THE HANF NUMBERS OF STATIONARY LOGIC II:
COMPARISON WITH OTHER LOGICS

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Aug 15, 1991

ABSTRACT. We show the ordering of the Hanf number of $\mathcal{L}_{\omega,\omega}(wo)$, (well ordering) $\mathcal{L}^c_{\omega,\omega}$ (quantification on countable sets), $\mathcal{L}_{\omega,\omega}(aa)$ (stationary logic) and second order logic, has no more restraints provable in ZFC than previously known (those independence proofs assume $CON(ZFC$ only). We also get results on corresponding logics for $\mathcal{L}_{\lambda,\mu}$.

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§0 Introduction.

The stationary logic, denoted by $\mathcal{L}(aa)$ was introduced by Shelah [Sh 43]. Barwise, Kaufman and Makkai [BKM] make a comprehensive research on it, proving for it the parallel of the good properties of $\mathcal{L}(Q)$. There has been much interest in this logic, being both manageable and strong, see [K] and [Sh 199].

Later some properties indicating its afinity to second order logic were discovered. It is easy to see that coutable cofinality logic is a sublogic of $\mathcal{L}(aa)$. By [Sh 199], for pairs φ, ψ of formulas in $\mathcal{L}_{\omega,\omega}(Q^{cf}_{\aleph_0})$, satisfying $\vdash \varphi \to \psi$ there is an interpolant in $\mathcal{L}(a,a)$. By Kaufman and Shelah [KfSh 150], for models of power $> \aleph_1$, we can express in $\mathcal{L}_{\omega,\omega}(aa)$ quantification on countable sets. Our main conclusion is (on the logics see Def 1.1 or the abstract, on h, The Hanf Numbers, see 1.2)

0.1 Theorem. *The only restiction on the Hanf numbers of $\mathcal{L}_{\omega,\omega}(wo), \mathcal{L}^c_{\omega,\omega}, \mathcal{L}_{\omega,\omega}(aa), \mathcal{L}^{II}_{\omega,\omega}$* ▮
are:
$$h\big(\mathcal{L}_{\omega,\omega}(wo)\big) \leq h\big(\mathcal{L}^c_{\omega,\omega}\big) \leq h\big(\mathcal{L}_{\omega,\omega}(aa)\big) \leq h\big(\mathcal{L}^{II}_{\omega,\omega}, h\big)$$
$$h(\mathcal{L}^c_{\omega,\omega}) < h(\mathcal{L}^{II}_{\omega,\omega}.$$

PROOF See **2.1** (neccessity), **2.2, 2.4, 2.5** and **3.3** (all six possibilities are consistent).

The independence results are proved assuming $CON(ZFC)$ only and the results are generalized to $\mathcal{L}_{\lambda^+,\omega}$. We do not always remember to write down the inequalities of the form $\mathcal{L}_{\lambda,\omega}(Q_1) < \mathcal{L}_{\mu,\omega}(Q_2)$. For some of the results when we generalize them to $\mathcal{L}_{\lambda^+,\omega}$ or $\mathcal{L}_{\lambda,\kappa}$ we need a stronger hypothesis. The proofs of the results on $h(\mathcal{L}_1) \leq h(\mathcal{L}_2)$ give really stronger information: we can interpret $\mathcal{L}_1$ in $\mathcal{L}_2$, usually here by using extra predicates, i.e.,every formula in $\mathcal{L}_1$ is equivalent to a formula in $\triangle(\mathcal{L}_2)$; remember $\triangle(\mathcal{L}_2)$ is defined by: $\theta \in \triangle(\mathcal{L}_2)(\tau)$ is represented by (θ_1, θ_2), $\theta_e \in \mathcal{L}_2(\tau_e), \tau_1 \cap \tau_2 = \tau$, $M \vDash \theta$ iff M can be expanded to a model of θ_1 iff M cannot be expanded to a model of θ_2 (so the requirement on (θ_1, θ_2) is strong). Note that this has two interpretation: one in which we allow τ_1, τ_2 to have new sorts hence new elements, the other in which we do not allow it. We use an

intermediate course, we allow this but the number of new elements are the power set of the old. But for $\mathcal{L}^c_{\omega,\omega} \leq \mathcal{L}_{\omega,\omega}(aa)$, for models of power $\lambda = \lambda^{\aleph_0}$ we do not need new elements.

We thank Matt Kaufman for discussions on this subject.

Notation. Let cardinals be denoted by $\lambda, \kappa, \mu, \chi$

Ordinals are denoted by $\alpha, \beta, \gamma, \xi, \zeta, i, \jmath$. δ is a limit ordinal.

Let $H(\lambda)$ be the family of sets whose transitive closure has cardinality $< \lambda$ (so for λ regular it is a model of ZFC , i.e., ZFC^- except the power set axiom: and for a strong limit a model of ZC.

Let Lévy$(\lambda, \kappa) = \{f : f$ a function from some $\alpha < \lambda$ into $\kappa\}$

Lévy $(\lambda, <\kappa) = \{f : f$ a partial function from $\lambda \times \kappa$ to $\kappa, |Dom f| < \lambda, f(\alpha, \beta) < 1 + \beta\}$.

Notation on Logics. : $\mathcal{L}$ will be a logic, τ a vocabulary (i.e., set of predicates and fuction symbols, always with a fixed arity, usually finite). We assume that $\mathcal{L}(\tau)$ is a set of formulas, each with $< Oc_1(\mathcal{L})$ free variables and $< Oc(\mathcal{L})$ predicates and function symbols; $\mathcal{L}(\tau)$ is closed under first order operations, substitutions and relativizations and $\mathcal{L}(\tau)$ is a set (with τ and the the family of variables sets)

Two formulas are isomorphic if some mapping from the set of predicates, function symbols and free variables of one onto those of another is one-to-one and map one formula to the other.

We are assuming that up to isomorphism there is a set of $\mathcal{L}$-formulas, this number is denoted by $|\mathcal{L}|$.

Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$ mean $\mathcal{L}_1(\tau) \subseteq \mathcal{L}_2(\tau)$ for every vocabulary τ.

§1 Preliminaries.

1.1 Definition.

(1) $\mathcal{L}_{\lambda,\kappa}$ is the logic in which $\wedge_{i\in I}(|I| < \lambda)$ and $(\exists x_0, \ldots, x_i \ldots)_{i\in J}(|J| < \blacksquare$
κ) are allowed, with $Oc_1(\mathcal{L}_{\lambda,\kappa}) = \kappa$ (so $\mathcal{L}_{\omega,\omega}$ is first order logic)

(2) For a logic $\mathcal{L}, \mathcal{L}(wo)$ extends $\mathcal{L}$ by allowing the quantifier $(wo\, x, y)\varphi(x, y)\blacksquare$
saying $\langle\{x : \exists\varphi(x, y)\}, \varphi(x, y)\rangle$ is well ordering

(3) For a logic $\mathcal{L}, \mathcal{L}^c = \mathcal{L}(\exists^c)$ extends $\mathcal{L}$ by allowing a monadic predicate
as free variable and the quantifier $(\exists^c X)\varphi(X)$ saying "there is a
countable set X such that $\varphi(X)$"

(4) For a logic $\mathcal{L}, \mathcal{L}(aa)$ extends $\mathcal{L}$ by allowing monadic predicates as
free variables and the quantifiers $(aaX)\varphi(X)$ saying that the collec-
tion of countable X satisfying φ contains a closed unbounded family
of countable subsets of the model

(5) For a Logic $\mathcal{L}, \mathcal{L}^{II} = \mathcal{L}(\exists^{II})$ extends $\mathcal{L}$ by allowing binary predicates
as free variables and the quantifiers $\exists R\varphi(R)$ saying there is a two-
place relation R on the model satisfying R

(6) For $Q \in \{\exists^c, aa, \exists^{II}\}, \mathcal{L}'(Q)$ is defined similarly allowing a string
$(Qx_1 \ldots Qx_i \ldots)_{i<\alpha}, |\alpha| < 0c_1(\mathcal{L})$

(7) Let $\mathcal{L}^c = \mathcal{L}(\exists^c), \mathcal{L}^{wo} = \mathcal{L}(wo), \mathcal{L}^{II} = \mathcal{L}(\exists^{II}), \mathcal{L}^{aa} = \mathcal{L}(aa)$

1.2 Definition.

(1) For a sentence ψ Let $h(\psi) = sup\{|M|^+ : M \models \psi\}$
(so it is a cardinal (or infinity)) and it is the first λ such that ψ
has no model $\geq \lambda$)

(2) For a theory $T, h(T) = h(\wedge_{\psi\in T}\psi)$

(3) For a logic $\mathcal{L}$ let $h(\mathcal{L}) = sup\{h(\psi) : h(\psi) < \infty, \psi \in \mathcal{L}(\tau)$ for some
vocabulary $\tau\}$

(4) For a logic $\mathcal{L}$ and cardinal λ let $h(\mathcal{L}, \lambda) = sup\{h(\psi) :$ for some
vocabulary τ of power $< \lambda, \psi \in \mathcal{L}(\tau), h(\psi) < \infty\}$

(5) For a logic $\mathcal{L}$ and cardinal λ let $hth(\mathcal{L}, \lambda) = sup\{h(T) :$ for some
vocabulary τ of power $< \lambda, T \subseteq \mathcal{L}(\tau), h(\tau) < \infty\}$
$hth(\mathcal{L}) = H(\mathcal{L}, \infty)$

1.3 Claim.

(1) *for every $\psi \in \mathcal{L}$ for some $\varphi \in \mathcal{L}$, $[h(\psi) < \infty \to h(\psi) < h(\varphi) < \infty]$*

(2) *$h(\mathcal{L})$ is strong limit*

(3) *If $\mathcal{L}$ is closed under $\wedge_{\alpha < \alpha_0}$ for $\alpha_0 < \lambda$ then $cf[h(\mathcal{L})] \geq \lambda$*

(4) *If the number of sentences in $\mathcal{L}$ (up to isomorphism) is $\leq \lambda$ then $cf[h(\mathcal{L})] \leq \lambda$*

1.4 Lemma. *assume $\mathcal{L}$ is a logic $\subseteq \mathcal{L}^{II}_{\omega,\omega}$ and there is a function f from Card to Card such that:*

(a) *f is definable in $\mathcal{L}^{II}_{\omega,\omega}$, i.e., the class of two sorted models $\langle \kappa, f(\kappa) \rangle$ is definable by some sentence of $\mathcal{L}^{II}_{\omega,\omega}$ or even just*

(a)⁻ *For some $\lambda^* < h(\mathcal{L}^{II}_{\omega,\omega})$ and $\varphi^* \in \mathcal{L}^{II}_{\omega,\omega}$ for $\kappa, \mu, \geq \lambda^*, \kappa < h(\mathcal{L}^{II}_{\omega,\omega})$ we have $\langle \kappa, \mu \rangle \vDash \varphi^*$ iff $\mu = f(\kappa)$*

(b) *If $\psi \in \mathcal{L}$ has a model of power $\geq \kappa$ then ψ has a model $M, \kappa \leq \|M\| \leq f(\kappa)$*

(c) *$\mathcal{L}$ is definable in $\mathcal{L}^{II}_{\omega,\omega}$ i.e., the class $\{(\psi, \tau, M) : \psi \in \mathcal{L}(\tau), M \, a \, \tau\text{-}$ model, $M \vDash \psi\}$ is definable by a sentence in $\mathcal{L}^{II}_{\omega,\omega}$*

(d) *For $\mu < h(\mathcal{L}), f(\mu) < h(\mathcal{L})$*

   $\underline{Then} \ h(\mathcal{L}) < h(\mathcal{L}^{II}_{\omega,\omega})$

PROOF Easy. Let $\psi_0 \in \mathcal{L}^{II}_{\omega,\omega}$ be such that $\lambda^* < h(\psi_0) < \infty$, where λ^*, φ^* are as in (a)⁻. We can assume $h(\psi_0) < h(\mathcal{L})$ (otherwise the conclusion is trivial). Let $\psi \in \mathcal{L}^{II}_{\omega,\omega}$ say that for some λ, μ_0:

(i) the model M is isomorphic to some $(H(\lambda), \in), \lambda$ strong limit,

(ii) for every $\kappa < \lambda, M \vDash (\exists \mu \geq \kappa)[\psi_0$ has a model of cardinality $\mu] \vee (\exists \mu \geq \kappa)[\langle \kappa, \mu \rangle \vDash \varphi^*]$

(iii) $\mu_0 < \lambda, \psi_0$ has a model of power whose cardinality is in the interval $\in (\mu_0, \lambda)$

(iv) for every $\kappa < \lambda, \kappa \geq \mu_0$, there is $\theta \in \mathcal{L}$ which has a model of cardinality in the interval (κ, λ), but for some $\kappa' \in (\kappa, \lambda)$ has no model of cardinality in the interval (κ', λ)

Now $(H(h(\mathcal{L})), \in)$ is a model of ψ and it has no models of larger cardinality.

$\boxdot_{1.4}$

We can prove similarly:

1.5 Lemma. *Suppose $\mathcal{L}_1, \mathcal{L}_2$ are logics and there is $f : Card \to Card$ such that*

(a) *for some $\lambda^* < h(\mathcal{L}_2)$ and $\varphi^* \in \mathcal{L}$ for $\kappa, \mu \geq \lambda^*$ we have: $\langle \kappa, \mu \rangle \models \varphi^*$ iff $\mu = f(\kappa)$*

(b) *if $\psi \in \mathcal{L}_1$ has a model of cardinality $\geq \kappa$ then ψ has a model $M, \kappa \leq \|M\| \leq f(\kappa)$*

(c) *$\mathcal{L}_1$ is definable in $\mathcal{L}_2$ just in the following weaker sense: for $K_1 = \{(\psi, \tau) : \psi \in \mathcal{L}_1(\tau)\}$, $K_2 = \{M, \psi, \tau) : M \models \psi, \psi \in \mathcal{L}_1(\tau)\}$ there are $\psi_e \in \mathcal{L}_2$.*

$(\forall x)[x \in K_e \Leftrightarrow$ for some λ, some expansion of $(H(\lambda), \in, x)$ satisfies $\psi_e]$ and for every x $\{\lambda :$ some expansion of $(H(\lambda), \in, x)$ satisfies $\psi_e\}$ is a bounded family of cardinals

(d) *For $\mu < h(\mathcal{L}_1), f(\mu) < h(\mathcal{L}_2)$*

(e) *$\|\mathcal{L}\| < h(\mathcal{L}_2), \mathcal{L}_{\omega,\omega}^{II} \subseteq \mathcal{L}_2$*
<u>Then</u> $h(\mathcal{L}_1) < h(\mathcal{L}_2)$

Remark. Of course if 1.5 is hypothesis holds for $\mathcal{L}_1$ (and $\mathcal{L}_2$) then the conclusion holds for $\mathcal{L}_1', \mathcal{L}_2'$ whenever $\mathcal{L}_1' \subseteq \mathcal{L}_1$ and $\mathcal{L}_2 \subseteq \mathcal{L}_2'$.

1.7 Lemma.

(1) *If $M \models \psi, \psi \in \mathcal{L}^{wo}$ then this is preserved by any forcing, this holds even for $\psi \in \mathcal{L}_{\infty,\omega}^{wo}$*

(2) *If $M \models \psi, \psi \in \mathcal{L}_{\omega,\omega}^{aa}$ then this is preserved by any $\aleph_1$-complete forcing this holds even for $\psi \in \mathcal{L}_{\infty,\omega}^{aa}$*

(3) *If $M \models \psi, \psi \in \mathcal{L}_{\omega,\omega}^c$ this is preseved by forcing not adding new countabale subsets of $|M|$ (this holds even for $\psi \in \mathcal{L}_{\infty,\omega_1})$*

(4) *If $M \models \psi, \psi \in \mathcal{L}_{\infty,\lambda}, \lambda$ regular, then this is preserved by forcing by P where P does not add sequences of ordinals of lenght $< \lambda$. If P is $\aleph_1$-complete this holds for $\psi \in \mathcal{L}_{\infty,\lambda}^{aa}$.*

(5) *Suppose V_1, V_2 are models of set theory (with the same ordinals), $V_1 \subseteq V_2$, and letting $\lambda = h(\mathcal{L})^{V_1}$ where $\mathcal{L}$ is $\mathcal{L}_{\omega,\omega}^{wo}$ or $\mathcal{L}_{\mu,\omega}^c$ or $\mathcal{L}_{\mu,\omega}^c$,*

> *(just a suitable downward Lowenheim Skolem theorem is needed).*
>
> If $\{A \subseteq \lambda : A \text{ bounded}, A \in V_1\} = \{A \subseteq \lambda : A \text{ bounded}, A \in V_2\}$
> then $h((\mathcal{L})^{V_1} = h(\mathcal{L})^{V_2}$.

PROOF Left to the reader.

§2 Independence for $\mathcal{L}^c_{\omega,\omega}, \mathcal{L}^{II}_{\omega\omega}$.

In this section we shall deal with the indepedence of the cases where $h(\mathcal{L}^{wo}_{\omega,\omega}) = h(\mathcal{L}^c_{\omega,\omega})$.

2.1 Lemma.

(1) *For any logic* $\mathcal{L}$: $h(\mathcal{L}(wo)) \leq h(\mathcal{L}^c) \leq h(\mathcal{L}(aa)) \leq h(\mathcal{L}^{II})$

(2) *For any logic* $\mathcal{L}$ *we have* $h(\mathcal{L}^c_{\omega\omega}) < h(\mathcal{L}^{II}_{\omega,\omega})$

(3) *For any logic* $\mathcal{L}$ *we have* $h(\mathcal{L}^c_{\lambda^+,\omega}) < h(\mathcal{L}^c_{\lambda^+,\omega})$, *moreover:*

   if $\lambda < h(\mathcal{L}^{II}_{\mu,\omega})$ *then* $h(\mathcal{L}^c_{\lambda^+,\omega}) < h(\mathcal{L}^{II}_{\mu,\omega})$

PROOF

(1) By Kaufman and Shelah [KfSh 150, Theorem 4.1]; only $\mathcal{L} = \mathcal{L}_{\omega,\omega}$ is discussed there, but it makes no difference, the non trivial part is $h(\mathcal{L}^c) \leq h(\mathcal{L}^{aa})$;

(2) See [KfSh 150];

(3) Use **1.5** for the function $f : f(\kappa) = (\kappa^{\aleph_0})^+$

2.2 Lemma.

(1) *If* $V = L$ *then* $h(\mathcal{L}^{wo}_{\omega,\omega}) = h(\mathcal{L}^c_{\omega,\omega}) < h(\mathcal{L}^{aa}_{\omega,\omega}) = h(\mathcal{L}^{II}_{\omega,\omega})$

(2) *If* $V = L$, *then for any logic* $\mathcal{L}$, $h(\mathcal{L}^{wo}) = h(\mathcal{L}^c) \leq h(\mathcal{L}^{aa}) = h(\mathcal{L}^{II})$.

PROOF

(1) See [KfSh 150]

(2) Same proof.

2.3 Fact. *For a regular cardinal λ and $\psi \in \mathcal{L}^{aa}_{\lambda,\lambda}$ the following are equivalent:*

(i) *for every μ large enough $\Vdash_{\text{Lévy}(\lambda,\mu)}$ "ψ has a model of power λ"*

(ii) *for some λ-complete forcing Q we have: $\Vdash_Q$ "ψ has a model of power $\geq$ ▉ λ".*

PROOF Easy; (i)$\Rightarrow$ (ii): as Lévy(λ, μ) is a λ-complete forcing notion, (i) is a particular case of (ii). (ii)$\Rightarrow$(i) let Q be a λ-complete forcing notion such that $\Vdash_Q$ "ψ has a model of cardinality $\geq \lambda$". Let μ be such that $\mu > |Q|, \Vdash_Q$ "ψ has a model of cardinality $\geq \lambda$ but $\leq \mu$" and $\mu = \mu^\lambda$. In $(V^Q)^{\text{Lévy}(\lambda,\mu)} \psi$ has a model of cardinality λ by **1.7(4)**.

But $(V^Q)^{\text{Lévy}(\lambda,\mu)}$ is $V^{\text{Lévy}(\lambda,\mu)}$. (see e.g. [Kun]).

2.3A Notation. Let $\mu_0[\psi, \lambda]$ be the first cardinal μ satisfying **2.3(i)**, if one exists, and λ otherwise.

2.4 Lemma.

(1) *In some forcing extension of L, $h(\mathcal{L}^{wo}_{\omega,\omega}) = h(\mathcal{L}^{c}_{\omega,\omega}) < h(\mathcal{L}^{aa}_{\omega,\omega}) < h(\mathcal{L}^{II})$*

(2) *Moreover for $\lambda < h(\mathcal{L}^{II}_{\mu,\kappa})$, we have $h(\mathcal{L}^{aa}_{\lambda,\lambda}) < h(\mathcal{L}^{II}_{\mu,\kappa})$*

2.4A Remark. If we want to have: $\lambda < h(\mathcal{L}^{aa}_{\mu,\lambda}) \Rightarrow h(\mathcal{L}^{c}_{\lambda,\omega}) < h(\mathcal{L}^{aa}_{\mu,\omega})$, we should define $\lambda_{i+1} = h(\mathcal{L}^{c}_{\mu_i^+,\omega})^+$.

PROOF Start with $V = L$. Let $\psi^* \in \mathcal{L}^{aa}_{\omega,\omega}$ a sentence such that $h(\mathcal{L}^{c}_{\omega,\omega}) < h(\psi^*) < \infty$ be chosen later. Let $\lambda_0 > h(\psi^*)$ be regular, $\lambda_\circ < h(\mathcal{L}^{II}_{\omega,\omega})$. We define an iterated forcing $\langle P_i, \underset{\sim}{Q}_j : i \leq \infty, j < \infty \rangle$ and cardinals λ_i such that:

(a) the iteration is with set support (so P_∞ is a class forcing)

(b) λ_i is regular cardinal

(c) $\lambda_i \geq \sum_{j<i} \lambda_j$, and λ_i is the first regular cardinal $\geq \sum_{j<i}(\lambda_j + \mu_j)^+$ (when $i > 0$)

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(d) $Q_i (\in V^{P_i})$ is λ_i-complete

(e) Let $\{\psi^i_\alpha : \alpha < \lambda_i\}$ be the set of all $\mathcal{L}^{aa}_{\lambda_i, \lambda_i}$ sentences (up to isomor-
phism) in V^{P_i}.

We define in V^{P_i}, Q_i to be Lévy(λ_i, μ_i) where μ_i is the successor of $sup\{\mu_0[\psi, \lambda_i]^{V^{P_i}} : $ ■
$\psi \in \mathcal{L}_{\lambda_i,\lambda_i}\}$ and so $\lambda_{i+1} = \mu_i^+$.

Our model is V^{P_∞}. Clearly the λ_i are not collapsed (as well as limits
of λ_i and $\chi < \lambda_0$) and other successor cardinals $\geq \lambda_0$ are collapsed. So in
V^{P_∞}, for regular $\chi \geq \lambda_0$, if $\psi \in \mathcal{L}^{aa}_{\chi,\chi}$ has a model of cardinality $\geq \chi$ then
it has a model of cardinality χ. As clearly $h(\mathcal{L}^{II}_{\omega,\omega}) > \lambda_0$, we get by **1.4**
$h(\mathcal{L}^{aa}) < h(\mathcal{L}^{II})$ (as well as (**2**)).

By the Lowenheim Skolem theorem, using **1.7(5)** for $\psi \in \mathcal{L}^{wo}_{\omega,\omega}$ or $\psi \in$
$\mathcal{L}^c_{\omega,\omega}, h(\psi)$ does not change (being ∞ or $< \lambda_0$) hence (in V^{P_∞}) $h(\mathcal{L}^{wo}_{\omega,\omega}) =$
$h(\mathcal{L}^{wo}_{\omega,\omega})^V ; h(\mathcal{L}^c_{\omega,\omega}) = h(\mathcal{L}^c_{\omega,\omega})^V$. Hence (in V^{P_∞}) $h(\mathcal{L}^{wo}_{\omega,\omega}) = h(\mathcal{L}^c_{\omega,\omega})$ as this
holds in L.

We still have to choose $\psi^* \in \mathcal{L}^{aa}_{\omega,\omega}$ and prove that in V^{P_∞} we have
$h(\mathcal{L}^c_{\omega,\omega}) < h(\mathcal{L}^{a,a}_{\omega,\omega})$. There is $\psi^* \in \mathcal{L}^{aa}_{\omega,\omega}, L \vDash$ "$h(\mathcal{L}^c_{\omega,\omega}) < h(\psi^*) < \infty$" (by
2.2).

Clearly for any such $\psi^*, V^{P_\infty} \vDash$ "$h(\mathcal{L}^c_{\omega,\omega}) < h(\psi^*)$" (as no new subset
of $h(\psi^*)$ is added), but we need also $V^{P_\infty} \vDash$ "$h(\psi^*) < \infty$"; but checking
the sentneces produced in [KfSh 150] proof of **Theorem 4.3** (for proving
$L \vDash h(\mathcal{L}^{aa}) = h(\mathcal{L}^{II})$), they are like that. So $V^{P_\infty} \vDash$ "$h(\mathcal{L}^c_{\omega,\omega}) < h(\mathcal{L}^{aa}_{\omega,\omega})$".

2.5 Lemma.

(1) *In some forcing expresion of L we have* $h(\mathcal{L}^{wo}_{\omega,\omega}) = h(\mathcal{L}^c_{\omega,\omega}) =$
$h(\mathcal{L}^{aa}_{\omega,\omega}) < h(\mathcal{L}^{II}_{\omega,\omega})$

(2) *In fact for any logic $\mathcal{L}$ we have $h(\mathcal{L}^{wo}) = h(\mathcal{L}^c) = h(\mathcal{L}^{aa})$*

(3) *For $\lambda < h(\mathcal{L}^{II}_{\mu,\kappa})$ then, $h(\mathcal{L}^{aa}_{\lambda,\lambda}) = h(\mathcal{L}^{II}_{\mu,\kappa})$.*

PROOF We start with $V = L$. We define a (full set support) iteration,
$\bar{Q} = \langle P_i, \underset{\sim}{Q_i} : i$ an ordinal $\rangle (\underset{\sim}{Q_i} - a \ P_i$ name) and cardinals λ_i such that

(a) λ_i is regular $\geq \aleph_1 + |P_i|$ for i limit $\lambda_i = (\sum_{j<i} \lambda_i)^+$

(b) $\underset{\sim}{Q_i}$ is λ_i-complete

(c) if i is even, $G_i \subseteq \underset{\sim}{P_i}$ the generic set (remember $Q_i \in V^{P_i}$) then let
the set of elements of P_i be listed as $\{p_\alpha^i : \alpha < \lambda_i\}$, and $\underset{\sim}{Q_i}$ will be
the product of the Lévy collapses of $\aleph_{\lambda_i\omega+4\alpha+2+m}$ to $\aleph_{\lambda_i\omega+4\alpha+1+m}$
for $\alpha < \lambda_i$ such that: $[p_\alpha^i \in G_i \Rightarrow m = 0]$ and $[p_\alpha^i \notin G_i \Rightarrow m = 1]$.
Let $\lambda_{i+1} = \aleph_{\lambda_i\omega+\lambda_i+1}$

(d) if i is odd, let $\{\psi_\alpha^i : \alpha < \lambda_i\}$ list all sentneces of $\mathcal{L}_{\lambda_i,\lambda_i}^{aa}$ in a rich
enough vocabulary of cardinality λ_i). For each α if there is a λ_i-
complete forcing notion Q (which is a set) and (in V^{P_i}) $\Vdash_Q$ "there
is a model of ψ_α^i of cardinality $\geq \lambda_i$" then let μ_α^i be such that
$\Vdash_{\text{Lévy}(\lambda_i,\mu_\alpha^i)}$ "ψ_α^i has a model of cardinalityλ_i"; otherwise $\mu_\alpha^i = \lambda_i$.
    Note that μ_α^i exists by **2.3**.
    Let $Q_i = \text{Lévy}(\lambda_i, < \lambda_{i+1})$ where $\lambda_{i+1} = (\lambda_i + \sum \alpha < \lambda_i \mu_\alpha^i)^{++}$.

Let $G_\infty \subseteq P_\infty$ be generic over V and $V[G_\infty]$ be our model. Note in $V[G_\infty]$,

(*) $[i \, \text{odd} \Rightarrow \lambda_{i+1} = \lambda_i^+]$

$[i \, \text{even} \Rightarrow \lambda_{i+1} = \lambda_i^{+(\lambda_i\omega+1)}]$

$[i \, \text{limit} \to \lambda_i = (\sum_{j<i} \lambda_j)^+]$.

For $\lambda = \lambda_{2j+1}$, if $\psi \in \mathcal{L}_{\lambda,\lambda}^{aa}$ has a model of cardinality $\geq \lambda$ then it
has a model of cardinality λ (by **2.3** + **1.7(4)**). By (*) we deduce that
$V^{P_\infty} \vDash$ "if $\psi \in \mathcal{L}_{\lambda,\chi}^{aa}$ has a model of cardinality $> \lambda$ then it has a model
$M, \lambda < ||M|| < \aleph_{\lambda^+}$".

So **1.5** is applicable to show $h(\mathcal{L}_{\omega,\omega}^{aa}) < h(\mathcal{L}_{\omega,\omega}^{II})$ (and by **1.6** and **1.7**)
also **2.5(3)** holds.

Why $h(\mathcal{L}_{\omega,\omega}^{wo}) = \mathcal{L}_{\omega,\omega}^{a,a}$?. Let ψ^* describe $(L_\lambda \in, G_\infty \cap \cup_{i<\delta} P_i)$.

If $M \vDash \psi^*$, then for some α and $G, M \cong (L_\alpha, \in, G)$, so without loss of
generality equality holds. Now if $\lambda < |\alpha|, M \vDash$ "λ is a [regular] cardinal of
L" iff λ is a [regular] cardinal of $\mathcal{L}$. Also we know that for every ordinal ζ,
if in $L, \lambda_{2i} \leq \aleph_\zeta < \lambda_{2i+1}, \zeta$ divisible by four then forcing by P_∞ collapses
at most one of the cardinals $\aleph_{\zeta+1}, \aleph_{\zeta+2}, \aleph_{\zeta+3}, \aleph_{\zeta+4}$ of L; if $\lambda_{2i}\omega \leq \zeta <$
$\lambda_{2i}\omega + \lambda_{2i}$ then exactly one.

We assume ψ^* say so, and so when $\aleph_{\zeta+4}^L \leq |\alpha|$ the answer in M to the
question "which of $\aleph_{\zeta+1}, \aleph_{\zeta+2}, \aleph_{\zeta+3}, \aleph_{\zeta+4}$ is collapsed" is the right one. So
when $\lambda_{2i+1} < |\alpha|$, we can in M reconstruct $G_\infty \cap P_{2i}$ (see choice of Q_{2i}).

But $V^{P_\infty} \vDash$ "$\lambda_{2i+1} \leq \aleph_{\lambda_{2i}(\omega+1)+1}$ and $\lambda_{2i+2} = (\lambda_{2i+1})^+$ and for limit δ we have $\lambda_\delta = (\sum_{i<\delta} \lambda_i)^+$"

The rest is as in [KfSh 150] proof of **4.3**

§3 $h(\mathcal{L}^{wo}_{\lambda,\omega})$ is O.K. but for $h(\mathcal{L}_{\aleph_3,\omega})$ large cardinals are needed and sufficient.

In section **2** we deal with the three cases for which $h(\mathcal{L}^{wo}_{\omega,\omega}) = h(\mathcal{L}^c_{\omega,\omega})$. Here we deal with the three cases where $h(\mathcal{L}^{wo}_{\omega,\omega}) < h(\mathcal{L}^c_{\omega,\omega})$. The new part is **Lemma 3.2**, and then, in **3.3** we get the desired conclusion. For dealing with $\mathcal{L}_{\lambda^+,\omega}$ we do not assume $CON(ZFC)$ alone, we assume the existence of a class of large cardinals (weaker than measurability). By **3.4** at least if $\lambda \geq \aleph_3 + (2^{\aleph_0})^+$, something of this sort is necessary.

3.1 Fact. : *The following are equivalent for $\psi \in \mathcal{L}^{wo}_{\omega,\omega}$ or even $\psi \in \mathcal{L}^{wo}_{\infty,\omega}$:*

(i) *for every μ large enough $\Vdash_{L\acute{e}vy(\aleph_0,<\mu)}$ "$h((\psi)) = \infty$"*

(ii) *for some (set) forcing notion P we have $\Vdash_P$ "$h(\psi) = \infty$".*

PROOF similar to the proof of **2.3**

3.1A Notation. Let the first μ satisfying (i) be $\mu_1(\psi)$ (and $\aleph_0$ if there is no such μ).

3.2 Lemma. $(V = L)$.

(1) *For some (set) forcing notion P*

$$\Vdash_P \text{ "}h(\mathcal{L}^{wo}_{\omega,\omega}) < h(\mathcal{L}^c_{\omega,\omega})$$

   and this is preserved by $h(\mathcal{L}^{wo}_{\omega,\omega})^+$-complete forcing".

(2) *In (1) we can use $L\acute{e}vy(\aleph_0 < \mu)$ for some $\mu > cf\mu = \aleph_0$*

(3) *We can use instead $Cohen(\mu) = \{f : f$ a finite function from μ to $(0,1)\}$. So cardinals are not collapsed*

PROOF 1) Let $\mu^* = \sup\{\mu_1(\psi) : \psi \in \mathcal{L}^{wo}_{\omega,\omega}\}$

We now define a finite support iteration $\langle P_i, \underset{\sim}{Q_n} : n < \omega\rangle$ and μ_n as follows:

$\mu_0 = \mu^*$

$Q_0 = L\acute{e}vy(\aleph_0, \mu_0)$

for $n \geq 0, \mu_{n+1}$ is $h(\mathcal{L}^{wo}_{\omega,\omega})^{V^{P_n}}$

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$$Q_n = \text{Lévy}(\aleph_0, \mu_n).$$

Let $\mu = (\sum \mu_n)$. Note that P_ω satisfies the $\mu^+ - c.c.$

Now V^{P_ω} is our model. Note

(*) $V^{P_\omega} \vDash G.C.H. + \aleph_1 = \mu^+$, and $V = L[\mathbb{R}, <]$ for any well $<$ ordering of $\mathbb{R}$.

Note that in $\mathfrak{B} = (\omega \cup \mathcal{P}(\omega))^{V^{P_\omega}}; o, +, \times, \in)$ we can define by first order formulas (representing ordinals by well ordering of ω):

(a) $\cup_n \mu_n$ (maximal countable ordinal which is a cardinal in L_{μ^+}

(b) L_{μ^+} hence $\langle \mu_n : n < \omega \rangle$ (by induction remembering the Lowenheim Sholem theorem) hence the iteration (really we can omit this as P_ω is just Lévy $(\aleph_0, < \mu)$)

(c) the set $\mathbb{R}^- =^{def} \{r \in \mathbb{R} : \text{for some } n, \text{ and } G \subseteq P_n \text{ generic over } V, r \in V[G]\}$.

   And for $r \in \mathbb{R}^-$

(d) $H_r = \{\psi \in \mathcal{L}^{wo}_{\omega,\omega} : L[r] \vDash h(\psi) < \infty\}$ as it is equal to
   $$\{\psi \in \mathcal{L}^{wo}_{\omega,\omega} : L[r] \vDash h(\psi) < \cup_n \mu_n\}.$$

[Note that $P'_n s$ are homogeneous, hence $h(\psi)$ does not depend on $G \subseteq P_n$]

So by **3.1** and the choice of μ_0, we can define in that model $\mathfrak{B}$

$H^* = \{\psi \in \mathcal{L}^{wo}_{\omega,\omega} : h(\psi)^{V^{P_\omega}} < \infty\}$

[How ? it is $\cap \{H_r : r \in \mathbb{R}^-\}$,remembering **3.1**]

Let $\lambda = h(\mathcal{L}^{wo}_{\omega,\omega})$ (in V^{P_ω}).

Now we define a sentence $\varphi \in \mathcal{L}^c_{\omega,\omega}$: it just describes $(H(\lambda), \in)$: it says

(i) enough axioms of ZFC holds

(ii) every countable bounded set of ordinals is represented

(iii) on every infinite cardinal α there is a model M_α with universe α satisfying some $\psi \in H^*$ (which we have shown is definable in any model M of φ)

So we have proved the first assertion from **3.2**. Now λ-complete forcing, preserve trivially "$h(\psi) \geq \mu$" as it preserves satisfaction for $\mathcal{L}^{wo}_{\omega,\omega}$. It preserves "$h(\psi) < \infty$" as this is equivalent to "$h(\psi) < \lambda$", the forcing adds no new model power $< \lambda$, and Lowenheim Skolem Theorem finishes the argument.

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2) We have proved it in the proof of (1)

3) A similar proof, replacing $\mu_1(\psi)$ by $\mu_1' = $ first μ such that $\Vdash_{Cohen(\mu)}$ "$h|\psi| = \infty$" if there is one $\aleph_0$ otherwise.

$\square_{3.2}$

3.3 Conclusion. *for some forcing extensions of $\mathcal{L}$:*

(1) $h(\mathcal{L}^{wo}_{\omega,\omega}) < h(\mathcal{L}^{c}_{\omega,\omega}) < h(\mathcal{L}^{aa}_{\omega,\omega}) = h(\mathcal{L}^{II}_{\omega,\omega})$

(2) $h(\mathcal{L}^{wo}_{\omega,\omega}) < h(\mathcal{L}^{c}_{\omega,\omega}) = h(\mathcal{L}^{aa}_{\omega,\omega}) < h(\mathcal{L}^{II}_{\omega,\omega})$

(3) $h(\mathcal{L}^{wo}_{\omega,\omega}) < h(\mathcal{L}^{c}_{\omega,\omega}) < h(\mathcal{L}^{aa}_{\omega,\omega}) < h(\mathcal{L}^{II}_{\omega,\omega})$.

PROOF: Combine **3.2** with §2.

3.4 Claim. $(\neg 0^{\#})$*: For $\lambda \geq \aleph_3 + (2^{\aleph_0})^+$ we have $h(\mathcal{L}^{wo}_{\lambda,\omega}) = h(\mathcal{L}^{c}_{\lambda,\omega})$.*

Remark. : The logics are essentially equivalent.

PROOF If $\psi \in \mathcal{L}^{wo}_{\lambda,\omega}$ says M is, for some $\alpha, (L_\alpha[A], \in)$ (up to isomorphism), $\alpha > 2^{\aleph_0}, A \subseteq 2^{\aleph_0}$, every subset of ω is in $L_{(2^{\aleph_0})}[A]$, and $\alpha \geq \omega_2$, and $\{\delta < \aleph_2 : cf\delta = \aleph_0 \text{ in } L_{\omega_2}[A]\} = \{\delta < \aleph_2 : cf\delta = \aleph_0\}$ then by Jensen's covering lemma $[\beta < |\alpha| \Rightarrow$ every countable subset of β is represented in the model $\square_{3.4}$

3.5 Claim. *Suppose that:*

(*) *for every χ for some μ, $\mu \to (\omega_1)^{<\omega}_{\chi}$ or even just*

(**) *for every χ for some μ, $\mu \to_{BG} (c)^{<\omega}_{\chi}$, which means: for every*
$f : [\mu]^{<\omega} \to \chi$ *for some $\langle \gamma_n : n < \omega \rangle$ for every $\alpha < \omega_1$, for some*
$Y \subseteq \mu$, Y *has order type α and $\wedge_n (\forall w \in [Y]^n)[\gamma_n = f(w)]$.*

Then for every λ, $h(\mathcal{L}^{wo}_{\lambda^+,\omega})$.

3.5A Remark.

(1) The property (**) was discoverd by Baumgartner and Galvin [BG]
such that:

$$\mu \to_{BG} (c)^{<\omega}_{\chi} \text{ iff } \mu \geq h(\mathcal{L}^{wo}_{\chi^+,\omega}).$$

(2) See [KfSh 150, 4.2] (for $\lambda = \omega$)

PROOF There is a sentence $\psi \in \mathcal{L}^{c}_{\lambda,\omega}$ such that for $\chi \leq \mu$: there is a model
$M, ||M|| = \mu, |P^M| = \lambda$, iff $(\forall \alpha < \mu)\alpha \to_{BG} (c)^{<\omega}_{\chi}$. $\qquad \qquad \square_{3.5}$

On $K = K^V$ (the core model of V) see Dodd and Jensen [DJ].

3.6 Claim. *Suppose $V = K$, and (**) (from **3.5**),then*

(1) *for every λ we have $h(\mathcal{L}^{wo}_{\lambda^+,\omega}) < h(\mathcal{L}^{c}_{\lambda^+,\omega}) < h(\mathcal{L}^{aa}_{\lambda^+,\omega}) = h(\mathcal{L}^{II}_{\lambda^+,\omega})$*

(2) *for every $\mathcal{L}$, $h(\mathcal{L}^{aa}) = h(\mathcal{L}^{II})$.*

PROOF 1) First inequality by the observation above, the second inequlity
follows from last equality **Th 2.1**, last equality see **(2)**

(note: if $cf\delta > \aleph_0$ in $\mathcal{L}^{aa}_{\omega,\omega}$ we can say for $A \subseteq \delta$ whether $\{\alpha < \delta : cf\alpha = \aleph_0, \alpha \in A\}$ is a stationary subset of δ).

2) As in [KfSh 150]

3.7 Observation. There is $\psi \in \mathcal{L}^{c}_{\omega,\omega}$ such that $M \vDash \psi$ iff M is isomorphic
to K_α for some α.

It is known see (see [BG], [DJ])

3.8 Fact. If in V there are, e.g., measurable cardinals in Card, then $K \vDash$ (**).

3.9 Claim. : *Suppose $V = K$ and (**) holds.*

For some forcing extension $V[G_\infty]$ of V, $V[G_\infty] \vDash$ (**) and for every λ, $h(\mathcal{L}^{wo}_{\lambda^+,\omega}) < h(\mathcal{L}^c_{\lambda^+,\omega}) < h(\mathcal{L}^{aa}_{\lambda^+,\omega}) < h(\mathcal{L}^{II}_{\lambda^+,\omega})$.

PROOF Similar ot **2.4(1)** except that we want to preserve (**). We define by induction on α an iterated forcing, $\langle P_i, Q_j \leq \alpha, \jmath < \alpha \rangle$ with set support and cardinals λ_i increasing such that:

  (i) $\lambda_0 = \aleph_2$

  (ii) $\lambda_\delta = (\sum_{i<\delta} \lambda_i + |P_\delta|)^+$

  (iii) if λ_i, P_i are defined, let μ_i be $\lambda_i^+ + \cup\{\mu_0[\psi, \lambda_i] : \psi \in \mathcal{L}^{aa}_{\lambda_i,\lambda_i}\}$.

$\qquad Q_i = $ Lévy (λ_i^+, μ_i^+)(in V^{P_i}) and λ_{i+1} is minimal such that

$\qquad \lambda_{i+1} \to_{BG} (c)^{<\omega}_{\mu_i^{++}}$ and $\lambda_{i+1} \leq h(\mathcal{L}^c_{\mu_i^+}, \omega)$.

We leave the rest to the reader.

3.10 Claim. *Suppose $V = K$ and (**) holds.*

*For some forcing extension $V[G_\infty]$ of V, $V[G_\infty] \vDash$ (**) (hence the conclusion of **3.7**) and for every λ*

$$h(\mathcal{L}^{wo}_{\lambda^+,\omega}) < h(\mathcal{L}^c_{\lambda^+,\omega}) = h(\mathcal{L}^{aa}_{\lambda^+,\omega}) < h(\mathcal{L}^{II}_{\lambda^+,\omega})$$

PROOF Combine the proofs of **3.9** and **2.5**.

§4 Lowering consistency strength.

We present here some alternative proofs with lower consistency strength than in §3. Specifically **4.1, 4.3** and **3.2(3)** justfy the restriction $\lambda \geq \aleph_3 + (2^{\aleph_0})^+$ in **3.4**].

4.1 Lemma. *Let $V = L$. Then there is a forcing notion $P \in L$, not adding reals, such that for $G \subseteq P$ generic over V, in $V[G]$:*

a) $h(\mathcal{L}^{wo}_{\omega_1,\omega}) < h(\mathcal{L}^c_{\omega_1,\omega})$

b) *No $\aleph_1$-complete forcing notion (or even forcing notion satisfying the $\mathbb{I}$-condition $\mathbb{I}$ a set of $\aleph_2^{V[G]}$-complete ideals from L changes the truth value of "$h(\psi) < \infty$" for $\psi \in L^{wo}_{\omega_1,\omega}$*

c) *There is a sentence $\psi \in \mathcal{L}^c_{\omega,\omega}$ whose class of models of power $\geq \aleph_2$ is just $\{L_\alpha[G] : \alpha \geq \aleph_2\}$ (and note $P \in L_{\aleph_2}[V[G]]$)*

d) $h(\mathcal{L}^c_{\lambda^+,\omega}) < h(\mathcal{L}^{aa}_{\lambda^+,\omega}) = h(\mathcal{L}^{II}_{\lambda^+,\omega})$

4.1A Remark. In the proof below, coding generic sets by the decision which L-cardinals are collapsed is replaced here by "which L-regular cardinal have in V cardinality $\aleph_0$ and which cardinality $\aleph_1$

PROOF Let $\mathbb{I}(\mu,\kappa)$ be, e.g. the calss of filters D which are λ-complete over some λ (this in V), where $\mu \leq \lambda < \kappa, |\cup D| < \kappa$

We define by induction on $n, \alpha_n, \beta_n, \lambda_{i,j}, \mu_{i,j}, \langle P_i, Q_\jmath : i \leq \alpha_n, \jmath < \alpha_n \rangle$ and f_n such that

(A) $\alpha_0 = 0, \alpha_{n+1} > \alpha_n$

(B) $\langle P_i, Q_\jmath, \mu_\jmath : i \leq \alpha_n, \jmath < \alpha_n \rangle$ is an RCS iteration suitable for $x_{\alpha_n} = \langle \mathbb{I}_{i,j}, \lambda_{i,j}, \mu^i_{i,j}, i < j \leq \alpha_n, i$ not strongly inaccessible $\rangle$.

   See [Sh-b Ch.XI] or [Sh-f (Ch XI)] particularly Def. 6.1

(C) f_n is a one-to-one function from P_i onto some ordinal β_n, extending $\cup_{e<n} f_e$.

   G_α will denote a generic subset of P_α.

   For $n = 0$ there is nothing to do.

   For $n + 1$, note that forcing by P_{α_n} does not add new reals. So $(\mathcal{L}^{wo}_{\omega_1,\omega})^V = (\mathcal{L}^{wo}_{\omega_1,\omega})^{V[G_{\alpha_n}]}$ and let $\{\psi_i : i < \omega_1\}$ be a lsit of the sentences (up to isomorphism).

We now (i.e for defining α_{n+1} etc.) define by induction on $\zeta < \omega_1, Q_{\alpha_n+\zeta}, x_{\alpha_n+\zeta+1}∎$ as follows:

(a) $\langle P_i, Q_\jmath : i \leq \alpha_n + \zeta \rangle$ is $x_{\alpha_n+\zeta+1}$-suitable RCS iteration

(b) If there is $Q_{\alpha_n+\zeta}$, a $P_{\alpha_n+\zeta}$-name of a forcing notion sattisfying the
$\mathbb{I}((|P_{\alpha_n+\zeta}| + sup\{\lambda_{i,j} : i < j \le \alpha_n + \zeta\})^+, \kappa$ -condition for some κ
then $\Vdash_{P_{\alpha_n+\zeta+1}*\underset{\sim}{Q}} \psi_\zeta$ has arbitrarily large models then $Q_{\alpha_n+\zeta}$ is like
that, otherwise it is, e.g., Lévy $(\aleph_1, 2^{\aleph_1})$.

Next let $\mu_\zeta = h(\mathcal{L}^{wo}_{\omega_1,\omega})^{V[G_{\alpha_n+\omega_1}]}, Q_{\alpha_n+\omega_1} =$ Lévy $(\aleph_1, \mu_\zeta^+)$. Now (where
$<,>$ is Godel's pairing function on ordinals), let in $V[G_{\alpha_n+\omega_1}]$: $A_n = |\{\langle f_n(p), f_n(q)\rangle : p, q \in P_{\alpha_n} \vDash p \le q$ and $p \ne q\} \cup \{\langle f_n(p), f_n(p)\rangle : q \in G_{\alpha_n+\omega}\}$ and let $\gamma_n = sup\{\langle f_n(p), f_n(q)\rangle : p, q \in P_{\alpha_n+\omega_1}\}$. Now we define
$Q_{\alpha_n+\omega_1+i}$ by induction on $i \le \gamma_n$:

$Q_{\alpha_n+\omega_1}$ is Lévy $(\aleph_1, \aleph_2)^{V[G_{\alpha_n+\omega_1}]}$,

$Q_{\alpha_n+\omega_1+1+2i+1}$ is Lévy $(\aleph_1, \aleph_2[V[G_{\alpha_n+\omega_1+1+2i+1}]]$,

$Q_{\alpha_n+\omega_1+1+2i}$ is Namba forcing (of $V[G_{\alpha_n+\omega_1+1+i}]$) if $i \in A_n$ and Lévy
$(\aleph_1, \aleph_\zeta)^{V(n,i)}$ where $V(n,i) = V[G_{\alpha_n+\omega_1+1+2i}]$ if $i \notin A_n$.

Now let $\alpha_{n+1} = \alpha_n + \omega_1 + 2\gamma_n, \lambda_{n+1} = |P_{\alpha_n+\omega_1+2\gamma_n}|$, and define f_{n+1}.

We leave the rest to the reader $\square_{4.1}$

4.2 Conclusion.

(1) *We can do the forcing from* **2.4, 2.5** *to the universe we got in* **4.1**
getting corresponding results (for $\mathcal{L}_{\omega_1,\omega}(Q)$'s, with CH and G.C.H):
so we need $CON(ZFC)$ *only.*

(2) *the same holds for* **4.3** *for the* $\mathcal{L}_{\omega_2,\omega}(Q)$'s (so we use $CON(ZFC+$
"the class of ordinals in Mahlo") only.

4.3 Lemma. *Suppose* $V = L$, *(for simplicity) and* ∞ *is a Mahlo cardinal*
(i.e., every closed unbounded class of cardinals has a regular member). then
there is an inaccessible cardinal λ *and a forcing notion* $P \subseteq H(\lambda)$, *such that*
:

(a) *P satisfies the* λ-c.c., *does not add reals and collapse every* $\mu \in$
$(\aleph_1, \lambda)$: *and* $\Vdash_P$ *"G.C.H.*λ *is* $\aleph_2$*" and* $|P| = \lambda$

(b) $h(\mathcal{L}^{wo}_{\omega_2,\omega}) < h(\mathcal{L}^c_{\omega,\omega})$

(c) *there is a sentence* $\psi \in \mathcal{L}^c_{\omega,\omega}$ *whose class of models of power* $\ge \aleph_2$
is just suitable expansions of $\{L_\alpha[G] : \alpha \ge \aleph_2\}$.

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PROOF Like **4.1**, but instead of induction on $n < \omega$ we do induction on $\gamma < \infty$, and in the induction only we first do the coding $(Q_{\alpha_n + \omega_1 + i}, i < \gamma)$ (so that for c), we say that for some club of C of ω_2, for $\delta \in C$ we are coding the set of sentence in $\mathcal{L}_{|\delta|^+}[G \cap P_\delta]$.

Do we really need the large cardinal hypothesis in **4.3** (and so in **4.2(2)**)?

4.4 Claim. *Suppose $0^{\#} \notin V$ and $\aleph_2^V$ is a successor cardinal in L and $2^{\aleph_0} = \aleph_1$ then for some sentence $\psi \in \mathcal{L}^{wo}_{\omega_2,\omega}$, its models are exactly suitable expansions of $(L_\alpha, \mathcal{P}_{<\aleph_1}(\alpha))$, where α is the last L-cardinal $< \aleph_2^V$.*

Hence $h(\mathcal{L}^{wo}_{\omega_2,\omega}) = h(\mathcal{L}^{c}_{\omega_2,\omega})$.

PROOF Should be clear

4.4 Concluding Remarks. : Still we do not settle the exact consistency strength. In fact e.g. if $\aleph_2^V$ is the first L-inaccessible, we can still prove the last sentence of **4.4**.

For $h(\mathcal{L}^{wo}_{\omega_2,\omega}) < h(\mathcal{L}^{c}_{\omega_1,\omega})$ with $2^{\aleph_0} = \aleph_1$ we can generalize **Lemma 4.3** to this case (using [Sh-f, XV]).

Also there is a gap in consistency strength in §**3** for $\lambda > \aleph_3 + (2^{\aleph_0})^+$.

It is not hard to show that if $\lambda \geq \aleph_2 + 2^{\aleph_0}, cf\lambda > \aleph_0$ and for some $A \subseteq \lambda$ does not exists, then $h(\mathcal{L}^{wo}_{\lambda^+,\omega}) = h(\mathcal{L}^{c}_{\lambda^+,\omega})$

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