x_\alpha, \rho_\alpha) : \alpha < 2^\kappa\rangle$ list all the pairs $(x, \rho)$ such that $q \in \mathbb{Q} \cup C$, and $\rho \in {}^{\kappa\geq}2$, without loss of generality for each $\alpha$

$$\{(x_\alpha, \nu) : \nu \in {}^{\lg(\rho)>}2\} \subseteq \{(x_\beta, \rho_\beta) : \beta < \alpha\}.$$

This is possible as $2^{<\kappa} < 2^\kappa$ (see Definition 2.3(1)(a) recalling that $\mathrm{cf}(2^\kappa) > \kappa$) Now we choose $\bar{p}^\alpha$ by induction on $\alpha < 2^\kappa$ such that

(1) $\bar{p}^\alpha = \langle p^\alpha_\eta : \eta \in \mathrm{suc}(\mathcal{T}^\alpha)\rangle \in K_{|\alpha|+\kappa}$.
(2) $\langle\bar{p}^\beta : \beta \leq \alpha\rangle$ is $\leq_{|\alpha|+\kappa}$-increasing continuous.
(3) if $\alpha = 2\beta + 1$ and $x_\beta = q_\beta \in \mathbb{Q}$:
    (a) If $\lg(\rho_\beta) < \kappa$ then $\rho_\beta \in S_{\bar{p}^\alpha,q_\beta}$, and $p^\alpha$ is full with respect to $q_\beta$.
    (b) If $\lg(\rho_\beta) = \kappa$ then for every $\gamma < |\alpha| + \kappa$ there is $\varrho_\gamma \in {}^\kappa 2$ so that $\eta_{\varrho_\gamma} = \eta_{p^{2\beta},q_{2\beta},\varrho_\gamma} \in \mathcal{T}^\alpha \setminus \mathcal{T}^{2\beta}$ and
    $p^\alpha_{\eta_{\varrho_\gamma}{}^\frown\langle 0\rangle} = F_\gamma(q_\beta, \langle p_{\eta_{\varrho_\gamma}\restriction(\varepsilon+1)} : \varepsilon < \lg(\eta_{\varrho_\gamma})\rangle)$.

(4) If $\alpha = 2\beta + 2$, $x_\beta \in C$ and $\eta \in \mathcal{T}^{2\beta}$ such that for every $\varepsilon < \lg(\eta)$ we have $(p^{2\beta}_{\eta\restriction(\varepsilon+1)} \leq_{\mathbb{P}} x_\beta)$, and for every $\gamma$ if $\eta^\frown\langle\gamma\rangle \in \mathcal{T}^{2\beta}$ then also $p^{2\beta}_{\eta^\frown\langle\gamma\rangle} \perp_0 x_\beta$ then there is $r \in \mathbb{P}$, $x_\beta \leq_\P r$ and $r \in \{p^\alpha_{\eta^\frown\langle\gamma\rangle} : \eta^\frown\langle\gamma\rangle \in \mathcal{T}^\alpha\}$.

For $\alpha = 0$ choose any $\bar{p} \in K_\mu$.

For $\alpha$ a limit ordinal we take $\bar{p}^\alpha$ to be the $\ell ub$ of $\langle p^\beta : \beta < \alpha\rangle$.

For $\alpha$ a successor ordinal there are a few cases:

case 1: $\alpha = 2\beta + 1$, $x_\beta = q_\beta \in \mathbb{Q}$ and $\lg(\rho_\beta) = \varepsilon < \kappa$.

Let $\bar{p}'$ be such that $\bar{p}^{2\beta} \leq_{\kappa+|\alpha|} \bar{p}'$ and $\bar{p}'$ is full with respect to $q_\beta$ (this can be done by Lemma 2.12). By our assumption on $\bar{y}$ for every $\zeta < \varepsilon$ there is $\gamma < \beta$ such that $x_\gamma = q_\beta$ and $\rho_\gamma = \rho_\beta\restriction\zeta$, therefore by the induction hypothesis $\rho_\gamma \in S_{\bar{p}^{2\gamma+1},q_\beta}$ and $\bar{p}^{2\gamma+1}$ is full with respect to $q_\beta$, therefore by 2.12 we have $S_{\bar{p}^{2\gamma+1},q_\beta} \subseteq S_{\bar{p}',q_\beta}$. If also $\rho_\beta \in S_{\bar{p}',q_\beta}$ define $\bar{p}^\alpha = \bar{p}'$, else, if $\varepsilon$ is a limit ordinal let $\eta = \cup\{\eta_{\rho_\beta\restriction\zeta} : \zeta < \varepsilon\}$ and if $\varepsilon = \zeta + 1$ let $\eta = \cup\{\eta_{\bar{p}',q_\beta,\rho_\beta\restriction\zeta,i} : i < \mathbf{i}(\bar{p}', q_\beta, \rho_\beta\restriction\zeta)\}$. The reason $\rho_\beta \notin S_{\bar{p}',q_\beta}$ is that for every $\gamma$ such that $\eta^\frown\langle\gamma\rangle \in \mathcal{T}[\bar{p}']$ we have $p'_{\eta^\frown\langle\gamma\rangle} \perp_1 q_\beta$. Define $\mathcal{T}^\alpha = \mathcal{T}[\bar{p}'] \cup \{\eta^\frown\langle\beta_\eta\rangle, \eta^\frown\langle\beta_\eta + 1\rangle, \eta\}$, note we need to add $\eta$ only if $\lg\eta$ is a limit ordinal. All that is left is to define $p^\alpha_{\eta^\frown\langle\beta_\eta+\ell\rangle}$ for $\ell \in \{0, 1\}$.

Now $q_\beta, \langle p'_{\eta_{\rho_\beta}\restriction(\zeta+1)} : \zeta < \lg(\eta)\rangle$ satisfy condition $(m)$ of definition 2.3 (see Remark 2.11) so there is $p \in \mathbb{P}$ which is an upper bound of the sequence and it is $\not\perp_1$ with $q_\beta$. For every $r \in C_\eta$ (recall $C_\eta = \{p_{\eta^\frown\langle\gamma\rangle} : \eta^\frown\langle\gamma\rangle \in \mathcal{T}[\bar{p}^{2\beta}]\}$, see Definition 2.6) we have $r \perp_1 q_\beta$ and $|C_\eta| < 2^\kappa$ so we can use clause $j$ of Definition 2.3 [3] to get $r_0, r_1$ above $p$ so that for every $r \in C_\eta$ $r_\ell \perp_0 r$, $r_0 \perp_0 r_1$ and $r_\ell \not\perp_1 q_\beta$. Using smoothness and clause $(i)$ in the mentioned definition we can choose $r_0, r_1$ in $C$. If $\lg(\eta) = \xi + 1$ then there are $r_{\eta\ell} \in \mathcal{D}_\xi \cap \mathbb{P}_{\geq r_\ell} \cap C$ such that $r_{\eta\ell} \not\perp_1 q_\beta$, and otherwise let $r\eta_\ell = r_\ell$ $(\ell = 0, 1)$. In either case $r_{\eta\ell} \not\perp_1 q_\beta$ and $r_{\eta_1} \perp_0 r_{\eta_1}$, and from smoothness for every $q \in C_\eta$ we have $r_{\eta\ell} \perp_0 q$. By the definition of $S_{\bar{p}^\alpha,q_\beta}$ it follows that $\rho_\beta \in S_{\bar{p}^\alpha,q_\beta}$.

case 2: $\alpha = 2\beta + 1$, $x_\beta = q_\beta \in \mathbb{Q}$ and $\lg(\rho_\beta) = \kappa$.

As in the previous case we take $\bar{p}^{2\beta} \leq_{\kappa+|\alpha|} \bar{p}'$ such that $\bar{p}'$ is full with respect to $q_\beta$. First we will show that $S_{\bar{p}',q_\beta} = {}^{\kappa>}2$. By our choice of $\bar{y}$, for every $\rho \in {}^{\kappa>}2$ there is $\gamma < \beta$ such that $x_\gamma = q_\beta$ and $\rho_\gamma = \rho$ therefore by the induction hypothesis $\bar{p}^{2\gamma+1}$ is full with respect to $q_\beta$ and $\rho \in S_{\bar{p}^{2\gamma+1}q_\beta}$

[3] this is the only place where this clause is used. try to change this condition to something more natural

and therefore by lemma 2.12 $\rho \in S_{\bar{p}',q_\beta}$. Now$|\mathcal{T}^{2\beta}| < 2^\kappa$ therefore exist $2^\kappa$ sequences $\varrho \in {}^\kappa 2$ such that $\eta_\varrho = \eta_{\bar{p}^{2\beta},q_\beta,\varrho} \notin \mathcal{T}^{2\beta}$, but for every $\varepsilon < \kappa$ $\varrho{\restriction}\varepsilon \in S_{\bar{p}',q_\beta}$ and therefore $p'_{\eta_\varrho \restriction(\varepsilon+1)} \not\leq_1 q_\beta$. Let $\Lambda_\alpha \subseteq {}^\kappa 2$ be a set of such $\rho$s of cardinality $|\alpha|+\kappa$, and $\langle \rho_\gamma : \gamma < |\alpha|+\kappa\rangle$ be a list of its elements, and let $p^\alpha_{\eta_{\rho_\gamma}\frown\langle 0\rangle} = F_\gamma(q_\beta, \langle p'_{\eta_{\rho_\gamma}\restriction\varepsilon} : \varepsilon < \kappa\rangle)$. We choose $\mathcal{T}^\alpha = \mathcal{T}^{2\beta} \cup \{\eta_{\rho\frown\langle 0\rangle} : \rho \in \Lambda_\alpha\}$, and $\bar{p}^\alpha = \bar{p}' \cup \{p^\alpha_{\eta_{\rho_\gamma}\frown\langle 0\rangle} : \gamma < |\alpha|+1\}$.

<u>case 3:</u> $\alpha = 2\beta+2$ and $x_\beta \in C$.

Let $E = \{\eta \in \mathcal{T}^{2\beta} : (\forall \varepsilon < \lg(\eta))(p^{2\beta}_{\eta\restriction(\varepsilon+1)} \leq_{\mathbb{P}} x_\beta) \wedge (\forall\gamma)(\eta\frown\langle\gamma\rangle \in \mathcal{T}^{2\beta} \Rightarrow p^{2\beta}_{\eta\frown\langle\gamma\rangle} \perp_0 x_\beta)\}$. Define $\mathcal{T}^\alpha = \mathcal{T}^{2\beta} \cup \{\eta\frown\langle\beta_\eta\rangle : \eta \in E\}$. If $\lg(\eta)$ is a limit ordinal we can simply define $p^\alpha_{\eta\frown\langle\beta_\eta\rangle} = x_\beta$, and if $\lg(\eta) = \varepsilon+1$, from maximality there is $q \in A_{p_\eta}$ such that $q \not\perp_o x_\beta$ and therefore (Definition **??**(e) and (i)) there is $p^\alpha_{\eta\frown\langle\beta_\eta\rangle} \in \mathcal{D}_\varepsilon \cap C \cap \mathbb{P}_{\geq q} \cap \mathbb{P}_{\geq x_\beta}$ such that $p^\alpha_{\eta\frown\langle\beta_\eta\rangle} \not\perp_0 x_\beta$. Using smoothness $p^\alpha_{\eta\frown\langle\beta_\eta\rangle} \perp_0 p^{2\beta+1}_{\eta\frown\langle\gamma\rangle}$ for every $\gamma < \beta_\eta$.

We claim that the limit $\bar{p}$ of $\langle \bar{p}^\alpha : \alpha < 2^\kappa\rangle$ is as required.

Clause $(b)$ of 2.5 holds by definition.

For every $q \in \mathbb{P}$ let $\Gamma_q = \{\beta : x_\beta = q,\ \textit{and}\ \lg(\rho_\beta) = \kappa\}$, since $\Gamma_q$ isn't empty $\gamma = \gamma_q = \min(\Gamma_q)$ is defined. For every $\beta \in \Gamma_q$, $\bar{\eta}_{\bar{p}^{2\beta},x_\beta} = \bar{\eta}_{\bar{p}^{2\gamma},x_\gamma}$ satisfies $(\alpha)$-$(\varepsilon)$ in demand $(c)$ ($(\gamma)$ is satisfied by remark **??**), and we took care of clause $\zeta$ in stage $\alpha = 2\beta+1$. As $|\Gamma_q| = 2^\kappa$ clause $(c)$ of 2.5 is satisfied. All that is left is to show that $\bar{p}$ is an $\mathfrak{h}(\mathbb{P})$-solution.

Clauses (a)-(d) of 2.4 hold by definition, we need to show clauses $(b^+)$, $(d^+)$, and $(e)$ there.

Notation: recall that for $\eta \in \mathcal{T}$, $C_\eta = \{p_{\eta\frown\langle\gamma\rangle} : \eta\frown\langle\gamma\rangle \in \mathcal{T}[\bar{p}]\}$, and let $C_{\eta,\beta} = \{p_{\eta\frown\langle\gamma\rangle} : \eta\frown\langle\gamma\rangle \in \mathcal{T}[\bar{p}^\beta]\} = C_\eta \cap \mathcal{T}^\beta$.

<u>Clause $(d^+)$:</u>

Let us assume towards contradiction that there are $\eta \in \mathcal{T}$ and $r \in \mathbb{P}$ such that

(1) $r \notin C_\eta$

(2) $r \perp_0 p$ for all $p \in C_\eta$

(3) $r \geq_{\mathbb{P}} p_{\eta\restriction(\varepsilon+1)}$ for $\varepsilon < \lg(\eta)$

without loss of generality $r \in C$, then for the $\beta$ such that $x_\beta = r$ and at stage $\alpha = 2\beta+2$ of the induction we would have added $p$ such that $x_\beta \leq_{\mathbb{P}} p$ and $p \in C_\eta \cap \mathcal{T}^\alpha$ in contradiction.

<u>Clause $(b^+)$:</u>

Let $\eta \in \mathcal{T}_{\varepsilon+1}$. First we show that $|C_\eta| = 2^\kappa$.
For every $p \in \mathcal{A}(p_\eta)$ there is $\beta < 2^\kappa$ such that $x_\beta = p$. and $\eta \in \mathcal{T}^{2\beta+1}$, therefore after stage $\alpha = 2\beta + 2$ of the construction there is $r_p$ such that $r_p \not\leq_0 p$ and $r_p \in \{p^\alpha_{\eta^\frown\langle\gamma\rangle}\}$, but this means that there is $p' \in A_{p_\eta}$ such that $r_p \in D_\varepsilon \cap \mathbb{P}_{\geq p'}$ but in that case, necessarily $p = p'$. We have shown that above every element of $A_{p_\eta}$ there is an element of $C_\eta$ and therefore $|C_\eta| = 2^\kappa$.
<u>Clause $(e)$</u>:
$\{p_\eta < \eta \in \mathrm{suc}(\mathcal{T})\}$ is an $\mathbf{x}$-dense subset of $\mathbb{P}$:
For $q \in \mathbb{P}$, let $\beta = \min\{\beta : x_\beta = q \text{ and } \lg(\rho_\beta) = \kappa\}$. For this $\beta$ there are by the induction construction $\rho \in {}^\kappa 2$ and $\alpha$ (in fact there are $|2\beta+1|$ such pairs) such that $\eta_\rho = \eta_{\bar{p}^{2\beta}, x_\beta, \rho} \in \mathcal{T}^{2\beta+1} \backslash \mathcal{T}^{2\beta}$ and $p^{2\beta+1}_{\eta_\rho{}^\frown\langle 0\rangle} = F_\alpha(q, \langle p^{2\beta}_{\eta_{\rho\restriction(\varepsilon+1)}} : \varepsilon < \kappa\rangle)$, and so in particular $p_{\eta_\rho{}^\frown\langle 0\rangle} R q$.

$\square$

## 3. Comparison To The Base Tree Theorem

For the readers convenience, let us recall the three parameter distributivity and other notions used in Balcar and Simon's Base Tree Theorem. In The following definition $\mathbf{B}$ is a Boolean algebra, $\tau, \kappa, \lambda$ are cardinals.

**Definition 3.1** (Boolean matrix)**.** A Boolean matrix $\mathscr{P} \subseteq P(\mathbf{B})$ is a collection of maximal disjoint subsets of $\mathbf{B}^+$.

**Definition 3.2** (distributivity and nowhere distributivity)**.** The Boolean algebra $\mathbf{B}$ is $(\tau, \kappa, \lambda)$- distributive if for every matrix $\mathscr{P} = \langle P_\alpha : \alpha < \tau\rangle$ with each $|P_\alpha| \leq \kappa$ there is some maximal disjoint system $Q \subseteq \mathbf{B}^+$such that for each $q \in Q$ and $\alpha < \tau$ we have $|\{p \in P_\alpha : p \cdot q \neq 0\}| < \lambda$.
$\mathbf{B}$ is nowhere distributive if called $(\tau, \kappa, \lambda)$ - nowhere distributive if for every $x \in \mathbf{B}^+$ the algebra $\mathbf{B}\restriction x$ is not $(\tau, \kappa, \lambda)$- distributive.

If we omit the condition $|P_\alpha|\kappa$ in the above definitions we denote this by $(\tau, \cdot, \lambda)$ distributivity or nowhere distributivity (for more on this see [Kop89] chapter 5, section 14).

**Theorem 3.3** (Balcar & Simon)**.** *Let $\tau, \kappa, \lambda$ be cardinals, $\tau, \kappa \geq \aleph_0$ $\lambda \geq 2$, $\mathbf{B}$ a $(\tau, \cdot, \lambda)$-nowhere distributive Boolean algebra having a $\kappa$-closed dense subset $C$. Let $\mathbf{B}$ be $(\rho, \cdot, 2)$-distributive for every $\rho < \tau$. If $\pi(\mathbf{B}) = \lambda^{<\kappa}$, then there is a dense subset $D \subseteq C$ of $\mathbf{B}$ such that $(D, \geq)$ is a tree of height $\tau$ and each $d \in D$ has $\lambda^{<\kappa}$ immediate successors.*

We would like to apply this theorem with the algebra $P(\omega)/\text{fin}$. Recall that the algebra $P(\omega)$ has very natural and simple properties, it is complete, atomic (all singletons are atoms) and the cardinality of a maximal disjoint set is at most $\aleph_0$. None of these properties are inherited by $\mathbf{B} = P(\omega)/\text{fin}$. $\mathbf{B}$ is atomless and therefore not $(2^{\aleph_0}, \cdot, 2)$ distributive, on the other hand it is $\aleph_1$ closed and therefore $(\aleph_0, \cdot, 2)$ distributive, so there is a minimal cardinal $\aleph_0 < \tau \leq 2^{\aleph_0}$ for which it isn't $(\tau, \cdot, 2)$ distributive. this leads to the following important cardinal invariant:

**Definition 3.4** (The cardinal invariant $\mathfrak{h}$)**.** The height of the algebra $\mathbf{B} = P(\omega)/\text{fin}$ is defined by $\mathfrak{h} = \min\{\tau : \mathbf{B} \text{ isn't } (\tau, \cdot, 2) \text{ distributive}\}$

*Remark* 3.5. $\mathbf{B} = P(\omega)/\text{fin}$ is a homogeneous algebra (i.e $\mathbf{B}\restriction b$ is isomorphism to $\mathbf{B}$ for every $b \in \mathbf{B}^+$) therefore we could have defined $\mathfrak{h}$ in terms of nowhere distributivity,

**Theorem 3.6** (Base Tree)**.** *There exists a tree $T \subseteq P(\omega)/\text{fin}$ such that:*

(1) *$T$ is dense in $P(\omega)/\text{fin}$.*
(2) *$(T, \geq)$ is a tree of height $\mathfrak{h}$.*
(3) *every level of the tree is a maximal disjoint set.*
(4) *every $x \in T$ has $2^{\aleph_0}$ immediate successors.*

*Proof.* choose $\mathbf{B} = P(\omega)/\text{fin}$, $\kappa = \aleph_1$, $\lambda = 2$, $\tau = \mathfrak{h}$ and apply Theorem 3.3. $\square$

We now compare this with out theorem, Theorem 2.5.
Given a preorder $\mathbb{P}$ and a cardinal $\kappa$ such that $\mathbb{P}$ contains a dense subset $C$ of cardinality $2^\kappa$ where $\kappa = \min\{\theta : 2^\theta = 2^\kappa\}$ we examine the simple case where $\perp_0, \perp_1$ are interpreted as $\perp$ (the usual incompatibility), $R$ is interpreted as $\leq_{\mathbb{P}}$ and $\mathbb{Q} = \mathbb{P}$. If $\mathbb{P}$ is $\kappa^+$ complete, and above every $p \in \mathbb{P}$ there are two elements which aren't compatible then we can conclude that there are $2^\kappa$ such elements which are incompatible in pairs. Let $\mathscr{A} = \langle A_p : p \in \mathbb{P}\rangle$ such that for every $p \in \mathbb{P}$ the set $A_p$ is of cardinality $2^\kappa$ and is a maximal disjoint set in $\mathbb{P}_{\geq p}$. For $\mathbf{x} = \langle \mathbb{P}, \mathbf{C}, \kappa, \mathbb{P}, \perp\perp \mathscr{A}, \leq_{\mathbb{P}}\rangle$, which in this case we will simply denote by $\mathbf{x} = \langle \mathbb{P}, \mathbf{C}, \kappa\rangle$, all most all the properties of definition 2.3 hold, with the exception perhaps of $(d)$ and $(n)$. By Theorem 2.5 we can conclude that if $\mathbf{x}$ is indeed a good candidate (i.e the mentioned properties also hold) then there is a tree $\mathcal{T}$ and $\bar{p} = \bar{p}[\mathcal{T}]$ such that $\bar{p}$ is and $\mathfrak{h}(\mathbb{P})$-solution.

Lets check the above conclusion for the order $\mathbb{P} = \langle \mathbf{B}, \geq_{\mathbf{B}} \rangle$ where $\mathbf{B}$ is the Boolean algebra $P(\omega)/\mathrm{fin}$. As $\mathbf{B}$ is a Boolean algebra requirement $(n)$ in the definition of a candidate is satisfied, also, as $\mathbf{B}$ is homogeneous, in particular $\mathfrak{h}(\mathbb{P}) = \mathfrak{h}(\mathbb{P}_{\geq p})$ for every $p \in \mathbb{P}$. If we define $F$ to be a function which for increasing sequences of length $\aleph_0$ it returns an upper bound we can deduce from Theorem 2.5 the existence of a tree $\mathcal{T}$ such that

(1) $\mathcal{T}$ is dense in $\mathbf{B}$.
(2) $(\mathcal{T}, \geq_{\mathbf{B}})$ is a tree of height $\mathfrak{h}$.
(3) Every successor level is a maximal disjoint set.
(4) Every $x \in \mathcal{T}$ has $2^{\aleph_0}$ immediate successors.

If we compare this with the conclusion of the base tree theorem of Balcar and Simon in [BS89] (and cited above in 3.3) it is basically the same conclusion (the difference in (3) above can easily be removed by looking at a tree $\mathcal{T}'$ where $\mathcal{T}'_\alpha = \mathcal{T}_{\alpha+1}$ for $\alpha \geq \omega$). The above conclusion isn't a coincidence. Given a good candidate $\mathbf{x} = \langle \mathbb{P}, \mathbf{C}, \kappa \rangle$, there is a Boolean algebra $\mathbf{B}$ such that $\mathbb{P}$ can be densely embedded in it so that the embedding preserves compatibility, denote $C$'s image under this embedding by $D$, so $D$ is dense in $\mathbf{B}$ and meets the requirements of the Balcar and Simon Theorem, so our theorem can be deduced from if. In certain cases also the opposite is correct. First we note that given a Boolean algebra $\mathbf{B}$ which realizes the requirements of Theorem 3.3 with $\lambda = 2$, then $\mathfrak{h}(\mathbf{B}) = \tau$ and also for every $b \in \mathbf{B}^+$ $\mathfrak{h}(\mathbf{B}{\restriction}b) = \tau$. $\mathbf{B}$ is a boolean algebra and therefore separative and so meets requirement (n) of being a good candidate, if $\kappa$ is a successor cardinal we can apply 2.5.

## References

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