RANKS FOR STRONGLY DEPENDENT THEORIES [COSH:E65]

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Abstract. There is much more known about the family of superstable theories when compared to stable theories. This calls for a search of an analogous "super-dependent" characterization in the context of dependent theories. This problem has been treated in [Shea, Sheb], where the candidates "Strongly dependent", "Strongly dependent ^{2}" and others were considered. These families generated new families when we considering intersections with the stable family. Here, continuing [Sheb, §2, §5E,F,G], we deal with several candidates, defined using dividing properties and related ranks of types. Those candidates are subfamilies of "Strongly dependent". fulfilling some promises from [Sheb] in particular [Sheb, 1.4(4)], we try to make this self contained within reason by repeating some things from there. More specifically we fulfil some promises from [Sheb] to to give more details, in particular: in §4 for [Sheb, 1.4(4)], in §2 for [Sheb, 5.47(2)=Ldw5.35(2)] and in §1 for [Sheb, 5.49(2)]

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1. Strongly dependent theories

Discussion 1.1. *The basic property from which this work is derived is strongly dependent*[1]*, it has been studied extensively in* [Sheb]*. For proofs and more we refer to that article. We quote the necessary minimum in order to build on that.*

Definition 1. We say that $\kappa^{\text{ict},1}(T) := \kappa^{\text{ict}}(T) > \kappa$ if the set

$$\Gamma_{\overline{\varphi}} := \left\{ \varphi_i(\overline{x}_\eta, \overline{y}_i^j)^{\mathbf{if}(\eta(i)=j)} : i < \kappa,\ j < \omega, \eta \in {}^{\kappa}\omega \right\}$$

is consistent with T, for some sequence of formulas $\overline{\varphi} = \langle \varphi_i(\overline{x}, \overline{y}_i) : i < \kappa \rangle$. We will say that $\kappa^{\text{ict}}(T) = \kappa$ iff $\kappa^{\text{ict}}(T) > \lambda$ holds for all $\lambda < \kappa$ but $\kappa^{\text{ict}}(T) > \kappa$ does not.

T is called strongly dependent [1] if $\kappa^{\text{ict}}(T) = \aleph_0$.

Discussion 1.2. *The following properties are used in connecting the new properties with the original.*

Claim 2. T is not strongly dependent [1] iff there exist sequences $\overline{\varphi} = \langle \varphi_i(\overline{x}, \overline{y}_i) : i < \kappa \rangle$ and $\langle \overline{a}^i_k : i < \kappa, k < \omega \rangle$ such that $\lg \overline{y}_i = \lg \overline{a}^i_k$, $\langle \overline{a}^i_k : k < \omega \rangle$ an indiscernible sequence over $\cup \left\{ \overline{a}^j_k : j \neq i,\ j < \kappa,\ k < \omega \right\}$ for all $i < \kappa$ it holds that $\left\{ \varphi_i(\overline{x}, \overline{a}^i_0) \wedge \neg\varphi_i(\overline{x}, \overline{a}^i_1) : i < \kappa \right\}$ is a type in $\mathfrak{C}$.

Theorem 3. *For a given (or any) $\alpha \geq \omega$ the following are equivalent*

(1) *T is strongly dependent* [1]

${}_\alpha(2)$ *For every $\overline{c} \subseteq \mathfrak{C}$ and indiscernible sequence $\langle \overline{a}_t : t \in I \rangle$ where $\lg(\overline{a}_t) = \alpha$ the function $t \mapsto \operatorname{tp}(\overline{a}_t, \overline{c})$ divides I to finitely many convex components.*

${}_\alpha(2)'$ *Same as ${}_\alpha(2)$ but with $\lg(\overline{c}) = 1$*

${}_\alpha(2)''$ *Same as ${}_\alpha(2)'$ but with $I = \omega$.*

Discussion 1.3. *Now we turn to discuss the new properties: strongly dependent$_\ell$ and strongly dependent$_{\mathcal{A}}$.*

1.1. **The dividing properties.**

Order-based indiscernible structures, forms and dividing.

Convention 1. *We fix a set $\mathcal{A} \subseteq \mathcal{P}(\mathcal{M}_{\mu_1,\mu_2}(\mu_3))$, such that all $\mathtt{A} \in \mathcal{A}$ contains at least one n-ary term, for $n > 0$.*

Definition 4. We call $\mathtt{A} \in \mathcal{A}$ a form, and we define

$$\mathtt{A}(I) := \left\{ \tau(\overline{t}) :\ \overline{t} = \langle t_i : i < \mu \rangle \in \operatorname{incr}(I, \mu),\ \tau(\mu) \in \mathtt{A},\ \mu < \mu_3 \right\}$$

for a linear order I.

Definition 5. We call $\overline{s}_0, \overline{s}_1$ equivalent in $\mathtt{A}(I)$ iff there exist a term $\overline{\tau} \subseteq A$ and increasing sequences $\overline{t}_0, \overline{t}_1$ such that $\overline{s}_i = \overline{\tau}(\overline{t}_i),\ (i = 0, 1)$.

Let E a convex equivalence relation on I we say that $\overline{s}_0, \overline{s}_1$ are equivalent in $\mathtt{A}(I, E)$ iff $\overline{s}_0, \overline{s}_1$ are equivalent in $\mathtt{A}(I)$ and also $\overline{t}_0, \overline{t}_1$ are equivalent relative to E.

Convention 2. *We will limit the discussion to the case $\mathcal{A} \subseteq \mathcal{P}(\mathcal{M}_{\omega\omega}(\omega))$.*

Remark 1. *Note that a form restricts both the terms which can be used as well as the assignable tuples to those which preserve the same order structure.*

Discussion 1.4. *We now turn to define the structure classes.*

Definition 6. $\mathfrak{k}^{\mathrm{or}}$ Denotes the class of linear orders with the dictionary $(I, <)$.

$\mathfrak{k}^{\mathrm{or}+\mathrm{or}(<n)}$ Denotes the class of structures $\mathtt{M}(I)$ whose universe is the disjoint union of a linear order $|I|$ with the set of increasing sequences of length $< n$ in I , and the dictionary is

$$(I \cup \mathrm{incr}(I, < n), <, S_0 \dots S_{n-1}, R_0 \dots R_{n-1})$$

where $<$ is binary, S_i is unary, and R_i binary such that $(I, <)$ is a linear order. $S_i = \left\{\overline{t} \in \mathrm{incr}(I, < n) : \lg(\overline{t}) = i\right\}$ for all $i < n$, $S_i(\overline{t})$ holds iff $\lg(\overline{t}) = i$. Also $R_i(\overline{t}, t_i)$ for all $i < \lg\left(\overline{t}\right)$ $(t_i \in I,\ \overline{t} \in \mathrm{incr}(I, < n))$.

Convention 3. *In the above notation, $< n$ can be replaced with $\leq n$ to mean $< n + 1$.*

Discussion 1.5. *We now turn to define the main properties with which we deal*

Definition 7. We say that the type $p(\overline{x})$ does $\mathrm{ict}^{\ell} - (\Delta, n)$-divide over A if

- **For $\ell = 1$::** *There exist an indiscernible sequence $\langle \overline{a}_t : t \in I \rangle = \overline{\mathbf{a}} \in \mathrm{Ind}_{\Delta}(\mathfrak{k}^{\mathrm{or}}, A)$ and $s_0 <_I t_0 \leq_I s_1 <_I t_1 <_I \dots s_{n-1} <_I t_{n-1}$ such that for any $\overline{c}$ which realizes p, $\mathrm{tp}_{\Delta}(\overline{c}^\frown \overline{a}_{s_i}, A) \neq \mathrm{tp}_{\Delta}(\overline{c}^\frown \overline{a}_{t_i}, A)$ holds for all $i < n$*
- **For $\ell = 2$::** *There exist an indiscernible sequence $\langle \overline{a}_t : t \in I \rangle = \overline{\mathbf{a}} \in \mathrm{Ind}_{\Delta}(\mathfrak{k}^{\mathrm{or}}, A)$ and $s_0 <_I t_0 \leq_I s_1 <_I t_1 <_I \dots s_{n-1} <_I t_{n-1}$ such that for any $\overline{c}$ which realizes p,*

  $$\mathrm{tp}_{\Delta}(\overline{c}^\frown \overline{a}_{s_\ell}, A \cup \left\{\overline{a}_{s_j} : j < \ell\right\}) \neq \mathrm{tp}_{\Delta}(\overline{c}^\frown \overline{a}_{t_\ell}, A \cup \left\{\overline{a}_{s_j} : j < \ell\right\})$$

  holds for all $\ell < n$
- **For $\ell = 3$::** *There exist an indiscernible structure $\langle \overline{a}_t : t \in I \cup \mathrm{incr}(< n, I) \rangle = \overline{\mathbf{a}} \in \mathrm{Ind}_{\Delta}(\mathfrak{k}^{\mathrm{or}+\mathrm{or}(<\mathrm{n})}, A)$ and $s_0 <_I t_0 \leq_I s_1 <_I t_1 <_I \dots s_{n-1} <_I t_{n-1}$ such that for any $\overline{c}$ realizing p and $\ell < n$:*

  $$\mathrm{tp}_{\Delta}(\overline{c}^\frown \overline{a}_{s_\ell}, A \cup \overline{a}_{\langle s_0 \dots s_{\ell-1} \rangle}) \neq \mathrm{tp}_{\Delta}(\overline{c}^\frown \overline{a}_{t_\ell}, A \cup \overline{a}_{\langle s_0 \dots s_{\ell-1} \rangle})$$

  holds.

For $\ell = \mathcal{A}$:: *For some form $\mathtt{A} \in \mathcal{A}$ and indiscernible structure $\bar{\mathbf{a}} = \langle \bar{a}_t : t \in \mathtt{A}(I) \rangle$ over A, $\langle \bar{a}_t : t \in \mathtt{A}(I, E) \rangle$ is not indiscernible over $A \cup \bar{c}$, for any $\bar{c}$ realizing p and convex equivalence relation E on I with $\leq n$ equivalence classes.*

Observation 1.6. *$p(\bar{x})$ does $\mathrm{ict}^4 - (\Delta, n)$-divide over A iff $p(\bar{x})$ does $\mathcal{A} - (\Delta, n)$-divide over A for $\mathcal{A} = \{\mathtt{A}_n = \{f_i(0, \dots, i-1) : 1 < i < n\} : n < \omega\}$.*

Observation 1.7. *If $\mathcal{A} \subseteq \mathcal{A}'$ and $p(\bar{x})$ does $\mathrm{ict}^{\mathcal{A}} - (\Delta, n)$-divide over A, then $p(\bar{x})$ does $\mathrm{ict}^{\mathcal{A}'} - (\Delta, n)$-divide over A.*

Observation 1.8. *If a type p does $\mathrm{ict}^1 - n(*)$-divide over A then p does $\mathrm{ict}^{\mathcal{A}} - n(*)$-divide over A.*

Observation 1.9. *If the type p does $\mathrm{ict}^{\ell} - n(*)$-divide over A then p does $\mathrm{ict}^{\ell+1} - n(*)$-divide over A ($1 \leq \ell \leq 3$).*

Definition 8. We say that the type $p(\bar{x})$ does $\mathrm{ict}^{\ell} - (\Delta, n)$-fork over A if there exist formulas $\varphi_i(\bar{x}, \bar{c}_i)$, $(i < m)$ such that $p(\bar{x}) \vdash \bigvee_{i<m} \varphi_i(\bar{x}, \bar{c}_i)$ and each φ_i does $\mathrm{ict}^{\ell} - (\Delta, n)$ divide over A.

Definition 9. We say that $\kappa_{\mathrm{ict},\ell}(T) > \kappa$ if some type p of T does $\mathrm{ict}^{\ell} - n$-fork over A, for all $n < \omega$ and $A \subseteq \mathrm{Dom}(p)$ of power $< \kappa$.

Definition 10. We call T strongly dependent$_{\ell}$ $(_{\mathcal{A}})$ iff $\kappa_{\mathrm{ict},\ell}(T) = \aleph_0$ $(\kappa_{\mathrm{ict},\mathcal{A}}(T) = \aleph_0)$

Observation 1.10. *If $p(\bar{x})$ does $\mathrm{ict}^{\ell} - (\Delta, n)$-fork over A then $p(\bar{x})$ does $\mathrm{ict}^{\ell} - (\Delta, k)$ fork over A for all $k < n$.*

Observation 1.11. *(finite character) if the type $p(\bar{x})$ does $\mathrm{ict}^{\ell} - (\Delta, n(*))$-divide over A then q does $\mathrm{ict}^{\ell} - (\Delta, n)$-divide over A for some finite $q \subseteq p$.*

Claim 11. If $p(\bar{x})$ does $\mathrm{ict}^{\ell} - (\Delta, n(*))$-divide over A, it is possible to find witnesses as follows:

Case $\ell = 1$:: *There exist $\bar{\mathbf{a}} = \langle \bar{a}_n : n < \omega \rangle \in \mathrm{Ind}(\mathfrak{k}^{\mathrm{or}}, A)$, $\bar{s}$ a sequence of length $n(*)$ from ω such that $s_0 = 0$, $1 \leq s_{n+1} - s_n \leq 2$ and formulas $\langle \varphi_i(\bar{y}, \bar{x}, \bar{c}) : i < i(*) \rangle$ $\bar{c} \in A$ such that*

$$p(\bar{x}) \vdash \bigvee_{i<i(*)} (\varphi_i(\bar{a}_{s_n}, \bar{x}, \bar{c}) \wedge \neg\varphi_i(\bar{a}_{s_n+1}, \bar{x}, \bar{c}))$$

for all $n < n(*)$.

Case $\ell = 2$:: *There exist* $\mathbf{\bar{a}} = \langle \bar{a}_n : n < \omega \rangle \in \mathrm{Ind}(\mathfrak{k}^{\mathrm{or}}, A)$, $\bar{s}$ *as in* $\ell = 1$ *and formulas* $\langle \varphi^n_i(\bar{y}_0 \ldots \bar{y}_{n-1}, \bar{x}, \bar{c}) : i < i(*), n < n(*) \rangle$ $\bar{c} \in A$ *such that*

$$p(\bar{x}) \vdash \bigvee_{i<i(*)} \left(\varphi^n_i(\bar{a}_{s_0} \ldots \bar{a}_{s_{n-1}} \bar{a}_{s_n}, \bar{x}, \bar{c}) \wedge \neg\varphi_i(\bar{a}_{s_0} \ldots \bar{a}_{s_{n-1}} \bar{a}_{s_n+1}, \bar{x}, \bar{c}) \right)$$

for all $n < n(*)$.

Case $\ell = 3$:: *There exist* $\mathbf{\bar{a}} = \langle \bar{a}_t : t \in \omega \cup \mathrm{incr}(< n(*), \omega) \rangle \in \mathrm{Ind}(\mathfrak{k}^{\mathrm{or}+\mathrm{or}(<\mathrm{n}(*))}, A)$, $\bar{s}$ *as in* $\ell = 1$ *and formulas* $\langle \psi^n_i(\bar{y}, \bar{z}, \bar{x}, \bar{c}) : i < i(*), n < n(*) \rangle$ *such that*

$$p(\bar{x}) \vdash \bigvee_{i<i(*)} \left(\psi^n_i(\bar{a}_{\langle s_0 \ldots s_{n-1} \rangle}, \bar{a}_{s_n}, \bar{x}, \bar{c}) \wedge \neg\psi^n_i(\bar{a}_{\langle s_0 \ldots s_{n-1} \rangle}, \bar{a}_{s_n+1}, \bar{x}, \bar{c}) \right)$$

for all $n < n(*)$.

Case $\mathcal{A}$:: *There exist* $\mathtt{A} \in \mathcal{A}$, $m_* < \omega$, $\mathbf{\bar{a}} = \langle \bar{a}_t : t \in \mathtt{A}(\omega) \rangle$ *indiscernible over* A, *sequences* $\langle \bar{s}_{0,E}, \bar{s}_{1,E} \in \mathtt{A}(m_*, E) : E \in \mathrm{ConvEquiv}(m_*, n(*)) \rangle$, $\bar{b} \in A$ *and formulas* $\langle \psi_{E,i}(\bar{x}, \bar{y}_{E,i}, \bar{b}) : E \in \mathrm{ConvEquiv}(m_*, n(*)),\ i < i_E \rangle$ *such that*

$$\bar{p}(\bar{x}) \vdash \bigvee_{i<i_E} \psi_{E,i}(\bar{x}, \bar{a}_{\bar{s}_{0,E}}, \bar{b}) \equiv \neg\psi_{E,i}(\bar{x}, \bar{a}_{\bar{s}_{1,E}}, \bar{b})$$

holds for all $E \in \mathrm{ConvEquiv}(m_*, n(*))$.

Proof.

For $\ell = 1, 2, 3$:: Easy, so we only give a summary. By 29 it follows that there exists a dense extension I' of I without endpoints such that $\langle \bar{a}_t : t \in I' \rangle$ is an indiscernible structure (for the corresponding ℓ) over A. Let $s_0 < t_0 \leq \ldots \leq s_{n-1} < t_{n-1}$ from I witness the dividing as in the definition. These indices can also be used to show that I' is a witness of dividing. Similarly we can choose an increasing $\langle r_n : n < \omega \rangle$ from I' such that $\{s_i, t_i : i < n-1\} \triangleleft \langle r_n : n < \omega \rangle \subseteq I$, to get a witness based on ω.

For $\mathcal{A}$:: Assume towards contradiction that the claim does not hold. So we can choose

(1) A type p which does $(\Delta, n(*))$-fork over A
(2) A linear order I.
(3) An indiscernible structure $\langle \bar{a}_t : t \in \mathtt{A}(I) \rangle$ over A witnessing 1.
(4) $\bar{c}$ realizing p

Such that for every finite $S \subseteq I$ there exists a convex equivalence relation E_S on I with $\leq n(*)$ equivalence classes such that $\mathrm{tp}_\Delta(\bar{a}_{\bar{s}_0}, A \cup \bar{c}) = \mathrm{tp}_\Delta(\bar{a}_{\bar{s}_1}, A \cup \bar{c})$ holds for any equivalent $\bar{s}_0, \bar{s}_1 \in \mathtt{A}(S, E_S)$.

Now let $\mathcal{D}$ an ultrafilter on $[I]^{<\omega}$ extending $\left\{G_S : S \in [I]^{<\omega}\right\}$, where $G_S := \left\{T \in [I]^{<\omega} : S \subseteq T\right\} \in \mathcal{D}$. For all S define the 2-sort model (with the sorts M, I) $M_S := \langle M, I, E, \langle f_{\tau,i} : \tau(\overline{x}_\tau) \in \mathtt{A},\ i < n_\tau \rangle, \overline{c}\rangle$ where

(1) M, I as defined
(2) $E^{M_S} = E_S$ an equivalence relation.
(3) Since $\langle \overline{a}_t : t \in \mathtt{A}(I) \rangle$ is indiscernible, for every term $\tau(\overline{x}_\tau) \in \mathtt{A}(\overline{x})$ we can define $n_\tau < \omega$ such that $n_\tau = \lg\left(\overline{a}_{\tau(\overline{u})}\right)$ for all $\overline{u} \in^{\lg(\overline{x}_\tau)} [I]$. We define for each term $\tau(\overline{x}_\tau) \in \mathtt{A}(\overline{x})$ and $i < n_\tau$:

$$\begin{aligned} f_{\tau(\overline{x}_\tau),i} : {}^{\lg(\overline{x}_\tau)}[I] &\to M \\ \tau(\overline{u}) &\mapsto \left(a_{\tau(\overline{u})}\right)_i \end{aligned}$$

Now, consider $N = \left(\prod_{S \in [I]^{<\omega}} M_S\right) / \mathcal{D}$. From the properties of ultraproducts it is easy to show that the functions

$$\begin{aligned} h : M \oplus I &\to N \\ a &\mapsto \langle a \rangle_{S \in [I]^{<\omega}} / \mathcal{D} \end{aligned}$$

fulfill

(1) $h \restriction \langle M, \overline{c} \rangle : \langle M, \overline{c} \rangle \to N \restriction \mathcal{L}_T \cup \{\overline{c}\}$ is elementary.
(2) $h(f^{M_S}_{\tau,i}(\overline{u})) = f^N_\tau(h(\overline{u}))$.
(3) $E^N \circ h$ is a convex equivalence relation on I^N with $\leq n(*)$ classes.
(4) $\operatorname{tp}_\Delta(\overline{a}_{\overline{s}_0}, A \cup \overline{c}, M) = \operatorname{tp}_\Delta(\overline{a}_{\overline{s}_1}, A \cup \overline{c}, M)$ holds for every pair of equivalent $\overline{s}_0, \overline{s}_1 \in \mathtt{A}(I, E^N \circ h)$.

Contradicting that $\langle \overline{a}_t : t \in \mathtt{A}(I) \rangle$ witnesses that p does $(\Delta, n(*))$-divide over A.

Now we show that it is possible to choose $I = \omega$. From 29 there exists an extension J of I without endpoints, such that $\langle \overline{a}_t : t \in \mathtt{A}(J) \rangle$ is indiscernible, extending $\langle \overline{a}_t : t \in \mathtt{A}(I) \rangle$. Let $\langle s_i : i < \omega \rangle$ increasing in J such that $\left\langle s_0 \dots s_{|S|} \right\rangle$ enumerates S above. We define $\overline{b}_{\tau(\overline{u})} = \overline{a}_{\tau(\overline{s}_{\overline{u}})}$ for all $\overline{u}, \tau \in \mathtt{A}$. by the conclusion of the claim it is easy to verify that $\left\langle \overline{b}_t : t \in \mathtt{A}(\omega) \right\rangle$ is a witness as required.

Now, since for any $\overline{s}_0, \overline{s}_1 \in \mathtt{A}(S)$ it holds that $\overline{s}_0, \overline{s}_1$ are equivalent in $\mathtt{A}(I, E)$ iff they are equivalent in $\mathtt{A}(S, E \restriction S)$, so for some $m_* < \omega$ such that $S \subseteq m_*$ we can choose two equivalent (in $\mathtt{A}(\omega, E)$) $\overline{s}_0, \overline{s}_1 \in \mathtt{A}(m_*)$ with $\overline{b}_{\overline{s}_0}$, $\overline{b}_{\overline{s}_1}$ having different types over A based only on $E \restriction m_*$.

$\square$

We use the following freely

Observation 1.12. *If $p(\overline{x})$ does $\mathrm{ict}^{\ell} - n$ divide over A then $p(\overline{x})$ does $\mathrm{ict}^{\ell} - n$-divide over B for every $B \subseteq A$.*

1.2. Strongly dependent $_1$ $\Rightarrow$ Strongly dependent ^{1}.

Discussion 1.13. *Claim 12 is a connection to* [Sheb].

Claim 12. T is strongly dependent$_1$ (Definition 10) $\Rightarrow$ T is strongly dependent ^{1} (Definition 1)

Definition 13. For a set of formulas $\mathcal{Q}$, define the formula

$$\mathrm{Even}\mathcal{Q} := \bigvee \left\{ \bigwedge_{q \in \mathcal{Q}} q^{\mathrm{if}(q \in u)} : u \in [\mathcal{Q}]^r \, , \ 2|r, \ r \leq |\mathcal{Q}| \right\}$$

Remark 2. *Even$\mathcal{Q}$ is true iff the number of true sentences in $\mathcal{Q}$ is even.*

Proof. Assume that T is not strongly dependent ^{1}: by $_{\alpha}(2)''$ of theorem 3 there exist an indiscernible sequence $\langle \overline{a}_n : n < \omega \rangle$ ($\lg \overline{a}_n = \omega$) and an element c such that $\mathrm{tp}(\overline{a}_n, c) \neq \mathrm{tp}(\overline{a}_{n+1}, c)$ for all $n < \omega$. consider $p(x) := \mathrm{tp}(c, \cup \{\overline{a}_n : n < \omega\})$. Fix a finite $A \subseteq \mathrm{Dom}(p)$. We need to show that p does $\mathrm{ict}^1 - n(*)$-fork over A for some $n(*)$, however we can prove this for any $1 < n(*) < \omega$. Fix $n(*)$ and let $\overline{u} \subseteq I$ increasing and finite such that $A \subseteq \cup \{\overline{a}_{u_i} : i < \lg \overline{u}\}$. Let $m = \max \overline{u} + 1$. So $\langle \overline{a}_n : m \leq n < \omega \rangle$ is indiscernible over A. since for all $n \geq m$ there exists $\varphi_n(\overline{x}, y)$ such that $\models \varphi_n(\overline{a}_n, c) \wedge \neg\varphi_n(\overline{a}_{n+1}, c)$, we get that $\varphi_n(\overline{a}_n, x) \wedge \neg\varphi_n(\overline{a}_{n+1}, x) \in p(x)$. Define a map $f : [\omega]^2 \to \{\mathbf{t}, \mathbf{f}\}^4$ as follows $f(\{i, j\}) = (s_0, s_1, s_2, s_3)$ where w.l.o.g $i < j$ and $s_k (k < 4)$ are truth values such that

$$\models \varphi_{m+2i}(\overline{a}_{m+2j})^{s_0} \wedge \varphi_{m+2i}(\overline{a}_{m+2j+1})^{s_1} \wedge \varphi_{m+2j}(\overline{a}_{m+2i})^{s_2} \wedge \varphi_{m+2j}(\overline{a}_{m+2i+1})^{s_3}$$

By Ramsey's theorem, there exists an infinite $S \subseteq \omega$ such that $f \restriction [S]^2$ is constant with value (s_0, s_1, s_2, s_3). Let $\langle i_n : n < n(*) \rangle$ enumerate S in increasing order.

Define $\psi(x, \overline{y})$ as follows:

if $s_0 = s_1 \wedge s_2 = s_3$ let $\psi(x, \overline{y}) := Even \{\varphi_{m+2i_n}(\overline{y}, x) : n < n(*)\}$.
if $s_0 \neq s_1$ let $\psi(x, \overline{y}) := \varphi_m(\overline{y}, x)$.
if $s_0 = s_1 \wedge s_2 \neq s_3$ let $\psi(x, \overline{y}) := \varphi_{m+2i_{n-1}}(\overline{y}, x)$.

Now let $\vartheta(x) := \bigwedge_{n<n(*)} \psi(x, \overline{a}_{m+2i_n}) \Delta \psi(x, \overline{a}_{m+2i_n+1})$. It is easy to verify that $\models \psi(c, \overline{a}_{m+2i_n)}) \equiv \neg\psi(c, \overline{a}_{m+2i_n+1})$ holds for any $n < n(*)$, so

$p \vdash \vartheta$. Now ϑ does $\mathrm{ict}^1 - (\psi, n(*))$-divide over A:

Choose a finite $u \subseteq \lg \overline{a}$ and let $\psi'(x, \overline{y} \restriction u) = \psi(x, \overline{y})$. So $\vartheta(x) \vdash \psi'(x, \overline{a}_{m+2i_n} \restriction u) \equiv \neg\psi'(x, \overline{a}_{m+2i_n+1} \restriction u)$ holds for the indiscernible sequence $\langle \overline{a}_n \restriction u : m \leq n < \omega \rangle$ and elements $s_n = m + 2i_n, t_n = m + 2i_n + 1$. $\square$

2. Ranks

Definition 14. We define the ranks $\mathrm{ict}^{\ell} - \mathrm{rk}_{P}^{m}$ ($P \in \{fork, div\}$) on the class of m-types of T ($m < \omega$) as follows:

- $\mathrm{ict}^{\ell} - \mathrm{rk}_{P}^{m}(p(\overline{x})) \geq 0$ *for all m-types.*
- *For a given ordinal α, $\mathrm{ict}^{\ell} - \mathrm{rk}_{P}^{m}(p(\overline{x})) \geq \alpha$ if for all $q \subseteq p$, $A \subseteq \mathrm{Dom}(p)$ and $n < \omega$ (q, A finite) and $\beta < \alpha$, for some extension $q' \supseteq q$ it holds that $\mathrm{ict}^{\ell} - \mathrm{rk}_{P}^{m}(q') \geq \beta$ and also:*
  For $P = fork$: q' does $\mathrm{ict}^{\ell} - (\mathcal{L}, n)$-fork over A.
  For $P = div$: q' does $\mathrm{ict}^{\ell} - (\mathcal{L}, n)$-divide over A.
- *If $P = fork$ we omit P.*

Observation 2.1. $\mathrm{ict}^{\ell} - \mathrm{rk}^{m}(p) \geq \mathrm{ict}^{\ell} - \mathrm{rk}_{div}^{m}(p)$ *for any m-type p.*

Observation 2.2. *For an m-type p over B such that $\mathrm{ict}^{\ell} - \mathrm{rk}^{m}(p) = \alpha$ there exists an extension $p \subseteq q \in \mathbf{S}^{m}(B)$, a complete type of the same rank .*

Proof. Identical to [She90, Theorem II.1.6, p.24]. □

Convention 4. *We denote for the rest of this section*

$$\begin{aligned} \lambda_{\ell} &= |T| \\ \lambda_{\mathcal{A}} &= |T| + \sum_{\mathtt{A} \in \mathcal{A}} \aleph_0^{|\mathtt{A}|} \end{aligned}$$

Lemma 15. *If $\mathrm{ict}^{\ell} - \mathrm{rk}^{m}(\overline{x} = \overline{x}) \geq \lambda_{\ell}^{+}$ then there exists $p \in \mathbf{S}^{m}(A)$ which does $\mathrm{ict}^{\ell} - n(*)$-divide over B for all $n(*) < \omega$, $B \in [A]^{<\omega}$.*

Proof. We prove for $\ell = 1$ and $\ell = \mathcal{A}$ (the cases $\ell = 2, 3$ are analogous to $\ell = 1$).

We choose, for each $\eta \in \mathrm{ds}(\lambda_{\ell}^{+})$, by induction on $\lg(\eta)$ the following objects:

Case $\ell = 1$::

$$p_{\eta},\ k_{\eta},\ \overline{b}_{\eta},\ \overline{c}_{\eta}$$

$$\left\langle \varphi_{\eta,k}(\overline{x}, \overline{y}_{\eta}), \mathbf{a}_{\eta,k} = \langle \overline{a}_{\eta,k,t} : t \in \omega \rangle,\ \overline{s}_{\eta,k} : k < k_{\eta} \right\rangle$$

$$\left\langle \overline{\psi}_{\eta,k,i}(\overline{z}_{\eta,k,i}, \overline{y}_{\eta}, \overline{x}) : k < k_{\eta}, i < \lg(\overline{s}_{\eta,k}) \right\rangle$$

Case $\ell = \mathcal{A}$::

$$p_\eta,\ k_\eta,\ \bar{b}_\eta,\ \bar{c}_\eta$$

$$\left\langle \varphi_{\eta,k}(\bar{x},\bar{y}_\eta), \bar{\mathbf{a}}_{\eta,k} = \langle \bar{a}_{\eta,k,t} : t \in \mathtt{A}_{\eta,k}(\omega)\rangle\,,\ m_{\eta,k} : k < k_\eta \right\rangle$$

$$\Big\langle \bar{s}^{\eta,k}_{E,0},\ \bar{s}^{\eta,k}_{E,1},\ \bar{\psi}_{\eta,k,E}(\bar{z}_{\eta,k,E},\bar{y}_\eta,\bar{x}) : \qquad k < k_\eta,\ E \in \mathrm{ConvEquiv}(m_{\eta,k},\lg(\eta)) \Big\rangle$$

such that

- $p_{\langle\,\rangle} = \emptyset$, $\bar{b}_{\langle\,\rangle} = \langle\,\rangle$, $k_{\langle\,\rangle} = 0$.
- $\bar{c}_\eta$ realizes p_η.
- p_η is a finite type, $\mathrm{ict}^\ell - \mathrm{rk}^m(p_\eta) \geq \min\left(\mathrm{Rang}(\eta) \cup \left\{\lambda_\ell^+\right\}\right)$ for all $\eta \in \mathrm{ds}(\lambda_\ell^+)$.
- $p_\eta \vdash \bigvee_{k<k_\eta} \varphi_{\eta,k}(\bar{x},\bar{b}_\eta)$.
- For $\eta = \nu^\frown\langle\alpha\rangle$:
  - $p_{\eta^\frown\langle\alpha\rangle} \supseteq p_\eta$
  - $\bar{b}_\nu \prec \bar{b}_\eta$.
  - p_η does $\mathrm{ict}^\ell - \lg(\eta)$-fork over $\bar{b}_\nu$. In particular $\varphi_{\eta,k}(\bar{x},\bar{b}_\eta)$ does $\mathrm{ict}^\ell - \lg(\eta)$-divide over $\bar{b}_\nu$ for $k < k_\eta$. Moreover,
    - Case $\ell = 1$: $\bar{\psi}_{\eta,k,i}$ is a finite sequence of formulas, and

$$\varphi_{\eta,k}(\bar{x},\bar{b}_\eta) \vdash \bigvee_{\psi \in \bar{\psi}_{\eta,k,i}} \left[\psi(\bar{a}_{\eta,k,s_i},\bar{b}_\eta,\bar{x}) \equiv \neg\psi(\bar{a}_{\eta,k,s_i+1},\bar{b}_\eta,\bar{x})\right]$$

holds for $i < \lg(\eta) = \lg(\bar{s}_{\eta,k})$.

- Case $\ell = \mathcal{A}$: $\bar{\psi}_{\eta,k,E}$ is a finite sequence of formulas, and

$$\varphi_{\eta,k}(\bar{x},\bar{b}_\eta) \vdash \bigvee_{\psi \in \bar{\psi}_{\eta,k,E}} \left[\psi(\bar{a}_{\eta,k,\bar{s}^{\eta,k}_{0,E}},\bar{b}_\eta,\bar{x}) \equiv \neg\psi(\bar{a}_{\eta,k,\bar{s}^{\eta,k}_{1,E}},\bar{b}_\eta,\bar{x})\right]$$

holds for every $E \in \mathrm{ConvEquiv}(m_{\eta,k},\lg(\eta)$ for some equivalent sequences $\bar{s}^{\eta,k}_{0,E}, \bar{s}^{\eta,k}_{1,E} \in \mathtt{A}_{\eta,k}(m_{\eta,k},E)$.

Choice of a tree of types with descending ranks. For $\eta = \langle\,\rangle$ - clear. Now let $\eta \in \mathrm{ds}(\lambda_\ell^+)$, $\alpha < \min\left(\mathrm{Rang}(\eta) \cup \left\{\lambda_\ell^+\right\}\right)$, and p_η a finite rank such that $\mathrm{ict}^\ell - \mathrm{rk}^m(p_\eta) \geq \min\left(\mathrm{Rang}(\eta) \cup \left\{\lambda_\ell^+\right\}\right)$. By the definition of rank and since p_η, $\mathrm{Dom}(p_\eta)$ are finite, there exists $q \supseteq p_\eta$ which does $\mathrm{ict}^\ell - (\lg\eta + 1)$-fork over $\mathrm{Dom}(p_\eta)$ with rank $\geq \alpha$. By the finite character of forking, there exists a finite $p_{\eta^\frown\langle\alpha\rangle} \subseteq q$ which does $\mathrm{ict}^\ell - \lg\eta$-fork over $\bar{b}_\eta$, extending p_η. On the other hand,

$$\mathrm{ict}^\ell - rk^m(p_{\eta^\frown\langle\alpha\rangle}) \geq \mathrm{ict}^\ell - rk^m(q) \geq \alpha$$

holds, since $q \supseteq p_{\eta^\frown\langle\alpha\rangle}$. By the definition of forking and 11 we get $\left\langle \varphi_{\eta^\frown\langle\alpha\rangle,k}(\overline{x},\overline{b}_{\eta^\frown\langle\alpha\rangle}) : k < k_{\eta^\frown\langle\alpha\rangle}\right\rangle$ (We choose w.l.o.g $\overline{b}_{\eta^\frown\langle\alpha\rangle} \succ \overline{b}_\eta$) and the witnesses for $\mathrm{ict}^\ell - \lg(\eta)$-dividing of each formula. This completes the iterated choice.

Choosing an infinite sequence. We define for every $\eta \neq \langle\ \rangle$:

Case $\ell = 1$:

$$\varrho_\eta := \left(k_\eta,\ \left\langle \varphi_{\eta,k}(\overline{x},\overline{y}_\eta),\ l_{\eta,k},\ \overline{s}_{\eta,k}, \overline{\psi}_{\eta,k,i}(\overline{z}_{\eta,k,i},\overline{y}_\eta,\overline{x}) : k < k_\eta\right\rangle\right)$$

where $l_{\eta,k} = \lg\left(\overline{a}_{\eta,k,n}\right)$ for all $n \in \omega$.

Case $\ell = \mathcal{A}$:

$$\varrho_\eta := \Bigg(k_\eta,\ \left\langle \varphi_{\eta,k}(\overline{x},\overline{y}_\eta),\ l_{\eta,k} : \mathtt{A}_{\eta,k} \to \omega,\ m_{\eta,k} : k < k_\eta\right\rangle$$
$$\left\langle \overline{s}^{\eta,k}_{0,E}, \overline{s}^{\eta,k}_{1,E}, \overline{\psi}_{\eta,k,E}(\overline{z}_{\eta,k,E},\overline{y}_\eta,\overline{x}) : E \in \mathrm{ConvEquiv}(m_{\eta,k},\lg(\eta))\right\rangle\Bigg)$$

where $l_{\eta,k}$ is a function, mapping to each term $\tau(\overline{v}) \in \mathtt{A}_{\eta,k}$ the length of $\overline{a}_{\eta,k,\tau(\overline{v})}$.

Now, there are at most λ_ℓ possibilities for the choice of ϱ_η since:

Case $\ell = 1$: $k_\eta,\ l_{\eta,k},\ \overline{s}_{\eta,k},\ \lg\left(\overline{y}_\eta\right),\ \lg\left(\overline{z}_{\eta,k,i}\right),\ \lg\left(\overline{\psi}_{\eta,k,i}\right) < \omega$ and so ϱ_η has at most $|T|$ possibilities.

Case $\ell = \mathcal{A}$: $k_\eta,\ m_{\eta,k} < \omega$. $l_{\eta,k}$ has at most $\sum_{\mathtt{A}\in\mathcal{A}} \aleph_0^{|\mathtt{A}|}$ possibilities and $\overline{s}^{\eta,k}_{0,E}, \overline{s}^{\eta,k}_{1,E}$ have at most $\sum_{\mathtt{A}\in\mathcal{A}} |\mathtt{A}|$ possibilities. The formulas contain a finite number of variables, so there are at most $|T|$ possibilities.

So by claim 28 it follows that we can find a sequence $\langle \varrho_j : j < \omega\rangle$ such that for any $j_* < \omega$ there exists $\eta_{j_*} \in \mathrm{ds}(\lambda_\ell^+)$ and $\varrho_{\eta_{j_*}\restriction j} = \varrho_j$ holds for all $j \leq j_*$. We denote the chosen objects as follows:

Case $\ell = 1$:

$$\varrho_j := \left(k_j,\ \left\langle \varphi_{j,k}(\overline{x},\overline{y}_j),\ l_{j,k},\ \overline{s}_{j,k}, \overline{\psi}_{j,k,i}(\overline{z}_{j,k,i},\overline{y}_j,\overline{x}) : k < k_j\right\rangle\right)$$

Case $\ell = \mathcal{A}$:

$$\varrho_j := \Bigg(k_j,\ \left\langle \varphi_{j,k}(\overline{x},\overline{y}_j),\ l_{j,k} : \mathtt{A}_{j,k} \to \omega,\ m_{j,k} : k < k_j\right\rangle$$
$$\left\langle \overline{s}^{j,k}_{0,E},\ \overline{s}^{j,k}_{1,E},\ \overline{\psi}_{j,k,E}(\overline{z}_{j,k,E},\overline{y}_j,\overline{x}) : E \in \mathrm{ConvEquiv}(m_{j,k},j)\right\rangle\Bigg)$$

Using compactness to choose a new object. We define a new dictionary τ_* by adding the constant symbols to τ_M: $\lg \bar{b}^*_j = \lg \bar{b}_j$, $\lg(\bar{c}^*) = \lg(\bar{x})$ and also

Case $\ell = 1$: $\lg(\bar{a}^*_{j,k,t}) = l_{j,k}$

$$\tau_* = \tau_M \cup \left\{\bar{a}^*_{j,k,t} : t \in \omega,\, k < k_j,\, j < \omega\right\} \cup \left\{\bar{b}^*_j : j < \omega\right\} \cup \bar{c}^*$$

Case $\ell = \mathcal{A}$: $\lg(\bar{a}^*_{j,k,\tau(\bar{v})}) = l_{j,k}(\tau(\bar{v}))$

$$\tau_* = \tau_M \cup \left\{\bar{a}^*_{j,k,t} : t \in \mathtt{A}_{j,k}(\omega),\, k < k_j,\, j < \omega\right\} \cup \left\{\bar{b}^*_j : j < \omega\right\} \cup \bar{c}^*$$

We now define families of formulas in $\mathcal{L}(\tau_*)$, for every $1 \le j < \omega$:

$$\Delta^{\text{type}}_j = \left\{\bigvee_{k<k_j} \varphi_{j,k}(\bar{c}^*, \bar{b}^*_j)\right\}$$

Case $\ell = 1$:

$$\Delta^{\text{div}}_j := \cup\left\{\operatorname{Ind}(\bar{\mathbf{a}}^*_{j,k}, \bar{b}^*_{j-1}) : k < k_j\right\} \cup \Bigg\{(\forall \bar{x})\, \varphi_{j,k}(\bar{x}, \bar{b}^*_j) \to \bigvee_{i<\lg(\bar{\psi}_{j,k,E})} \left(\psi_{j,k,i}(\bar{a}^*_{j,k,s_{j,k,i}}, \bar{b}^*_{j-1}, \bar{x}) \equiv \neg\psi_{j,k,i}(\bar{a}^*_{j,k,s_{j,k,i}+1}, \bar{b}^*_{j-1}, \bar{x})\right) : E \in \operatorname{ConvEquiv}(m_{j,k}, j),\ k < k_j\Bigg\}$$

Case $\ell = \mathcal{A}$:

$$\Delta^{\text{div}}_j := \cup\left\{\operatorname{Ind}(\bar{\mathbf{a}}^*_{j,k}, \bar{b}^*_{j-1}) : k < k_j\right\} \cup \Bigg\{(\forall \bar{x})\, \varphi_{j,k}(\bar{x}, \bar{b}^*_j) \to \bigvee_{i<\lg(\bar{\psi}_{j,k,E})} \left(\psi_{j,k,E,i}(\bar{a}^*_{j,k,\bar{s}^{j,k}_{0,E}}, \bar{b}^*_{j-1}, \bar{x}) \equiv \neg\psi_{j,k,E,i}(\bar{a}^*_{j,k,\bar{s}^{j,k}_{1,E}}, \bar{b}^*_{j-1}, \bar{x})\right) : E \in \operatorname{ConvEquiv}(m_{j,k}, j),\ k < k_j\Bigg\}$$

And define $\Delta_j = \Delta^{\text{type}}_j \cup \Delta^{\text{div}}_j$. The collection $\Delta := \bigcup_{j<\omega} \Delta_j$ is consistent with T, since for all $j_* < \omega$, the assignment

$$\bar{\mathbf{a}}_{\eta_{j_*}\restriction j,k},\ \bar{b}_{\eta_{j_*}\restriction j},\ \bar{c}_{\eta_{j_*}\restriction j} \mapsto \bar{\mathbf{a}}^*_{j,k},\ \bar{b}^*_j,\ \bar{c}^* \quad (j \le j_*)$$

realizes $\bigcup_{j<j_*} \Delta_j$.

Proving the chosen object is a counterexample, finishing the proof. Now, let $\bar{\mathbf{a}}^*_{j,k}$, $\bar{b}^*_j \subseteq \mathfrak{C}_T$ realizing Δ (recall that $\mathfrak{C}$ is sufficiently saturated) and work again in τ_T. To complete the proof we note the following:

- $p_0(\overline{x}) = \left\{ \bigvee_{k<k_j} \varphi_{j,k}(\overline{x}, \overline{b}^*_j) : k < k_j \right\}$ is a type in T.
- The formula $\varphi_{j,k}(\overline{x}, \overline{b}^*_j)$ does $\mathrm{ict}^\ell - \langle \Delta, j \rangle$-divide over $\overline{b}^*_{j-1}$ for all $k < k_j$, $0 < j < \omega$.
- For $\mathbf{S}^m(\bigcup_{j<\omega} \overline{b}^*_j) \ni p \supseteq p_0$, $n < \omega$ and finite $A \subseteq \mathrm{Dom}(p)$, there exists $n \leq j < \omega$ such that $A \subseteq \overline{b}^*_{j-1}$. Since p is complete, $p \vdash \bigvee_{k<k_j} \varphi_{j,k}(\overline{x}, \overline{b}^*_j)$ and $\mathrm{Dom}(p)$ contains the constants on the right hand, there exists $k < k_j$ such that $p \vdash \varphi_{j,k}$. Since Δ^{div}_j is realized, we get that $\varphi_{j,k}(\overline{x}, \overline{b}^*_j)$ does $\mathrm{ict}^\ell - j$-divide over $\overline{b}^*_{j-1}$, and by monotonicity of dividing we get that $\varphi_{j,k}$ does $\mathrm{ict}^\ell - n$-divide over A. Therefore p does also $\mathrm{ict}^\ell - n$ divide over A.

$\square$

Corollary 16. $\mathrm{ict}^\ell - \mathrm{rk}^m(\overline{x} = \overline{x}) \geq \infty \Rightarrow \mathrm{ict}^\ell - \mathrm{rk}^m_{div}(\overline{x} = \overline{x}) \geq \infty$.

Theorem 17. *For a first-order complete T, TFAE:*

(1) $\kappa_{\mathrm{ict},\ell}(T) > \aleph_0$
(2) $\mathrm{ict}^\ell - \mathrm{rk}^m(\overline{x} = \overline{x}) = \infty$.
(3) $\mathrm{ict}^\ell - \mathrm{rk}^m(\overline{x} = \overline{x}) \geq \lambda^+_\ell$.
(4) *There exists a type $p(\overline{x})$ such that for all finite $A \subseteq \mathrm{Dom}(p)$, $n_* < \omega$ it holds that p does $\mathrm{ict}^\ell - n_*$ divide over A.*

Proof.

$4 \Rightarrow 1$:: Directly by the definitions.

$1 \Rightarrow 2$:: For some type $p(\overline{x})$ for all finite $A \subseteq \mathrm{Dom}(p)$, $n < \omega$ it holds that p does $\mathrm{ict}^\ell - n$-fork over A. $\mathrm{ict}^\ell - \mathrm{rk}^m(p) \geq 0$. Assume that $\mathrm{ict}^\ell - \mathrm{rk}^m(p) \geq \alpha$ and we will show that $\mathrm{ict}^\ell - \mathrm{rk}^m(p) \geq \alpha + 1$. Let $q \subseteq p$, $A \subseteq \mathrm{Dom}(p)$, $n < \omega$, then p extends q and does $\mathrm{ict}^\ell - n$-fork over A. Therefore $\mathrm{ict}^\ell - \mathrm{rk}^m(p) \geq \alpha + 1$.

$2 \Rightarrow 3$:: Clearly.

$3 \Rightarrow 4$:: By Lemma 15.

$\square$

3. Equivalent definitions of "strongly dependent$_\ell(\mathcal{A})$" using automorphisms

Discussion 3.1. *It is useful to have an equivalent characterization of the strongly dependent$_\ell(\mathcal{A})$ properties using automorphisms. This enables to work in a "pure model theoretic" environment when possible. What enables this equivalent characterization is a sufficiently strongly saturated model where equivalence of types implies existence of automorphisms of the model.*

Definition 18. The model M is strongly κ-saturated if $\mathrm{tp}(\overline{a}, M) = \mathrm{tp}(\overline{b}, M)$, implies that $f(\overline{a}) = \overline{b}$ for some $f \in \mathrm{Aut}(M)$, for all $\overline{a}, \overline{b} \in^{\gamma} |M|$, $\gamma < \kappa$.

Claim 19. Let M be strongly $(\kappa + |\mathcal{L}_M|)^+$-saturated. Then $\mathrm{Th}(M)$ is strongly independent[1] iff for some finite sequence $\overline{c}$ and $\langle \overline{a}_{\alpha,i} : i < \omega, \alpha < \kappa \rangle$ it holds that $\langle \overline{a}_{\alpha(*),i} : i < \omega \rangle$ is indiscernible over $\{\overline{a}_{\alpha,i} : i < \omega, \alpha \neq \alpha(*)\}$ but $\pi(\overline{a}_{\alpha,0}) \neq \overline{a}_{\alpha,1}$ for all $\pi \in \mathrm{Aut}(M/\overline{c})$, $\alpha < \kappa$.

Proof. We use claim 2. Indeed, assume that $\mathrm{Th}(M)$ is not strongly dependent[1]. Therefore we can find $\overline{\varphi} := \langle \varphi_i(\overline{x}, \overline{y}_i) : i < \kappa \rangle$ such that the union of the set of formulas in the variables $\langle \overline{x}_{\alpha,i} : i < \omega, \alpha < \kappa \rangle$, saying that $\langle \overline{x}_{\alpha(*),i} : i < \omega \rangle$ is an indiscernible sequence over $\{\overline{x}_{\alpha,i} : i < \omega, \alpha \neq \alpha(*)\}$ and $\{\varphi_\alpha(\overline{x}, \overline{x}_{\alpha,0}) \wedge \varphi_\alpha(\overline{x}, \overline{x}_{\alpha,1}) : \alpha < \kappa\}$ is consistent. this is a family of formulas in κ which is realized in M, by saturation. Clearly no elementary map over $\overline{c}$ maps $\overline{a}_{\alpha,0}$ to $\overline{a}_{\alpha,1}$, for any $\alpha < \kappa$. Conversely, if we can find $\langle \overline{a}_{\alpha,i} : i < \omega, \alpha < \kappa \rangle$ as above, it clearly follows by the strong saturation that $\mathrm{tp}(\overline{a}_{\alpha,0}, \overline{c}, M) \neq \mathrm{tp}(\overline{a}_{\alpha,1}, \overline{c}, M)$ for all $\alpha < \kappa$. □

Discussion 3.2. *We now turn to strongly dependent*$_{\ell}$, $(_{\mathcal{A}})$. *By Theorem 17, being strongly independent*$_{\ell}(_{\mathcal{A}})$ *is equivalent to existence of* $A, \overline{a}$ *such that* $\mathrm{tp}(\overline{a}, B, \mathfrak{C})$ *does* $\mathrm{ict}^{\ell} - n$*-divide over* B *for any finite* $B \subseteq A$, $n < \omega$. *From this it follows that finding a characterization by automorphisms for dividing is sufficient.*

Claim 20. Let M be a strongly κ-saturated model. For some $\overline{a}, A \subset M$, $|\lg \overline{a}| + |A| < \kappa$ it holds that $\mathrm{tp}(\overline{a}, A, M)$ does $\mathrm{ict}^{\ell} - n$-divide ($\mathrm{ict}^{\mathcal{A}} - n$-divide) strongly over B if and only if:

- **Case** $\ell = 1$**::** *There exists an indiscernible sequence* $\langle \overline{a}_t : t \in \omega \rangle$ *over* B *and a sequence* $\overline{s}$ *of length* n *such that* $1 \leq s_{i+1} - s_i \leq 2$ *and for all* $f \in \mathrm{Aut}(M/A)$, $g \in \mathrm{Aut}(M/B \cup f(\overline{a}))$ *and* $i < n$, *it holds that* $g(\overline{a}_{s_i}) \neq \overline{a}_{s_i+1}$.
- **Case** $\ell = 2$**::** *There exists an indiscernible sequence* $\langle \overline{a}_t : t \in \omega \rangle$ *over* B *and a sequence* $\overline{s}$ *of length* n *such that* $1 \leq s_{i+1} - s_i \leq 2$ *and for all* $f \in \mathrm{Aut}(M/A)$, $i < n-1$ *and* $g \in \mathrm{Aut}(M/B \cup f(\overline{a}) \cup \overline{a}_{s_0} \ldots \overline{a}_{s_{i-1}})$ *it holds that* $g(\overline{a}_{s_i}) \neq \overline{a}_{s_i+1}$.
- **Case** $\ell = 3$**::** *There exists an indiscernible structure* $\langle \overline{a}_t : t \in \omega \cup \mathrm{incr}(< n, \omega) \rangle = \mathbf{a} \in \mathrm{Ind}(\mathfrak{k}^{\mathrm{or+or(<n)}}, A)$ *and a sequence* $\overline{s}$ *of length* n *such that* $1 \leq s_{i+1} - s_i \leq 2$ *and for all* $f \in \mathrm{Aut}(M/A)$, $i < n-1$ *and* $g \in \mathrm{Aut}(M/B \cup f(\overline{a}) \cup \overline{a}_{\langle s_0 \ldots s_{i-1} \rangle})$ *it holds that* $g(\overline{a}_{s_i}) \neq \overline{a}_{s_i+1}$.
- **Case** $\mathcal{A}$**::** *There exist an indiscernible structure* $\langle \overline{a}_t : t \in \mathtt{A}(\omega) \rangle$ *over* B, $m < \omega$ *and equivalent sequences* $\overline{s}_{E,0}, \overline{s}_{E,1} \in \mathtt{A}(\omega)$ *for all* $E \in \mathrm{ConvEquiv}(m, n)$ *such that for all* $f \in \mathrm{Aut}(M/A)$ *and* $g \in \mathrm{Aut}(M/B \cup f(\overline{a}))$ *it holds that* $g(\overline{a}_{\overline{s}_{E,0}}) \neq \overline{a}_{\overline{s}_{E,1}}$.

4. Preservation of strongly dependent under sums

Fact 21. *For a cardinal κ, there exist a cardinal μ and ultrafilter $\mathcal{D}$ on μ such that for any model M, the ultrapower $M^{\mu}/\mathcal{D}$ is strongly κ^{+}-saturated.*

Definition 22. Let M, N be models in the same relational dictionary (i.e. no functions or constants) τ. We define new models $M \oplus N$ and $M + N$ as follows

- *The universe of $M \oplus N$ is $|M| \cup |N|$ (w.l.o.g $|M| \cap |N| = \emptyset$). the dictionary $\tau \cup \{L, R\}$ where L, R are unary relation, interpreting $S^{M \oplus N} = S^M \cup S^N$ for every relation $S \in \tau$, and $L^{M \oplus N} = |M|$, $R^{M \oplus N} = |N|$.*
- $M + N = M \oplus N \restriction \tau$

Claim 23. For $\mathcal{D}$ an ultrafilter on I it holds that $(M \oplus N)^I / \mathcal{D} \simeq M^I/\mathcal{D} \oplus N^I/\mathcal{D}$

Theorem 24. *Let M_1, M_2 models in a relational dictionary τ. If $\mathrm{Th}(M_1), \mathrm{Th}(M_2)$ are strongly dependent^1, then $\mathrm{Th}(M_1 \oplus M_2)$ is also strongly dependent^1.*

Proof. By claim 23 and 21 it follows that w.l.o.g $M_1, M_2, M_1 \oplus M_2$ are strongly κ^{+}-saturated. By claim 19 there exist $\langle \overline{a}_{\alpha,i} : \alpha < \kappa, j < \omega \rangle, \overline{c}$ witnessing $\kappa^{\mathrm{ict}}(\mathrm{Th}(M_1 \oplus M_2)) > \kappa$. W.l.o.g $\overline{c} = \overline{c}^1{}^\frown\overline{c}^2$, $\overline{a}_{\alpha,j} = \overline{a}^1_{\alpha,j}{}^\frown\overline{a}^2_{\alpha,j}$ such that $\overline{a}^i_{\alpha,j}, \overline{c}^i \in M_i$. Recall that $\langle \overline{a}_{\alpha(*),j} : j < \omega \rangle$ is an indiscernible sequence over $\{\overline{a}_{\alpha j} : j < \omega, \alpha \neq \alpha(*)\}$ for $\alpha(*) < \kappa$, therefore $\left\langle \overline{a}^i_{\alpha(*),j} : j < \omega \right\rangle$ is indiscernible over $\left\{\overline{a}^i_{\alpha j} : i < \omega, \alpha \neq \alpha(*)\right\}$. Also, $f \in \mathrm{Aut}(M_1 \oplus M_2)$ iff there exist $f_i \in \mathrm{Aut}(M_i)$ such that $f = f_1 \cup f_2$ (as functions). Therefore, for some $i = 1, 2$ and unbounded $S \subseteq \kappa$ it holds for all $\alpha \in S$ and for all $f_i \in \mathrm{Aut}(M_i/\overline{c}^i)$ that $f_i(\overline{a}^i_{\alpha,0}) \neq \overline{a}^i_{\alpha,1}$. By Claim 19 it follows that the sequences $\{\overline{a}_{\alpha j} : j < \omega, \alpha \in S\}$ are witnesses for $\kappa^{\mathrm{ict},1}(M_i) > \mathrm{otp}(S) = \kappa$. $\square$

Theorem 25. *(Case $\ell = 1, 2, 3$) $\mathrm{T}h(M^1 \oplus M^2)$ is strongly dependent$_\ell$ iff $\mathrm{Th}(M^1)$, $\mathrm{Th}(M^2)$ are strongly dependent$_\ell$.*

Proof. "only if" direction - assume w.l.o.g that $\mathrm{Th}(M^1)$ is not strongly dependent$_\ell$. By lemma 15 there exist $\overline{a} \in M^1$ and a set $A \subseteq M^1$ such that $\mathrm{tp}(\overline{a}, A, M^1)$ does $\mathrm{ict}^\ell - n$ divide over B for any finite $B \subseteq A$ and $n < \omega$. This easily implies that $\mathrm{tp}(\overline{a}, A, M^1 \oplus M^2)$ does $\mathrm{ict}^\ell - n$ divide over B for any finite $B \subseteq A$ and $n < \omega$, and so, $\mathrm{T}h(M^1 \oplus M^2)$ is not strongly dependent$_\ell$.

"if" direction - By 15, there exist $\overline{a}^i \in M^i$ and sets $A^i \subseteq M^i$ such that $\mathrm{tp}(\overline{a}^1{}^\frown\overline{a}^2, A^1 \cup A^2, M^1 \oplus M^2)$ does $\mathrm{ict}^\ell - 2 \cdot n$ divide over $B^1 \cup B^2$ for all finite $B^i \subseteq A^i$ and $n < \omega$. If $\mathrm{tp}(\overline{a}, A^1, M^1)$ does $\mathrm{ict}^\ell - n$ divide over B^1 for all $n < \omega$ and finite $B^1 \subseteq A^1$ this concludes the proof. Otherwise, there exist $n_0 < \omega$ and finite $B^1 \subseteq A^1$ such that $\mathrm{tp}(\overline{a}, A^1, M^1)$ does not $\mathrm{ict}^\ell - n_0$ divide over B^1. Since for all finite $B^2 \subseteq A^2$, $n > n_0$ it holds that $\mathrm{tp}(\overline{a}^1{}^\frown\overline{a}^2, A^1 \cup A^2, M^1 \oplus M^2)$ does $\mathrm{ict}^\ell - 2 \cdot n$ divide over $B^1 \cup B^2$, we get by claim 27 that $\text{tp}(\overline{a}, A^2, M^2)$ does necessarily $\text{ict}^\ell - n$ divide over B^2. Thus, again by 15, $\text{Th}(M^2)$ is not strongly dependent$_\ell$. $\square$

Fact 26. *$M \oplus N \equiv M' \oplus N'$ for models $M \equiv M', N \equiv N'$.*

Claim 27. (Cases $\ell = 1, 2, 3$) Let $\overline{a}^i, A^i, B^i \subseteq |M^i|$, $(i \in \{1, 2\})$, then $\text{tp}(\overline{a}^1 {}^\frown \overline{a}^2, A^1 \cup A^2, M^1 \oplus M^2)$ does $\text{ict}^\ell - 2n$-divide over $B^1 \cup B^2$ iff $\text{tp}(\overline{a}^i, A^i, M^i)$ does $\text{ict}^\ell - n$-divide over B^i, for some $i \in \{1, 2\}$.

Proof. The proof for all the cases is analogous and the "if" direction is easy so we only give here the "only if" of case $\ell = 1$: w.l.o.g $M^1, M^2, M^1 \oplus M^2$ are strongly κ^+-saturated and $|A^1 \cup A^2| \leq \kappa$. By 20 we can find $\langle \overline{a}^1_t {}^\frown \overline{a}^2_t : t \in \omega \rangle$, an indiscernible sequence over $B^1 \cup B^2$ and a sequence $\overline{s}$ of length $2n$ such that $1 \leq s_{j+1} - s_j \leq 2$ for all $j < 2n$ and that $g(\overline{a}^1_{s_j} {}^\frown \overline{a}^2_{s_j}) \neq \overline{a}^1_{s_j+1} {}^\frown \overline{a}^2_{s_j+1}$ holds for all $f \in \text{Aut}\left(M^1 \oplus M^2 / A^1 \cup A^2\right)$, $g \in \text{Aut}(M / B^1 \cup B^2 \cup f(\overline{a}^1 \cup \overline{a}^2))$ and $j < 2n$.

Now, assume towards contradiction that $f_i \in \text{Aut}(M^i / A^i)$ $(i = 1, 2)$ and that $g_i \in \text{Aut}(M^i / B^i \cup f^i(\overline{a}^i))$ are such that $g_i(\overline{a}^i_{s_j}) = \overline{a}^i_{s_j+1}$ holds for some $j < 2n$. By the bijection $\Phi : \text{Aut}(M^1) \times \text{Aut}(M^2) \to \text{Aut}(M^1 \oplus M^2)$, we get that $f = f_1 \cup f_2 \in \text{Aut}\left(M^1 \oplus M^2 / A^1 \cup A^2\right)$ and that $g = g_1 \cup g_2 \in \text{Aut}(M / B^1 \cup B^2 \cup f(\overline{a}^1 \cup \overline{a}^2))$ - a contradiction. Thus, for all $j < 2n$ there exists $i \in \{1, 2\}$ such that $g(\overline{a}^i_{s_j}) \neq \overline{a}^i_{s_j+1}$ holds for all $f \in \text{Aut}(M^i / A^i)$, $g \in \text{Aut}(M^i / B^i \cup f^i(\overline{a}^i))$. Denote by $i(j)$, the appropriate i for every $j < 2n$, . Let $i_0 \in \{1, 2\}$ be such that $S_{i_0} = \{i(j) = i_0 : j < 2n\}$ has at least n elements. It now follows easily from 20 that $\langle \overline{a}^{i_0}_t : t \in \omega \rangle$ are witnessing that $\text{tp}(\overline{a}^{i_0}, A^{i_0}, M^{i_0})$ does $\text{ict}^\ell - n$-divide over B^{i_0}. $\square$

5. Appendix - various claims.

Claim 28. Let κ be a cardinal, $f : \text{ds}(\kappa^+) \to \kappa$. We can find a sequence $\langle \alpha_k : k < \omega \rangle \subseteq \kappa$ such that for every $k_* < \omega$ there exists $\eta \in \text{ds}(\kappa^+)$ of length k_* such that $f(\eta \restriction k) = \alpha_k$ holds for all $k < k_*$.

Corollary 29. *If M is κ-homogeneous and κ-saturated, and $I' \supseteq I$ are linear orders such that $|I'| < \kappa$, $A \subseteq M, |A| < \kappa$ then:*

1. *Every $\langle \overline{a}_t : t \in I \rangle \in \text{Ind}(\mathfrak{k}^{\text{or}}, A, M)$ can be extended to $\langle \overline{a}_t : t \in I' \rangle \in \text{Ind}(\mathfrak{k}^{\text{or}}, A, M)$.*
2. *Every $\langle \overline{a}_t : t \in I \cup {}^{<n}I \rangle \in \text{Ind}(\mathfrak{k}^{\text{or}+\text{or}(<n)}, A, M)$ can be extended to $\langle \overline{a}_t : t \in I' \cup {}^{<n}I' \rangle \in \text{Ind}(\mathfrak{k}^{\text{or}}, A, M)$..*
3. *Every $\langle \overline{a}_t : t \in {}^{\leq n}I \rangle \in \text{Ind}(\mathfrak{k}^{\text{or}(\leq n)}, A, M)$ can be extended to $\langle \overline{a}_t : t \in {}^{\leq n}I' \rangle \in \text{Ind}(\mathfrak{k}^{\text{or}(\leq n)}, A, M)$.*
4. *Every structure $\langle \overline{a}_t : t \in \mathtt{A}(I) \rangle$ indiscernible over A can be extended to $\langle \overline{a}_t : t \in \mathtt{A}(I') \rangle$, also indiscernible over A.*

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