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STRONGLY BOUNDED GROUPS OF VARIOUS CARDINALITIES

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ABSTRACT. Strongly bounded groups are those groups for which every action by isometries on a metric space has orbits of finite diameter. Many groups have been shown to have this property, and all the known infinite examples so far have cardinality at least $2^{\aleph_0}$. We produce examples of strongly bounded groups of many cardinalities, including $\aleph_1$, answering a question of Yves de Cornulier [Comm. Algebra 34 (2006), no. 7, 2337–2345]. In fact, any infinite group embeds as a subgroup of a strongly bounded group which is, at most, two cardinalities larger.

1. INTRODUCTION

In geometric group theory one extracts information regarding groups via actions on metric spaces. Little knowledge can be gleaned from a group action which has bounded orbits, and so one often uses nongeometric approaches for the study of, say, a finite group. Interestingly, there are infinite groups which are similarly not suited for study using geometric techniques. A group G is *strongly bounded* if every action of G by isometries on a metric space has bounded orbits [4] (this is sometimes referred to as *the Bergman property*). We emphasize that we are considering all abstract actions of G on all metric spaces, regardless of any natural topology which G may carry. Examples of infinite strongly bounded groups were produced by the second author in [11] using extra set theoretic assumptions, and more recently Bergman showed that the full symmetric group on a set is strongly bounded [1]. The group of self-homeomorphisms of the Cantor set and of the irrational numbers [5], ω_1-existentially closed groups, and arbitrary powers of a finite perfect group are also strongly bounded [4].

All infinite strongly bounded groups are necessarily uncountable (see [4, Remark 2.5]), and all known infinite examples so far have cardinality at least $2^{\aleph_0}$. It is natural to ask whether there exists a strongly bounded group of cardinality $\aleph_1$ (see [4, Question 4.16]). We give an affirmative answer to this and many other such questions (see Section 2 for set theoretic definitions).

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Theorem A. *Let λ be a cardinal of uncountable cofinality, and let K be a group such that $|K| < \lambda$. Then there exists a strongly bounded group $G \geqslant K$ which is of cardinality λ, except possibly when $\lambda = \mu^+$ where $\operatorname{cof}(\mu) = \omega$ and μ is a limit of weakly inaccessible cardinals.*

Thus, for example, there exist strongly bounded groups of cardinality $\aleph_1$, $\aleph_2$, $\aleph_{\omega+1}$, and $\aleph_{\aleph_1}$. Moreover, if an infinite group is of cardinality κ, then it embeds as a subgroup of a strongly bounded group of cardinality κ^{++}, though often the strongly bounded group can be made to have cardinality κ^+ instead. The proof utilizes small cancellation over free products. One cannot drop the assumption regarding uncountable cofinality: a group which is countably infinite, or uncountable of cardinality which is ω-cofinal, cannot be strongly bounded. It is already known that any group K embeds in a strongly bounded group of cardinality $|K|^{\aleph_0}$ (see [4, Corollary 3.2]).

By assuming some extra set theory we can produce other examples of strongly bounded groups of cardinality $\aleph_1$ which seem slightly more tame. In the next theorem, the hypothesis $\operatorname{cof}(LM) = \aleph_1$ is equivalent to the assertion that there exists an increasing sequence $\{X_\alpha\}_{\alpha<\aleph_1}$ of sets of Lebesgue measure zero such that any set of measure zero is eventually included in the elements of the sequence.

Theorem B. *Suppose that $\operatorname{cof}(LM) = \aleph_1$ and that H is a nontrivial finite perfect group. Then there exists a strongly bounded group of cardinality $\aleph_1$ which is a subgroup of $\prod_\omega H$.*

Such groups will be constructed by producing a special type of Boolean algebra and applying a result of de Cornulier. The group $\prod_\omega H$ mentioned in Theorem B is itself already known to be strongly bounded [4, Theorem 4.1]. Assuming that ZF is consistent, one can produce models of ZFC in which $\operatorname{cof}(LM) = \aleph_1$ and also $2^{\aleph_0}$ is any cardinal which is not ruled out by the classical theorems of set theory [8]. Thus we obtain the following corollary.

Corollary 1.1. *If κ is a cardinal of uncountable cofinality, then there exists a model of ZFC in which $2^{\aleph_0} = \kappa$ and there is a strongly bounded group of cardinality $\aleph_1$ which is a subgroup of $\prod_\omega H$, where H is any nontrivial finite perfect group. (Assuming, of course, that ZF is consistent.)*

In Section 2 we prove Theorem A and in Section 3 we prove Theorem B.

2. Proof of Theorem A

In this section we will first quote an alternative characterization for a group to be strongly bounded and then review small cancellation over free products. Then we review some set theory and furnish the proof of Theorem A.

If G is a group, 1_G denotes the identity element of G, and $Z \subseteq G$, we denote

$$\mathcal{G}(Z) = Z \cup \{1_G\} \cup \{g^{-1} \mid g \in Z\} \cup \{gh \mid g, h \in Z\}.$$

Lemma 2.1 ([4, Proposition 2.7]). *A group G is strongly bounded if and only if for every sequence $\{Z_m\}_{m\in\omega}$ of subsets of G such that $\mathcal{G}(Z_m) \subseteq Z_{m+1}$ and $\bigcup_{m\in\omega} Z_m = G$, there exists an $m \in \omega$ for which $Z_m = G$.*

Now for the review of free products (see [10, V.9]). Recall that elements of a free product $F = *_{i\in I} H_i$ are naturally viewed as words whose letters are nontrivial elements of $\bigcup_{i\in I} H_i$. We will write $w \equiv u$ to say that two such words are equal as words, letter for letter, and write $w = u$ if the group element given by the product w is equal in F to the group element given by the product u. Concatenation of words w and u will be denoted as usual by wu, meaning that one writes the word w and then to the right of this one writes the word u.

Each element g of F has a unique writing as a word $g = w \equiv g_1 \cdots g_k$ which is of minimal length (the *normal form*) in which no two consecutive letters in the word are elements of the same H_i, and we let $L(g) = k$ denote the length of such an expression. Given two normal forms $w \equiv g_1 \cdots g_k$ and $u \equiv h_1 \cdots h_j$ one computes the normal form of the group element wu in the following way. First we find $s \in \omega$ which is maximal such that $g_{k+1-r} = h_r^{-1}$ for all $1 \leqslant r \leqslant s$ (we allow s to be 0). In case $k - s \geqslant 1$ and $s + 1 \leqslant j$ we get $g_{k-s} \neq h_{s+1}^{-1}$. If g_{k-s} is in the same H_i as h_{s+1}, then we let $g_{k-s}h_{s+1} = h \in H_i$ and obtain the normal form $g_1 \cdots g_{k-s-1}hh_{s+2} \cdots h_j$ for the group element wu. Otherwise we get $g_1 \cdots g_{k-s}h_{s+1} \cdots h_j$ as the normal form. We say that a group element $w \in F$ has *semireduced form* uv if both u and v are normal forms, $w = uv$, and the number s used in the computation for the normal form for uv is 0.

An element in F with normal form $w \equiv g_1 \cdots g_k$ is *cyclically reduced* if either $L(w) \leqslant 1$, or g_1 and g_k are in different H_i. More generally, we say that w is *weakly cyclically reduced* if either $L(w) \leqslant 1$ or $g_1 \neq g_k^{-1}$. A subset $R \subseteq F$ is *symmetrized* if every $w \in R$ is weakly cyclically reduced and every weakly cyclically reduced conjugate of w and of w^{-1} is also in R. From a set Γ of weakly cyclically reduced elements of F one obtains a symmetrized set by taking all weakly cyclically reduced conjugates of Γ and then taking their inverses. Given a symmetrized set R, a word u is a *piece* if there exist distinct $w_1, w_2 \in R$ with semi-reduced forms $w_1 = uv_1$ and $w_2 = uv_2$.

Definition 2.2. A symmetrized set R for the free product $F = *_{i \in I} H_i$ satisfies the $C'(\eta)$ *condition*, where $\eta > 0$, if for each $w \in R$ we have

(1) $L(w) > \frac{1}{\eta}$; and
(2) whenever $w = uv$ is a semi-reduced form, with u a piece, we have $L(u) < \eta L(w)$.

We use the following:

Lemma 2.3 (see [10, Corollary V.9.4])**.** *Let $F = *_{i \in I} H_i$ be a free product, and let R be a symmetrized subset of F which satisfies $C'(\frac{1}{6})$. Let N be the normal closure of R in F. Then the natural map $F \to F/N$ embeds each factor H_i of F.*

Lemma 2.4. *For each $n \geqslant 1$ there is a group word $w(x_0, x_1, \ldots, x_{n-1}, y)$ such that the following holds: if G is a group and $f : (G \backslash \{1_G\})^n \to G$, then there exist group H and $c \in H$ such that*

(a) *$G \leqslant H$;*
(b) *$c \in H \backslash G$;*
(c) *for all $\overline{g} \in (G \backslash \{1_G\})^n$ we have $w(\overline{g}, c) = f(\overline{g})$;*
(d) *$H = \langle G \cup \{c\} \rangle$.*

Proof. Let $u(x_0, x_1, \ldots, x_{n-1}, y)$ be given by

$$x_0 y x_1 y x_2 y \cdots x_{n-2} y x_{n-1},$$

and let $w(x_0, \ldots, x_{n-1}, y)$ be given by

$$y^k u y^{k-1} u y^{k-2} u \cdots y^3 u y^2 u y u,$$

where $k = 32$. Let F be the free product given by $F = \langle c \rangle * G$, where c has infinite order. Let $\Gamma_0 = \{(f(\overline{g}))^{-1}w(\overline{g}, c) \mid \overline{g} \in (G\backslash\{1_G\})^n\}$. Notice that the elements of Γ_0 are weakly cyclically reduced unless $g_{n-1} = f(\overline{g})$, in which case we replace the word $(f(\overline{g}))^{-1}w(\overline{g}, c)$ with the weakly cyclically reduced word obtained by reducing the word $w(\overline{g}, c)(f(\overline{g}))^{-1}$. By performing all these replacements we obtain a new set Γ.

Notice that the symmetrization R of Γ satisfies $C'(\frac{1}{6})$ over the free product F. More specifically, each element of Γ is weakly cyclically reduced and of length $(2n-1)k + k + 1 = 2nk + 1$ in case $f(\overline{g}) \neq 1_G, g_{n-1}$; of length $2nk + 1 - 2 = 2nk - 1$ in case $g_{n-1} = f(\overline{g})$; or of length $2nk$ in case $f(\overline{g}) = 1_G$. Weakly cyclically reduced conjugates of elements of Γ will have length at least $2nk - 2$, similarly for the inverses of such elements. It is clear that no normal form which has form

$$v_1 c^{m_1}(u(\overline{g}, c))^{\pm 1} c^{m_2}(u(\overline{g}, c))^{\pm 1} c^{m_3} v_3,$$

where $v_1, v_3 \in F$ and $m_1, m_2, m_3 \in \mathbb{Z}\backslash\{0\}$, can be a piece. Thus we can use, for example, $10n$ as a very naïve upper bound on the length of a piece. For any $w \in R$ we have

$$L(w) \geqslant 2nk - 2 = 64n - 2 > 6$$

as well as

$$10n < \frac{1}{6}(64n - 2) = \frac{1}{6}(2nk - 2) \leqslant \frac{1}{6}L(w),$$

and so R indeed satisfies $C'(\frac{1}{6})$.

Let N be the normal subgroup in F generated by R and by Lemma 2.3 that the homomorphism $F \to F/N = H$ embeds each of G and $\langle c \rangle$. The claim is immediate. $\square$

As is usual, we shall consider each ordinal number to be the set of ordinal numbers below itself (e.g., $0 = \varnothing$, $1 = \{0\}$, $\omega + 1 = \{0, 1, \ldots, \omega\}$) and the cardinal numbers to be the ordinals which cannot inject to a proper initial subinterval of themselves. The notation $|Y|$ denotes the cardinality of the set Y. A subset X of ordinal α is *bounded* if there is an upper bound $\beta < \alpha$ for all elements of X. The *cofinality* of an ordinal α (denoted $\mathrm{cof}(\alpha)$) is the least cardinality of an unbounded subset of α. An infinite cardinal λ is *regular* if $\mathrm{cof}(\lambda) = \lambda$, and is *singular* otherwise. We use κ^+ to denote the smallest cardinal which is strictly greater than κ, and similarly $\kappa^{++} = (\kappa^+)^+$. An infinite cardinal λ is a *successor cardinal* if $\lambda = \kappa^+$ for some cardinal κ, and is a *limit cardinal* otherwise. An uncountable cardinal which is a limit regular cardinal is *weakly inaccessible*. For any infinite cardinal κ the successor cardinal κ^+ is regular.

Next we remind the reader of some notation from Ramsey coloring theory.

Definitions 2.5. If X is a set and $n \in \omega$, we let $[X]^n$ denote the set of subsets of X of cardinality n. If κ, λ, and μ are cardinals and $n \in \omega$, then we write

$$\lambda \to [\mu]^n_\kappa$$

to mean that if $f : [\lambda]^n \to \kappa$ is any function, then for some $A \subseteq \lambda$ with $|A| = \mu$ we have that $f([A]^n)$ is a proper subset of κ (see [6]). The negation of this relation is denoted $\lambda \nrightarrow [\mu]^n_\kappa$. The reader should take care not to confuse this square bracket partition relation with the parenthetical notation $\lambda \to (\mu)^n_\kappa$.

Proof of Theorem A. The relation $\bigoplus_{\lambda,n}$, where λ is an infinite cardinal and $n \in \omega$, will mean that there exists some $f : [\lambda]^n \to \lambda$ such that if $h : \lambda \to \omega$ is any function, then for some $m \in \omega$ we have

$$\lambda = \{f(Z) \mid Z \subseteq \{\alpha < \lambda \mid h(\alpha) < m\} \text{ and } |Z| = n\}.$$

Clearly $\lambda \twoheadrightarrow [\lambda]^n_\lambda$ implies $\bigoplus_{\lambda,n}$. We note that if λ is a successor of a regular cardinal, or if $\lambda = \mu^+$ where μ is singular and not a limit of weakly inaccessible cardinals, then $\lambda \twoheadrightarrow [\lambda]^2_\lambda$, and therefore $\bigoplus_{\lambda,2}$, holds (see [12, Theorems 3.1, 3.3(3)] and [13]). We consider three cases.

Case 1 ($\bigoplus_{\lambda,n}$ holds, $\mathrm{cof}(\lambda) > \omega$). In this case we let K be a group, without loss of generality infinite, with $|K| < \lambda$. The construction is by induction. First we define an increasing sequence of ordinals $\{\beta_\alpha\}_{\alpha<\lambda}$ by letting $\beta_0 = |K|$, $\beta_{\alpha+1} = \beta_\alpha + \beta_\alpha$, and $\beta_\alpha = \bigcup_{\gamma<\alpha} \beta_\gamma$ when α is a limit ordinal.

Next we let $f : [\lambda\backslash\{0\}]^n \to \lambda\backslash\{0\}$ witness $\bigoplus_{\lambda,n}$. We can without loss of generality assume that $f(W) \in \beta_\alpha$ for all $\alpha < \lambda$ and $W \in [\beta_\alpha]^n$. To see this, let U be a set such that $|U| = \lambda$, and by assumption let $g : [U]^n \to U$ be such that for any $h : U \to \omega$ there exists $m \in \omega$ for which

$$U = \{g(W) \mid W \subseteq \{x \in U \mid h(x) < m\} \text{ and } |W| = n\}.$$

Pick a well order $U = \{x_\epsilon\}_{\epsilon<\lambda}$. Given a subset $W \subseteq U$ we let $g''(W) = \bigcup_{k\in\omega} W_k$ where $W_0 = W$ and $W_{k+1} = W_k \cup \{g(W) \mid W \subseteq W_k \text{ and } |W| = n\}$. Let $U_0' \subseteq U$ be such that $|U_0'| = |K|$. Let $U_0 = g''(U_0')$. If $\alpha < \lambda$ is a limit ordinal we let $U_\alpha = \bigcup_{\gamma<\alpha} U_\gamma$. If $\alpha = \gamma + 1$, then we pick $U \supseteq U_\alpha' \supseteq U_\gamma$ such that the minimal element of $U\backslash U_\gamma$ is in U_α' and $|U_\alpha'| = |U_\alpha'\backslash U_\gamma| = |U_\gamma|$. Let $U_\alpha = g''(U_\alpha')$. Notice that $g([U_\alpha]) \subseteq U_\alpha$ for each $\alpha < \lambda$. By the induction we also have $U = \bigcup_{\alpha<\lambda} U_\lambda$. Taking $p : U \to \lambda\backslash\{0\}$ to be any bijection such that $p(U_\alpha) = \beta_\alpha$ for all α and defining $f = p \circ g \circ p^{-1}$, we obtain the required f.

We define the group G to have a set of elements λ and give it a group structure as an increasing union of subgroups G_α, with G_α having β_α as its underlying set of elements. Define G_0 to have the group structure of K on the set of elements β_0 with 0 identified with the trivial group element 1_K. If we have defined the group structure G_γ for all $\gamma < \alpha \leqslant \lambda$ and α is a limit ordinal, then we let G_α have the unique group structure imposed by the G_γ with $\gamma < \alpha$. If $\lambda > \alpha = \gamma + 1$, then by Lemma 2.4 we define G_α to have group structure such that

(a) $G_\gamma \leqslant G_\alpha$;
(b) for all $\overline{g} \in (G_\gamma\backslash\{1_{G_\gamma}\})^n$ such that $g_0 < g_1 < \cdots < g_{n-1}$, we have $w(\overline{g}, \beta_\gamma) = f(\{g_0, \ldots, g_{n-1}\})$;
(c) $G_\alpha = \langle G_\gamma \cup \{\beta_\gamma\}\rangle$

(here we use the fact that $|\beta_\alpha| = |\beta_\alpha\backslash\beta_\gamma| = |\beta_\gamma|$).

Now $G = G_\lambda$, and we let $X = \{\beta_\alpha\}_{\alpha<\lambda}$. Suppose that $\{Z_m\}_{m\in\omega}$ is a sequence of subsets of G such that $G = \bigcup_{m\in\omega} Z_m$ and $Z_{m+1} \supseteq \mathcal{G}(Z_m)$. Select $m \in \omega$ large enough such that

$$\lambda\backslash\{0\} = \{f(W) \mid W \subseteq \{0 \neq \alpha < \lambda \mid \alpha \in Z_m\} \text{ and } |W| = n\}$$

and that $1_G \in Z_m$ and that $X \cap Z_m$ is unbounded in λ. Given arbitrary $g \in G\backslash\{1_G\}$ we select nontrivial $g_0 < \cdots < g_{n-1}$ in Z_m such that $f(\{g_0, \ldots, g_{n-1}\}) = g$. Pick $g_n \in X \cap Z_m$, which is larger than all $g_0, \ldots, g_{n-1}$. Then we have $w(g_0, \ldots, g_{n-1}, g_n) = f(\{g_0, \ldots, g_{n-1}\}) = g$, and so $G = Z_{m+j}$ where j is the length of the word w. Case I is proved.

Case 2 (λ is a limit cardinal, $\mathrm{cof}(\lambda) > \omega$). In this case we let $\lambda = \bigcup_{\alpha<\mathrm{cof}(\lambda)} \lambda_\alpha$, where $\{\lambda_\alpha\}_{\alpha<\mathrm{cof}(\lambda)}$ is a strictly increasing sequence of cardinals below λ such that $\lambda_0 \geqslant |K|$. Notice that each cardinal λ_α^{++} satisfies Case I. Let λ_0 be given any group structure such that K is a subgroup in λ_0. By Case I we let λ_0^{++} be given a group structure G_0 which is strongly bounded. For each $\alpha < \mathrm{cof}(\lambda)$ we endow λ_α^{++} with a group structure G_α which extends the group structure on $\bigcup_{\gamma<\alpha} \lambda_\gamma^{++}$ and such that G_α is strongly bounded (by Case I). Now let λ be given the group structure G inherited from all the G_α. Let $\{Z_m\}_{m\in\omega}$ be such that $\mathcal{G}(Z_m) \subseteq Z_{m+1}$ and $\bigcup_{m\in\omega} Z_m = G$, and notice that for each $\alpha < \mathrm{cof}(\lambda)$ there exists some minimal $m_\alpha \in \omega$ such that $Y_{m_\alpha} \cap G_\alpha = G_\alpha$. Then $\alpha \mapsto m_\alpha$ is a nondecreasing sequence from $\mathrm{cof}(\lambda)$ to ω, so it eventually stabilizes, and so G is strongly bounded.

Case 3 ($\lambda = \mu^+$ where $\mathrm{cof}(\mu) > \omega$ and μ is singular). Let K be a group of cardinality $< \lambda$. We let $\mu = \bigcup_{\alpha<\mathrm{cof}(\mu)} \mu_\alpha$ with $\{\mu_\alpha\}_{\alpha<\mathrm{cof}(\mu)}$ being a strictly increasing sequence of cardinals. We have $\mu_\alpha^{++} \twoheadrightarrow [\mu_\alpha^{++}]^2_{\mu_\alpha^{++}}$. Let $f_\alpha : [\mu_\alpha^{++}]^2 \to \mu_\alpha^{++}$ witness this. Let $f : [\mu]^2 \to \mu$ be defined by $f(W) = f_\alpha(W)$, where $\alpha < \mathrm{cof}(\mu)$ is minimal such that $W \in [\mu_\alpha^{++}]^2$.

For each ordinal $\mu \leqslant \gamma < \mu^+ = \lambda$ we let $j_\gamma : \gamma \to \mu$ be any bijection and define $h_\gamma : [\gamma]^2 \to \gamma$ by $j_\gamma^{-1} \circ f \circ j_\gamma$ (here $j_\gamma(W)$, where $W \in [\gamma]^2$, means the 2-element set obtained by applying j_γ to the elements of W). Define $h : [\lambda]^3 \to \lambda$ by $h_{\max(W)}(W\backslash\{\max(W)\})$.

We define a group structure on λ by induction. Let $\beta_0 = \mu$, let $\beta_{\delta+1} = \beta_\delta + \beta_\delta$, and let $\beta_\delta = \bigcup_{\epsilon<\delta} \beta_\epsilon$, when δ is a limit ordinal. Let G_0 be any group structure on β_0 which includes K as a subgroup. If G_δ has been defined for all $\epsilon < \delta \leqslant \lambda$ and δ is a limit ordinal, then we let G_δ be the induced group structure on β_δ. If $\lambda > \delta = \epsilon + 1$, then we let G_δ be the group given by

(a) $G_\epsilon \leqslant G_\delta$,
(b) for all $\overline{g} \in (G_\epsilon\backslash\{1_{G_\epsilon}\})^2$ with $j_{\beta_\epsilon}(g_0) < j_{\beta_\epsilon}(g_1)$ we have $w(\overline{g}, \beta_\epsilon) = h(\{g_0, g_1, \beta_\epsilon\})$,
(c) $G_\delta = \langle G_\epsilon \cup \{\beta_\epsilon\}\rangle$,

where w is as in Lemma 2.4. Now we have our group structure G on λ. Let $\{Z_m\}_{m\in\omega}$ be as usual.

We claim that for each $\delta < \lambda$ there exists some $m \in \omega$ such that $G_\delta \subseteq Z_m$. Fix $\delta < \lambda$ and select $m_0 \in \omega$ large enough that $\beta_\delta \in Z_{m_0}$. Notice that for each $\alpha < \mathrm{cof}(\mu)$ there is some natural number $m > m_0$ for which $|\mu_\alpha^{++} \cap j_{\beta_\delta}(Z_m \cap \beta_\delta)| = \mu_\alpha^{++}$ (since μ_α^{++} is necessarily of cofinality $> \omega$). As $\mathrm{cof}(\mu) > \omega$ there must exist some $m_1 > m_0$ for which

$$\{\alpha < \mathrm{cof}(\mu) \mid |\mu_\alpha^{++} \cap j_{\beta_\delta}(Z_{m_1} \cap \beta_\delta)| = \mu_\alpha^{++}\}$$

is unbounded in $\mathrm{cof}(\mu)$. Let $g \in G_\delta$ be given. Select $\alpha < \mathrm{cof}(\mu)$ large enough that $j_{\beta_\delta}(g) \in \mu_\alpha^{++}$ and such that $|\mu_\alpha^{++} \cap j_{\beta_\delta}(Z_{m_1} \cap \beta_\delta)| = \mu_\alpha^{++}$. Select elements $\mu_\alpha^+ < \zeta_0 < \zeta_1 < \mu_\alpha^{++}$ which are elements of $j_{\beta_\delta}((Z_{m_1}\backslash\{1_G\}) \cap \beta_\delta)$ for which $f_\alpha(\{\zeta_0, \zeta_1\}) = j_{\beta_\delta}(g)$. Then $f(\{\zeta_0, \zeta_1\}) = j_{\beta_\delta}(g)$, and by construction it follows that

$$w(j_{\beta_\delta}^{-1}(\zeta_0), j_{\beta_\delta}^{-1}(\zeta_1), \beta_\delta) = g,$$

and so $G_\delta \subseteq Z_{m_1+j}$ where j is the length of the word w.

Now letting m_δ be minimal such that $G_\delta \subseteq Z_{m_\delta}$ we get a nondecreasing function $\delta \mapsto m_\delta$ from λ to ω, which must stabilize. Thus G is strongly bounded. $\square$

3. Proof of Theorem B

We assume that the reader is familiar with the definition of a Boolean algebra (see [7, I. 7]). We shall use the notation $x \wedge y$ and $x \vee y$ for the meet and join of x and y in a Boolean algebra, x^c for the complement of x, $x - y = x \wedge y^c$, and 1 and 0 for the top and bottom elements. Given a subset Z of a Boolean algebra $\mathcal{A}$ we let $\mathcal{R}(Z)$ equal the following set:

$$Z \cup \{0,1\} \cup \{x^c \mid x \in Z\} \cup \{x \vee y \mid x, y \in Z\} \cup \{x \wedge y \mid x, y \in Z\} \cup \{x - y \mid x, y \in Z\}.$$

Definition 3.1. A *proper $\mathcal{R}$-filtration* of a Boolean algebra $\mathcal{A}$ is a sequence $\{Z_n\}_{n\in\omega}$ such that Z_n properly includes in Z_{n+1}, in $\mathcal{R}(Z_n) \subseteq Z_{n+1}$, and also in $\mathcal{A} = \bigcup_{n\in\omega} Z_n$. A proper $\mathcal{R}$-filtration induces a function $f : \mathcal{A} \to \omega$ by letting $f(x) = \min\{n \in \omega \mid x \in Z_n\}$.

Definition 3.2 (see [4, Remark 4.5])**.** An infinite Boolean algebra has *strong uncountable cofinality* if it has no proper $\mathcal{R}$-filtration.

We shall be especially interested in a specific type of algebra.

Definition 3.3. An *algebra on a set* X is a collection $\mathcal{A}$ of subsets of X for which

- $X \in \mathcal{A}$;
- $Z, Z' \in \mathcal{A}$ implies $Z \cap Z' \in \mathcal{A}$; and
- $Z \in \mathcal{A}$ implies $X \setminus Z \in \mathcal{A}$.

Intersection, union, and set theoretic complementation answer for the meet, join, and complementation which endow $\mathcal{A}$ with a natural Boolean algebra structure.

Given an algebra $\mathcal{A}$ on X and a function $f : X \to Y$, we shall say that f is *measurable* if each preimage $f^{-1}(y)$ is in $\mathcal{A}$ for each $y \in Y$. If Y is a finite group, then it is easy to check that the set of measurable functions from X to Y forms a group under componentwise multiplication: $(f_0 * f_1)(x) = f_0(x)f_1(x)$.

The following was essentially proved by Yves de Cornulier in [4].

Theorem 3.4. *Suppose $\mathcal{A}$ is an algebra of sets on a set X which is of strong uncountable cofinality and H is a finite perfect group. Then the group of measurable functions from X to H is strongly bounded.*

Proof. See the proof of [4, Thm. 4.1]. □

Thus to prove Theorem B it suffices to prove the following.

Proposition 3.5. *If* $\mathrm{cof}(LM) = \aleph_1$*, then there exists an algebra of sets on* ω *of cardinality* $\aleph_1$ *which is of strong uncountable cofinality.*

This is a slight refinement of the main result of [3] in which Cielsielski and Pawlikowski construct from the assumption $\mathrm{cof}(LM) = \aleph_1$ an algebra of cardinality $\aleph_1$ which is of uncountable cofinality (i.e., an algebra that is not the union of a strictly increasing ω sequence of subalgebras). Models of ZFC + $\aleph_1 < 2^{\aleph_0}$ in which such an algebra exists were first constructed by Just and Koszmider [8]. Under Martin's axiom the existence of an algebra of cardinality $\aleph_1$ of uncountable cofinality implies the continuum hypothesis [9, Prop. 5]. Thus one cannot hope to prove the conclusion of Proposition 3.5 without extra set theoretic assumptions.

The proof of Proposition 3.5 will follow a slight modification to the lovely proof used in [3]. Given a set X we let $[X]^{\leqslant n}$ denote the set of all subsets of X of cardinality at most n. Consistent with [3] we let $\mathcal{CH}$ denote the collection of all subsets $T \subseteq \omega^\omega$ of form $T = \prod_{n\in\omega} T_n$ where $T_n \in [\omega]^{\leqslant n+1}$. The cardinal $\mathrm{cof}(LM)$ is equal to the cardinal

$$\min(\{|\mathcal{F}| \ \mid \mathcal{F} \subseteq \mathcal{CH} \text{ and } \bigcup \mathcal{F} = \omega^\omega\})$$

(see [2]), and for our construction we will use this latter formulation.

Lemma 3.6. *For each $T \in \mathcal{CH}$ there exists a strictly increasing $g \in \omega^\omega$ such that for every strictly increasing $f \in T$ we have $f(n) < g(n)$ for all $n \in \omega$, and whenever $g(n) \geqslant f(m)$ we have $g(n+1) > f(m+2)$.*

Proof. Let $g(0) = \max(T_0)+1$, and generally let $g(n+1) = \max(T_{g(n)+2} \cup \{g(n)\})+1$. Clearly, $g(n+1) \geqslant g(n)+1$ for all $n \in \omega$, and so g is strictly increasing and, moreover, $g(n) > n$. Given a strictly increasing $f \in T$ we notice that $f(0) < g(0)$ since $f(0) \in T_0$ and $f(n+1) < f(g(n)+2) < g(n+1)$ for all n. Finally, suppose that $m, n \in \omega$ are such that $g(n) \geqslant f(m)$. Then $g(n) \geqslant f(m) \geqslant m$, and so $g(n)+2 \geqslant m+2$. Now

$$f(m+2) \leqslant f(g(n)+2) \leqslant \max(T_{g(n)+2}) < g(n+1),$$

and we are finished. □

The argument for the next lemma follows that of [4, Proposition 4.4].

Lemma 3.7. *If $f : \mathcal{A} \to \omega$ corresponds to a proper $\mathcal{R}$-filtration of Boolean algebra $\mathcal{A}$, then there exists a sequence $\{a_n\}_{n\in\omega}$ for which $a_n \wedge a_m = 0$ whenever $m \neq n$ and such that $f(a_0) < f(a_1) < \cdots$.*

Proof. Let $\mathcal{L} = \{a \in \mathcal{A} \mid f(\downarrow a) \text{ is unbounded in } \omega\}$, where $\downarrow a$ denotes the set of elements in $\mathcal{A}$ below a. We know that $1 \in \mathcal{L}$ and that if $a \in \mathcal{L}$ and $a' \leqslant a$, then either a' or $a - a'$ is in $\mathcal{L}$. Let $c_0 = 1$ and select $a_0 \in \mathcal{A}$ such that $c_1 = c_0 - a_0 \in \mathcal{L}$. Suppose that we have selected disjoint $a_0, \ldots, a_n \in \mathcal{A}$ as well as decreasing $c_0, \ldots, c_{n+1} \in \mathcal{L}$ with $c_m = c_{m+1} \vee a_m$ and $c_{m+1} \wedge a_m = 0$ and $f(a_0) < f(a_1) \cdots < f(a_n)$. Select $a'_{n+1} \leqslant c_{n+1}$ such that $f(a'_{n+1}) \geqslant \max(\{f(a_n), f(c_{n+1})\})+2$. Notice that $f(c_{n+1})+2 \leqslant f(a'_{n+1}) \leqslant \max(\{f(c_{n+1}), f(c_{n+1} - a'_{n+1})\}) + 1$, and so $f(c_{n+1} - a'_{n+1}) + 1 \geqslant f(a'_{n+1})$ and $f(c_{n+1} - a'_{n+1}) \geqslant f(a_n) + 1$. Thus $f(a'_{n+1}), f(c_{n+1} - a'_{n+1}) > f(a_n)$. If $c_{n+1} - a'_{n+1} \in \mathcal{L}$, then let $a_{n+1} = a'_{n+1}$ and $c_{n+2} = c_{n+1} - a'_{n+1}$, else $a'_{n+1} \in \mathcal{L}$ and we let $c_{n+2} = a'_{n+1}$ and $a_{n+1} = c_{n+1} - a'_{n+1}$. Now it is clear that the produced sequence $\{a_n\}_{n\in\omega}$ consists of disjoint elements and $f(a_0) < f(a_1) < \cdots$. □

For the following, cf. [3, Lemma 3].

Lemma 3.8. *If $\mathrm{cof}(LM) = \aleph_1$, then for every countably infinite Boolean algebra $\mathcal{A}$ there exists a family of sequences $\{a_n^\zeta\}_{n\in\omega,\zeta<\aleph_1}$ in $\mathcal{A}$ such that*

(1) *$a_n^\zeta \wedge a_m^\zeta = 0$ whenever $\zeta < \aleph_1$ and $n \neq m$; and*
(2) *for every proper $\mathcal{R}$-filtration f of $\mathcal{A}$ there exists $\zeta < \aleph_1$ for which $f(a_n^\zeta) > n$ for all $n \in \omega$.*

Proof. Since $\mathcal{A}$ is countably infinite, and finitely generated Boolean algebras are finite, we can write $\mathcal{A}$ as the union of a strictly increasing chain $A_0 \subsetneq A_1 \subsetneq \cdots$ of finite Boolean subalgebras. By $\mathrm{cof}(LM) = \aleph_1$ we select a subset $\{T_\theta\}_{\theta<\aleph_1} \subseteq \mathcal{CH}$ such that $\omega^\omega = \bigcup_{\theta<\aleph_1} T_\theta$. For each T_θ select a function g_θ as in Lemma 3.6.

We notice that if f is a proper $\mathcal{R}$-filtration of $\mathcal{A}$ there exist $\theta_0, \theta_1 < \aleph_1$ such that both of the following hold for all $n \in \omega$:

(a) $g_{\theta_0}(n) > f(b)$ for every $b \in A_n$;

(b) there is an antichain $b_{n,0},\dots,b_{n,2n}\in A_{g_{\theta_1}(n+1)}$ such that

$$g_{\theta_0}(g_{\theta_1}(n))+4(n+1)+1<f(b_{n,0})<\cdots<f(b_{n,2n}).$$

To see this we select a strictly increasing $h_0\in\omega^\omega$ such that $f(b)<h_0(n)$ for each $b\in A_n$. Since $h_0\in\omega^\omega=\bigcup_{\theta<\aleph_1}T_\theta$, we select θ_0 such that $h_0\in T_{\theta_0}$. Then g_{θ_0} satisfies property (a) since $g_{\theta_0}(n)>h_0(n)$ for all $n\in\omega$. Select a sequence $\{a_n\}_{n\in\omega}$ as in Lemma 3.7. Define $h_1\in\omega^\omega$ by first letting $h_1(0)=0$ and then selecting $a_{0,0}$ in $\{a_n\}_{n\in\omega}$ for which $g_{\theta_0}(h_1(0))+5<f(a_{0,0})$. Suppose we have already selected $h_1(0),\dots,h_1(m)$ and $\{a_{r,i}\}_{0\leqslant r\leqslant m,0\leqslant i\leqslant 2r}$ such that for each $0\leqslant r<m$ we have

$$g_{\theta_0}(h_1(r))+4(r+1)+1<f(a_{r,0})<f(a_{r,1})<\cdots<f(a_{r,2r})$$

and $a_{r,0},a_{r,1},\dots,a_{r,2r}\in A_{h_1(r+1)}$, and so that

$$g_{\theta_0}(h_1(m))+4(m+1)+1<f(a_{m,0})<f(a_{m,1})<\cdots<f(a_{m,2m}).$$

Select $h_1(m+1)$ such that $a_{m,0},\dots,a_{m,2m}\in A_{h_1(m+1)}$, and select further elements $a_{m+1,0},\dots,a_{m+1,2(m+1)}$ among $\{a_n\}_{n\in\omega}$ so that

$$g_{\theta_0}(h_1(m+1))+4(m+2)+1<f(a_{m+1,0})<\cdots<f(a_{m+1,2(m+1)}).$$

Such an h_1 is obviously strictly increasing and $h_1\in T_{\theta_1}$ for some $\theta_1<\aleph_1$.

We know that for each $n\in\omega$ there exists a maximal $m_n\in\omega$ such that $h_1(m_n)\leqslant g_{\theta_1}(n)$ and certainly $m_n\geqslant n$; moreover, by Lemma 3.6 we know $m_{n+1}\geqslant m_n+2$. We select the antichains $b_{n,0},\dots,b_{n,2(n+1)}$ for each $n\in\omega$ by letting $b_{n,i}=a_{m_n+1,i}$. Thus (a) and (b) are both satisfied.

Formally setting $A_{g_{\theta_1}(-1)}=\varnothing$ we know that the proper $\mathcal{R}$-filtration f is an element of the set

$$X_{\theta_0,\theta_1}=\prod_{n\in\omega}(g_{\theta_0}(g_{\theta_1}(n))^{A_{g_{\theta_1}(n)}\setminus A_{g_{\theta_1}(n-1)}}.$$

Each set $\omega^{A_{g_{\theta_1}(n)}\setminus A_{g_{\theta_1}(n-1)}}$ is countably infinite and can therefore be bijected with ω. This bijection extends to a bijection of ω^ω with $\prod_{n\in\omega}\omega^{A_{g_{\theta_1}(n)}\setminus A_{g_{\theta_1}(n-1)}}$. Since $\operatorname{cof}(LM)=\aleph_1$ there exists a covering of cardinality $\aleph_1$ of $\prod_{n\in\omega}\omega^{A_{g_{\theta_1}(n)}\setminus A_{g_{\theta_1}(n-1)}}$ by sets of form $S^\eta=\prod_{n\in\omega}S^\eta_n$, where $S^\eta_n\in[\omega^{A_{g_{\theta_1}(n)}\setminus A_{g_{\theta_1}(n-1)}}]^{\leqslant n+1}$. Let $J_{\theta_0,\theta_1,\eta}$ denote the set of all proper $\mathcal{R}$-filtrations of $\mathcal{A}$ which are in $S^\eta\cap X_{\theta_0,\theta_1}$ and satisfy (a) and (b) for the parameters θ_0,θ_1.

There are only $\aleph_1$-many choices for θ_0,θ_1, and each X_{θ_0,θ_1} can be covered by $\aleph_1$-many sets S^η. Therefore it is now sufficient to construct a sequence $\{a_n\}_{n\in\omega}$ for which

(1) $a_n\wedge a_m=0$ whenever $n\neq m$; and
(2) $f(a_n)>n$ for any $f\in J_{\theta_0,\theta_1,\eta}$.

Let $\{f_i: A_{g_{\theta_1}(n)}\setminus A_{g_{\theta_1}(n-1)}\to g_{\theta_1}(g_{\theta_0}(n))\}_{0\leqslant i\leqslant n}$ be the set of restrictions $f\restriction A_{g_{\theta_1}(n)}\setminus A_{g_{\theta_1}(n-1)}$, where $f\in J_{\theta_0,\theta_1,\eta}$. We inductively define a set of elements $\{d^i_n\}_{0\leqslant i\leqslant n}\subseteq A_{g_{\theta_1}(n)}$ such that for all $j\leqslant i$ we have $f_j(d^i_n)>g_{\theta_0}(g_{\theta_1}(n-1))+4n-2i$.

For $i=0$ we can select by (b) an element $d^0_n\in A_{g_{\theta_1}(n)}$ for which $f_0(d^0_n)>g_{\theta_0}(g_{\theta_1}(n-1))+4n+1$. Suppose that we have selected $d^i_n\in A_{g_{\theta_1}(n)}$, where $i<n$, such that for all $0\leqslant j\leqslant i$ we have $f_j(d^i_n)>g_{\theta_0}(g_{\theta_1}(n))+4n-2i$. If also

$$f_{i+1}(d^i_n)>g_{\theta_0}(g_{\theta_1}(n-1))+4n-2(i+1),$$

then we set $d_n^{i+1} = d_n^i$. Else we have

$$f_{i+1}(d_n^i) \leqslant g_{\theta_0}(g_{\theta_1}(n-1)) + 4n - 2(i+1).$$

By (b) we select an antichain $b_{n-1,0}, \ldots, b_{n-1,2n} \in A_{g_{\theta_1}(n)}$ such that

$$g_{\theta_0}(g_{\theta_1}(n-1)) + 4n + 1 < f_{i+1}(b_{n-1,0}) < \cdots < f_{i+1}(b_{n-1,2n}).$$

Notice that

$$\max(\{f_{i+1}(d_n^i \wedge b_{n-1,k}), f_{i+1}((d_n^i)^c \wedge b_{n-1,k})\}) \geqslant f_{i+1}(b_{n-1,k}) - 1,$$

since otherwise $f_{i+1}(b_{n-1,k}) = f_{i+1}((d_n^i \wedge b_{n-1,k}) \vee ((d_n^i)^c \wedge b_{n-1,k})) \leqslant f_{i+1}(b_{n-1,k}) - 1$. Thus we may select $d \in \{d_n^i, (d_n^i)^c\}$ for which $f_{i+1}(d \wedge b_{n-1,k}) \geqslant f_{i+1}(b_{n-1,k}) - 1$ for $n+1$ elements of $\{0, \ldots, 2n\}$ by the pigeon hole principle. Let $K \subseteq \{0, \ldots, 2n\}$ denote the set of all k for which $f_{i+1}(d \wedge b_{n-1,k}) \geqslant f_{i+1}(b_{n-1,k}) - 1$. Letting $e_k = d \wedge b_{n-1,k}$ for each $k \in K$ we have $f_{i+1}(e_k) \geqslant f_{i+1}(b_{n-1,k}) - 1 > g_{\theta_0}(g_{\theta_1}(n-1)) + 4n$. This means that for all $k \in K$ we have

$$f_{i+1}(d^c \vee e_k) \geqslant g_{\theta_0}(g_{\theta_1}(n-1)) + 4n - 2(i+1) + 1,$$

for otherwise we would have

$$\begin{aligned} g_{\theta_0}(g_{\theta_1}(n-1)) + 4n &< f_{i+1}(e_k) \\ &= f_{i+1}((d^c \vee e_k) \wedge d) \\ &\leqslant g_{\theta_0}(g_{\theta_1}(n-1)) + 4n - 2(i+1) + 2 \\ &\leqslant g_{\theta_0}(g_{\theta_1}(n-1)) + 4n. \end{aligned}$$

Next we notice that for every $0 \leqslant j \leqslant i$ there is at most one $k \in K$ for which

$$f_j(d^c \vee e_k) \leqslant g_{\theta_0}(g_{\theta_1}(n-1)) + 4n - 2(i+1),$$

for if distinct $k, k' \in K$ satisfied this inequality we would have

$$\begin{aligned} g_{\theta_0}(g_{\theta_1}(n-1)) + 4n - 2(i+1) + 1 &\geqslant f_j((d^c \vee e_k) \wedge (d^c \vee e_{k'})) \\ &= f_j(d^c) \\ &> g_{\theta_0}(g_{\theta_1}(n-1)) + 4n - 2i - 1, \end{aligned}$$

which is absurd. Thus by the pigeonhole principle, since $i < n$, there exists some $k \in K$ for which $f_j(d^c \vee e_k) > g_{\theta_0}(g_{\theta_1}(n-1)) + 4n - 2(i+1)$ for all $0 \leqslant j \leqslant i+1$, and we let $d_n^{i+1} = d^c \vee e_k$. The construction of the d_n^i is now complete.

Letting $\{d_n\}_{n \geqslant 1}$ be given by $d_n = d_n^n$ we notice that for every $f \in J_{\theta_0,\theta_1,\eta}$ we have $f(d_n) = f(d_n^n) > g_{\theta_0}(g_{\theta_1}(n-1)) + 2n$ for each $n \geqslant 1$. Thus letting $c_n = d_{n+1}$ we get for all $f \in J_{\theta_0,\theta_1,\eta}$

$$g_{\theta_0}(g_{\theta_1}(n+1)) > f(c_n) > g_{\theta_0}(g_{\theta_1}(n)) + 2(n+1)$$

and

$$g_{\theta_0}(g_{\theta_1}(n+1)) \geqslant f(c_n^c) \geqslant g_{\theta_0}(g_{\theta_1}(n)) + 2(n+1),$$

since $c_n \in A_{g_{\theta_1}(n+1)}$.

For each $n \in \omega$ let $c_n^0 = c_n$ and $c_n^1 = c_n^c$. We define a sequence $n_0 < n_1 < \cdots$ of natural numbers, a sequence σ of 0s and 1s, as well as a sequence of subsets $\omega \supseteq Z_0 \supseteq Z_1 \supseteq \cdots$. Let $n_0 = 0$. Notice that it is either the case that there are infinitely many k for which

$$f(c_{n_0} \wedge c_k) \geqslant f(c_k) - 1 \text{ for all } f \in J_{\theta_0,\theta_1,\eta}$$

or infinitely many k for which

$$f(c_{n_0}^c \wedge c_k) \geqslant f(c_k) - 1 \text{ for all } f \in J_{\theta_0,\theta_1,\eta}.$$

Select $\sigma(0)$ so that for infinitely many $k \in \omega$ we have $f(c_{n_0}^{\sigma(0)} \wedge c_k) \geqslant f(c_k) - 1$ for $f \in J_{\theta_0,\theta_1,\eta}$, and let

$$Z_0 = \{k > n_0 \mid f(c_{n_0}^{\sigma(0)} \wedge c_k) \geqslant f(c_k) - 1 \text{ for all } f \in J_{\theta_0,\theta_1,\eta}\}.$$

Let $n_1 = \min(Z_0)$. Select $\sigma(1)$ so that the set

$$Z_1 = \{k > n_1, k \in Z_0 \mid f(c_{n_0}^{\sigma(0)} \wedge c_{n_1}^{\sigma(1)} \wedge c_k) \geqslant f(c_k) - 2 \text{ for all } f \in J_{\theta_0,\theta_1,\eta}\}$$

is infinite. Continuing in this manner we construct a sequence $l_m = c_{n_0}^{\sigma(0)} \wedge \cdots \wedge c_{n_m}^{\sigma(m)}$ in $\mathcal{A}$ such that $f(l_m) \geqslant f(c_{n_m}) - m - 1$, $l_m \geqslant l_{m+1}$, and $l_m \in A_{g_{\theta_1}(n_m+1)}$. Since

$$\begin{aligned} f(l_m) &\geqslant f(c_{n_m}) - m - 1 \\ &\geqslant f(c_{n_m}) - n_m - 1 \\ &> g_{\theta_0}(g_{\theta_1}(n_m)) \\ &> f(l_{m-1}) \end{aligned}$$

for all $f \in J_{\theta_0,\theta_1,\eta}$ we get that

$$\begin{aligned} f(l_m - l_{m+1}) &\geqslant f(l_{m+1}) - 1 \\ &> f(c_{n_{m+1}}) - m - 2 \\ &> g_{\theta_0}(g_{\theta_1}(n_{m+1})) + 2(n_{m+1} + 1) - m - 2 \\ &\geqslant g_{\theta_0}(g_{\theta_1}(n_{m+1})) + 2(m+2) - m - 2 \\ &> m. \end{aligned}$$

Thus letting $a_m = l_m - l_{m+1}$ we are done. □

The construction for Proposition 3.5 now follows that used for [3, Theorem 1] with almost no alteration. For completeness we provide the construction and proof below.

Proof of Proposition 3.5. As $\mathrm{cof}(LM) = \aleph_1$ we have by Lemma 3.6 a set $\{g_\theta\}_{\theta<\aleph_1}$ of strictly increasing functions $g_\theta : \omega \to \omega$ such that for each $f : \omega \to \omega$ there is some $\theta < \aleph_1$ for which $f(n) < g_\theta(n)$ for all $n \in \omega$. Let $\{X_m\}_{m\in\omega}$ be a partition of ω into infinite pairwise disjoint sets. For each $\theta < \aleph_1$ we let $g'_\theta(m) = \min(X_m \cap (g_\theta(m), \infty))$. Given any sequence $\overline{a} = \{a_n\}_{n\in\omega}$ of pairwise disjoint subsets of ω we let

$$(\overline{a})^m = \bigcup_{n\in X_m} a_n$$

and

$$(\overline{a})^\theta = \bigcup_{m\in\omega} a_{g'_\theta(m)}.$$

Moreover, we let

$$F(\overline{a}) = \{(\overline{a})^m \mid m \in \omega\} \cup \{(\overline{a})^\theta \mid \theta < \aleph_1\}.$$

The Boolean algebra $\mathcal{A}$ will be constructed by induction over the ordinals less than $\aleph_1$. Let $\mathcal{A}_0$ be a Boolean algebra on ω of cardinality $\aleph_1$. Whenever $\epsilon < \aleph_1$ is a limit ordinal we let $\mathcal{A}_\epsilon = \bigcup_{\delta<\epsilon} \mathcal{A}_\delta$. Construct $\mathcal{A}_{\delta+1}$ from $\mathcal{A}_\delta$ by letting $\mathcal{A}_\delta = \{b_\gamma\}_{\gamma<\aleph_1}$ be an enumeration, and for each $\omega \leqslant \alpha < \aleph_1$ let $\mathcal{A}_{\delta,\alpha}$ be the Boolean subalgebra generated by $\{b_\gamma\}_{\gamma<\alpha}$. Since $\mathcal{A}_{\delta,\alpha}$ is countably infinite we select, by Lemma 3.8, sequences $\{\overline{a}^{\zeta,\alpha}\}_{\zeta<\aleph_1}$ such that $a_n^{\zeta,\alpha} \wedge a_m^{\zeta,\alpha} = 0$ when $m \neq n$, and for any proper $\mathcal{R}$-filtration f of $\mathcal{A}_{\delta,\alpha}$ there exists $\zeta < \aleph_1$ for which $f(a_n^{\zeta,\alpha}) > n$ for all $n \in \omega$. Let $\mathcal{A}_{\delta+1}$ be the Boolean algebra generated by $\mathcal{A}_\delta \cup \bigcup_{\omega \leqslant \alpha < \aleph_1, \zeta < \aleph_1} F(\overline{a}^{\zeta,\alpha})$. Let $\mathcal{A} = \bigcup_{\delta<\aleph_1} \mathcal{A}_\delta$.

We check that $\mathcal{A}$ is as required. Certainly the cardinality of $\mathcal{A}$ is correct. To see that $\mathcal{A}$ is of strong uncountable cofinality we suppose for contradiction that $f : \mathcal{A} \to \omega$ is a proper $\mathcal{R}$-filtration. Select elements $b_n \in \mathcal{A}$ such that $f(b_n) > n$. Then $\{b_n\}_{n\in\omega} \subseteq \mathcal{A}_\delta$ for some $\delta < \aleph_1$, and therefore $\{b_n\}_{n\in\omega} \subseteq \mathcal{A}_{\delta,\alpha}$ for some $\alpha < \aleph_1$. The restriction $f \upharpoonright \mathcal{A}_{\delta,\alpha}$ is therefore a proper $\mathcal{R}$-filtration of $\mathcal{A}_{\delta,\alpha}$. Letting $\{\overline{a}^{\zeta,\alpha}\}_{\zeta<\aleph_1}$ be the sequence selected for $\mathcal{A}_{\delta,\alpha}$, we know for some $\zeta < \aleph_1$ that $f(a_n^{\zeta,\alpha}) > n$ for all $n \in \omega$.

Now $F(\overline{a}^{\zeta,\alpha}) \subseteq \mathcal{A}_{\delta+1} \subseteq \mathcal{A}$. Select $\theta < \aleph_1$ for which $f((\overline{a}^{\zeta,\alpha})^m) + m + 1 < g_\theta(m)$ for all $m \in \omega$. Now $f((\overline{a}^{\zeta,\alpha})^\theta) = m$ for some $m \in \omega$. We notice that $(\overline{a}^{\zeta,\alpha})^\theta \cap (\overline{a}^{\zeta,\alpha})^m = a_{g'_\theta(m)}$, whence

$$\begin{aligned} g'_\theta(m) &< f(a_{g'_\theta(m)}) \\ &\leqslant \max(\{m, f((\overline{a}^{\zeta,\alpha})^m)\}) + 1 \\ &< g_\theta(m) \\ &< g'_\theta(m), \end{aligned}$$

which is a contradiction. □

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