Vive la Différence I: Nonisomorphism of ultrapowers of countable models

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Abstract: We show that it is not provable in ZFC that any two countable elementarily equivalent structures have isomorphic ultrapowers relative to some ultrafilter on ω.

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This paper owes its existence to Annalisa Marcja's hospitality in Trento, July 1987; van den Dries' curiosity about Kim's conjecture; the willingness of Hrushovski and Cherlin to look at §3 through a dark glass; and most of all to Cherlin's insistence that this is one of the fundamental problems of model theory.
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0. Introduction.

Any two elementarily equivalent structures of cardinality λ have isomorphic ultrapowers (by [Sh 13], in 1971) with respect to an ultrafilter on 2^λ. Earlier, as the culmination of work in the sixties, Keisler showed, assuming $2^\lambda = \lambda^+$, that the ultrafilter may be taken to be on λ [Keisler]. In particular, assuming the continuum hypothesis, for countable structures any nonprincipal ultrafilter on ω will do. As a special case, the continuum hypothesis implies that an ultraproduct of power series rings over prime fields F_p is isomorphic to the ultrapower of the corresponding rings of p-adic integers ; this has number-theoretic consequences [AxKo]. Kim has conjectured that the isomorphism $\prod_p F_p[[t]]/\mathcal{F} \simeq \prod_p \mathbb{Z}_p/\mathcal{F}$ is valid for any nonprincipal ultrafilter over ω, regardless of the status of the continuum hypothesis. In fact it has not previously been clear what could be said about isomorphism of nonprincipal ultrapowers or ultraproducts over ω *in general*, in the absence of the continuum hypothesis; it has long been suspected that such questions do involve set theoretic issues going beyond ZFC, but there have been no concrete results in this area. For the case of two different ultrafilters and on higher cardinals, see [Sh a VI]. In particular, ([Sh a VI, 3.13]) if $M = (\omega, <)^\omega/D$ (D an ultrafilter on ω), the cofinality of $(\{a \in M : a > n$ for every natural number $n\}, >)$ can be any regular $\kappa \in (\aleph_0, 2^{\aleph_0}]$.

It does follow from the results of [Sh 13] that there is always an ultrafilter $\mathcal{F}$ on λ such that for any two elementarily equivalent models M, N of cardinality λ, $M^\omega/\mathcal{F}$ embeds elementarily into $N^\omega/\mathcal{F}$. On the other hand, we show here that it is easy to find some countable elementarily equivalent structures with nonisomorphic ultrapowers relative to a certain nonprincipal ultrafilter on ω: given enough Cohen reals, some ultrafilter will do the trick (§3), and with more complicated forcing any ultrafilter will do the trick (§1, refined in §2). The (first order theories of the) models involved have the independence property but do not have the strict order property. Every unstable theory either has the independence property or the strict order property (or both) (in nontechnical terms, in the theory we can interprate in a way the theory of the random graph or the theory of a linear order), and our work here clearly makes use of the independence property. The rings occurring in the Ax-Kochen isomorphism are unstable, but do not have the independence property, so the results given here certainly do not apply directly to Kim's problem. However it does appear that the methods used in §3 can be modified to refute Kim's conjecture, and we intend to return to this elsewhere [Sh 405].

A final technical remark: the forcing notions used here are $<\omega_1$-proper, strongly proper, and Borel. Because of improvements made in the iteration theorems for proper forcing [Sh 177, Sh f], we just need the properness; in earlier versions ω-properness was somehow used.

In the appendix we give a full presentation of a less general variant of the preservation theorem of [Sh f] VI §1.

The forcing notions introduced in §1, §2 here (see 1.15, 1.16) are of interest per se. Subsequently specific cases have found more applications; see Bartoszynski, Judah and Shelah [BJSh 368], Shelah and Fremlin [ShFr 406].

§1. All ultrafilters on ω can be inadequate

Starting with a model V of CH, in a generic extension we will make $2^{\aleph_0} = \aleph_2$ and find countable elementarily equivalent graphs Γ, Δ such that for any pair of ultrafilters $\mathcal{F}, \mathcal{F}'$ on ω, $\Gamma^\omega/\mathcal{F} \not\simeq \Delta^\omega/\mathcal{F}'$. More precisely:

1.1 Theorem

Suppose $V \models$CH. Then there is a proper forcing notion $\mathcal{P}$ with the $\aleph_2$-chain condition, of cardinality $\aleph_2$ (and hence $\mathcal{P}$ collapses no cardinal and changes no cofinality) which makes $2^{\aleph_0} = \aleph_2$ and has the following effects on ultraproducts:

(i) There are countable elementarily equivalent graphs Γ, Δ such that no ultrapowers $\Gamma^\omega/\mathcal{F}_1$, $\Delta^\omega/\mathcal{F}_2$ are isomorphic.

(ii) There is a nonprincipal ultrafilter $\mathcal{F}$ on ω such that for any two sequences Γ_n, Δ_n of finite models for a countable language, if their ultrapowers with respect to $\mathcal{F}$ are elementarily equivalent, then these ultrapowers are isomorphic, and in fact saturated.

1.2 Remark

The two properties *(i,ii)* are handled quite independently by the forcing, and in particular *(ii)* can be obtained just by adding random reals.

1.3 Notation

We work with the language of bipartite graphs (with a specified bipartition P, Q). $\Gamma_{k,l}$ is a bipartite graph with bipartition $U = U_{k,\ell}, V = V_{k,\ell}$, $|U| = k$ and $V = \dot{\bigcup}_{m<l} \binom{U}{m}$, where $\binom{U}{m}$ denotes the set of all subsets of U of cardinality m. The edge relation is membership. We also let Γ_∞ be the bipartite graph with $|U| = \aleph_0$ specifically $U = \omega$ and V the set of all finite subsets of U. The theory of the $\Gamma_{k,l}$ converges to that of Γ_∞ as $l, k/l \longrightarrow \infty$.

1.4 Remark

Our construction will ensure that for any sequence (k_n, l_n) with $l_n, k_n/l_n \longrightarrow \infty$ and any ultrafilters $\mathcal{F}_1, \mathcal{F}_2$ the ultraproducts $\prod_i \Gamma_{k_{n_i}, l_{n_i}}/\mathcal{F}_1$ and $\Gamma_\infty^\omega/\mathcal{F}_2$ are nonisomorphic. In particular, if Γ_{fin} is the disjoint union of the graphs $\Gamma_{2^n,n}$, and Γ is the disjoint union of Γ_{fin} and Γ_∞, then Γ_{fin} and Γ are elementarily equivalent, but any isomorphism of $\Gamma^\omega/\mathcal{F}$ and $\Gamma_{\text{fin}}^\omega/\mathcal{F}$ would induce an isomorphism of an ultrapower of Γ_∞ with some ultraproduct $\prod_i \Gamma_{2^{n_i}, n_i}/\mathcal{F}$. (Note that the graphs under consideration have connected components of diameter at most 4.)

1.5 The model

We will build a model N of ZFC by iterating suitable proper forcing notions with countable support [Sh b], see also [Jech]. The model N will have the following combinatorial properties:

P1. If $(A_n)_{n<\omega}$ is a collection of finite sets with $|A_n| \longrightarrow \infty$, and $g : \omega \longrightarrow \omega$ with $g(n) \longrightarrow \infty$, and f_i $(i < \omega_1)$ are functions from ω to ω with $f_i \in \prod_n A_n$ for all $i < \omega_1$, then there is a function H from ω to finite subsets of ω such that: $H(n)$ has size at most $g(n)$; $H(n) \subseteq A_n$; and for each i, $H(n)$ contains $f_i(n)$ if n is sufficiently large (depending on i).

P2. ${}^\omega\omega$ has true cofinality ω_1, that is: there is a sequence $(f_i)_{i<\omega_1}$ which is cofinal in ${}^\omega\omega$ with respect to the partial ordering of eventual domination (given by "$f(n) < g(n)$ for sufficiently large n").

P3. For every sequence $(A_k : k < \omega)$ of finite sets, for any collection $B_i(i < \omega_1)$ of infinite subsets of ω, and for any collection $(g_i)_{i<\omega_1}$ of functions in $\prod_k A_k$, there is a function $f \in \prod_k A_k$ such that for all $i, j < \omega_1$, the set $\{n \in B_i : f(n) = g_j(n)\}$ is infinite.

P4. $2^{\aleph_1} = \aleph_2$.

Note that (P3,P4) imply $2^{\aleph_0} = \aleph_2$.

1.6 Proposition

Any model N of ZFC with properties (P1-P2) satisfies part (i) of Theorem 1.1. More precisely, the following weak saturation property holds for any ultraproduct $\Gamma^* = \prod_n \Gamma_{k_n,l_n}/\mathcal{F}$ for which $l_n \longrightarrow \infty$, $(\ell_n < k_n)$ and fails in any countably indexed ultrapower of Γ_∞:

(†) *Given ω_1 elements of U^{Γ^*}, some element of V^{Γ^*} is linked to each of them.*

Proof:

Our discussion in Remark 1.4 shows that it suffices to check the claim regarding (†). First consider an ultraproduct $\Gamma^* = \prod_n \Gamma_{k_n,l_n}/\mathcal{F}$ for which $l_n \longrightarrow \infty$, $l_n < k_n$.

Given $\aleph_1$ elements $a_i = f_i/\mathcal{F} \in \Gamma^*$ we apply (P1) with $g(n) = l_n - 1, A_n = U_{k_n,l_n}$. H picks out a sequence of small subsets of U_{k_n,l_n}, and if $b \in V^{\Gamma^*}$ is chosen so that its n-th coordinate is linked to all the elements of $H(n)$, then this does the trick.

Now let Γ^* be of the form $\Gamma_\infty^\omega/\mathcal{F}$. We will show that (†) fails in this model. Let $(f_i : i < \omega_1)$ be a cofinal increasing sequence in ${}^\omega\omega$, under the partial ordering given by eventual domination. Remember $U^{\Gamma_\infty} = \omega$. Let $a_i = f_i/\mathcal{F}$ for $i < \omega_1$. Let $b \in V^{\Gamma^*}$ be represented by the sequence b_n of elements of V in Γ_∞. Let B_n be the subset of U^{Γ_∞} coded by b_n; we may suppose it is never empty. Define $g(n) = \sup B_n$ and let i be chosen so that f_i dominates g eventually. Then off a finite set we have $f_i(n) \notin B_n$, and hence in Γ^*, a_i and b are unlinked. ∎

1.7 Proposition

Any model N of ZFC with the properties (P3,P4) satisfies part (ii) of Theorem 1.1.

Proof:

We must construct an ultrafilter $\mathcal{F}$ on ω such that any ultraproduct of finite structures with respect to $\mathcal{F}$ is saturated. The construction takes place in $\aleph_2$ steps; at stage $\alpha < \aleph_2$ we have a filter $\mathcal{F}_\alpha$ generated by a subfilter of at most $\aleph_1$ sets $(B_i)_{i<\omega_1}$ containing the cobounded subsets of ω, and we have a type $p = (\varphi_i)_{i<\omega_1}$ over some ultraproduct $\prod_k A_k/\mathcal{F}$ of finite structures to realize. (More precisely, since the filter $\mathcal{F}$ has not yet been constructed, the "type" p is given as a set of pairs $(\varphi_i(x;\bar y), \bar f^{(i)})$ where $\bar f^{(i)} = \langle f_1^{(i)}, \ldots\rangle$ with $f_j^{(i)} \in \prod_k A_k$, p is closed under conjunction, and p is consistent in a strong sense: for each φ_i there is a function g_i such that $\varphi_i(g_i(n); f_1^{(i)}(n), \ldots)$ holds for all n in some set which has already been put into $\mathcal{F}$.) By (P4) we can arrange the construction so that at a given stage α we only have to deal with one such type.

By (P3) there is a function $f \in \prod_k A_k/\mathcal{F}$ such that for all $i, j < \omega_1$, the set $\{n \in B_j : f(n) = g_i(n)\}$ is infinite, where g_i witnesses the consistency of φ_i. We adjoin to $\mathcal{F}$ all of the sets $X_i = \{n \in \omega : \varphi_i(f(n); f_1^{(i)}(n), \ldots)\}$. The resulting filter is nontrivial, and is again generated by at most $\aleph_1$ sets. Furthermore our construction ensures that $f/\mathcal{F}$ will realize the type $p = \{\varphi_i(x; f_1^{(i)}/\mathcal{F}, \ldots)\}$ in the ultraproduct.

One may also take care as one proceeds to ensure that the filter which is being constructed will be an ultrafilter. ∎

1.8 Outline of the Construction

In the remainder of this section we will manufacture a model N of ZFC with the properties P1-P4 specified in 1.5. We will use a countable support iteration of length ω_2 of ${}^\omega\omega$-bounding proper forcing notions of cardinality at most $\aleph_1$, starting from a model M of GCH. (See the Appendix for definitions and an outline of relevant results.) By [Sh 177] or [Sh f] VI§2 or A2.3 here, improving the iteration theorem of [Sh b, Theorem V.4.3], countable support iteration preserves the property:

"${}^\omega\omega$-bounding and proper".

Thus every function $f : \omega \longrightarrow \omega$ in N is eventually dominated by one in M, and property P2 follows: ${}^\omega\omega$ has true cofinality ω_1 in N. Our construction also yields P4: $2^{\aleph_1} = \aleph_2$. The other two properties are more specifically combinatorial, and will be ensured by the particular choice of forcing notions in the iteration. The next two propositions state explicitly that suitable forcing notions exist to ensure each of these two properties; it will then remain only to prove these two propositions.

1.9 Proposition

Suppose that $(A_n)_{n<\omega}$ is a collection of finite sets with $|A_n| \longrightarrow \infty$, and $g : \omega \longrightarrow \omega$ with $g(n) \longrightarrow \infty$. Then there is a proper ${}^\omega\omega$-bounding forcing notion $\mathcal{P}$ such that for some $\mathcal{P}$-name $\underset{\sim}{H}$ the following holds in the corresponding generic extension:

$\underset{\sim}{H}$ is a function with domain ω with $\underset{\sim}{H}(n) \subseteq A_n$ and $|\underset{\sim}{H}(n)| \leq g(n)$ for all relevant n, and for every $f \in \prod_n A_n$ in the ground model, we have $f(n) \in \underset{\sim}{H}(n)$ if n is sufficiently large (depending on f).

1.10 Proposition

Suppose M is a model of ZFC, and $(A_k : k < \omega)$ is a sequence of finite sets in M. Then there is an ${}^{\omega}\omega$-bounding proper forcing notion such that in the corresponding generic extension we have a function $\underset{\sim}{\eta} \in \prod_k A_k$ satisfying: for all $f \in \prod_k A_k$ and infinite $B \subseteq \omega$, both in M, $\underset{\sim}{\eta}$ agrees with f on an infinite subset of B.

We give the proof of Proposition 1.10 first.

1.11 Definition

For $\mathcal{A} = (A_k : k < \omega)$ a sequence of finite sets of natural numbers, for simplicity $|A_k| \geq 2$ for every k, let $\mathcal{Q}(\mathcal{A})$ be the set of pairs (T, K) where $T \subseteq {}^{\omega >}\omega$ is a tree and $K : T \longrightarrow \omega$, such that for all η in T we have:

1. $\eta(l) \in A_l$ for $l < \operatorname{len}(\eta)$.
2. For any $k \geq K(\eta)$ and $x \in A_k$ there is ρ in T extending η with $\rho(k) = x$.

We take $(T', K') \geq (T, K)$ iff T' is a subtree of T. By abuse of notation, we may write "T" for "(T, K)" with $K(\eta)$ the minimal possible value, and we may ignore the presence of K in other ways.

We use $\mathcal{Q}(\mathcal{A})$ as a forcing notion: the intersection of a generic set of conditions defines a function $\eta \in \prod_k A_k$, called the *generic branch*.

We also define partial orders $\leq_m$ on $\mathcal{Q}(\mathcal{A})$ as follows. $T \leq_m T'$ iff $T \leq T'$ and:

1. $T \cap {}^{m\geq}\omega = T' \cap {}^{m\geq}\omega$;
2. $K(\eta) = K'(\eta)$ for $\eta \in T \cap {}^{m\geq}\omega$.

Note the *fusion* property: if (T_n) is a sequence of conditions with $T_n \leq_n T_{n+1}$ for all n, then $\sup T_n$ exists (and is a condition). We pay attention to K in this context.

1.12 Remark

With the notation of 1.11, $\mathcal{Q}(\mathcal{A})$ forces:

For any $f \in \prod_k A_k$ and infinite $B \subseteq \omega$, both in the ground model, the generic branch $\underset{\sim}{\eta}$ agrees with f on an infinite subset of B.

1.13 Proof of Proposition 1.10

It suffices to check that $\mathcal{Q}(\mathcal{A})$ is an ${}^{\omega}\omega$-bounding proper forcing notion. We claim in fact:

(*) Let $(T, K) \in \mathcal{Q}(\mathcal{A})$, $m < \omega$, and let $\underset{\sim}{\alpha}$ be a $\mathcal{Q}(\mathcal{A})$-name for an ordinal. Then there is $T', T \leq_m T'$ such that for some finite set w of ordinals, $T' \Vdash$ "$\underset{\sim}{\alpha} \in w$".

This condition implies that $\mathcal{Q}(\mathcal{A})$ is ${}^{\omega}\omega$-bounding, since given a name $\underset{\sim}{f}$ of a function in ${}^{\omega}\omega$, we can find a sequence of conditions T_n and finite sets w_n of integers such that (T_n) is a fusion sequence (i.e. $T_n \leq_n T_{n+1}$ for all n) and $T_n \Vdash$ "$\underset{\sim}{f}(n) \in w_n$"; then $T = \sup\ T_n$ forces "$\underset{\sim}{f}(n) \leq \max\ w_n$ for all n".

At the same time, the condition $(*)$ is stronger than Baumgartner's Axiom A, which implies α-properness for all countable α.

It remains to check $(*)$. We fix T (and the corresponding function $K : T \longrightarrow \omega$), $\underset{\sim}{\alpha}$, m as in $(*)$. For $\nu \in T$ let T^{ν} be the restriction of T to the set of nodes comparable with ν. For ν in T, pick a condition (T_{ν}, K_{ν}) by induction on $\mathrm{len}\,(\nu)$ such that $T_{\nu} \geq T^{\nu}$ and $\eta \lhd \nu \& \nu \in T_{\eta} \Longrightarrow T_{\nu} \geq T_{\eta}$ and $T_{\nu} \Vdash$ "$\underset{\sim}{\alpha} = \alpha_{\nu}$" for some α_{ν}. We may suppose $K_{\nu} \geq K$ on T_{ν}. Set $k_0 = m$, and define k_l inductively by

$$k_{l+1} =: \max\,(k_l + 1, \max\,\{K_{\eta}(\eta) + 1 : \eta \in T \cap {}^{k_l}\omega\}).$$

Let $(\eta_j)_{j=2,\ldots,N}$ be an enumeration of $T \cap {}^{\leq k_1}\omega$. (It is convenient to begin counting with 2 here.) For $\nu \in T$ with $\nu \restriction k_1 = \eta_j$, we will write $j = j(\nu)$.

Let T' be:

$$\{\eta : \exists \nu \in T \text{ extending } \eta,\ \mathrm{len}\,(\nu) \geq k_N,\ \text{ and } \nu \in T_{\nu \restriction k_{j(\nu)}}\}$$

Observe that for η of length at least k_N, the only relevant ν in the definition of T' is η itself. That is, $\eta \in T'$ if and only if $\eta \in T_{\eta \restriction k_{j(\eta)}}$. In particular T' is a condition (with $K'(\eta) \leq K_{\eta \restriction k_{j(\eta)}}(\eta)$ for $\mathrm{len}\,(\eta) \geq k_N$). Also, since $T' \cap {}^{k_N \geq}\omega \subseteq \bigcup \{T_{\nu} : \nu \in T \cap {}^{k_N \geq}\omega\}$, we find $T' \Vdash$ "$\underset{\sim}{\alpha} \in \{\alpha_{\nu} : \nu \in T \cap {}^{k_N \geq}\omega\}$". Notice also that $T' \restriction k_1 = T \restriction k_1$.

The main point, finally, is to check that we can take $K' = K$ on $T' \cap {}^{m \geq}\omega$. Fix $\eta_j \in T' \cap {}^{m \geq}\omega$, $k \geq K(\eta_j)$, and $x \in A_k$; we have to produce an extension ν of η_j in T', with $\nu(k) = x$. Let η_h be an extension of η_j of length k_1, such that η_h has an extension $\nu \in T$ with $\nu(k) = x$. If $k < k_h$, then $\nu \restriction (k+1) \in T'$, as required.

Now suppose $k \geq k_{h+1}$, and let η be an extension of η_h of length k_h. Then $T_{\eta} \subseteq T'$, and $k \geq K_{\eta}(\eta)$. Thus a suitable ν extending η exists.

We are left only with the case: $k \in [k_h, k_{h+1})$. In particular $k \geq k_2$, so $k > K(\eta_h)$ for all η_h in $T \cap {}^{k_1 \geq}\omega$. This means that any extension of $\eta_{h'}$ of η of length k_1 could be used in place of our original choice of η_h. Easily there is such $h' \neq h$ (remember $|A_k| \geq 2$ and demand on K). But k cannot lie in two intervals of the form $[k_h, k_{h+1})$, so we must succeed on the second try. ∎

1.14 Logarithmic measures

We will define the forcing used to prove Proposition 1.9 in 1.16 below. Conditions will be perfect trees carrying extra information in the form of a (very weak) "measure" associated with each node. These measures may be defined as follows.

For a a set, we write $P^+(a)$ for $P(a) \setminus \{\emptyset\}$. A *logarithmic measure* on a is a function $\| \ \| : P^+(a) \longrightarrow \mathbb{N}$ such that:

1. $x \subseteq y \Longrightarrow \|x\| \leq \|y\|$;
2. If $x = x_1 \cup x_2$ then for some $i = 1$ or 2, $\|x_i\| \geq \|x\| - 1$.

By (1), $\| \ \|$ has finite range. If a is finite (as will generally be the case in the present context), one such logarithmic measure is $\|x\| = \lfloor \ln_2 |x| \rfloor$.

1.15 The forcing notion $\mathcal{LT}$

We will force with trees such that the set of successors of any node carries a specified logarithmic measure; the measures will be used to prevent the tree from being pruned too rapidly. The formal definition is as follows.

1. $\mathcal{LT}$ is the set of pairs (T, t) where:

1.1. T is a subtree of ${}^{\omega>}\omega$ with finite stem; this is the longest branch in T before ramification occurs. We call the set of nodes of T which contain the whole stem the *essential part* of T; so T will consist of its essential part together with the proper initial segments of its stem. We denote the essential part of T by $\mathrm{ess}\,(T)$.

1.2. t is a function defined on the essential part of T, with $t(\eta)$ a logarithmic measure on the set $\mathrm{succ}_T(\eta)$ of all successors of η in T; we often write $\| \ \|_\eta$ (or possibly $\| \ \|^T_\eta$) for $t(\eta)$. For η a proper initial segment of the stem of T, we stipulate $t(\eta)[\mathrm{succ}(\eta)] = 0$.

2. The partial order on $\mathcal{LT}$ is defined by: $(T_2, t_2) \geq (T_1, t_1)$ iff $T_2 \subseteq T_1$, and for $\eta \in T_2$ $t_2(\eta)$ is the restriction of $t_1(\eta)$ to $P^+(\mathrm{succ}_{T_2}(\eta))$.
3. We define $\mathcal{LT}^{[(T,t)]}$ to be $\{(T', t') \in \mathcal{LT} : (T', t') \geq (T, t)\}$ with the induced order. Similarly for $\mathcal{LT}^f$, $\mathcal{LT}_d$, and $\mathcal{LT}^f_d$ (see below).

1.16 The forcing notion $\mathcal{LT}^f_d$

$\mathcal{LT}^f$ is the set of pairs $(T, t) \in \mathcal{LT}$ in which T has only *finite* ramification at each node.

$\mathcal{LT}_d$ is the set of pairs $(T, t) \in \mathcal{LT}$ such that for any m, every branch of T is almost contained in the set $\{\eta \in T : \forall \nu \geq \eta \ \|\mathrm{succ}_T(\nu)\|_\nu \geq m\}$ (i.e. the set difference is finite).

$\mathcal{LT}^f_d$ is $\mathcal{LT}^f \cap \mathcal{LT}_d$. For $T \in \mathcal{LT}^f$, an equivalent condition for being in $\mathcal{LT}^f_d$ is: $\lim_k \inf\{\|\mathrm{succ}_T(\eta)\|_\eta : \mathrm{len}\,(\eta) = k\} = \infty$. Note: $\mathcal{LT}^f_d$ is an upward closed subset of $\mathcal{LT}_d$.

We make an observation concerning fusion in this connection. Define:

1. $(T_1, t_1) \leq^* (T_2, t_2)$ if $(T_1, t_1) \leq (T_2, t_2)$ and in addition for all $\eta \in \mathrm{ess}\, T_2$, $\|\mathrm{succ}_{T_2}(\eta)\|^{T_2}_\eta \geq \|\mathrm{succ}_{T_1}(\eta)\|^{T_1}_\eta - 1$.
2. $(T_1, t_1) \leq_m (T_2, t_2)$ if $(T_1, t_1) \leq (T_2, t_2)$ and for all $\eta \in T_2$ with $\|\mathrm{succ}_{T_1}(\eta)\|_\eta \geq m$, (so $\eta \in \mathrm{ess}\,(T_1)$) we have $\|\mathrm{succ}_{T_2}(\eta)\|_\eta \geq m$ (hence $\eta \in \mathrm{ess}\,(T_2)$ when $m > 0$).

3. $(T_1, t_1) \leq^*_m (T_2, t_2)$ if $(T_1, t_1) \leq_m (T_2, t_2)$ and for all $\eta \in T_2$ with $\|\mathrm{succ}(\eta)\|^{T_1}_\eta \leq m$, we have $\mathrm{succ}_{T_1}(\eta) \subseteq T_2$.

If (T_n, t_n) is a sequence of conditions in $\mathcal{LT}^f_d$ with $(T_n, t_n) \leq^*_n (T_{n+1}, t_{n+1})$ for all n, then $\sup(T_n, t_n)$ exists in $\mathcal{LT}^f_d$.

We also mention in passing that a similar statement holds for $\mathcal{LT}_d$, with a more complicated notation. Using arguments like those given here one can show that $\mathcal{LT}_d$ is also proper. This will not be done here.

For $\eta \in T$, $(T, t) \in \mathcal{LT}$ we let T^η be the set of $\nu \in T$ comparable with η, $t^\eta = t \restriction \mathrm{ess}\,(T^\eta)$: so $(T, t) \leq (T^\eta, t^\eta)$; we may write $(T, t)^\eta$ or (T^η, t) instead of (T^η, t^η).

We will now restate Proposition 1.9 more explicitly, in two parts.

1.17 Proposition

Suppose that $(A_n)_{n<\omega}$ *is a collection of finite sets with* $|A_n| \longrightarrow \infty$, *and that* $g : \omega \longrightarrow \omega$ *with* $g \longrightarrow \infty$. *Then there is a condition* (T_0, t_0) *in* $\mathcal{LT}_d^f$ *such that* (T_0, t_0) *forces:*

> *There is a function* $\underset{\sim}{H}$ *such that* $|\underset{\sim}{H}(n)| < g(n)$ *for all* n *[more exactly,* $|\underset{\sim}{H}(n)| < \max\{g(n), 1\}]$ *, and for every* f *in the ground model,*
>
> $f(n) \in \underset{\sim}{H}(n)$ *for* n *sufficiently large.*

Proof:

Without loss of generality $g(n) > 1$ and A_n is nonempty for every n. Let $a_n = \{A \subseteq A_n : |A| = g(n) - 1\}$, $T_0 = \bigcup_N \prod_{n<N} a_n$, and define a logarithmic measure $\| \ \|_n$ on a_n by $\|x\|_n = \max\{l :$ if $A' \subseteq A_n$ has cardinality 2^l, then there is $A \in x$ containing $A'\}$. Set $t_0(\eta) = \| \ \|_{\operatorname{len} \eta}$.

Obviously $(T_0, t_0) \in \mathcal{LT}_d^f$, (a pedantic reader will note $T_0 \not\subseteq {}^{\omega>}\omega$ and rename) For a generic branch $\underset{\sim}{\eta}$ of T_0:

$(T_0, t_0) \Vdash_{\mathcal{LT}_d^f}$ "$|\underset{\sim}{\eta}(n)| < g(n)$ for all n;"

$(T_0, t_0) \Vdash_{\mathcal{LT}_d^f}$ "For f in the ground model, $f(n) \in \underset{\sim}{\eta}(n)$ for all large n."

∎

1.18 Proposition

The forcing notion $\mathcal{LT}_d^f$ *is* ${}^\omega\omega$*-bounding and proper.*

It remains only to prove this proposition.

1.19 Lemma

If $(T, t) \in \mathcal{LT}_d$ *and* W *is a subset of* T, *then there is some* $(T', t') \in \mathcal{LT}_d$ *with* $(T, t) \leq^* (T', t')$ *such that either:*

(+) *every branch of* T' *meets* W; *or else*

(−) T' *is disjoint from* W.

Proof:

Let T^W be the set of all $\eta \in T$ for which there is a condition (T', t') such that T' has stem η, $(T^\eta, t) \leq^* (T', t')$, and every infinite branch of T' meets W. (T^η is the set of $\nu \in T$ comparable to η; so it is a tree whose stem contains η.)

If the stem of T is in T^W we get (+). Otherwise we will construct $(T', t') \in \mathcal{LT}_d$ such that (−) holds, $(T, t) \leq^* (T', t')$, and $T' \cap T^W = \emptyset$. For this we define $T' \cap {}^n\omega$ (and $t' = t \restriction \operatorname{ess}(T')$) inductively.

If $n \leq \text{len}\,(\text{stem}(T))$ then we let $T' \cap {}^n\omega$ be $\{\text{stem}(T)\lceil n\}$.

So suppose that $n \geq \text{len}\,(\text{stem}\ T)$ and that we have defined everything for $n' \leq n$. Let $\nu \in T' \cap {}^n\omega$, and in particular, $\nu \notin T^W$. Let $a = \text{succ}_T(\nu)$, $a_1 = a \cap T^W$, $a_2 = a \backslash a_1$. Then for some $i = 1$ or 2, $\|a_i\|_\nu \geq \|a\|_\nu - 1$.

Since $\nu \notin T^W$, it follows easily that $\|a_1\|_\nu < \|a\|_\nu - 1$; otherwise one pastes together the conditions $(T_{\nu'}, t_{\nu'})$ associated with $\nu' \in a_1$ to show $\nu \in T^W$. Thus $\|a_2\|_\nu \geq \|a\|_\nu - 1$. Let $T' \cap (\text{succ}_T(\nu))$ be a_2. As we can do this for all $\nu \in T' \cap {}^n\omega$, this completes the induction step. ∎

1.20 Lemma

If $\underset{\sim}{\alpha}$ is an $\mathcal{LT}_d^f$-name of an ordinal, $(T, t) \in \mathcal{LT}_d^f$, $m < \omega$, and $\|\text{succ}_T \eta\|_\eta > m$ for $\eta \in \text{ess}\,(T)$, then there is $(T', t') \in \mathcal{LT}_d^f$ with $(T, t) \leq_m (T', t')$, and a finite set w of ordinals, such that $(T', t') \Vdash_{\mathcal{LT}_d^f}$ "$\underset{\sim}{\alpha} \in w$".

Proof:

Let W be the set of nodes ν of T for which there is a condition (T_ν, t_ν) with $(T_\nu, t_\nu) \,{}_m\!\geq (T^\nu, t^\nu)$ such that (T_ν, t_ν) forces a value on $\underset{\sim}{\alpha}$. We claim that for any $(T_1, t_1) \;{}^*\!\geq (T, t)$, T_1 must meet W. Indeed, fix $(T_2, t_2) \geq (T_1, t_1)$ forcing "$\underset{\sim}{\alpha} = \beta$" for some β. Then for some $\nu \in T_2$, all extensions η of ν in T_2 will satisfy $\|\text{succ}_{T_2}(\eta)\|_\eta \geq m$, and $(T_2, t_2)^\nu$ witnesses the fact that $\nu \in W$. Thus if we apply Lemma 1.19, the alternative $(-)$ is not possible.

Accordingly we have some $(T_1, t_1) \;{}^*\!\geq (T, t)$ such that every branch of (T_1, t_1) meets W. Let W_0 be the set of minimal elements of W in T_1. Then W_0 is finite. For $\nu \in W_0$ select (T_ν, t_ν) with $(T_\nu, t_\nu) \,{}_m\!\geq (T, t)^\nu$ and $(T_\nu, t_\nu) \Vdash$ "$\underset{\sim}{\alpha} = \alpha_\nu$" for some α_ν. Form $T' = \bigcup\{T^\nu : \nu \in W_0\}$. ∎

1.21 Lemma

If $(T, t) \in \mathcal{LT}_d^f$, $\underset{\sim}{\alpha}$ is an $\mathcal{LT}_d^f$-name of an ordinal, $m < \omega$, then there is $(T', t') \in \mathcal{LT}_d^f$ with $(T, t) \leq_m^* (T', t')$, and a finite set of ordinals w, such that $(T', t') \Vdash$ "$\underset{\sim}{\alpha} \in w$".

Proof:

Fix k so that $\|\text{succ}(\eta)\|_\eta > m$ for $\text{len}\,(\eta) \geq k$. Apply 1.20 to each T^ν for $\nu \in T$ of length $k + 1$. ∎

1.22 Proof of 1.18

As in 1.13, using 1.21. ∎

This completes the verification that the desired model N can be constructed by iterating forcing.

§2. Nonisomorphic ultraproducts of finite models.

We continue to use the bipartite graphs $\Gamma_{k,l}$ introduced in 1.3. Varying the forcing used in §1, we will get:

2.1 Theorem

Suppose that V satisfies CH, and that $(k_n, l_n), (k'_n, l'_n)$ are monotonically increasing sequences of pairs (and $2 < l'_n < k'_n < l_n < k_n < l'_{n+1}$) such that:

(1) $k'_n/l'_n \longrightarrow \infty$;

(2) $(k_n/l_n) > (k'_n)^{ndl'_n}$, *for each* $d > 0$, *for* n *large enough;*

(3) $\ln l'_n > k^n_{n-1}$.

Then there is a proper forcing $\mathcal{P}$ satisfying the $\aleph_2$-cc, of size $\aleph_2$, such that in $V^{\mathcal{P}}$ no two ultraproducts $\prod \Gamma_{k_i,l_i}/\mathcal{F}_1$, $\prod \Gamma_{k'_i,l'_i}/\mathcal{F}_2$ are isomorphic.

More precisely, we will call a bipartite graph with bipartition (U, V) $\aleph_1$-*complete* if every set of ω_1 elements of U is linked to a single common element of V (property (†) of Proposition 1.6), and then our claim is that in $V^{\mathcal{P}}$, no nonprincipal ultraproduct of the first sequence Γ_{k_n,l_n} is $\aleph_1$-complete, and every nonprincipal ultraproduct of the second sequence $\Gamma_{k'_n,l'_n}$ is; furthermore, as indicated, this phenomenon can be controlled by the rates of growth of k and of l/k.

2.2 Definition

Let f, g be functions in ${}^\omega\omega$. A model N of ZFC is (f,g)-*bounded* if for any sequence $(A_n)_{n<\omega}$ of finite sets with $|A_n| = f(n)$, there are $\aleph_1$ sequences $\mathcal{B}_i = (B_{i,n} : n < \omega)$, indexed by $i < \omega_1$, with:

(1) $B_{i,n} \subseteq A_n$ for all n;

(2) For all $i < \omega_1$, $|B_{i,n}| < g(n)$ eventually;

(3) $\bigcup_i \prod_n B_{i,n} = \prod_n A_n$ in N.

2.3 Lemma

Let $(k_n), (l_n)$ be sequences with $l_n, k_n/l_n \longrightarrow \infty$, and let $f(n) = \binom{k_n}{l_n}$, $g(n) = k_n/l_n$. Suppose that N is a model of ZFC which is (f,g)-bounded. Then no ultraproduct $\prod_n \Gamma_{k_n,l_n}/\mathcal{F}$ can be $\aleph_1$-complete.

Proof:

Let $\mathcal{B}_i$ have properties (1-3) of 2.2 with respect to $A_n = V_{k_n,l_n}$. For each i, choose $a_i \in \prod_n U_{k_n,l_n}$ so that $a_i(n)$ is not linked to any $b \in B_{i,n}$, as long as $|B_{i,n}| < g(n)$ (so $l_n|B_{i,n}| < k_n$). Then $a_i/\mathcal{F}(i < \omega_1)$ cannot all be linked to any single b in $\prod_n \Gamma_{k_n,l_n}/\mathcal{F}$, for any ultrafilter $\mathcal{F}$. ∎

2.4 Definition

For functions $f, g \in {}^{\omega}\omega$ we say that a forcing notion $\mathcal{P}$ has the (f, g)-*bounding property* provided that:

(∗) For any sequence $(A_k : k < \omega)$ in the ground model, with $|A_k| = f(k)$, and any $\underset{\sim}{\eta} \in \prod_k A_k$ in the generic extension, there is a "cover" $\mathcal{B} = (B_k : k < \omega)$ in the ground model with $B_k \subseteq A_k$, $|B_k| < g(k)$ (more exactly, $< \mathrm{Max}\{g(k), 2\}$), and $\underset{\sim}{\eta}(k) \in B_k$ for each k.

Similarly a forcing notion has the $(\mathbf{F}, g)$-bounding property, for $\mathbf{F}$ a collection of functions, if it has the (f, g^{ε})-bounding property for each $f \in \mathbf{F}$ and each $\varepsilon > 0$. In this terminology, notice that $(\{f\}, g)$-bounding is a stronger condition than (f, g)-bounding.

2.4A Definition

Call a family $\mathbf{F}$ *g-closed* if it satisfies the following two closure conditions:

1. For $f \in \mathbf{F}$, the function $F(n) = \prod_{m<n}(f(m) + 1)$ lies in $\mathbf{F}$;
2. For $f \in \mathbf{F}$, f^g is in $\mathbf{F}$.

Proof of 2.1

We build a model N of ZFC by an iteration of length ω_2 with countable support of proper forcing notions with the $(\mathbf{F}, g)$ bounding property for a suitable family $\mathbf{F}$, all of which are of the form $(\mathcal{LT}_d^f)^{[(T,t)]}$; and we arrange that all of the forcing notions of this form which are actually $(\mathbf{F}, g)$-bounding will occur cofinally often. (In order to carry this out one actually makes use of auxiliary functions (f_1, g_1) with f_1 eventually dominating $\mathbf{F}$ and g_1 eventually dominated by any positive power of g, but these details are best left to the discussion after 2.5.)

One can show that a countable support iteration of proper $(\mathbf{F}, g)$-bounding forcing notions is again $(\mathbf{F}, g)$-bounding. This is an instance of a general iteration theorem of [Sh f, VI] but we make our presentation self-contained by giving a proof in the appendix-A2.5. If we force over a ground model with CH (so that CH holds at intermediate points in the iteration) then our final model is $(\mathbf{F}, g)$-bounded, and by 2.3 no ultraproduct of the Γ_{k_n, l_n} can be $\aleph_1$-complete.

One very important point still remains to be checked. It may be formulated as follows.

2.5 Proposition

Let $f_0, g_0, h : \omega \longrightarrow \omega \setminus \{0,1\}$ and suppose that $(A_n)_{n<\omega}$ is a sequence of finite nonempty sets with $|A_n| \longrightarrow \infty$. Assume:

(1) $$\prod_{m \le n} |A_m|^{h(m)} < g_0(n) \text{ for every } n \text{ large enough;}$$

(2) $$\frac{\ln h(n)}{\ln \prod_{i<n} f_0(i)} \longrightarrow \infty.$$

Then there is a condition $(T,t) \in \mathcal{LT}_d^f$ such that $(\mathcal{LT}_d^f)^{[(T,t)]}$ is (f_0,g_0)-bounding and (T,t) forces:

There is a function $\underset{\sim}{H}$ such that $\underset{\sim}{H}(n) \subseteq A_n$, $|\underset{\sim}{H}(n)| < h(n)$ for all n, and for every f in the ground model, $f(n) \in \underset{\sim}{H}(n)$ for n sufficiently large.

Continuation of the proof of 2.1:

We will now check that the proof of theorem 2.1 can be completed using this proposition.

We set $f^*(n) = \binom{k_n}{l_n}$, $g(n) = k_n/l_n$, $h(n) = l'_n$, and $A_n = U_{k'_n,l'_n}$. (So $|A_n| = k'_n$.) Let $\mathbf{F}_0$ be the set of increasing functions f satisfying

$$\lim_{n \longrightarrow \infty} \ln h(n)/(g^d(n-1) \ln f(n-1)) \longrightarrow \infty \qquad \text{for all } d > 0.$$

If $f_0 \in \mathbf{F}_0$ and g_0 is a positive power of g, then conditions (1,2) of 2.5 hold by condition (2) of 2.1 (for (2) of 2.5 note for $d = 2$ that $g^d(n-1) > n$). Furthermore $\mathbf{F}_0$ is g-closed (this uses the fact that $g(n) \ge n$ eventually by (2) of 2.1), and $f^* \in \mathbf{F}_0$. By diagonalization find f_1, g_1 satisfying $(1,2)$ of 2.5 so that f_1 eventually dominates any function in the g-closure of f^*, and g_1 is eventually dominated by any positive power of g. Apply the proposition to (f_1, g_1, h) and observe that an (f_1, g_1)-bounding forcing notion is (g-closure of f^*, g)-bounding. We let $\mathbf{F} = g$-closure of $\{f^*\}$.

Forcing with the corresponding $(\mathcal{LT}_d^f)^{[(T,t)]}$ produces a branch $\underset{\sim}{H}$ so that if $\underset{\sim}{H}(n)$ is thought of as an element $b_n \in V_{k'_n,l'_n}$, then for all $f \in \prod_n A_n$ in the ground model, and any ultrafilter $\mathcal{F}$ on ω, $f/\mathcal{F}$ is linked to $\underset{\sim}{H}(n)/\mathcal{F}$ in $\prod_n \Gamma_{k'_n,l'_n}/\mathcal{F}$.

2.6 Terminology

A logarithmic measure $\|\ \|$ on a is called *m-additive* if for every choice of $(a_i)_{i<m}$ with $\bigcup_i a_i = a$, there is $i < m$ with $\|a_i\| \ge \|a\| - 1$.

2.7 Lemma

Suppose $f, g : \omega \longrightarrow \omega \setminus \{0,1\}$, $(T,t) \in \mathcal{LT}_d^f$, and:

i. for every $\eta \in \mathrm{ess}\,(T)$, $t(\eta)$ is $\prod_{i<\mathrm{len}\,\eta} f(i)$-additive;

ii. for every n we have $|T \cap {}^{(n+1)}\omega| < g(n)$.

Then $(\mathcal{LT}_d^f)^{[(T,t)]}$ is (f,g)-bounding.

Proof:

Let $F(n) = \prod_{i<n} f(i)$. Suppose that $(A_n)_{n<\omega} \in V$, $|A_n| \le f(n)$, and $(T,t) \Vdash$ "$\underset{\sim}{\eta} \in \prod_n A_n$". By fusion as in 1.19–1.22 there is $(T',t') \in \mathcal{LT}_d^f$ with $(T',t') \ge (T,t)$ such that for every n the set

$$W =: \{\nu \in T' : (T'^{\nu}, t') \text{ forces a value on } \underset{\sim}{\eta}(n)\}$$

meets every branch of (T',t').

For each n, choose $N(n)$ large enough that (T'^{ν}, t') forces a value η^n_ν on $\underset{\sim}{\eta}\restriction n$ for each $\nu \in T' \cap {}^{N(n)}\omega$. Thus $\eta^n_\nu \in \prod_{i<n} A_i$. By downward induction on $k < N(n)$, for $\nu \in T' \cap {}^k\omega$ choose $\eta^n_\nu \in {}^n\omega$ and $s(\nu,n) \subseteq \mathrm{succ}_{T'}(\nu)$ so that:

$$\|s(\nu,n)\|_\nu \ge \|\mathrm{succ}_{T'}(\nu)\|_\nu - 1; \quad \eta^n_\nu \restriction \min\{k,n\} = \eta^n_{\nu'} \restriction \min\{k,n\} \text{ for } \nu' \in s(\nu,n).$$

Since $|\{\eta_{\nu'} \restriction \min(k,n) : \nu' \in \mathrm{succ}_{T'}(\nu)\}| \le F(k)$ and $\|\ \|_\nu$ is $F(k)$-additive, this is easily done. Let $T'_n = \{\nu \in T' : (\forall l < \mathrm{len}\,(\nu) \cap N(n))\ \nu\restriction(l+1) \in s(\nu\restriction l, n)\}$.

We now define $T'' \subseteq T'$ so that for all k the set X_k of n for which $T'' \cap {}^{k\ge}\omega = T'_n \cap {}^{k\ge}\omega$ is infinite. For this we proceed by induction on k. If $T'' \cap {}^{k\ge}\omega$ has been defined, then we can select $X \subseteq X_k$ infinite such that for $n \in X$ and $\nu \in T'' \cap {}^k\omega$, $s(\nu,n) = s(\nu)$ is independent of n. We then define

$$T'' \cap {}^{(k+1)}\omega = \{\nu \in T' \cap {}^{k+1}\omega : \nu\restriction k \in T'' \cap {}^k\omega \text{ and } \nu \in s(\nu\restriction k)\}$$

Observe that $(T'', t\restriction T'')\ {}^*\!\ge (T',t')$, and $(T'', t\restriction T'')$ forces:

"For any k, if $n \in X_{N(k)}$ and $n \ge k$, then $\underset{\sim}{\eta}\restriction k = \eta^n_\nu \restriction k$ for some $\nu \in T'' \cap {}^k\omega$".

Indeed, for any ν' of length $N(k)$ in T'', if $\nu' \in T'_n$ then $\eta^k_{\nu'} = \eta^n_{\nu'} \restriction k = \eta^n_{\nu'\restriction k} \restriction k$. Since $|T'' \cap {}^{k+1}\omega| \le |T \cap {}^{k+1}\omega| < g(k)$, this yields the stated bounding principle. ∎

2.8 Proof of 2.5

Let $F_0(n) = \prod_{i<n} f_0(i)$. Let $a_n = \{A \subseteq A_n : |A| = h(n) - 1\}$, $T_0 = \bigcup_N \prod_{n<N} a_n$, and define a logarithmic measure $\| \, \|_n$ on a_n by: for $a \subseteq a_n$

$$\|a\|_n = \max\{l : \text{ for all } A' \subseteq A_n \text{ of cardinality} \le F_0(n)^l, \text{ there is } A \in a \text{ containing } A'\}.$$

Set $t_0(\eta) = \| \, \|_{\operatorname{len}\eta}$.

Obviously $\| \, \|_n$ is $F_0(n)$-additive and $|T \cap {}^{(n+1)}\omega| = \prod_{m \le n}(|A_m|)^{(h(m)-1)}$ which is (by condition (1) of 2.5) $< g_0(n)$, so $(\mathcal{LT}^f_d)^{[(T_0,t_0)]}$ is (f_0, g_0)-bounding by lemma 2.7.

We need to check that $\|a_n\|_n \longrightarrow \infty$:

$$\|a_n\|_n = \max\{l : F_0(n)^l < h(n)\} \sim \frac{\ln h(n)}{\ln F_0(n)}.$$

So (2) from 2.5 guarantees it. ∎

§3. Adding Cohen reals creates a bad ultrafilter.

In this section we show that a weaker form of the results in §§1, 2 is obtained just by adding $\aleph_3$ Cohen reals to a suitable ground model. This result was actually the first one obtained in this direction. This construction is also used in [Sh 345] and again in [Sh 405].

3.1 Theorem

If we add $\aleph_3$ Cohen reals to a model of $[2^{\aleph_i} = \aleph_{i+1}\ (i=1,2)\ \&\ \diamondsuit_{\{\delta<\aleph_3 : \mathrm{cof}\,\delta=\aleph_2\}}]$, then there will be a nonprincipal ultrafilter $\mathcal{F}$ on ω and two sequences of pseudorandom finite graphs (Γ^1_n), (Γ^2_n) such that $\prod_n \Gamma^1_n/\mathcal{F} \not\simeq \prod_n \Gamma^2_n/\mathcal{F}$. In fact the same result will apply if the sequences Γ^1_n, Γ^2_n are replaced by any subsequences.

Here we call a sequence (Γ^1_n) of finite graphs *pseudorandom* if the theory of Γ^1_n converges fairly rapidly to the theory of the random infinite graph; cf. 3.4 below. The only condition needed on the two sequences in Theorem 3.1 is that the Γ^1_m and Γ^2_n are of radically different sizes (3.5 below). As a variant (with very much the same proof) we can take all Γ^2_n equal to the random infinite graph, keeping (Γ^1_n) a sequence of pseudorandom finite graphs, and obtain the same result for a suitable ultrafilter.

3.2 Corollary

Under the hypotheses of Theorem 3.1 there are elementarily equivalent countable graphs Γ^1_ω, Γ^2_ω and a nonprincipal ultrafilter $\mathcal{F}$ on ω with $(\Gamma^1_\omega)^\omega/\mathcal{F} \not\simeq (\Gamma^2_\omega)^\omega/\mathcal{F}$.

This is proved much as in Remark 1.4, noting that large pseudorandom graphs are connected of diameter 2.

3.3 Remark

With more effort we can replace the hypotheses on the ground model in Theorem 3.1 by:

$$2^{\aleph_i} = \aleph_{i+1}\ (i=0,1)\ \&\ \diamondsuit_{\{\delta<\aleph_2 : \mathrm{cof}\,\delta=\aleph_1\}},$$

adding only $\aleph_2$ Cohen reals. In the definition of $\mathcal{AP}$ below, $\underset{\sim}{\mathcal{F}}$ would then not be an arbitrary name of an ultrafilter; instead $\mathcal{AP}$ would be replaced by a family of $\aleph_1$ isomorphism types of members of $\mathcal{AP}$, (using $\aleph_0$ in place of $\aleph_1$ in clause 3.8 (i) below) which is closed under the operations used in the proof.

The same approach allows us to eliminate the $\diamondsuit$ from Theorem 3.1. With the modified version of $\mathcal{AP}$ and $\aleph_3$ Cohen reals, we can replace $\diamondsuit_{\{\delta<\aleph_3 : \mathrm{cof}\,\delta=\aleph_2\}}$ by $\diamondsuit_{\{\delta<\aleph_3 : \mathrm{cof}\,\delta=\aleph_1\}}$, which in fact follows from the other hypotheses [Gregory, Sh 82].

We will not enlarge on these remarks any further here.

3.4 Definition

A finite graph Γ on n vertices is *sufficiently random* if:

i. For any two disjoint sets of vertices V_1, V_2 with $|V_1 \cup V_2| \leq (\log n)/3$, there is a vertex v linked to all vertices of V_1, and none in V_2;

ii. For any sets of vertices V_1, V_2 with $|V_i| > 3 \log n$ there are adjacent and nonadjacent pairs of vertices in $V_1 \times V_2$.

iii. If V_1, V_2, V are three disjoint sets of vertices and $P \subseteq V_1 \times V_2$, with $|P|, |V| > 5 \log n$, and if all pairs in P have distinct first entries, then some $v \in V$ *separates* some pair $(v^1, v^2) \in P$ in the sense that: $[R(v^1, v) \iff \neg R(v^2, v)]$. Here R is the edge relation (in the appropriate graph).

For sufficiently large n most graphs of size n are sufficiently random. We call any sequence of sufficiently random graphs of size tending to infinity a sequence of pseudorandom graphs.

(See [Bollobas] for background on random graphs.)

3.5 Notation

i. (Γ^1_n), (Γ^2_n) are two sequences of sufficiently random graphs such that for any m, n we have $\|\Gamma^1_m\| > \|\Gamma^2_n\|^5$ or $\|\Gamma^2_n\| > \|\Gamma^1_m\|^5$. ($\|\Gamma\|$ is the number of vertices of Γ.) These sequences are kept fixed. Γ is the infinite random (homogeneous) graph. If we replace $\prod \Gamma^2_n/\mathcal{F}$ by $\Gamma^\omega/\mathcal{F}$ throughout, the argument is much the same, with slight simplifications.

ii. $\mathbb{P}$ is the forcing notion that adds $\aleph_3$ Cohen reals to V. $\underset{\sim}{x}_\alpha$ is the name of the α-th Cohen real as an element of ${}^\omega\omega$. For $\mathcal{A} \subseteq \aleph_3$, $\mathbb{P}\restriction\mathcal{A}$ denotes $\{\mathtt{p} \in \mathbb{P} : \operatorname{dom} \mathtt{p} \subseteq \mathcal{A}\}$.

3.6 Discussion

Working in the ground model we will build a $\mathbb{P}$-name for a suitable nonprincipal ultrafilter $\underset{\sim}{\mathcal{F}}$. We will view the reals $\underset{\sim}{x}_\alpha$ as (for example) potential members of the ultraproduct $\prod \Gamma^1_n$. We will consider candidates $\underset{\sim}{y}_\alpha$ for (representatives of) their images under a putative isomorphism, and defeat them by arranging (for example) that the set of n for which

$$R(\underset{\sim}{x}_\alpha(n), \underset{\sim}{x}_\beta(n)) \text{ iff } \neg R(\underset{\sim}{y}_\alpha(n), \underset{\sim}{y}_\beta(n))$$

gets into $\underset{\sim}{\mathcal{F}}$.

Note however that this must be done for every two potential sequences $(\underset{\sim}{k}^1(n))$ and $(\underset{\sim}{k}^2(n))$ indexing the ultraproducts $\prod_n \Gamma^1_{\underset{\sim}{k}^1(n)}/\underset{\sim}{\mathcal{F}}$, $\prod_n \Gamma^2_{\underset{\sim}{k}^2(n)}/\underset{\sim}{\mathcal{F}}$ to be formed. At stage α we deal with sequences $\underset{\sim}{k}^1_\alpha(n), \underset{\sim}{k}^2_\alpha(n) \in V^{\mathbb{P}\restriction\alpha}$ (which are guessed by the diamond). We require

$\{n : \underset{\sim}{x}_\alpha(n) \in \Gamma^{\varepsilon_\alpha}_{\underset{\sim}{k}^{\varepsilon_\alpha}_\alpha(n)}\} \in \underset{\sim}{\mathcal{F}}$ where $\varepsilon_\alpha \in \{1,2\}$ is a label, and another very important requirement is that for any sequence $(\underset{\sim}{A}_n : n < \omega) \in V^{\mathbb{P}\restriction\alpha}$ with $\underset{\sim}{A}_n \subseteq \Gamma^{\varepsilon_\alpha}_{\underset{\sim}{k}^{\varepsilon_\alpha}_\alpha}(n)$ and $|\underset{\sim}{A}_n|/\|\Gamma^{\varepsilon_\alpha}_{\underset{\sim}{k}^{\varepsilon_\alpha}_\alpha(n)}\|$ small enough, the set $\{n : \underset{\sim}{x}_\alpha(n) \notin \underset{\sim}{A}_n\} \in \underset{\sim}{\mathcal{F}}$. (This sort of condition is an analog of the notion of a Γ-*big* type in [Sh 107].) It will be used in combination with clause (ii) in the definition of sufficient randomness.

The name $\underset{\sim}{\mathcal{F}}$ is built by carefully amalgamating a large set of approximations to the final object, using the combinatorial principle $\diamondsuit_{\aleph_2}$, which follows from the cardinal arithmetic [Gregory]; this method, which was illustrated in [Sh 107], is based on the theorem from [ShHL 162]. (The comparatively elaborate tree construction of [ShHL 162] can be simplified in the presence of $\diamondsuit$; it is designed to work when $\aleph_2$ is replaced by a limit cardinal and $\diamondsuit$ is weakened to the principle Dl_λ.) In what follows, the connection with [ShHL 162] is left somewhat vague; the details will be found in §A3 of the Appendix. In particular, in §A3.5 we show how the present $\mathcal{AP}$ fits the framework of §A3.1-3.

3.7 A notion of smallness

If $\mathcal{F}$ is a filter on ω, $k \in {}^\omega\omega$, $\varepsilon \in \{1,2\}$, then a sequence $(A_n : n < \omega)$ of subsets of the $\Gamma^\varepsilon_{k(n)}$ (i.e. $A_n \subseteq \Gamma^\varepsilon_{k(n)}$) is $(\mathcal{F}, k, \varepsilon)$-*slow* if there is some d such that $\mathcal{F}$-lim $\left[|A_n|/\left(\sqrt{\|\Gamma^\varepsilon_{k(n)}\|} \cdot (\log\|\Gamma^\varepsilon_{k(n)}\|)^d\right)\right] = 0$. Later on we will deal primarily with the case $\varepsilon = 1$, to lighten the notation, and we will then write "$(\mathcal{F}, k)$-slow" in place of "$(\mathcal{F}, k, 1)$-slow".

It should perhaps be emphasized that here (as opposed to §2) ε is merely a label.

3.8 Definition

We define the partially ordered set $\mathcal{AP}$ of approximations as follows. The intent is that the approximations should build the name of a suitable ultrafilter $\underset{\sim}{\mathcal{F}}$. Recall that the sequences (Γ^ε_n) (with $\varepsilon \in \{1,2\}$) are fixed (3.5(i)). Also bear in mind that the ultrafilter must eventually "defeat" a potential isomorphism between two ultraproducts $\prod_n \Gamma^\varepsilon_{\underset{\sim}{k}^\varepsilon(n)}/\underset{\sim}{\mathcal{F}}$.

1. An element $q \in \mathcal{AP}$ is a quadruple $(\mathcal{A}, \underset{\sim}{\mathcal{F}}, \boldsymbol{\varepsilon}, \underset{\sim}{\mathbf{k}}) = (\mathcal{A}^q, \underset{\sim}{\mathcal{F}}^q, \boldsymbol{\varepsilon}^q, \underset{\sim}{\mathbf{k}}^q)$ where

i. $\mathcal{A} \subseteq \aleph_3$ has cardinality $\aleph_1$; $\boldsymbol{\varepsilon} = (\varepsilon_\alpha : \alpha \in \mathcal{A})$ with each ε_α an element of $\{1,2\}$;

ii. $\underset{\sim}{\mathcal{F}}$ is a $\mathbb{P}\restriction\mathcal{A}$-name of a nonprincipal ultrafilter on ω, and if we set $\underset{\sim}{\mathcal{F}}\restriction(\mathcal{A}\cap\alpha) =: \underset{\sim}{\mathcal{F}}\restriction\{\underset{\sim}{X} : \underset{\sim}{X}$ is a $\mathbb{P}\restriction(\mathcal{A}\cap\alpha)$-name for a subset of $\omega\}$, then $\underset{\sim}{\mathcal{F}}\restriction(\mathcal{A}\cap\alpha)$ is a $\mathbb{P}\restriction(\mathcal{A}\cap\alpha)$-name for all α;

iii. $\underset{\sim}{\mathbf{k}} = (\underset{\sim}{k}_\alpha : \alpha \in \mathcal{A})$ with $\underset{\sim}{k}_\alpha$ a $\mathbb{P}\restriction(\mathcal{A}\cap\alpha)$-name of a function from ω to ω;

iv. For each $\alpha \in \mathcal{A}$, and each $\mathbb{P}\restriction(\mathcal{A}\cap\alpha)$-name $(\underset{\sim}{A}_n : n < \omega)$;

if $\Vdash_{\mathbb{P}\restriction(\mathcal{A}\cap\alpha)}$ "$(\underset{\sim}{A}_n)_{n<\omega}$ is $(\underset{\sim}{\mathcal{F}}\restriction\alpha,\underset{\sim}{k}_\alpha,\varepsilon_\alpha)$-slow" then
$\Vdash_{\mathbb{P}}$ "$\{n : \underset{\sim}{x}_\alpha(n) \in \Gamma^{\varepsilon_\alpha}_{\underset{\sim}{k}_\alpha(n)} \setminus \underset{\sim}{A}_n\} \in \underset{\sim}{\mathcal{F}}$".

We write $\mathcal{A} = \mathcal{A}^q$, $\mathcal{F} = \mathcal{F}^q$, and so on, when necessary.

2. We take $q \le q'$ if $\mathcal{A}^q \subseteq \mathcal{A}^{q'}$ and $q'\restriction\mathcal{A}^q = q$.

Some further comment is in order here. When we begin to check that $\underset{\sim}{\mathcal{F}}$ is indeed the name of an ultrafilter such that for any pair of sequences $\underset{\sim}{k}^1(n), \underset{\sim}{k}^2(n)$, the ultraproducts $\prod \Gamma^{\varepsilon}_{\underset{\sim}{k}^\varepsilon(n)}/\underset{\sim}{\mathcal{F}}$ are nonisomorphic, we will notice that there is an automatic asymmetry because the sequences (Γ^1_n) and (Γ^2_n) are so different: on some set in $\underset{\sim}{\mathcal{F}}$ we will have $|\Gamma^{\varepsilon}_{\underset{\sim}{k}^\varepsilon(n)}| > |\Gamma^{\varepsilon^*}_{\underset{\sim}{k}^{\varepsilon^*}(n)}|^5$ holding with $\{\varepsilon, \varepsilon^*\} = \{1, 2\}$ in some order. The parameter ε_α in an approximation can be viewed as a guess as to the direction in which this asymmetry goes (after adding Cohen reals); the notion of an approximation includes a clause (iv) designed to be useful when $\underset{\sim}{k}_\alpha$ coincides with a particular $\underset{\sim}{k}^\varepsilon$ in the context just described.

On the other hand, we could first use $\diamondsuit$ to guess ε_α, $\underset{\sim}{k}^{\varepsilon_\alpha}_\alpha$, and many other things; in this case we do not actually need to include these kinds of data in the approximations themselves, though it would still be necessary to mention them in clause (iv). Alternatively, the set $\mathcal{AP}$ could also be used as a forcing notion, without $\diamondsuit$, and in this case the ε and $\underset{\sim}{k}$ would have to be included. So the version given here is the most flexible one.

3.9 Claim (Amalgamation)

1. Suppose that $q_0, q_1, q_2 \in \mathcal{AP}$, $\mathcal{A}^{q_1} \subseteq \delta$, $\mathcal{A}^{q_2} = \mathcal{A}^{q_0} \cup \{\delta\}$, and $q_0 \le q_1, q_2$. Then we can find $r \ge q_1, q_2$ in $\mathcal{AP}$.

2. If $q_1, q_2 \in \mathcal{AP}$, $\alpha < \aleph_3$, $\operatorname{dom} q_1 \subseteq \alpha$, and $q_2\restriction\alpha \le q_1$, then there is $r \ge q_1, q_2$ in $\mathcal{AP}$.

Proof:

1: Let $\mathcal{A}_i = \mathcal{A}^{q_i}$, $\underset{\sim}{\mathcal{F}}^i = \underset{\sim}{\mathcal{F}}^{q_i}$, $\mathcal{A} = \mathcal{A}_1 \cup \{\delta\}, \varepsilon = \varepsilon^{q_2}_\delta$ and $\underset{\sim}{k} = \underset{\sim}{k}^{q_2}_\delta$. In particular $\underset{\sim}{\mathcal{F}}^0 \subseteq \underset{\sim}{\mathcal{F}}^1, \underset{\sim}{\mathcal{F}}^2$, and we have to combine them into one ultrafilter $\underset{\sim}{\mathcal{F}}$ in $V^{\mathbb{P}\restriction\mathcal{A}}$. The point is to preserve 3.8(iv), that is to ensure that $\mathbb{P}\restriction\mathcal{A}$ forces the relevant family of sets (namely, $\underset{\sim}{\mathcal{F}}^1, \underset{\sim}{\mathcal{F}}^2$, and sets imposed on us by 3.8(iv)) to have the finite intersection property.

If $\mathsf{p} \in \mathbb{P}\restriction\mathcal{A}$ forces the contrary, then after extending p suitably we may suppose that there is a $(\mathbb{P}\restriction\mathcal{A}_1)$-name $\underset{\sim}{a}$ of a member of $\underset{\sim}{\mathcal{F}}^1$, a $(\mathbb{P}\restriction\mathcal{A}_2)$-name $\underset{\sim}{b}$ of a member of $\underset{\sim}{\mathcal{F}}^2$, and – since $\mathcal{A}_1 = \mathcal{A} \cap \delta$ – a $(\mathbb{P}\restriction\mathcal{A}_1)$-name $(\underset{\sim}{A}_n : n < \omega)$ forced by p to be $(\underset{\sim}{\mathcal{F}}^1, \underset{\sim}{k}, \varepsilon)$-slow (as in (iv) of 3.8) so that letting $\underset{\sim}{c} = \{n < \omega : \underset{\sim}{x}_\delta(n) \in \Gamma^{\varepsilon_\delta}_{\underset{\sim}{k}_\delta(n)} \setminus \underset{\sim}{A}_n\}$ we have:

$$\mathsf{p} \Vdash_{\mathbb{P}\restriction\mathcal{A}} \text{"}\underset{\sim}{a} \cap \underset{\sim}{b} \cap \underset{\sim}{c} = \emptyset\text{"}.$$

(i.e. we used the fact that there are three kinds of requirements of the form "a set belongs to $\mathcal{F}$ ", each kind is closed under finite intersections).

Let $\mathtt{p}_i = \mathtt{p}\restriction\mathcal{A}_i$ for $i = 0, 1, 2$. To clarify the matter choose $\mathtt{H}^0 \subseteq \mathbb{P}\restriction\mathcal{A}_0$ generic over V so that $\mathtt{p}_0 \in \mathtt{H}^0$. Note that $\underset{\sim}{k}$ is a $(\mathbb{P}\restriction\mathcal{A}_0)$-name (3.8(iii)).

In $V[\mathtt{H}^0]$, for each $n < \omega$ let

$$\underset{\sim}{B}_n[\mathtt{H}^0] = \{v \in \Gamma^{\varepsilon}_{\underset{\sim}{k}(n)}[\mathtt{H}^0] : \text{ For some } \mathtt{p}'_2 \in \mathbb{P}\restriction\mathcal{A}_2 \text{ with } \mathtt{p}'_2 \geq \mathtt{p}_2 \text{ and } \mathtt{p}'_2\restriction\mathcal{A}_0 \in \mathtt{H}^0,$$

$$\mathtt{p}'_2 \Vdash_{\mathbb{P}\restriction\mathcal{A}_2} \text{“}\underset{\sim}{x}_\delta(n) = v \text{ and } n \in \underset{\sim}{b}\text{”}\}.$$

Then $(\underset{\sim}{B}_n : n < \omega)$ is not $(\underset{\sim}{\mathcal{F}}^0\restriction\delta, \underset{\sim}{k}, \varepsilon)$-slow, since $(\underset{\sim}{B}_n : n < \omega)$ is a $\mathbb{P}\restriction\mathcal{A}_0$-name, $q_2 \in \mathcal{AP}$, and $\mathtt{p}_2 \Vdash$ “For $n \in \underset{\sim}{b}$, $\underset{\sim}{x}_\delta(n) \in \underset{\sim}{B}_n$” (and (iv) of 3.8(1)).

Also in $V[\mathtt{H}^0]$, let $\underset{\sim}{b}^+[\mathtt{H}^0] = \{n :$ for every $\mathtt{p}'_0 \in \mathtt{H}^0$, $\mathtt{p}'_0 \cup \mathtt{p}_2 \nVdash$ “$n \notin \underset{\sim}{b}$”$\}$. As $q_2 \in \mathcal{AP}$, we have $\underset{\sim}{b}^+ \in \underset{\sim}{\mathcal{F}}^0[\mathtt{H}^0]$. For each $n \in \underset{\sim}{b}^+[H^\delta]$ let

$$\underset{\sim}{A}^1_n[\mathtt{H}^0] =: \{v \in \Gamma^{\varepsilon}_{\underset{\sim}{k}(n)}[\mathtt{H}^0] : \text{ For no } \mathtt{p}'_1 \geq \mathtt{p}_1 \text{ in } \mathbb{P}\restriction\mathcal{A}_1 \text{ with } \mathtt{p}'_1\restriction\mathcal{A}_0 \in \mathtt{H}^0,$$

$$\mathtt{p}'_1 \Vdash_{\mathbb{P}\restriction\mathcal{A}_1} \text{“}n \in \underset{\sim}{a} \text{ and } v \notin \underset{\sim}{A}_n.\text{”}\}$$

Let $\underset{\sim}{A}^1_n[\mathtt{H}^0] = \emptyset$ if $n \notin \underset{\sim}{b}^+$.

Easily $(\underset{\sim}{A}^1_n : n < \omega)$ is $\underset{\sim}{\mathcal{F}}^0$-slow. Hence in $V[\mathtt{H}^0]$ the sequence $(\underset{\sim}{B}_n \setminus \underset{\sim}{A}^1_n : n < \omega)$ is not $(\underset{\sim}{\mathcal{F}}^0[\mathtt{H}^0])$-slow. We can compute the values of $\underset{\sim}{B}_n$ and $\underset{\sim}{A}^1_n$ in $V[\mathtt{H}^0]$. So we can find $n \in \underset{\sim}{b}^+[H^0]$ with $\underset{\sim}{B}_n \setminus \underset{\sim}{A}^1_n \neq \emptyset$, and choose $v \in \underset{\sim}{B}_n \setminus \underset{\sim}{A}^1_n$. Then there are $\mathtt{p}'_1 \in \mathbb{P}\restriction\mathcal{A}_1/\mathtt{H}^0$, $\mathtt{p}'_1 \geq \mathtt{p}_1$, and $\mathtt{p}'_2 \in \mathbb{P}\restriction\mathcal{A}_2/\mathtt{H}^0$, with $\mathtt{p}'_2 \geq \mathtt{p}_2$, so that:

$$\mathtt{p}'_1 \Vdash \text{“}n \in \underset{\sim}{a} \text{ and } v \notin \underset{\sim}{A}_n\text{”}.$$

$$\mathtt{p}'_2 \Vdash \text{“}n \in \underset{\sim}{b} \text{ and } \underset{\sim}{x}_\delta(n) = v\text{”}$$

Now $\mathtt{p} \leq \mathtt{p}'_1 \cup \mathtt{p}'_2 \in \mathbb{P}\restriction\mathcal{A}$ and $\mathtt{p}'_1 \cup \mathtt{p}'_2$ forces “$n \in \underset{\sim}{a} \cap \underset{\sim}{b} \cap \underset{\sim}{c}$” (over $\mathtt{H}^0$), contradicting the choice of $\mathtt{p}$. This completes the proof of 3.9 (1).

2: Let $[(\mathcal{A}^{q_2} \setminus \alpha) \bigcup \{\sup \mathcal{A}^{q_2}\}] = \{\delta_i : i \leq \gamma\}$ in increasing order. Define inductively $r_i \in \mathcal{AP}$, increasing in i, with $q_2\restriction(\mathcal{A} \cap \delta_i) \leq r_i$, $\operatorname{dom} r_i \subseteq \delta_i$, $r_0 = q_1$; then let $r = r_\gamma$.

At successor stages $i = j + 1$ we apply 3.9 (1) to $q_2\restriction(\mathcal{A}^{q_2} \cap \delta_j)$, r_j, $q_2\restriction[\mathcal{A}^{q_2} \cap (\delta_j + 1)]$.

If i is a limit of uncountable cofinality, we just take unions:

$$\mathcal{A}^{r_i} = \bigcup_{\zeta < i} \mathcal{A}^{r_\zeta}; \underset{\sim}{\mathcal{F}}^{r_i} = \bigcup_{\zeta < i} \underset{\sim}{\mathcal{F}}^{r_\zeta}; \boldsymbol{\varepsilon}^{r_i} = \bigcup_{\zeta < i} \varepsilon^{r_\zeta}; \underset{\sim}{\mathbf{k}}^{r_i} = \bigcup_{\zeta < i} \underset{\sim}{\mathbf{k}}^{r_\zeta};$$

while if i is a limit of cofinality $\aleph_0$, we have actually to extend $\bigcup_{\zeta<i} \underset{\sim}{\mathcal{F}}^{r_\zeta}$ to a $\mathbb{P}\restriction\mathcal{A}^{r_i}$-name of an ultrafilter in $V^{\mathbb{P}\restriction\mathcal{A}^{r_i}}$. However, in $V^{\mathbb{P}\restriction\mathcal{A}^{r_i}}$, $\bigcup_{\zeta<i} \underset{\sim}{\mathcal{F}}^{r_\zeta}$ is interpreted as a filter including all cofinite subsets of ω, hence can be completed to an ultrafilter. ∎

3.10 Claim

1. *If q_i $(i < \delta)$ is an increasing sequence of members of $\mathcal{AP}$, with $\delta < \aleph_2$, then for some $q \in \mathcal{AP}$, $q \geq q_i$ for all $i < \delta$.*
2. *If $q_1, q_2 \in \mathcal{AP}$, $\alpha < \aleph_3$, $q_2 \restriction \alpha \leq q_1$, and $\operatorname{dom} q_1 \cap \operatorname{dom} q_2 = \operatorname{dom} q_1 \cap \alpha$, then there is $r \geq q_1, q_2$ in $\mathcal{AP}$.*

Proof:

1: We may suppose $\delta = \aleph_0$ or $\aleph_1$. Let $\mathcal{A} =: \bigcup_i \mathcal{A}^{q_i}$ be enumerated in increasing order as $\{\alpha_j : j < \gamma\}$ for the appropriate γ, and set $\alpha_\gamma = \sup \mathcal{A}$. We define an increasing sequence of members r_j of $\mathcal{AP}$ for $j \leq \gamma$ by induction on j so that:

$$\mathcal{A}^{r_j} = \{\alpha_\zeta : \zeta < j\};$$

$$q_i \restriction \alpha_j \leq r_j \text{ for all } i < \delta.$$

In all cases we proceed as in the proof of Claim 3.9. The only difference is that we deal with several q_i, but as they are linearly ordered there is no difficulty.

2: This is proved similarly to part (1): let $\gamma = \sup(\operatorname{dom} q_1 \cup \operatorname{dom} q_2)$. Choose by induction on $\beta \in (\operatorname{dom} q_1 \cup \operatorname{dom} q_2 \cup \{\gamma\}) \setminus \alpha$ an upper bound r_β of $q_1 \restriction \beta$ and $q_2 \restriction \beta$, increasing with β, with $\operatorname{dom} r_\beta = \beta \cap (\operatorname{dom} q_1 \cup \operatorname{dom} q_2)$. The successor step is by 3.9(i). The limit is easy too. Note: if $\operatorname{dom} q_1/E$ has only finitely many classes, when $\beta_1 \, E \, \beta_2$ iff $\bigwedge_{\gamma \in \operatorname{dom} q_2} [\gamma < \beta_1 \Leftrightarrow \gamma < \beta_2]$, then 3.9(ii) suffices. ∎

3.11 Proof of Theorem 3.1: The construction

We define an increasing sequence $G^\alpha \subseteq \{q \in \mathcal{AP} : \mathcal{A}^q \subseteq \alpha\}$ of $\aleph_2$-directed sets increasing in α, and a set of at most $\aleph_2$ "commitments" which G^α will meet. In particular we require that $\forall \beta < \alpha \, \exists q \in G^\alpha \, (\beta \in \mathcal{A}^q)$, and at each stage α we may make new commitments to "enter some collection of dense sets" – in set theoretic terminology – or equivalently, to "omit some type" – in model theoretic terms. We make use of $\diamondsuit_{\{\delta < \aleph_3 : \operatorname{cof} \delta = \aleph_2\}}$ to choose the commitments. The combinatorics involved in meeting the commitments are treated in [ShLH 162], and are reviewed in §A3 of the Appendix. Our summary of the construction in the present section will be less formal.

At a stage $\delta < \aleph_3$ with $\operatorname{cof} \delta = \aleph_2$, we will "guess" $\mathbb{P}$-names $\underset{\sim}{k}^1_\delta, \underset{\sim}{k}^2_\delta, \underset{\sim}{F}_\delta$, a condition $\mathtt{p}^\delta \in \mathbb{P} \restriction \delta$ and a parameter $\varepsilon_\delta \in \{1, 2\}$, explained in connection with (4) below, and attempt to "kill" the possibility that $\mathtt{p}^\delta$ forces:

"$\underset{\sim}{F}_\delta : \prod_n \Gamma^1_{\underset{\sim}{k}^1_\delta(n)} \longrightarrow \prod_n \Gamma^2_{\underset{\sim}{k}^2_\delta(n)}$ induces a map which can be extended to an isomorphism:

$$\prod \Gamma^1_{\underset{\sim}{k}^1_\delta(n)} / \underset{\sim}{\mathcal{F}} \simeq \prod \Gamma^2_{\underset{\sim}{k}^2_\delta(n)} / \underset{\sim}{\mathcal{F}}".$$

(Here we have taken $\varepsilon_\delta = 1$; otherwise the roles of 1 and 2 in this – and in all that follows – must be reversed.)

We will refer to the genericity game of [ShHL 162], as described in §A3 of the Appendix. In that game the Ghibellines can accomplish the following. For $\delta < \aleph_3$, they determine a set of compatible approximations G^δ which together will determine an ultrafilter $\mathcal{F}\restriction\delta$ on ω in $V^{\mathbb{P}\restriction\delta}$ (specifically, G^α is a subset of $\{r \in \mathcal{AP} : \operatorname{Dom} r \subseteq \alpha\}$ which is directed, increasing in α). The Guelfs set them tasks which ensure that the ultrafilter $\mathcal{F}$ which is gradually built up by the Ghibellines has all the desired properties.

Let $\mathcal{F}_0$ be a fixed nonprincipal ultrafilter on ω, in the ground model and without loss of generality there is $q \in G^0$ with $\mathcal{F}^q = \mathcal{F}_0$. For $\delta < \aleph_3$ of cofinality $\aleph_2$, let q^*_δ be an approximation $(\{\delta\}, \underset{\sim}{\mathcal{F}}^\delta, (\varepsilon_\delta), (\underset{\sim}{k}^{\varepsilon_\delta}_\delta))$, where $\underset{\sim}{\mathcal{F}}^\delta$ is the $\mathbb{P}\restriction\{\delta\}$-name of some ultrafilter on ω extending $\mathcal{F}_0$ such that

(1) $\{n : \underset{\sim}{x}_\delta(n) \in \Gamma^{\varepsilon_\delta}_{\underset{\sim}{k}^{\varepsilon_\delta}_\delta(n)}\} \in \underset{\sim}{\mathcal{F}}^\delta$;

(2) $\{n : \underset{\sim}{x}_\delta(n) \notin A_n\} \in \underset{\sim}{\mathcal{F}}^\delta$ for any $(\mathcal{F}_0, \underset{\sim}{k}^{\varepsilon_\delta}_\delta, \varepsilon_\delta)$-slow sequence (A_n) in the universe V

The Ghibellines will be required (by the Guelfs) to put q^*_δ in $G^{\delta+1}$. The Ghibellines are also obliged to make commitments of the following form, which should then be respected throughout the rest of the construction. (These commitments involve a parameter $\alpha > \delta$ to be controlled by the Ghibellines as the play progresses: of course these commitments have to satisfy density requirements.)

$(*)^\delta_{q^*, \underset{\sim}{z}^*}$: For every $\alpha > \delta$, every $q \in G^\alpha$ with $\delta \in \operatorname{dom} q$, every $k^{1-\varepsilon_\delta}_\delta(n)$ (really a $(\mathbb{P}\restriction\delta)$-name) and every $(\mathbb{P}\restriction\mathcal{A}^\delta)$-name $\underset{\sim}{z}$ of a member of $\prod_n \Gamma^{1-\varepsilon_\delta}_{k_\delta(n)}$:

if $(q, \underset{\sim}{z}) \simeq (q^*, \underset{\sim}{z}^*)$ over $\delta + 1$, *then* there will be some r in G^α, some $\mathtt{p}' \in \mathbb{P}\restriction\mathcal{A}^r$, and some $\mathbb{P}\restriction(\mathcal{A}^r \cap \delta)$-name $\underset{\sim}{x}$ of a member of $\prod_n \Gamma^{\varepsilon_\delta}_{\underset{\sim}{k}^{\varepsilon_\delta}_\delta(n)}$, with $r \geq q$, $\mathtt{p}' \geq \mathtt{p}^\delta$, $\underset{\sim}{F}_\delta(\underset{\sim}{x})$ is a $\mathbb{P}\restriction(\mathcal{A}^r \cap \delta)$-name, and:

$$(\dagger)\qquad \mathtt{p}' \Vdash_{\mathbb{P}\restriction\mathcal{A}^r} \text{“}\{n : \Gamma^{\varepsilon_\delta}_{k^{\varepsilon_\delta}_\delta(n)} \models R(\underset{\sim}{x}(n), \underset{\sim}{x}_\delta(n)) \iff \Gamma^{1-\varepsilon_\delta}_{k^{1-\varepsilon_\delta}_\delta(n)} \models \neg R(\underset{\sim}{F}_\delta(\underset{\sim}{x})(n), \underset{\sim}{z}(n))\} \in \underset{\sim}{\mathcal{F}}^r\text{”}$$

There is such a commitment for each $q^*, \underset{\sim}{z}^*$ with $q^*_\delta \leq q^* \in \mathcal{AP}$, $q^*\restriction\delta \in G^\delta$, and $\underset{\sim}{z}^*$ a $(\mathbb{P}\restriction\mathcal{A}^{q^*})$-name of a member of $\prod_n \Gamma^2_{\underset{\sim}{k}^2_\delta(n)}$. So apparently we are making $\aleph_3$ commitments, which is not feasible, but as we are using isomorphism types this amounts to only $2^{\aleph_1} = \aleph_2$ commitments, and this *is* feasible. This is formalized in §A3.6 in the Appendix.

These commitments can only be met when the corresponding set of approximations is dense, but on the other hand we have a stationary set δ of opportunities to meet such a commitment, and we will show that for any candidate $\underset{\sim}{F}$ for an isomorphism, either we kill it off as outlined above (by making it obvious that $\underset{\sim}{F}(\underset{\sim}{x}_\delta)$ cannot be defined), or else – after failing to do this on a stationary set – that $\underset{\sim}{F}$ must be quite special (somewhat definable) and hence even more easily dealt with, as will be seen in detail in the next few sections.

After we have obtained G^α for all α, we will let $\underset{\sim}{\mathcal{F}}^\alpha$ be $\bigcup\{\underset{\sim}{\mathcal{F}}^q : q \in G^\alpha\}$ (that is, the appropriate $(\mathbb{P}\restriction\alpha)$-name of a uniform ultrafilter on ω). Letting $G =: G^{\aleph_3} =: \bigcup_\alpha G^\alpha$, also $\underset{\sim}{\mathcal{F}} = \underset{\sim}{\mathcal{F}}^{\aleph_3}$ is defined.

3.12 Proof of Theorem 3.1: The heart of the matter

Now suppose toward a contradiction that after $\underset{\sim}{\mathcal{F}}$ has been constructed in this way, there are $\mathbb{P}$-names $\underset{\sim}{F}$, $\underset{\sim}{k}^1$, $\underset{\sim}{k}^2$, and a condition $\mathtt{p} \in \mathbb{P}$ such that:

(3)

$\mathtt{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{F}$ is a function from $\prod_n \Gamma^1_{\underset{\sim}{k}^1(n)}$ onto $\prod_n \Gamma^2_{\underset{\sim}{k}^2(n)}$ which induces an isomorphism

of the corresponding ultraproducts with respect to $\underset{\sim}{\mathcal{F}}$".

Actually, we will want to assume in addition that $\mathtt{p}$ forces:

$$\text{(4)} \qquad \text{“}\{n : \|\Gamma^1_{\underset{\sim}{k}^1(n)}\| > \|\Gamma^2_{\underset{\sim}{k}^2(n)}\|\} \in \underset{\sim}{\mathcal{F}},\text{”}$$

which could force us to increase p and to switch the roles of 1 and 2 in all that follows; this is why we have carried along a parameter ε in our definition of $\mathcal{AP}$.

We will say that a set $\mathcal{A} \subseteq \aleph_3$ is $(\underset{\sim}{F}, \underset{\sim}{k}^1, \underset{\sim}{k}^2, \mathtt{p})$-*closed* if:

i. $\underset{\sim}{k}^1, \underset{\sim}{k}^2$ are $(\mathbb{P}\restriction\mathcal{A})$-names; $\underset{\sim}{F}\restriction\mathcal{A}$ is a $(\mathbb{P}\restriction\mathcal{A})$-name;

ii. $\mathtt{p} \Vdash_{\mathbb{P}\restriction\mathcal{A}}$:

" $\underset{\sim}{F}\restriction\mathcal{A}$ is a function from $\prod_n \Gamma^1_{\underset{\sim}{k}^1(n)}$ onto $\prod_n \Gamma^2_{\underset{\sim}{k}^2(n)}$ which (interpreted in $\mathbb{P}\restriction\mathcal{A}$) induces an isomorphism from $\prod_n \Gamma^1_{\underset{\sim}{k}^1(n)}/(\underset{\sim}{\mathcal{F}}\restriction\mathcal{A})$ onto $\prod_n \Gamma^2_{\underset{\sim}{k}^2(n)}/(\underset{\sim}{\mathcal{F}}\restriction\mathcal{A})$".

iii. $\mathtt{p} \Vdash_{\mathbb{P}\restriction\mathcal{A}}$: "$\{n : \|\Gamma^1_{\underset{\sim}{k}^1(n)}\| > \|\Gamma^2_{\underset{\sim}{k}^2(n)}\|\} \in \underset{\sim}{\mathcal{F}}\restriction\mathcal{A}$."

Properly speaking, the only actual closure condition here is clause (ii). Note that the condition in (iii) can be strengthened to:

$$\text{“}\left\{n : \|\Gamma^1_{\underset{\sim}{k}^1(n)}\| > \|\Gamma^2_{\underset{\sim}{k}^2(n)}\|^5\right\} \in \underset{\sim}{\mathcal{F}}\restriction\mathcal{A},\text{”}$$

by the choice of the sequences (Γ^i_n) $(i = 1, 2)$.

Let C be $\{\delta < \aleph_3 : \mathrm{cof}(\delta) = \aleph_2, \delta$ is $(\underset{\sim}{F}, \underset{\sim}{k}^1, \underset{\sim}{k}^2, \mathtt{p})$-closed$\}$. Clearly the set C is unbounded and is closed under $\aleph_2$-limits. By our construction, for a stationary subset S_C of C we may suppose that for $\delta \in S_C$: $\underset{\sim}{F}_\delta = \underset{\sim}{F}\restriction\delta$, $\mathtt{p}^\delta = \mathtt{p}$, $\varepsilon_\delta = 1$, $\underset{\sim}{k}_\delta = \underset{\sim}{k}^1$, and that δ was $(\underset{\sim}{F}, \underset{\sim}{k}^1, \underset{\sim}{k}^2, \mathtt{p})$-closed. So $q^*_\delta \in G^{\delta+1}$, and we can find $q \in G$ such that $\underset{\sim}{z} =: \underset{\sim}{F}(\underset{\sim}{x}_\delta)$ is a $(\mathbb{P}\restriction\mathcal{A}^q)$-name, $\delta \in \mathcal{A}^q$.

At stage δ in the construction, the Ghibellines had tried to make the commitment $(*)^\delta_{q^*,\underset{\sim}{z}^*}$, with $(q^*, \underset{\sim}{z}^*) = (q, \underset{\sim}{z})$. They later failed to meet this commitment, since otherwise there would be some $r \geq q$ in G, some $\mathtt{p}' \geq \mathtt{p}$ in $\mathbb{P}\restriction\mathcal{A}^r$, and some $[\mathbb{P}\restriction(\mathcal{A}^r \cap \delta)]$-name of a member $\underset{\sim}{x}$ of $\Gamma^1_{\underset{\sim}{k}^1(n)}$, for which (†) holds:

$$\mathtt{p}' \Vdash_{\mathbb{P}\restriction\mathcal{A}^r} \text{“}\{n : [\Gamma^1_{\underset{\sim}{k}^1_\delta} \models R(\underset{\sim}{x}(n), \underset{\sim}{x}_\delta(n)) \iff \Gamma^2_{\underset{\sim}{k}^2_\delta} \models \neg R(\underset{\sim}{F}_\delta(\underset{\sim}{x})(n), \underset{\sim}{z}(n))]\} \in \mathcal{F}^r\text{”}.$$

and $\underset{\sim}{z}$ is $\underset{\sim}{F}(\underset{\sim}{x}_\delta)$. But $\mathtt{p}$ forced $\underset{\sim}{F}$ to induce an isomorphism, so we have a contradiction.

The failure to make the commitment $(*)^\delta_{q,\underset{\sim}{z}}$, implies a failure of density, which means that for some $(q', \underset{\sim}{z}') \simeq (q, \underset{\sim}{z})$ over $\delta + 1$ – and hence also for $(q, \underset{\sim}{z})$ – taking $q_0 = q\restriction\delta$, we will have:

(i) δ is $(\underset{\sim}{F}, \underset{\sim}{k}^1, \underset{\sim}{k}^2, \mathtt{p})$-closed.

(ii) $\mathtt{p} \in \mathbb{P}\restriction\mathcal{A}^{q_0}$, $\delta \in \mathcal{A}^q$, $\varepsilon^q = 1$, $\underset{\sim}{k}^q_\delta = \underset{\sim}{k}^1$, $\underset{\sim}{F}_\delta = \underset{\sim}{F}\restriction\delta$;

(iii) $\underset{\sim}{z}$ is a $(\mathbb{P}\restriction\mathcal{A}^q)$-name for a member of $\prod_n \Gamma^2_{\underset{\sim}{k}^2_\delta(n)}$;

(iv) For all $r \geq q$ in $\mathcal{AP}$ such that $r\restriction\delta \in G^\delta$, and $\underset{\sim}{x}$ a $(\mathbb{P}\restriction\mathcal{A}^{r\restriction\delta})$-name, with $\underset{\sim}{y} =: \underset{\sim}{F}(\underset{\sim}{x})$ a $(\mathbb{P}\restriction\mathcal{A}^{r\restriction\delta})$-name, we have:

$(*)_{\underset{\sim}{x},\underset{\sim}{y}}$ $\mathtt{p} \Vdash$ “The set $\{n : \Gamma^1_{\underset{\sim}{k}^1(n)} \models R(\underset{\sim}{x}(n), \underset{\sim}{x}_\delta(n))$ iff $\Gamma^2_{\underset{\sim}{k}^2(n)} \models R(\underset{\sim}{y}(n), \underset{\sim}{z}(n))\}$ is in $\underset{\sim}{\mathcal{F}}^r$”.

(Note: another possibility of failure, $q \notin G^\alpha$, is ruled out by the choice of q).

Now we analyze the meaning of $(*)_{\underset{\sim}{x},\underset{\sim}{y}}$. Consider the following property of $(\mathbb{P}\restriction\delta)$-names $\underset{\sim}{x}, \underset{\sim}{y}$ for a fixed choice of $\delta \in C$, $q \in \mathcal{AP}$ with $\delta \in \mathcal{A}^q$, and $\underset{\sim}{z}$ a $(\mathbb{P}\restriction\mathcal{A}^q)$-name.

$(**)_{\underset{\sim}{x},\underset{\sim}{y}}$ For all $r \geq q$ in $\mathcal{AP}$ such that $r\restriction\delta \in G^\delta$ and $\underset{\sim}{x}$, $\underset{\sim}{y}$ are $(\mathbb{P}\restriction\mathcal{A}^{r\restriction\delta})$-names, $(*)_{\underset{\sim}{x},\underset{\sim}{y}}$ holds.

We explore the meaning of this property when $\underset{\sim}{y}$ is not necessarily $\underset{\sim}{F}(\underset{\sim}{x})$.

Clearly,

$(\otimes_1)$ If $\underset{\sim}{x}$ is a $(\mathbb{P}\restriction\delta)$-name, $\underset{\sim}{y} = \underset{\sim}{F}(\underset{\sim}{x})$, then $(**)_{\underset{\sim}{x},\underset{\sim}{y}}$.

To simplify the analysis, let $\mathtt{H}$ be generic for $\mathbb{P}\restriction\delta$. Let $\underset{\sim}{x}$ be a $\mathbb{P}\restriction\delta$-name of a real, $\mathcal{A} \subseteq \delta$. We say $\underset{\sim}{x}$ is *unrestricted* for $(\mathtt{H}, \mathcal{A}, \underset{\sim}{k}^1)$ if:

There is no $(\underset{\sim}{\mathcal{F}}\restriction\mathcal{A}, \underset{\sim}{k}^1)$-slow sequence $(\underset{\sim}{B}_n)_{n<\omega}$ in $V[\mathtt{H}\restriction\mathcal{A}]$ such that: $\{n : \underset{\sim}{x}(n) \in \Gamma^1_{\underset{\sim}{k}^1(n)} \setminus \underset{\sim}{B}_n\} \equiv \emptyset \mod \underset{\sim}{\mathcal{F}}^\delta[\mathtt{H}]$.

Observe that if $\sup\mathcal{A} < \gamma < \delta$ and $\underset{\sim}{k}_\gamma = \underset{\sim}{k}^1$, then the Cohen real $\underset{\sim}{x}_\gamma$ is forced (in $\mathbb{P}\restriction\delta$) to be unrestricted for $(\mathtt{H}, \mathcal{A}, \underset{\sim}{k}^1)$.

3.12A Claim

If $\underset{\sim}{x}^1, \underset{\sim}{x}^2$ are $(\mathbb{P}\restriction\delta)$-names of functions in $\prod_n \Gamma^1_{\underset{\sim}{k}^1(n)}$, $\underset{\sim}{y}$ is a $(\mathbb{P}\restriction\delta)$-name of a member of $\prod_n \Gamma^2_{\underset{\sim}{k}^2(n)}$, and both pairs $(\underset{\sim}{x}^1, \underset{\sim}{y})$ and $(\underset{\sim}{x}^2, \underset{\sim}{y})$ satisfy the condition (**) above, then:

(Clm) $\quad \mathtt{p} \Vdash_{\mathbb{P}\restriction\delta}$ "$\underset{\sim}{x}^1 = \underset{\sim}{x}^2 \bmod \underset{\sim}{\mathcal{F}}\restriction\delta[\mathtt{H}]$ or both are restricted for $(\mathtt{H}, \mathcal{A}^{q_0}, \underset{\sim}{k}^1)$."

We will give the proof of this, which contains one of the main combinatorial points, in paragraph 3.13. For the present we continue with the proof of the theorem. We first record a consequence of the claim.

$(\otimes_2)$ If $\underset{\sim}{x}, \underset{\sim}{y}$ are $(\mathbb{P}\restriction\delta)$-names with $\underset{\sim}{x}$ forced by $\mathbb{P}\restriction\delta$ to be unrestricted for $(\mathtt{H}, \mathcal{A}^{q_0}, \underset{\sim}{k}^1)$, and the pair $(\underset{\sim}{x}, \underset{\sim}{y})$ satisfies $(**)_{\underset{\sim}{x},\underset{\sim}{y}}$, then $p \Vdash_{\mathbb{P}\restriction\delta}$ "$\underset{\sim}{F}(\underset{\sim}{x}) = \underset{\sim}{y} \bmod \underset{\sim}{\mathcal{F}}^\delta$"

Indeed, if $\mathtt{H} \subseteq \mathbb{P}\restriction\delta$ is generic over V, and $\underset{\sim}{F}(\underset{\sim}{x})[\mathtt{H}] = \underset{\sim}{y}_1[\mathtt{H}] \neq \underset{\sim}{y}[\mathtt{H}] \bmod \underset{\sim}{\mathcal{F}}^\delta$, then since $\underset{\sim}{F}$ is *onto* (in $V[\mathtt{H}]$, as δ is $(\underset{\sim}{F}, \underset{\sim}{k}^1, \underset{\sim}{k}^2, \mathtt{p})$-closed), there is a $(\mathbb{P}\restriction\delta)$-name $\underset{\sim}{x}'$ with $\underset{\sim}{F}(\underset{\sim}{x}')[\mathtt{H}] = \underset{\sim}{y}[H]$, so $\underset{\sim}{x}'[\mathtt{H}] \neq \underset{\sim}{x}[\mathtt{H}] \bmod \underset{\sim}{\mathcal{F}}^\delta$. Now $\underset{\sim}{x}, \underset{\sim}{x}', \underset{\sim}{y}$ contradict (Clm). Thus $(\underset{\sim}{F})$ holds. As $(\otimes_1) + (\otimes_2)$ holds for stationarily many δ's, it holds for $\delta = \aleph_3$ (in the natural interpretation).

In what follows, we use the statements $(\otimes_1) + (\otimes_2)$ as a kind of "definability" condition on $\underset{\sim}{F}$; but we deal with the current concrete case, rather than seeking an abstract formulation of the situation.

Let $S = \{\gamma \in S_C : \underset{\sim}{F}(\underset{\sim}{x}_\gamma)$ is (forced by p to be equal to) a $[\mathbb{P}\restriction(\gamma+1)]$-name $\}$. We claim that S is stationary. Let $C' \subseteq \aleph_3$ be closed unbounded, and let $\delta \in S_C$ be taken with $C' \cap S_C$ unbounded below δ. Let $q \in G$ be chosen so that $\underset{\sim}{F}(\underset{\sim}{x}_\delta)$ is a $(\mathbb{P}\restriction\mathcal{A}^q)$-name, let $q_0 = q\restriction\delta$, and $\gamma_0 = \sup \mathcal{A}^{q_0}$. It suffices to check that for $\gamma_0 < \gamma < \delta$ with $\gamma \in S_C$, we have $\gamma \in S$. So let $r_1 \in G^\delta$ be chosen so that $\underset{\sim}{y}_1 =: \underset{\sim}{F}(\underset{\sim}{x}_\gamma)$ is a $(\mathbb{P}\restriction\mathcal{A}^{r_1})$-name. It suffices to show that $\underset{\sim}{y}_1$ is (forced by p to be equal to) a $(\mathbb{P}\restriction[\mathcal{A}^{r_1} \cap (\gamma+1)])$-name. Otherwise, by a density requirement (Appendix, §A3) we can find a 1-1 order preserving function h with domain $\mathcal{A}^{r_1}$, h is the identity on $\mathcal{A}^{r_1} \cap (\gamma+1)$, $h(\min(\mathcal{A}^{r_1} \setminus (\gamma+1))) > \sup \mathcal{A}^{r_1}$, with $r_2 =: h(r_1)$ in G^δ. Let $\underset{\sim}{y}_2 = h(\underset{\sim}{y}_1)$. Then $(**)_{\underset{\sim}{x}_\gamma, \underset{\sim}{y}_i}$ holds for $i = 1, 2$, so $p \Vdash_{\mathbb{P}\restriction\delta}$ "$\underset{\sim}{y}_1 = \underset{\sim}{y}_2 \bmod \underset{\sim}{\mathcal{F}}^\delta$", but by 3.14 below we can ensure that this is not the case (by making additional commitments, cf. §A3).

Now for $\gamma \in S$ let $q_\gamma \in G^{\gamma+1}$ be chosen so that $\underset{\sim}{z}_\gamma = \underset{\sim}{F}(\underset{\sim}{x}_\gamma)$ is a $(\mathbb{P}\restriction\mathcal{A}^{q_\gamma})$-name, and let $\hat{\gamma} = \sup(\mathcal{A}^{q_\gamma} \cap \gamma)$. By Fodor's lemma we can shrink S so that $\hat{\gamma}$ and $\mathcal{A}_0 = \mathcal{A}^{q_\gamma} \cap \hat{\gamma}$ and $q_\gamma\restriction\hat{\gamma}$ are constant for $\gamma \in S$. Now choose $\delta_1 < \delta_2$ in S, and let $q_i = q_{\delta_i}$, $\mathcal{A}_i = \mathcal{A}^{q_i}$ for $i = 1, 2$, so $\mathcal{A}_1 = \mathcal{A}^{q_1} = \mathcal{A}_0 \cup \{\delta_1\}$, $\mathcal{A}_2 = \mathcal{A}^{q_2} = \mathcal{A}_0 \cup \{\delta_2\}$; also let $\mathcal{A} =: \mathcal{A}_1 \cup \mathcal{A}_2$; we now let $q_i\restriction\hat{\gamma}$ be called q_0. Let $\underset{\sim}{\mathcal{F}}^i = \underset{\sim}{\mathcal{F}}^{q_i}$, and set

$(\underset{\sim}{d}) \qquad \underset{\sim}{d} =: \{n : \Gamma^1_{\underset{\sim}{k}^1(n)} \models R(\underset{\sim}{x}_{\delta_1}(n), \underset{\sim}{x}_{\delta_2}(n)) \iff \Gamma^2_{\underset{\sim}{k}^2(n)} \models \neg R(\underset{\sim}{z}_{\delta_1}(n), \underset{\sim}{z}_{\delta_2}(n))\}.$

We want to find $r \in \mathcal{AP}$ with $\mathcal{A}^r = \mathcal{A}$ so that $r \geq q_1, q_2$, and $\mathtt{p} \Vdash$ "$\underset{\sim}{d} \in \underset{\sim}{\mathcal{F}}^r$". This will then mean that $\underset{\sim}{F}$ could have been "killed", after all, and will complete the argument.

Suppose this is not possible, and thus as in 3.9 (1) for some $\mathtt{p}' \geq \mathtt{p}$ in $\mathbb{P}\restriction\mathcal{A}$, if $\mathtt{p}'_i = \mathtt{p}'\restriction\mathcal{A}_i$ for $i = 0, 1, 2$, we have:

a $(\mathbb{P}\restriction\mathcal{A}_1)$-name $\underset{\sim}{a}$ of a member of $\underset{\sim}{\mathcal{F}}^1$;

a $(\mathbb{P}\restriction\mathcal{A}_2)$-name $\underset{\sim}{b}$ of a member of $\underset{\sim}{\mathcal{F}}^2$; and

a $\mathbb{P}$-name $\underset{\sim}{c} =: \{n : \underset{\sim}{x}_{\delta_2}(n) \in \Gamma^1_{\underset{\sim}{k}^1(n)} \setminus \underset{\sim}{A}_n\}$ associated with a $(\mathbb{P}\restriction\mathcal{A}_1)$-name $(\underset{\sim}{A}_n)_{n<\omega}$ of an $(\underset{\sim}{\mathcal{F}}^1, \underset{\sim}{k}^1)$-slow sequence; with

$$\mathtt{p}' \Vdash_{\mathbb{P}\restriction\mathcal{A}} \text{“}\underset{\sim}{a} \cap \underset{\sim}{b} \cap \underset{\sim}{c} \cap \underset{\sim}{d} = \emptyset\text{”}$$

We shall get a contradiction. Let $H^0 \subseteq \mathbb{P}\restriction\mathcal{A}^0$ be generic over V.

We define for every n the following $(\mathbb{P}\restriction\mathcal{A}^0)$-names:

$$\underset{\sim}{can}^1_n[H^0] = \{(u,v) \in \Gamma^1_{\underset{\sim}{k}^1(n)} \times \Gamma^2_{\underset{\sim}{k}^2(n)} : \text{For some } \mathtt{p}''_1 \in \mathbb{P}\restriction\mathcal{A}_1 \text{ with } \mathtt{p}''_1 \geq \mathtt{p}'_1 \text{ and } \mathtt{p}''_1\restriction\mathcal{A}_0 \in \mathtt{H}^0,$$
$$\mathtt{p}''_1 \Vdash_{\mathbb{P}\restriction\mathcal{A}_1/\mathtt{H}^0} \text{“}[\underset{\sim}{z}_{\delta_1}(n) = u,\ u \notin \underset{\sim}{A}_n,\ n \in \underset{\sim}{a} \text{ and } \underset{\sim}{z}_{\delta_1}(n) = v]\text{”}\}$$

$$\underset{\sim}{can}^2_n[H^0] = \{(u,v) \in \Gamma^1_{\underset{\sim}{k}^1(n)} \times \Gamma^2_{k^2(n)} : \text{For some } \mathtt{p}''_2 \in \mathbb{P}\restriction\mathcal{A}_2 \text{ with } \mathtt{p}''_2 \geq \mathtt{p}'_2 \text{ and } \mathtt{p}''_2\restriction\mathcal{A}_0 \in \mathtt{H}^0,$$
$$\mathtt{p}''_2 \Vdash_{\mathbb{P}\restriction\mathcal{A}_2/\mathtt{H}^0} \text{“}[\underset{\sim}{z}_{\delta_2}(n) = u,\ n \in \underset{\sim}{b} \text{ and } \underset{\sim}{z}_{\delta_2}(n) = v]\text{”}\}$$

and for $i = 1, 2$ and $u \in \Gamma^1_{\underset{\sim}{k}^1(n)}$ we let

$$\underset{\sim}{can}^i_n(u) =: \{v \in \Gamma^2_{\underset{\sim}{k}^2(n)} : \ (u,v) \in \underset{\sim}{can}^i_n\}$$

$$\underset{\sim}{A}^i_n =: \{u : (\exists v)(u,v) \in \underset{\sim}{can}^i_n\}$$

Now in $V[\mathtt{H}^0]$, $(\underset{\sim}{A}^i_n : n < \omega)$ is not $(\underset{\sim}{\mathcal{F}}^i, \underset{\sim}{k}^1)$-slow, and thus the set:

$$\{n : |\underset{\sim}{A}^1_n| / \|\Gamma^1_{\underset{\sim}{k}^1(n)}\|, |\underset{\sim}{A}^2_n| / \|\Gamma^1_{\underset{\sim}{k}^1(n)}\| \text{ are greater than } \|\Gamma^1_{\underset{\sim}{k}^1(n)}\|^{-1/2}\}$$

belongs to $\underset{\sim}{\mathcal{F}}^0[\mathtt{H}]$. Choose any such n, and by finite combinatorics we shall derive a contradiction. Remember that we have assumed without loss of generality that $\|\Gamma^1_{\underset{\sim}{k}^1(n)}\| > \|\Gamma^2_{\underset{\sim}{k}^2(n)}\|^5$ for a large set of n modulo $\underset{\sim}{\mathcal{F}}\restriction\mathcal{A}_0$, so wlog our n satisfies this, too. Let $g_i : \underset{\sim}{A}^i_n \longrightarrow \Gamma^2_{\underset{\sim}{k}^2(n)}$ be such that $g_i(v) \in \underset{\sim}{can}^i_n(v)$. Now $|\text{range}\,(g_i)| \leq \|\Gamma^2_{\underset{\sim}{k}^2(n)}\|$, so there are $b_1, b_2 \in \Gamma^2_{\underset{\sim}{k}^2(n)}$ such that for $i = 1, 2$:

$$|g_i^{-1}(b_i)| \geq |\underset{\sim}{A}^i_n| / \|\Gamma^2_{\underset{\sim}{k}^2(n)}\| > \|\Gamma^1_{\underset{\sim}{k}^1(n)}\|^{1/5}.$$

Now by 3.4(ii) we find $a_i, a'_i \in g^{-1}(b_i)$ for $i = 1, 2$ with $\Gamma^1_{\underset{\sim}{k}^1(n)} \models R(a_1, a_2) \& \neg R(a'_1, a'_2)$. As either $\Gamma^1_{\underset{\sim}{k}^1(n)} \models R(b_1, b_2)$ or $\Gamma^1_{\underset{\sim}{k}^1(n)} \models \neg R(b_1, b_2)$, we can show that it is not forced by $\mathtt{p}'$ that $n \notin \underset{\sim}{a} \cap \underset{\sim}{b} \cap \underset{\sim}{c} \cap \underset{\sim}{d}$, a contradiction. ∎

3.13 Proof of the Claim 3.12A from 3.12

We first recall the situation. We had:

(i) δ is $(\underset{\sim}{F}, \underset{\sim}{k}^1, \underset{\sim}{k}^2, \mathtt{p})$-closed; $q_0 = q \restriction \delta$;

(ii) $\mathtt{p} \in \mathbb{P} \restriction \mathcal{A}^{q_0}$, $\delta \in \mathcal{A}^q$, $\varepsilon^q = 1$, $\underset{\sim}{k}^q_\delta = \underset{\sim}{k}^1$, $\underset{\sim}{F}_\delta = \underset{\sim}{F} \restriction \delta$;

(iii) $\underset{\sim}{z}$ is a $(\mathbb{P} \restriction \mathcal{A}^q)$-name for a real;

(iv) For all $r \geq q$ in $\mathcal{AP}$ such that $r \restriction \delta \in G^\delta$, and $\underset{\sim}{x}$ a $(\mathbb{P} \restriction \mathcal{A}^{r \restriction \delta})$-name, with $\underset{\sim}{y} =: \underset{\sim}{F}(\underset{\sim}{x})$ a $(\mathbb{P} \restriction \mathcal{A}^{r \restriction \delta})$-name, we have:

$(*)_{\underset{\sim}{x},\underset{\sim}{y}}$ $\mathtt{p} \Vdash$ "The set $\{n : \Gamma^1_{\underset{\sim}{k}^1(n)} \models R(\underset{\sim}{x}(n), \underset{\sim}{x}_\delta(n))$ iff $\Gamma^2_{\underset{\sim}{k}^2(n)} \models R(\underset{\sim}{y}(n), \underset{\sim}{z}(n))\}$ is in $\underset{\sim}{\mathcal{F}}^r$".

We defined the property $(**)_{\underset{\sim}{x},\underset{\sim}{y}}$ as follows:

$(**)_{\underset{\sim}{x},\underset{\sim}{y}}$ For all $r \geq q$ in $\mathcal{AP}$ such that $r \restriction \delta \in G^\delta$ and $\underset{\sim}{x}$, $\underset{\sim}{y}$ are $(\mathbb{P} \restriction \mathcal{A}^{r \restriction \delta})$-names, $(*)_{\underset{\sim}{x},\underset{\sim}{y}}$ holds.

Claim

If $\underset{\sim}{x}^1, \underset{\sim}{x}^2$ are $(\mathbb{P} \restriction \delta)$-names of functions in $\prod_n \Gamma^1_{\underset{\sim}{k}^1(n)}$, $\underset{\sim}{y}$ is a $(\mathbb{P} \restriction \delta)$-name of a member of $\prod_n \Gamma^2_{\underset{\sim}{k}^2(n)}$, and both pairs $(\underset{\sim}{x}^1, \underset{\sim}{y})$ and $(\underset{\sim}{x}^2, \underset{\sim}{y})$ satisfy the condition $(**)_{\underset{\sim}{x},\underset{\sim}{y}}$ above, then:

$$\mathtt{p} \Vdash_{\mathbb{P} \restriction \delta} \text{“}\underset{\sim}{x}^1 = \underset{\sim}{x}^2 \bmod \underset{\sim}{\mathcal{F}} \restriction \delta[\mathtt{H}] \text{ or both are restricted for } (\mathtt{H}, \mathcal{A}^{q_0}, \underset{\sim}{k}^1).\text{”}$$

Proof:

Suppose that $\mathtt{p} \leq \hat{\mathtt{p}} \in \mathbb{P} \restriction \delta$ and $\hat{\mathtt{p}}$ forces the contrary; so without loss of generality

$$(5) \qquad \hat{\mathtt{p}} \Vdash \text{“}\underset{\sim}{x}^1 \neq \underset{\sim}{x}^2 \bmod \underset{\sim}{\mathcal{F}} \restriction \delta[\underset{\sim}{\mathtt{H}}]\text{”};$$

$$(6) \qquad \hat{\mathtt{p}} \Vdash \text{“}\underset{\sim}{x}^1 \text{ is unrestricted for } (\mathtt{H}, \mathcal{A}^{q_0}, \underset{\sim}{k}^1).\text{”}$$

Choose any $q_1 \geq q_0$ with $q_1 \in G^\delta$ so that $\underset{\sim}{x}^1, \underset{\sim}{x}^2, \underset{\sim}{y}$ are $\mathbb{P} \restriction \mathcal{A}^{q_1}$-names. Now we will construct $r \geq q_1, q \restriction (\delta + 1)$, with r in $\mathcal{AP}$ and $\mathcal{A}^r = \mathcal{A}^{q_1} \cup \{\delta\}$, so that:

$$(7) \qquad \hat{\mathtt{p}} \Vdash \text{“}\{n : \Gamma^1_{\underset{\sim}{k}^1(n)} \models \text{‘}R(\underset{\sim}{x}^1(n), \underset{\sim}{x}_\delta(n)) \iff \neg R(\underset{\sim}{x}^2(n), \underset{\sim}{x}_\delta(n))]\} \in \underset{\sim}{\mathcal{F}}^r.\text{”}$$

By 3.9(2) we can also find $r' \geq r, q$, and then (7) contradicts $(**)_{\underset{\sim}{x}^1,\underset{\sim}{y}}$ & $(**)_{\underset{\sim}{x}^2,\underset{\sim}{y}}$. Thus to complete the proof of our claim, it suffices to find r.

This is the sort of problem considered in 3.9(1), with an additional set required to be in $\underset{\sim}{\mathcal{F}} \restriction (\mathcal{A}^{q_1} \cup \{\delta\})$. The q_0, q_1 under consideration here correspond to the q_0, q_1 of 3.9(1), and we let q_2 be $q \restriction (\delta + 1)$. Following the notation of 3.9(1), set $\underset{\sim}{\mathcal{F}}^i = \underset{\sim}{\mathcal{F}}^{q_i}$, $\mathcal{A}_i = \mathcal{A}^{q_i}$ for $i = 0, 1, 2$, and $\mathcal{A} = \mathcal{A}_1 \cup \{\delta\} = \mathcal{A}_1 \cup \mathcal{A}_2$. We need to find $r \geq q_1, q_2$ as in 3.9(1), with (7) holding.

Suppose on the contrary that $\hat{\mathtt{p}} \leq \mathtt{p}' \in \mathbb{P}\restriction\mathcal{A}$ and $\mathtt{p}'$ forces "There is no $\underset{\sim}{\mathcal{F}}$ as required". Then extending $\mathtt{p}'$, we may suppose that we have a $\mathbb{P}\restriction\mathcal{A}_1$-name $\underset{\sim}{a}$ for a member of $\underset{\sim}{\mathcal{F}}^1$, a $\mathbb{P}\restriction\mathcal{A}_2$-name $\underset{\sim}{b}$ for a member of $\underset{\sim}{\mathcal{F}}^2$, a $\mathbb{P}\restriction\mathcal{A}_1$-name for an $(\underset{\sim}{\mathcal{F}}^1, \underset{\sim}{k}^1)$-slow sequence $(\underset{\sim}{A}_n)$ (associated with a power $d < \omega$ – cf. 3.7), such that setting:

$$\underset{\sim}{c} = \{n : \underset{\sim}{x}_\delta(n) \in \Gamma^1_{\underset{\sim}{k}^1(n)} \setminus \underset{\sim}{A}_n\}$$

$$\underset{\sim}{d} = \{n : \Gamma^1_{\underset{\sim}{k}^1(n)} \models \text{“}R(\underset{\sim}{x}^1(n), \underset{\sim}{x}_\delta(n)) \iff \neg R(\underset{\sim}{x}^2(n), \underset{\sim}{x}_\delta(n))\text{”}\}$$

we have:

$$\mathtt{p}' \Vdash_{\mathbb{P}\restriction\mathcal{A}} \text{“}\underset{\sim}{a} \cap \underset{\sim}{b} \cap \underset{\sim}{c} \cap \underset{\sim}{d} = \emptyset\text{”}$$

Let $\mathtt{p}'_i = \mathtt{p}'\restriction\mathcal{A}_i$ for $i = 0, 1, 2$, and take $\mathtt{H}^0 \subseteq \mathbb{P}\restriction\mathcal{A}_0$ generic over V. Without loss of generality, for some natural number d:

$$\mathtt{p}'_1 \Vdash \text{“}n \in \underset{\sim}{a} \Longrightarrow \underset{\sim}{x}^1(n) \neq \underset{\sim}{x}^2(n) \text{ and } |\underset{\sim}{A}_n| \leq \sqrt{\|\Gamma^1_{\underset{\sim}{k}^1(n)}\|} \cdot (\log \|\Gamma^1_{\underset{\sim}{k}^1(n)}\|)^d \text{ (and } \underset{\sim}{A}_n \subseteq \Gamma^1_{\underset{\sim}{k}^1(n)}).\text{”}$$

We are interested in $\underset{\sim}{B}_n[\mathtt{H}^0]$=:

$$\{v \in \Gamma^1_{\underset{\sim}{k}^1(n)} : \text{ for some } \mathtt{p}''_2 \geq \mathtt{p}'_2 \text{ with } \mathtt{p}''_2\restriction\mathcal{A}_0 \in \mathtt{H}^0,\ \mathtt{p}''_2 \Vdash_{\mathbb{P}\restriction\mathcal{A}_2} \text{:“}n \in \underset{\sim}{b} \text{ and } \underset{\sim}{x}_\delta(n) = v\text{”}\}$$

(which is a $(\mathbb{P}\restriction\mathcal{A}_0)$-name). Clearly the sequence $(\underset{\sim}{B}_n)$ is not $(\underset{\sim}{\mathcal{F}}^1, \underset{\sim}{k}^1)$-slow in $V[\mathtt{H}^0]$.

For each n let us also consider the set $\underset{\sim}{Y}_n[\mathtt{H}^0]$ =:

$$\{(A, v_1, v_2) : A \cup \{v_1, v_2\} \subseteq \Gamma^1_{\underset{\sim}{k}^1(n)}, v_1 \neq v_2, \text{ and for some } \mathtt{p}''_1 \text{ with } \mathtt{p}''_1 \geq \mathtt{p}'_1, \mathtt{p}''_1\restriction\mathcal{A}_0 \in \mathtt{H}^0,$$

$$\mathtt{p}''_1 \Vdash \text{“}n \in \underset{\sim}{a}, \underset{\sim}{A}_n = A, \underset{\sim}{x}^1(n) = v_1, \underset{\sim}{x}^2(n) = v_2.\text{”}\}$$

For every $(A, v_1, v_2) \in \underset{\sim}{Y}_n$, we have:

(8) $$|A| \leq \sqrt{\|\Gamma^1_{\underset{\sim}{k}^1(n)}\|} \cdot (\log \|\Gamma^1_{\underset{\sim}{k}^1(n)}\|)^d, \text{ and } v_1 \neq v_2$$

As $\underset{\sim}{x}_\delta$ is unrestricted over $\mathcal{A}_0$ in $V[\mathtt{H}^0]$, for the $\underset{\sim}{\mathcal{F}}^0$-majority of n we have:

(9) $$|\underset{\sim}{B}_n| \geq \sqrt{\|\Gamma^1_{\underset{\sim}{k}^1(n)}\|} \cdot (\log \|\Gamma^1_{\underset{\sim}{k}^1(n)}\|)^{d+2}$$

Now (by (6)), also for the $\underset{\sim}{\mathcal{F}}^0$ majority of n we have:

(10) $$\underset{\sim}{C}_n =: \{v_1 \in \Gamma^1_{\underset{\sim}{k}^1(n)} : \text{There are } A, v_2 \text{ so that } (A, v_1, v_2) \in \underset{\sim}{Y}_n\}$$

$$\text{has at least } \sqrt{\|\Gamma^1_{\underset{\sim}{k}^1(n)}\|} \text{ members}$$

Now it will suffice to find n, $v \in \underset{\sim}{B}_n$ and $(A, v_1, v_2) \in \underset{\sim}{Y}_n$ so that

(11) $$\Gamma^1_{\underset{\sim}{k}^1(n)} \models [R(v_1, v) \iff \neg R(v_2, v)] \,\&\, v \notin A,$$

as we can then choose $\mathtt{p}''_1 \in \mathbb{P}\restriction\mathcal{A}_1$, $\mathtt{p}''_2 \in \mathbb{P}\restriction\mathcal{A}_2$ with $\mathtt{p}''_i \geq \mathtt{p}'_i$, $\mathtt{p}''_i\restriction\mathcal{A}_0 \in \mathtt{H}^0$ for $i = 1, 2$, so that:

$$\mathtt{p}''_1 \Vdash \text{“}n \in \underset{\sim}{a}, \underset{\sim}{A}_n = A, \underset{\sim}{x}^1(n) = v_1, \underset{\sim}{x}^2(n) = v_2\text{”}; \qquad \mathtt{p}''_2 \Vdash \text{“}n \in \underset{\sim}{b} \text{ and } \underset{\sim}{x}_\delta(n) = v\text{”}$$

and hence $\mathtt{p}''_1 \cup \mathtt{p}''_2 \Vdash$ “$n \in \underset{\sim}{a} \cap \underset{\sim}{b} \cap \underset{\sim}{c} \cap \underset{\sim}{d}$”, a contradiction.

So it remains to find n, v and (A, v_1, v_2). For n sufficiently large satisfying (8-10), we can choose triples $t_i = (A^i, v^i_1, v^i_2) \in \underset{\sim}{Y}_n$ for $i < 5 \log \|\Gamma^1_{\underset{\sim}{k}^1(n)}\|$ with all vertices v^i_1 distinct from each other and from all v^i_2. By the pseudorandomness of $\Gamma^1_{\underset{\sim}{k}^1(n)}$ (more specifically 3.4(iii)), the set

$\underset{\sim}{S} = \{v \in \Gamma^1_{\underset{\sim}{k}^1(n)} :$ For no $i < 5 \log \|\Gamma^1_{\underset{\sim}{k}^1(n)}\|$ do we have $R(v^i_1, v) \iff \neg R(v^i_2, v)\}$ has size at most $5 \log \|\Gamma^1_{\underset{\sim}{k}^1(n)}\|$. So if $\underset{\sim}{S}' =: \underset{\sim}{S} \cup \bigcup\{A^i : i < 5 \log \|\Gamma^1_{\underset{\sim}{k}^1(n)}\|\}$, then we will have: $|\underset{\sim}{S}'| \ll \sqrt{\|\Gamma^1_{\underset{\sim}{k}^1(n)}\|}(\log \|\Gamma^1_{\underset{\sim}{k}^1(n)}\|)^{d+2}$, so there is $v \in \underset{\sim}{B}_n \setminus \underset{\sim}{S}'$. Since $v \notin \underset{\sim}{S}'$, for some i (11) will hold with $(A, v_1, v_2) = (A^i, v^i_1, v^i_2)$. ∎

3.14 The last detail

The following was used in the proof of 3.12 (after 3.12A slightly before ($\underset{\sim}{d}$)).

Claim.

Assume $q_2\restriction\beta \leq q_1$, $\mathcal{A}^{q_1} \subseteq \beta$. Let $q_0 = q_2\restriction\beta$, and write $\mathcal{A}_i$ for $\mathcal{A}^{q_i}$, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, and $\mathcal{F}^i$ for $\mathcal{F}^{q_i}$. Let $\mathtt{p} \in \mathbb{P}\restriction\mathcal{A}$ and $\mathtt{p}_i = \mathtt{p}\restriction\mathcal{A}_i$. Then we can find r with $\mathcal{A}^r = \mathcal{A}$ and $r \geq q_1, q_2$, so that for any $(\mathbb{P}\restriction\mathcal{A}_i)$-names $\underset{\sim}{y}_i$ $(i = 1, 2)$ of members of $\prod_n \Gamma^2_{\underset{\sim}{k}^2(n)}$ if:

$$\mathtt{p}_i \Vdash_{\mathbb{P}\restriction\mathcal{A}_i} \text{“}\underset{\sim}{y}_i \neq \underset{\sim}{y}' \bmod \underset{\sim}{\mathcal{F}}^i\text{”}$$

for $(i = 1, 2)$ and for all $(\mathbb{P}\restriction\mathcal{A}_0)$-names $\underset{\sim}{y}'$, then we have:

$$\mathtt{p} \Vdash_{\mathbb{P}\restriction\mathcal{A}} \text{“}\underset{\sim}{y}_1 \neq \underset{\sim}{y}_2 \quad \bmod \underset{\sim}{\mathcal{F}}^r\text{”}$$

Hence $p \Vdash_{\mathbb{P}\restriction\mathcal{A}}$ “if $\underset{\sim}{y}_i \neq \underset{\sim}{y}' \bmod \underset{\sim}{\mathcal{F}}^i$ for $i = 1, 2$ and $\underset{\sim}{y}'$ a $(\mathbb{P}\restriction\mathcal{A}_0)$-name then $\underset{\sim}{y}_1 \neq \underset{\sim}{y}_2 \bmod \underset{\sim}{\mathcal{F}}^r$”.

Proof: We use induction construction. Much as in the proof of 3.9, we must deal primarily with the case in which $\mathcal{A}^{q_2} = \mathcal{A}^{q_0} \cup \{\beta\}$. Suppose toward a contradiction that $\mathtt{p} \le \mathtt{p}' \in \mathbb{P}\restriction\mathcal{A}$, and with $\mathtt{p}'_i = \mathtt{p}'\restriction\mathcal{A}_i$ for $i = 0, 1, 2$ we have:

i. a $(\mathbb{P}\restriction\mathcal{A}_1)$-name $\underset{\sim}{a}$ of a member of $\underset{\sim}{\mathcal{F}}^1$;

ii. a $(\mathbb{P}\restriction\mathcal{A}_2)$-name $\underset{\sim}{b}$ of a member of $\underset{\sim}{\mathcal{F}}^2$;

iii. a $(\mathbb{P}\restriction\mathcal{A})$-name $\underset{\sim}{c} = \{n : \underset{\sim}{x}_\beta(n) \in \Gamma^2_{\underset{\sim}{k}^2(n)} \backslash \underset{\sim}{A}_n\}$ associated with a $(\mathbb{P}\restriction\mathcal{A}_1)$-name $(\underset{\sim}{A}_n)_{n<\omega}$ of a $(\underset{\sim}{\mathcal{F}}^1, \underset{\sim}{k}^1)$-slow sequence; and

iv. a $(\mathbb{P}\restriction\mathcal{A})$-name $\underset{\sim}{d} = \bigcap_{j=1}^N \underset{\sim}{d}_j$, for a finite intersection of sets of the form $\underset{\sim}{d}_j =: \{n : \underset{\sim}{y}^1_j(n) \ne \underset{\sim}{y}^2_j(n)\}$, with each $\underset{\sim}{y}^i_j$ a $\mathbb{P}\restriction\mathcal{A}_i$-name of a member of $\prod_n \Gamma^2_{\underset{\sim}{k}^2(n)}$, such that for each $i = 1, 2$ and $j = 1, \ldots, N$:

$$\mathtt{p}_i \Vdash \text{“}\underset{\sim}{y}^i_j \ne \underset{\sim}{y}' \mod \underset{\sim}{\mathcal{F}}^i \text{ for any } (\mathbb{P}\restriction\mathcal{A}_0)\text{-name } \underset{\sim}{y}' \text{ of a member of } \textstyle\prod_n \Gamma^2_{\underset{\sim}{k}^2(n)}.\text{”}$$

and that $\mathtt{p}' \Vdash$ “$\underset{\sim}{a} \cap \underset{\sim}{b} \cap \underset{\sim}{c} \cap \underset{\sim}{d} = \emptyset$”. Let $\mathtt{H}^0$ be generic over V, $\mathtt{p}_0 \in \mathtt{H}^0$, and let us define in $V[\mathtt{H}^0]$:

$$\underset{\sim}{\Delta}^1_n[\mathtt{H}^0] =: \{(A, u_1, \ldots, u_N) : A \subseteq \Gamma^2_{\underset{\sim}{k}^2(n)},\ u_1, \ldots, u_N \in \Gamma^2_{\underset{\sim}{k}^2(n)},$$
$$\text{and there is } \mathtt{p}'_1 \in \mathbb{P}\restriction\mathcal{A}_1,\ \mathtt{p}'_1 \ge \mathtt{p}_1,\ \mathtt{p}'_1\restriction\mathcal{A}_0 \in \mathtt{H}^0, \text{ and}$$
$$\mathtt{p}'_1 \Vdash_{\mathbb{P}\restriction\mathcal{A}_1} \text{“}\underset{\sim}{A}_n = A,\ \underset{\sim}{y}^1_1(n) = u_1, \ldots, \underset{\sim}{y}^1_N(n) = u_N, \text{ and } n \in \underset{\sim}{a}\text{”}.\}$$

$$\underset{\sim}{\Delta}^2_n[\mathtt{H}^0] =: \{(v_0, v_1, \ldots, v_N) : \text{ all } v_j \in \Gamma^2_{\underset{\sim}{k}^2(n)} \text{ and there is}$$
$$\mathtt{p}'_2 \in \mathbb{P}\restriction\mathcal{A}_2,\ \mathtt{p}' \ge \mathtt{p}_2,\ \mathtt{p}'_2\restriction\mathcal{A}_0 \in \mathtt{H}^0 \text{ and}$$
$$\mathtt{p}'_2 \Vdash_{\mathbb{P}\restriction\mathcal{A}_2} \text{“}\underset{\sim}{x}_\beta(n) = v_0,\ \underset{\sim}{y}^2_1(n) = v_1 \ldots, \underset{\sim}{y}^2_N(n) = v_N \text{ and } n \in \underset{\sim}{b}\text{”}\}$$

Without loss of generality, for some d,

$$\mathtt{p}_1 \Vdash\text{: “For } n \in \underset{\sim}{a},\ |\underset{\sim}{A}_n| \le \sqrt{\|\Gamma^2_{\underset{\sim}{k}^2(n)}\|} \cdot (\log \|\Gamma^2_{\underset{\sim}{k}^2(n)}\|)^d.\text{”}$$

Thus:

(1) $$(A, u_1, \ldots, u_N) \in \underset{\sim}{\Delta}^1_n \Longrightarrow |A| \le \sqrt{\|\Gamma^2_{\underset{\sim}{k}^2(n)}\|} \cdot (\log \|\Gamma^2_{\underset{\sim}{k}^2(n)}\|)^d.$$

By the assumption on $\underset{\sim}{y}^1_1, \ldots, \underset{\sim}{y}^1_N$,

(2) If $e < \omega$, $C_n \subseteq \Gamma^2_{\underset{\sim}{k}^2(n)}$, $|\underset{\sim}{C}_n| \le e$, and $(\underset{\sim}{C}_n : n < \omega) \in V[\mathtt{H}^0]$, then

$$\{n : \text{ there is } (A, u_1, \ldots, u_N) \in \underset{\sim}{\Delta}^1_n,\ u_1, \ldots, u_N \notin \underset{\sim}{C}_n\ \} \in \underset{\sim}{\mathcal{F}}^0.$$

Hence without loss of generality:

For $n \in \underset{\sim}{a}$, there are $(A, u_1^j, \ldots, u_N^j) \in \underset{\sim}{\Delta}_n^1$, for $j \leq N+1$, with

(3) The sets $\{u_1^j, \ldots, u_N^j\}$ (for $j \leq N+1$) pairwise disjoint.

As $q_2 \in \mathcal{AP}$,

(4) If $(\underset{\sim}{C}_n : n < \omega) \in V[\mathtt{H}^0]$ is $(\underset{\sim}{\mathcal{F}}^0, \underset{\sim}{k})$-slow then

$$\{n : \text{There is } (v_0, v_1, \ldots, v_N) \in \underset{\sim}{\Delta}_n^2 \text{ with } v_0 \notin \underset{\sim}{C}_n\} \in \underset{\sim}{\mathcal{F}}^0$$

Let $\underset{\sim}{a}^+ =: \{n : \underset{\sim}{\Delta}_n^1 \neq \emptyset, \underset{\sim}{\Delta}_n^2 \neq \emptyset$, moreover, $\underset{\sim}{\Delta}_n^1$ satisfies (3)$\}$ (a $\mathbb{P}{\restriction}\mathcal{A}^0$-name of a member of $\mathcal{F}^0$). So for $n \in \underset{\sim}{a}^+$, there are $(N+1)$-tuples $(A^{n,j}, u_1^{n,j}, \ldots, u_N^{n,j})$ for $j \leq N+1$ with the sets $\{u_1^{n,j}, \ldots, u_N^{n,j}\}$ pairwise disjoint. Let $\underset{\sim}{C}_n = \bigcup_{j \leq N} A^{n,j}$ for $n \in \underset{\sim}{a}^+$, $\underset{\sim}{C}_n = \emptyset$ for $n \notin \underset{\sim}{a}^+$. So $(\underset{\sim}{C}_n)_{n<\omega} \in V[\mathtt{H}^0]$ is $(\underset{\sim}{\mathcal{F}}^0, \underset{\sim}{k}^1)$-slow, hence for some $n \in \underset{\sim}{a}^+$, there is $(v_0, v_1, \ldots, v_N) \in \underset{\sim}{\Delta}_n^2$, with $v_0 \notin \underset{\sim}{C}_n$. Now for some $j \leq N+1$ we have $\bigwedge_{i=1}^N v_i \neq u_i^{n,j}$. Choose $\mathtt{p}_2' \in \mathbb{P}{\restriction}\mathcal{A}_2$, $\mathtt{p}_2' \geq \mathtt{p}_2$, with $\mathtt{p}_2'{\restriction}\mathcal{A}_0 \in \mathtt{H}^0$ and $\mathtt{p}_2' \Vdash$ "$n \in \underset{\sim}{b}$, $\underset{\sim}{x}_\beta(n) = v_0, \bigwedge_{i=1}^N \underset{\sim}{y}_i^2(n) = v_i$". Choose $\mathtt{p}_1' \in \mathbb{P}{\restriction}\mathcal{A}_1$, $\mathtt{p}_1' \geq \mathtt{p}_1$, with $\mathtt{p}_1'{\restriction}\mathcal{A}_0 \in \mathtt{H}^0$ and $\mathtt{p}_1' \Vdash$ "$n \in \underset{\sim}{a}$, $\underset{\sim}{A}_n = A^{n,j}$, and for all $i = 1, \ldots, N$ $\underset{\sim}{y}_i^1(n) = u_i^{n,j}$." Now $\mathtt{p}_1' \cup \mathtt{p}_2' \Vdash$ "$n \in \underset{\sim}{a} \cap \underset{\sim}{b} \cap \underset{\sim}{c} \cap \underset{\sim}{d}$", a contradiction.

This finishes the case $\mathcal{A}_2 = \mathcal{A}_1 \cup \{\beta\}$. The general case follows as in 3.9(2). At successors we apply the case just treated. Limits of uncountable cofinality are handled by taking unions. At limits of cofinality ω we have to repeat the first argument with some variations; we do not have to worry about $\underset{\sim}{c}$, so the fact that there are several $\underset{\sim}{x}_\beta$ involved is not a problem. The problem in this case is of course to extend the union of the ultrafilters constructed so far to an ultrafilter in a slightly larger model of set theory, while retaining the main property for new names $\underset{\sim}{y}_i^2$. ∎

Appendix. Background material.

A1. Proper and α-proper forcing

A1.1. Proper Forcing

Let $\mathcal{P} = (P, \leq)$ be a partially ordered set. A cardinal λ is *$\mathcal{P}$-large* if the power set of P is in V_λ (the universe of all sets of rank less than λ). With $\mathcal{P}$ fixed and λ $\mathcal{P}$-large, let $\mathcal{V}_\lambda$ be the structure $(V_\lambda; \in, P, \leq)$.

1. For $\mathcal{M} \prec \mathcal{V}_\lambda$ and $p \in P$, p is *$\mathcal{M}$-generic* iff for each name of an ordinal $\underset{\sim}{\alpha}$ with $\underset{\sim}{\alpha} \in M$, $p \Vdash$ "$\underset{\sim}{\alpha} \in \mathcal{M}$".
2. $\mathcal{P}$ is *proper* iff for all $\mathcal{P}$-large λ and all countable elementary substructures $\mathcal{M}$ of $\mathcal{V}_\lambda$ with $\mathcal{P} \in \mathcal{M}$, each $p \in \mathcal{M}$ has an $\mathcal{M}$-generic extension in P.

A1.2. Axiom A

$\mathcal{P}$ satisfies *Axiom* A if there is a collection $\leq_n$ $(n = 1, 2, \ldots)$ of partial orderings on the set P with $\leq_1$ coinciding with the given ordering $\leq$, and $\leq_{n+1}$ finer than $\leq_n$ for each n, satisfying the following two conditions:

1. If $p_1 \leq_1 p_2 \leq_2 \leq p_3 \leq_3 \ldots$ then there is some $p \in P$ with $p_n \leq_n p$ for all n;
2. For all $p \in P$, any name $\underset{\sim}{\alpha}$ of an ordinal, and any n, there is a condition $q \in P$ with $p \leq_n q$, and a countable set B of ordinals, such that $q \Vdash \underset{\sim}{\alpha} \in B$.

The forcings used in §§1,2 were seen to satisfy Axiom A, and the following known result was then applied.

A1.3. Proposition

If $\mathcal{P}$ satisfies Axiom A then $\mathcal{P}$ is proper.

Proof:

Given a countable $\mathcal{M} \prec \mathcal{V}_\lambda$ and $p \in P \cap M$, let $\underset{\sim}{\alpha}_n$ be a list of all ordinal names in $\mathcal{M}$, and use clause (2) of Axiom A to find $q_n, B_n \in \mathcal{M}$ with $q_n \in P$, B_n countable. $p \leq_1 q_1 \leq_2 q_2 \leq \ldots$ and $q_n \Vdash$ "$\underset{\sim}{\alpha}_n \in B_n$. Then use clause (1) to find $q \geq$ all q_n; this q will be $\mathcal{M}$-generic. ∎

A1.4 Countable Support Iteration

Our notation for iterated forcing is as follows. $\underset{\sim}{\mathcal{Q}}_\alpha$ is the name of the α-th forcing in the iteration, and $\mathcal{P}_\alpha$ is the iteration up to stage α. The sequence $\mathcal{P}_\alpha$ is called the iteration, and the $\underset{\sim}{\mathcal{Q}}_\alpha$ are called the factors. It is assumed that $\underset{\sim}{\mathcal{Q}}_\alpha$ is a $\mathcal{P}_\alpha$-name for a partially ordered set with minimum element 0, and that $\mathcal{P}_{\alpha+1}$ is $\mathcal{P}_\alpha * \underset{\sim}{\mathcal{Q}}_\alpha$.

In general it is necessary to impose some further conditions at limit ordinals. We will be concerned exclusively with countable support iteration: at a limit ordinal δ, $\mathcal{P}_\delta$ consists of δ-sequences p such that $p \restriction \alpha \in \mathcal{P}_\alpha$ for $\alpha < \delta$, and $\Vdash_{\mathcal{P}_\alpha} p(\alpha) = 0$ for all but countably many $\alpha < \delta$.

A1.5 Proposition

Let $\mathcal{P}_\alpha$ be a countable support iteration of length λ with factors $\underset{\sim}{\mathcal{Q}}_\alpha$ such that for all $\alpha < \lambda$, $\Vdash_{\mathcal{P}_\alpha}$ "$\underset{\sim}{\mathcal{Q}}_\alpha$ is proper." Then $\mathcal{P}_\lambda$ is proper.

See [Sh b, Sh f, or Jech] for the proof.

In §§1,2 we need additional iteration theorems discussed in [Sh b] in the context of ω-proper forcing. Improvements in [Sh 177] or [Sh f] make this unnecessary, but we include a discussion of the relevant terminology here. This makes our discussion compatible with the contents of [Sh b].

A1.6 α-Proper Forcing

Let α be a countable ordinal. Then $\mathcal{P}$ is *α-proper* iff for every $\mathcal{P}$-large λ, every continuous increasing $\alpha+1$-sequence $(\mathcal{M}_i)_{i \le \alpha}$ of countable elementary substructures of $\mathcal{V}_\lambda$ with $\mathcal{P} \in \mathcal{M}_0$, every $p \in P \cap M_0$ has an extension $q \in P$ which is $\mathcal{M}_i$-generic for all $i \le \alpha$.

Axiom A implies α-properness for α countable. For example we check ω-properness. So we consider a condition p in M_0, where $(\mathcal{M}_i)_{i<\omega}$ is a sequence of suitable countable models satisfying, among other things, $\mathcal{M}_i \in \mathcal{M}_{i+1}$. There is an $\mathcal{M}_0$-generic condition p_1 above p, and we can take $p_1 \in \mathcal{M}_1$, since $\mathcal{M}_1 \prec \mathcal{V}_\lambda$. Similarly we can successively find $p_{n+1} \in P \cap M_{n+1}$ with p_{n+1} $\mathcal{M}_n$-generic, and $p_n \le_n p_{n+1}$. A final application of Axim A yields q above all the p_n.

Countable support iteration also preserves α-properness for each α [Sh b]. Furthermore it is proved in [Sh b, V4.3] that countable support iteration preserves the following conjunction of two properties: ω-properness and ${}^\omega\omega$-bounding. So [Sh b] contains most of the information needed in §§1,2, though we will need to add more concerning the iteration theorems below.

A2. Iteration theorems

A2.1 Fine* Covering Models

We recall the formalism introduced in [Sh b, Chap. VI] for proving iteration theorems. We consider collections of subtrees of ${}^{\omega>}\omega$ that cover ${}^{\omega}\omega$ in the sense that every function in ${}^{\omega}\omega$ represents a branch of one of the specified trees, and iterate forcings that do not destroy this property. Of course the precise formulation is considerably more restrictive. See discussion A2.6.

Weak covering models.

A structure $(D;R)$ consisting of a set D and a binary relation R on D is called a *weak covering model* if:

1. For $x,t\in D$, $R(x,t)$ implies that t is a (nonempty) subtree of ${}^{\omega>}\omega$, with no terminal nodes (leaves); we denote the set of branches of t by $\mathrm{Br}\,(t)$.
2. For every $\eta\in{}^{\omega}\omega$, and every $x\in\mathrm{dom}\,R$, there is some $t\in D$ with $R(x,t)$ and $\eta\in\mathrm{Br}\,(T)$. In this case, we say: (D,R) covers ${}^{\omega}\omega$.

$(D;R)$ should be thought of as a suitable small fragment of a universe of sets, and $R(x,t)$ is to be thought of intuitively as saying, in some manner, that the tree t has "size" at most x. In the next definition we introduce an ordering on the "sizes" and exploit more of our intutition, though certain intuitively natural axioms are omitted, as they are never needed in proofs.

Fine* Covering Models.

A structure $\mathcal{D}=(D;R,<)$ is called a *fine* covering model* if $(D;R)$ is a weak covering model, $<$ is a partial order on $\mathrm{dom}\,R$ with no minimal element, and:

(1) If $x,y\in\mathrm{dom}\,R$ with $x<y$, then there is $z\in\mathrm{dom}\,R$ with $x<z<y$ (and $D\neq\emptyset$ and for every $y\in D$ there is $x<y$ in D).

(2) $x<y\,\&\,R(x,t)$-implies $R(y,t)$.

(3) In any generic extension V^* in which $(D;R)$ is a weak covering model we have:

 (∗) for $x<y$ (from $\mathrm{dom}\,R$) and $t_n\in D$ with $R(x,t_n)$ for all n there is $t\in D$ with $R(y,t)$ holding and there are indices $n_0<n_1<\ldots$ such that: for all $\eta\in{}^{\omega}\omega$: if $\eta\restriction n_i\in\bigcup_{j\le i}t_j$ for all i then $\eta\in\mathrm{Br}\,(t)$.

 ⊗ if $\eta\in{}^{\omega}\omega$, $\eta_n\in{}^{\omega}\omega$, $\eta_n\restriction n=\eta\restriction n$ for $n<\omega$ and $x\in\mathrm{dom}\,R$ then for some t, $R(x,t)$, $\eta\in\mathrm{Br}\,(t)$ and for infinitely many n we have $\eta_n\in\mathrm{Br}\,(t)$.

In particular we require (∗) and ⊗ to hold in the original universe V. Observe also that in (3∗) we have in particular $t_0\subseteq t$.

Note that $(3)^+$ below implies (3).

$(3)^+$ In any generic extension V^* (of V) in which (D, R) is a weak covering model we have:

$(*)^+$ For $x < y$ and $t_n \in D$ with $R(x, t_n)$ for all n, there is $t \in D$ with $R(y, t)$ holding and there are indices $0 = n_0 < n_1 < \ldots$ such that for all $\eta \in {}^\omega\omega$ if $\eta \restriction n_i \in \bigcup_{j \le i} t_{n_j}$ for all i, then $\eta \in Br(t)$; we let $w = \{n_0, n_1, \ldots\}$.

[Why $(3)^+ \Rightarrow (3)$? assume $(3)^+$, so let a generic extension V^* of V in which (D, R) is a weak covering model be given, so in V^*, $(*)^+$ holds. First, for $\otimes$ of (3) let η, η_n, y be given, let $x < y$; as "(D, R) is a weak covering model in V^*" for each $n < \omega$ there is $t_n \in D$ such that $R(x, t_n) \& \eta_n \in \mathrm{Br}(t_n)$. Apply $(*)^+$ to x, y, t_n and get t which is as required there. Second, for $(*)$ of (3), let $x < y, t_n (n < \omega)$ be given. Choose inductively $y', x_n, x < x_n < y' < y$, $x_n < x_{n+1}$ (possible by condition (1)). Choose by induction on n, k_n, t^*_n such that: $t^*_0 = t^*, R(x_n, t^*_n), t^*_n \subseteq t^*_{n+1}$ and $[\nu \in t_{n+1} \& \nu \restriction k_n \in t^*_n \Rightarrow \nu \in t^*_{n+1}]$. For $n = 0$-trivial, for $n + 1$ use $(*)^+$ with $\langle x_n, x_{n+1}, t^*_n, t_{n+1}, t_{n+1}, \ldots \rangle$ here standing for $\langle x, y, t_0, t_1, t_2, \ldots \rangle$ there, and we get t^*_{n+1}, w_n (for t, w there), let $k_n = \mathrm{Min}(w_n \setminus \{0\})$, easily t^*_n as required. Now apply $(*)^+$ to $\langle y', y, t^*_0, t^*_1, \ldots \rangle$ and get $t, \langle n_i : i < \omega \rangle$; thining the n_i's we finish].

A forcing notion $\mathcal{P}$ is said to be *$\mathcal{D}$-preserving* if $\mathcal{P}$ forces: "$\mathcal{D}$ is a fine* covering model"; equivalently, $\mathcal{P}$ forces: "$(D; R)$ covers ${}^\omega\omega$." So this means that $\mathcal{P}$ does not add certain kinds of reals.

In this terminology, we can state the following general iteration theorem ([Sh 177],[Sh-f]VI§1, §2):

A2.2 Iteration Theorem

Let $\mathcal{D}$ be a fine covering model. Let $\langle \mathcal{P}_\alpha, \underset{\sim}{\mathcal{Q}}_\beta : \alpha \le \delta, \beta < \delta \rangle$ be a countable support iteration of proper forcing notions with each factor $\mathcal{D}$-preserving. Then $\mathcal{P}_\delta$ is $\mathcal{D}$-preserving.*

Proof:

We reproduce the proof given in [Sh b, pp. 199-202], with the modifications suggested in [Sh 177]. We note that in the present exposition we have suppressed some of the terminology in [Sh b] and made other minor alterations. In particular our statement of the main theorem is slightly weaker than the one given in [Sh f]. We have also suppressed the discussion of variants of condition $(3*)$ in the definition of fine* covering model, which occurs on pages 197-198 of [Sh b]; as a result we leave a little more to the reader.

By [Sh b, V4.4], if δ is of uncountable cofinality then there is no problem, as all new reals are added at some earlier point. So we may suppose that $\mathrm{cf}\,\delta = \aleph_0$ hence by associativity of CS iterations of proper forcing ([Sh-b], III) without loss of generality $\delta = \omega$.

We claim that $\Vdash_{\mathcal{P}_\omega}$ "$(D; R)$ covers ${}^\omega\omega$." (Note that this suffices for the proof of the iteration theorem.)

Fix $x \in \operatorname{dom} R$, $p \in \mathcal{P}_\omega$, $\underset{\sim}{f}$ a $\mathcal{P}_\omega$-name with $p \Vdash$ "$\underset{\sim}{f} \in {}^\omega\omega$." We need to find an extension p' of p and a tree $t \in D$ with $R(x,t)$ such that $p' \Vdash$ "$\underset{\sim}{f} \in \mathrm{Br}\,(t)$." As in the proof that countable support iteration preserves properness, we may assume without loss of generality (after increasing p) that $\underset{\sim}{f}(n)$ is a $\mathcal{P}_n$-name for all n.

By induction on n we define conditions $p^n \in \mathcal{P}_n$ and $\mathcal{P}_m$-names $\underset{\sim}{t}_{m,n}$ for $m \le n$ with the following properties:

(1) $\Vdash_{\mathcal{P}_i}$ "$p(i) \le p^n(i) \le p^{n+1}(i)$" for $i < n$;

(2) If $G_m \subseteq \mathcal{P}_m$ is generic with $m \le n$, then in $V[G_m]$ we have $(p^n(m), \dots, p^n(n-1)) \Vdash_{\mathcal{P}_n/\mathcal{P}_m}$ "$\underset{\sim}{f}(n) = \underset{\sim}{t}_{m,n}$."

This is easily done; for each n, we increase p^n n times, once for each possible m. By (1) we have $p \restriction n \le p^n \le p^{n+1}$.

We let $\underset{\sim}{f}_m$ be the $\mathcal{P}_m$-name for an element of ${}^\omega\omega$ satisfying: $\underset{\sim}{f}_m(n) = \underset{\sim}{t}_{m,n}$ for $n \ge m$, $\underset{\sim}{f}_m(n) = \underset{\sim}{f}(n)$ for $n < m$. Then we have:

(3) $(0, \dots, 0, p^n(m)) \Vdash_{\mathcal{P}_{m+1}}$ "$\underset{\sim}{f}_m \restriction n = \underset{\sim}{f}_{m+1} \restriction n$"

(4) $\Vdash_{\mathcal{P}_n}$ "$\underset{\sim}{f} \restriction n = \underset{\sim}{f}_n \restriction n$."

Choose $x_1 < x' < x$ and then inductively $x_1 < x_2 < \dots$ with all $x_n < x'$, and choose a countable $N \prec V_\lambda$ (with λ $\mathcal{P}$-large) such that all the data $(x_n)_{n<\omega}$, $(\mathcal{P}_n, \mathcal{Q}_n)_{n<\omega}$, $\underset{\sim}{f}$, $(p^n)_{n<\omega}$, $(\underset{\sim}{t}_{m,n})_{m \le n < \omega}$ lie in N. We will define conditions $q^n \in \mathcal{P}_n$ and trees $t_n \in D$ (not names!) by induction on n with $q^{n+1} \restriction n = q^n$ (hence we may write: $q^n = (q_0, q_1, \dots, q_{n-1})$) and $t_n \subseteq t_{n+1}$, satisfying the following conditions:

(A) $p \restriction n \le q^n$;

(B) q^n is $(N, \mathcal{P}_n)$-generic;

(C) $q^n \Vdash$ "$\underset{\sim}{f}_n \in \mathrm{Br}\,(t_n)$";

(D) $R(x_{3n}, t_n)$;

(E) For $m < n < \omega$ we have $q^m \Vdash_{\mathcal{P}_m}$ "q_m and $p^n(m)$ are compatible in $\underset{\sim}{\mathcal{Q}}_m$".

Suppose we succeed in this endeavour. Then we can let $q = \bigcup_n q^n$. By condition (2) in A2.1 for every $n < \omega$ $R(x', t_n)$ (as $x_{3n} < x'$). Let $(n_i : i < \omega)$ be a strictingly increasing sequence of natural numbers and t be as guaranteed by $(*)$ of condition (3) of A2.1 (for $\langle t_n : n < \omega \rangle, x', x$) so $R(x,t)$ and: if $\eta \restriction n_i \subseteq \bigcup_{j \le i} t_j$ for each $i < \omega$ then $\eta \in t$. Let $g(i) =: n_i$.

By (E) above there are conditions q'_m with $q^m \Vdash_{\mathcal{P}_m}$ "$q'_m \in \mathcal{Q}_m$, $q'_m \ge q_m, p^{g(m)}(m)$." Let $q' = (q'_0, q'_1, \dots)$. Then $q' \ge q \ge p$ and for $m \le n \le g(m)$ we will have (if we succeed in defining q_n, t_n) $q' \restriction n \Vdash_{\mathcal{P}_n}$ "$\underset{\sim}{f} \restriction n = \underset{\sim}{f}_m \restriction n$", hence:

$$q' \restriction n \Vdash_{\mathcal{P}_n} \text{“}\underset{\sim}{f} \restriction n \in \mathrm{Br}\ t_m\text{”}.$$

Now we have finished proving the existence of p', t (see before (1)) as required: $q' \Vdash$ "$\underset{\sim}{f} \in \mathrm{Br}\ (t)$", as t includes the tree: $\{\eta \in {}^{\omega>}\omega$: For all i, $\eta \restriction n_i \in \bigcup_{j \le i} t_j\}$; and $R(x,t)$

holds. Hence we have finished proving $\Vdash_{\mathcal{P}_\omega}$ "$(D;R)$ covers ${}^\omega\omega$". So it suffices to carry out the induction.

There is no problem for $n=0$ or 1. Assume that q^n and t_n are defined. Let $G_n \subseteq P_n$ be generic with $q_n \in G_n$. Then $\underset{\sim}{f}_{n+1}$ becomes a $\underset{\sim}{\mathcal{Q}}_n[G_n]$-name $\hat{\underset{\sim}{f}}_{n+1} = \underset{\sim}{f}_{n+1}/G_n$ for a member of ${}^\omega\omega$. As $\mathcal{P}_{n+1}$ preserves (D,R), for every $r \in \underset{\sim}{\mathcal{Q}}_n[G_n]$ and every $y \in \operatorname{dom} R$ there is a condition $r' \geq r$ in $\underset{\sim}{\mathcal{Q}}_n[G_n]$ such that

$$(*) \qquad r' \Vdash \text{“}\hat{\underset{\sim}{f}}_{n+1} \in \operatorname{Br}(t')\text{” for some } t' \in D \text{ with } R(y,t').$$

For each $m < \omega$, applying this to $r =: p^m(n)$, $y = x_{3n}$ we get $r' = r^n_m$, $t' = t^n_{m+1}$; we could have guaranteed $t^n_{m+1} \subseteq t^n_{m+2}$. Now choose by induction on $l < \omega$, $r^n_{m,l} \in \underset{\sim}{\mathcal{Q}}_n[G_n]$ such that: $r^n_{m,0} = r^n_m$, $r^n_{m,l} \leq r^n_{m,l+1}$, $r^n_{m,l+1}$ forces a value to $\hat{\underset{\sim}{f}}_{n+1}\restriction l$. So for some $\eta^n_m \in {}^\omega\omega[G_n]$, $r^n_{m,l} \Vdash$"$\hat{\underset{\sim}{f}}_{n+1}\restriction l = \eta^n_m\restriction l$". Note $\eta^n_m\restriction m = f_n\restriction m$. Without loss of generality, $\langle r^n_m, t^n_m, r^n_{m,\ell}, \eta^n_m : n,m,\ell < \omega\rangle$ belongs to N. Applying $(3\otimes)$ from A2.1 (to $\eta = \underset{\sim}{f}_n[G_n]$, $\eta_m = \eta^n_m$) we can find $T^I_n \in D \cap N[G_n] = D \cap N$ such that $R(x_{3n}, T^I_n)$, $\underset{\sim}{f}_n \in \operatorname{Br}(T^I_n)$ and $\eta^n_m \in \operatorname{Br}(T^I_n)$ for infinitely many $m < \omega$. Applying $(3*)$ from A2.1 (to $T^I_n, t^n_1, t^n_2, \ldots$ and x_{3n}, x_{3n+1}) we obtain a tree T^{II}_n. Returning to V, we have a $\mathcal{P}_n$-name $\underset{\sim}{T}$ for such a tree. For $s \in \mathcal{P}_n$, if $s \Vdash$ "$\underset{\sim}{T} = T$" for some tree T in V, let $T(s)$ be this tree. Let U be the open dense subset of $s \in \mathcal{P}_n$ for which $T(s)$ is defined. Some such function $T(\cdot)$ belongs to N, and $U \in N$. If q^n is in the generic set G_n, then some $s \in U \cap N$ is in G_n, by condition (2). Let $U \cap N = \{s_i : i < \omega\}$.

Applying $(3*)$ there is a tree t_{n+1} satisfying:

(a) $R(x_{3n+3}, t_{n+1})$.

(b) $t_n \subseteq t_{n+1}$.

(c) for every $T \in (\operatorname{Rang} R) \cap N$ such that $R(x_{3n+2}, T)$ for some $k_T < \omega$ we have:

$$\nu \in T \,\&\, \nu\restriction k_T \in t_n \Rightarrow \nu \in t_{n+1}$$

We shall prove now

(d) suppose $G_n \subseteq \mathcal{P}_n$ is generic over V with $q^n \in G_n$, and $k^* < \omega$.
Then there is q', $p^{k^*}(n) \leq q' \in \underset{\sim}{\mathcal{Q}}_n[G_n] \cap N[G_n]$, such that
$q' \Vdash$"$\underset{\sim}{f}_{n+1} \in \operatorname{Br}(t_{n+1})$"
(though t_{n+1} is generally not in N).

Proof of (d):

As $q^n \in G_n$ necessarily for some $s \in P_n \cap N$ we have $s \in G_n$ so (c) applies to T_s and $T_s = \underset{\sim}{T}^{II}_n[G_n]$ (as $T^{II}_n = \underset{\sim}{T}^{II}_n[G_n]$ is well defined and also T^I_n is well-defined and belongs to $N \cap D$ not only $N[G_n] \cap D$, as $D \subseteq V$). By the choice of T^I_n the following set is infinite

$$w = \{i < \omega : \eta^n_i \in \operatorname{Br}(T^I_n)\}$$

By the choice of t^n_{i+1}, for every $i \in w$ there exists $k_i < \omega$ such that $\eta \in t^n_{i+1} \& \eta \restriction k_i = \eta^n_i \restriction k_i \Longrightarrow \eta \in T^{II}_n$. To show (d), choose $i \in w \setminus k^*$ (exists as w is infinite, k^* will be shown to be as required in (d)).

Now $r^n_{i,k} \in N \cap \underset{\sim}{Q}_n[G_n]$ is well-defined, and any q', $p^i(n) \leq q' \in \underset{\sim}{Q}'_n[G_n]$ which is $(N, \underset{\sim}{Q}_n[G_n])$-generic is as required (note that $p^{k^*}(n) \leq p^i(n)$).

We can assume without loss of generality that Q_n is closed under countable disjunction, so we can find $\underset{\sim}{r}_n$ compatible with $p^n(m)$ for all m such that:

$$(q_0, \ldots, q_{n-1}, q'_n) \Vdash_{\mathcal{P}_{n+1}} \text{“}\underset{\sim}{f}_{n+1} \in \text{Br } (t_{n+1})\text{”}.$$

Now find $q_n \geq q'_n$ such that $(q_0, \ldots, q_{n-1}, q_n)$ is $(N, \mathcal{P}_{n+1})$-generic. This completes the induction step.

[If this infinite disjunction bothers you, define by induction on n sequences $\langle q^n_\eta : \eta \in {}^{n+1}\omega\rangle$ where $q^n_\eta \in \underset{\sim}{Q}_n$ is such that for every $\eta \in {}^m\omega$ the condition $\langle q^i_{\eta\restriction(i+1)} : i < n\rangle$ is generic for N and q^n_η is above $p^{\eta(n)}(n)$.] ∎

A2.3 The ${}^\omega\omega$-bounding property

We leave the successor case to the reader (see A2.6(2)).

A forcing notion $\mathcal{P}$ is ${}^\omega\omega$-bounding if it forces every function in ${}^\omega\omega$ in the generic extension to be bounded by one in the ground model. In §1 we quoted the result that a countable support iteration of proper ${}^\omega\omega$-bounding forcing notions is again ${}^\omega\omega$-bounding, which is almost Theorem V.4.3 of [Sh b]. In Chapter VI, §2 of [Sh b] this result is shown to fit into the framework just given. Here D is just a single collection $\mathcal{T}$ of trees; to fit D into the general framework given previously, we would let A be any suitable partial order, $D = A \dot\cup \mathcal{T}$, and $R = A \times \mathcal{T}$. The set $\mathcal{T}$ will consist of all subtrees of ${}^{\omega>}\omega$ with finite ramification (as we have no measure on how small $t \in \mathcal{T}$ is, so $<, R$ are degenerate).

In a generic extension of the universe, the set $\mathcal{T}$ (as defined in the ground model) will cover ${}^\omega\omega$ if and only if every function in ${}^\omega\omega$ is dominated by one in the ground model. In fact the only relevant trees are those of the form $T_f = \{\eta \in {}^{\omega>}\omega : \eta(i) \leq f(i) \text{ for } i < \text{len}\,\eta\}$ with f in the ground model. Thus the ${}^\omega\omega$-bounding property coincides with the property of being $\mathcal{D}$-preserving, where $\mathcal{D}$ is essentially $\mathcal{T}$, more precisely $\mathcal{D} = (A \times \mathcal{T}; R, <)$ for a suitable $R, <$ (which play no role in this degenerate case). Thus to see that the general iteration theorem applies, it suffices to check that such a $\mathcal{D}$ will be a fine* covering model. We have to check the final clause (3) of the definition of fine* covering model. In fact we will prove a strong version of $(3)^+$.

For any sequence of trees T_n in $\mathcal{T}$, there is a tree T such that for all $\eta \in {}^\omega\omega$, if $\eta \restriction i \in \bigcup_{j \leq i} T_j$ for all i, then $\eta \in \text{Br}\,(T)$.

We will verify that this property holds in any generic extension V^* of V in which $\mathcal{D}$ covers ${}^\omega\omega$. Let $T^* = \{\eta \in {}^{\omega>}\omega : \text{for all } i \leq \text{len}\,(\eta),\ \eta\restriction i \in \bigcup_{j \leq i} T_j\}$. If T^* is in V this will do, but since the sequence (T_n) came from a generic extension, this need not be the case. On the other hand the sequence $T^* \restriction n$ of finite trees is itself coded by a real $f \in {}^\omega\omega$, and as $\mathcal{D}$ covers ${}^\omega\omega$, there is a tree $T^\bullet$ in D which contains this code f; via a decoding, $T^\bullet$ can be thought of as a tree T^o whose nodes t are subtrees of ${}^{n\geq}\omega$ with no maximal nodes below level n, so that for any $s, t \in T^\bullet$ with $s \leq t$, s is the restriction of t to the level of s, and such that the sequence $T^* \restriction n$ actually is a branch of T^o. Let T be the subtree of ${}^{\omega>}\omega$ consisting of the union of all the nodes of T^o. Then T still has finite ramification, lies in the ground model, and contains T^*.

A2.4 Cosmetic Changes

(a) We may want to deal just with $\mathrm{Br}\,(T^*)$, where T^* a subtree ${}^{\omega>}\omega$ (hence downward closed). So D is a set of subtrees of T^*, so we can replace D by $\{\{\eta \in {}^{\omega>}\omega : \eta \in T$ or $(\exists \ell)[\eta\restriction\ell \in T \,\&\, \eta\restriction(\ell+1) \notin T^*\} : T \in D\}$.

(b) We may replace subtrees T^* of ${}^{\omega>}\omega$ by isomorphic trees.

(c) We may want to deal with some $(D_i; R_i, <_i)$ simultaneously; by renaming without loss of generality the D_i are pairwise disjoint, and even: $\bigwedge_{\ell=1,2} t_l \in D_{i_l} \& i_1 \neq i_2 \Longrightarrow \mathrm{Br}\, T_1 \cap \mathrm{Br}\, t_2 = \emptyset$. Then we use $(\bigcup D_i; \bigcup R_i, \bigcup <_i)$ to get the result.

(d) We may want to have $(D; R)$ (i.e. no $<$); just use $(D \cup \mathbb{Q} \times D; R', <)$ where $R'(x,t)$ iff $x = (q, y)$, $q \in \mathbb{Q}, y \in D, R(y,t), (q_1, y_1) < (q_2, y_2)$ iff $q_1 < q_2 \,\&\, y_1 = y_2$.

A2.5 The (f, g)-bounding property

We leave the successor case to the reader (see A2.6(2)).

Let $\mathbf{F}$ be a family of functions in ${}^\omega\omega$, and $g \in {}^\omega\omega$ with $1 < g(n)$ for all n. We say that a forcing notion $\mathcal{P}$ has the $(\mathbf{F}, g)$-*bounding property* if:

(∗) For any sequence $(A_k : k < \omega)$ in the ground model, with $|A_k| \in \mathbf{F}$ (as a function of k), and any $\underset{\sim}{\eta} \in \prod_k A_k$ in the generic extension and $\varepsilon > 0$, there is a "cover" $\mathcal{B} = (B_k : k < \omega)$ in the ground model with $B_k \subseteq A_k$, $[|B_k| > 1 \Rightarrow |B_k| < g(k)^{\varepsilon}]$ and $\underset{\sim}{\eta}(k) \in B_k$ for each k.

This notion is only of interest if $g(n) \longrightarrow \infty$ with n.

We will show that this notion is also covered by a case of the general iteration theorem of §A2.2.

Let $\mathcal{T}_{f,g}$ $[\mathcal{T}^{\varepsilon}_{f,g}]$ be the set of those subtrees T of $\bigcup_n \prod_{m<n} f(n)$ of the form $\bigcup_n \prod_{m<n} B_m$, such that $|B_n| < \max\{g(k), 2\}$ [such that $|B_n| \leq \max\{2, g(k)^{\varepsilon}\}$], where as usual $f(n)$ is thought of as the set $\{0, \ldots, f(n)-1\}$. Let $\mathcal{T}_{\mathbf{F},g}$ be $\bigcup_{f\in\mathbf{F},\varepsilon\in\mathbb{Q}^+} \mathcal{T}^{\varepsilon}_{f,g}$. Our fine* covering model is essentially $\mathcal{T}_{\mathbf{F},g}$, more accurately, it is the family of $\{(\mathcal{T}_{f,g} \cup \mathbb{Q}^+; R, <) : f \in \mathbf{F}\}$, where $\mathbb{Q}^+$ is the set of positive rationals, $<$ is the order on $\mathbb{Q}^+$, and $R(\varepsilon, t) =: \varepsilon \in \mathbb{Q}^+ \,\&\, t \in \mathcal{T}^{\varepsilon}_{f,g}$. See A2.4(c).

Call a family $\mathbf{F}$ *g-closed* if it satisfies the following two closure conditions:

1. For $f \in \mathbf{F}$, the function $F(n) = \prod_{m<n}(f(m)+1)$ lies in $\mathbf{F}$;
2. For $f \in \mathbf{F}$, f^g is in $\mathbf{F}$.

If $\mathbf{F}$ is g-closed, $f \in \mathbf{F}$, and $(A_n)_{n<\infty}$ are sets with $|A_n| = f(n)$, then the function $f'(n)$ = the number of trees of the form $\prod_{m<n} B_m$ with $B_m \subseteq A_m$ and $|B_m| < g(m)$ is dominated by a function in $\mathbf{F}$.

Using the formalism of §A2.2, we wish to prove:

Theorem

If $\mathbf{F}$ *is g-closed then a countable support iteration of* $(\mathbf{F}, g)$*-bounding proper forcing notions is again an* $(\mathbf{F}, g)$*-bounding proper forcing.*

Since the $\mathcal{D}$-preserving forcing notions are the same as the $(\mathbf{F}, g)$-bounding ones, we need only check that $\mathcal{D}$ is a fine* covering model. Again the nontrivial condition is $(3)^+$, i.e.,

Let $f \in \mathbf{F}$. For any sequence of trees T_n in $\mathcal{T}_{f,g}$, $R(\varepsilon', T_n)$, $\varepsilon' < \varepsilon$ (in $\mathbb{Q}^+$), there is a tree T in D satisfying $R(\varepsilon, T)$ and an increasing sequence n_i such that for all $\eta \in {}^{\omega}\omega$, if $\eta \restriction n_i \in \bigcup_{j \le i} T_{n_j}$ for all i, then $\eta \in \mathrm{Br}\,(T)$.

This must be verified in any generic extension V^* of V in which $\mathcal{D}$ covers ${}^{\omega}\omega$. Working in V, choose $(n_i)_{i<\omega}$ increasing so that $n_0 = 0$ and for $n_i \le n$ we have $\min_{n \ge n_i} g(n)^{(\varepsilon-\varepsilon')/2} > i+1$. For $n_i \le n < n_{i+1}$ set:

$$B_n = \{\eta(n) : \eta \in \cup\{T_j : n_j \le n\}.$$

(For $n < n_0$ let $B_n = \{\eta(n) : \eta \in T_0\}$.) If the sequence B_n was in the ground model, we could take $T = \bigcup_n \prod_{m<n} B_m$. Instead we have to think of the sequence B_n as a possible branch through the tree of finite sequences of subsets of $f(n)$ of size at most (say) $\max\{1, g(n)-1\}$. As $\mathbf{F}$ is g-closed, $\mathcal{T}_{\mathbf{F},g}$ contains a tree $T^{\bullet}$ which encodes a tree T^o of such subsets, for which the desired sequence B_n is a branch in V^*, so that the number of members of T^0 of level m is $\le g(m)^{(\varepsilon-\varepsilon')/2}$ (or is ≤ 1). Let $B^o_n = \bigcup_{b \in T^o} b(n)$. Then $B_n \subseteq B^o_n$, $\lim_{n \longrightarrow \infty} |B^o_n|/g^{\varepsilon}(n) = 0$ and $\bigcup_n \prod_{m<n}(B^o_m)$ is in V. ∎

A2.6 Discussion:

This was treated in [Sh-f,VI] [Sh-f, XVIII §3] too (the presentation in [Sh-b, VI] was inaccurate). The version chosen here goes for less generality (gaining, hopefully, in simplicity and clarity) and is usually sufficient. We consider below some of the differences.

A2.6(1) A technical difference

In the context as phrased here the preservation in the successor case of the iteration was trivial — by definition essentially. We can make the fine* covering model (in A2.1) more similar to [Sh-f, VI §1] by changing (3∗) to

$(*)'$ For $y_0 < y_1 < \ldots y < x$ in $\mathrm{dom}\, R$ and $t_n \in D$ such that $R(y_n, t_n)$ for all n, there is $t \in D$ with $R(x,t)$ holding and indices $n_0 < n_1 < \ldots$ such that $[\eta \in {}^{\omega>}\omega \,\&\, \bigwedge_i \eta \restriction \eta_i \in \bigcup_{j \le i} t_j \Rightarrow \eta \in t]$.

We can use this version here.

A2.6(2) Two-stage iteration

We can make the fine* covering model (in A2.1) more similar to [Sh-f, VI §1] by changing (3∗). In the context as presented here the preservation by two step iteration is trivial — by definition essentially. In [Sh-f VI, §2] we phrase our framework such that we can have: if $Q_0 \in V$ is x-preserving, $\underset{\sim}{Q}_1$ is X-preserving (over V^{Q_0}, $\underset{\sim}{Q}_1$ a Q_0-name) then $Q_0 * \underset{\sim}{Q}_1$ is x-preserving. The point is that X-preserving means $(D,R,<)^V$-preserving, i.e. $(D,R,<)$ is a definition (with a parameter in V_0). The point is that if $V_1 = V_0{}^{Q_0}$, $V_2 = V_1^{\underset{\sim}{Q}_1}$ *then* for $\eta \in ({}^{\omega}\omega)^{V_2}$ and $x \in \mathrm{dom}\, R$, we choose $y < x$ and $t \in D^{V_1}$ such that $\eta \in \mathrm{Br}\,(t)$, $R^{V_1}(y,t)$, then we look in V_0 at the tree of possible initial segments of t getting $T \in D^{V_0}$ such that $t \in \mathrm{Br}\,(T)$, $R^{V_0}(y,T)$. If y was chosen rightly, $\bigcup \mathrm{Br}\,(T)$ is as required. Here it may be advantegous to use a preservation of several $(D,R,<)$'s at once (see A2.4(c)).

A2.6(3) Several models — the real case

We may consider a (weak) (fine*) covering family of models $\langle (D_\ell, R_\ell, <_\ell) : \ell < \ell^* \rangle$ (actually a sequence) i.e. *not* that each one is a cover, but simultaneously.

(A) We say $(\overline{D}, \overline{R}) = \langle (D_\ell, R_\ell) : \ell < \ell^* \rangle$ is a weak c.f.m. if each D_ℓ is a set, R_ℓ a binary relation, $\ell^* < \omega$ and

1. $R_\ell(x,t)$ implies that t is a subtree of ${}^{\omega>}\omega$ (nonempty, no maximal models).
2. Every $\eta \in {}^{\omega}\omega$ is of kind ℓ for at least one $\ell < \ell^*$ which means: for every $x \in \mathrm{dom}\, R_\ell$ for some t, we have $R_\ell(x,t) \,\&\, \eta \in \mathrm{Br}\,(t)$.

(B) We say $(\overline{D}, \overline{R}, \overline{<})$ is a fine* c.f.m. if:

0. $(\overline{D}, \overline{R})$ is a weak family.
1. If $x \in \mathrm{dom}\, R_\ell \Rightarrow (\exists z) z <_\ell x$ and $\forall y <_\ell x\, \exists z (y <_\ell z <_\ell x)$ (and $D \neq \emptyset$).
2. $x <_\ell y \,\&\, R_\ell(x,t) \Rightarrow R_\ell(y,t)$.
3. For any generic extensions V^* in which $(\overline{D}, \overline{R})$ is a weak c.f.m.

(∗) for every $\ell < \ell^*$ and $y <_\ell x$ (from $\operatorname{dom} R_\ell$) and $t_n \in D_\ell$ with $R_\ell(y, t_n)$ for all n there is $t \in D$ with $R_\ell(x, t)$ and there are indices $n_0 < n_1 < \ldots$ such that for every $\eta \in {}^\omega\omega$: if $\eta \restriction n_i \in \bigcup_{j \le i} t_j$ for all i then $\eta \in \operatorname{Br}(t)$.

⊗ if $\ell < \ell^*$, $\eta \in {}^\omega\omega$, $\eta_n \in {}^\omega\omega$, $\eta_n \restriction n = \eta \restriction n$, $x \in \operatorname{dom} R_\ell$ and η, η_n are of kind ℓ, *then* for some t^*, $R_\ell(x, t^*)$, $\eta \in \operatorname{Br}(t^*)$ and for infinitely many $n < \omega$, $\eta_n \in \operatorname{Br}(t^*)$.

Theorem

If $(\overline{D}; \overline{R}, \overline{<})$ is a fine c.f.m., $\langle \mathcal{P}_\alpha, \underset{\sim}{\mathcal{Q}}_\beta : \alpha \le \delta, \beta < \alpha \rangle$ is a countable support iteration of proper forcing notions with each factor $(\overline{D}; \overline{R}, \overline{<})$-preserving. Then $\mathcal{P}_\delta$ is $(\overline{D}; \overline{R}, \overline{<})$-preserving.*

Proof:

Similar to the previous one, with the following change. After saying that without loss of generality $\delta = \omega$ and, above p, for every n, $\underset{\sim}{f}(n)$ as a P_n-name, and choosing x_n, x', we do the following. For clarity think that our universe V is countable in the true universe or at least $\beth_3(|P_\omega|)^V$ is. We let $K = \{(n, p, G) : n < \omega, p \in P_\omega,\ G \subseteq P_n$ is generic over V and $p \restriction n \in G_n\}$. On K there is a natural order $(n, p, G) \le (n', p', G')$ if $n \le n'$, $P_\omega \models p \le p'$ and $G \subseteq G'$. Also for $(n, p, G) \in G$ and $n' \in (n, \omega)$ there are p', G' such that $(n, p, G) \le (n', p', G')$. For $(n, p, G) \in K$ let $L_{(n,p,G)} = \{g : g \in ({}^\omega\omega)^{V[G]}$ and there is an increasing sequence $\langle p_\ell : \ell < \omega \rangle$ of conditions in P_ω / G, $p \le p_0$, such that $p_\ell \Vdash \underset{\sim}{f} \restriction \ell = g \restriction \ell\}$. So:

$g \in L_{(n,p,G)} \Rightarrow \underset{\sim}{f} \restriction n = g \restriction n$

$(n, p, G) \le (n', p', G') \Rightarrow L_{(n',p',G')} \subseteq L_{(n,p,G)}$.

Claim.

There are ℓ_* and $(n, p, G) \in K$ such that if $(n, p, G) \le (n', p', G') \in K$ then there is $g \in L_{(n',p',G')}$ which is of the ℓ_*'th kind.

Proof:

Otherwise choose by induction (n^ℓ, p^ℓ, G^ℓ) for $\ell \le \ell^*$, in K, increasing such that: $L_{(n^{\ell+1}, p^{\ell+1}, G^{\ell+1})}$ has no member of the ℓ'th kind. So $L_{(n^{\ell^*}, p^{\ell^*}, G^{\ell^*})} = \emptyset$ contradiction.

So without loss of generality for every $(n, p, G) \in K$, $L_{(n,p,G)}$ has a member of the ℓ_*'th kind. Now we choose by induction on n, A_n, $\langle p_\eta : \eta \in {}^{n+1}\omega, \eta \restriction n \in A_n \rangle$, $\langle \underset{\sim}{f}_\eta : \eta \in A_n \rangle$, $\langle q_\eta : \eta \in A_n \rangle$, and t_n such that

(A)′ $A_n \subseteq {}^n\omega, A_0 = \{\langle\rangle\}, \eta \in A_n \Rightarrow (\exists^{\aleph_0} \ell)(\eta\,\hat{}\,\langle \ell \rangle \in A_{n+1})$ $p_\eta \in \mathcal{P}_\omega \cap N, p_{<>} = p, p_\eta \le p_{\eta\,\hat{}\,\langle \ell \rangle}, p_\eta \restriction n \le q_n$.

(B)′ q_η is $(N, \mathcal{P}_{\lg \eta})$-generic, $q_\eta \in \mathcal{P}_{\lg \eta}$ and $[\ell < \lg \eta \Rightarrow q_\eta \restriction \ell = q_{\eta \restriction \ell}]$.

(C)′ $q_\eta \Vdash$ "$\underset{\sim}{f}_\eta \in \operatorname{Br}(t_n)$ is of the ℓ_*'th kind" when $\eta \in A_n$ and $\underset{\sim}{f}_\eta$ is a $\mathcal{P}_n$-name.

(D)′ $R_{3n}(x_{3n}, t_n), t_n \subseteq t_{n+1}$.

(E)′ $p_{\eta\,\hat{}\,\langle \ell \rangle} \Vdash_{\mathcal{P}_w}$ "$\underset{\sim}{f}_\eta \restriction \ell = \underset{\sim}{f}_{\eta\,\hat{}\,\langle \ell \rangle} \restriction \ell = \underset{\sim}{f} \restriction \ell$".

This suffices, as $x_n < x'$ so $\bigwedge_n R(x', t_n)$ hence for some $\langle n_i : i < \omega\rangle$ strictly increasing and t as guaranteed by $(*)$ of (3) we find $\nu \in {}^\omega\omega$ increasing fast enough and let $q = \bigcup_{n<\omega} q_{\nu\restriction n}$. In the induction there is no problem for $n = 0, 1$. For $n+1$; first for each $\eta \in {}^{n+1}\omega$ we choose ℓ, work in $V^{\mathcal{P}_{n+1}}$ and find $\langle p_{\eta\hat{}\langle\ell\rangle} : \ell < \omega\rangle, \underset{\sim}{f}_\eta$, and without loss of generality they are in N. For $\eta \in {}^n\omega$ there is a $\mathcal{P}_{n+1}$-name $\underset{\sim}{t}_\eta \in N$ of a member of D, $R_\ell(x_{3n}, \underset{\sim}{t}_\eta)$, $\underset{\sim}{f}_\eta \in \mathrm{Br}(\underset{\sim}{t}_\eta)$, $(\exists^\infty \ell)\underset{\sim}{f}_{\eta\hat{}\langle\ell\rangle} \in \mathrm{Br}(\underset{\sim}{t}_\eta)$. Now we can replace $p_{\eta\hat{}\langle 1\rangle}$ by $p_{\eta\hat{}\langle\ell'\rangle}$, $\ell' = \mathrm{Min}\{m : m \geq \ell, \underset{\sim}{f}_{\eta\hat{}\langle\ell\rangle} \in \mathrm{Br}(\underset{\sim}{t}_\eta)\}$. We continue as in A2.2. Note: it is natural to use this framework e.g. for preservation of P-points. ∎

A3. Omitting types

A3.1 Uniform partial orders.

In the proof of Theorem 3.1 given in §3 we used the combinatorial principle developed in [ShLH162]. (Cf. [Sh107] for applications published earlier.) This is a combinatorial refinement of forcing with $\mathcal{AP}$ to get a $\mathbb{P}_3$-name $\underset{\sim}{\mathcal{F}}$ with the required properties in a generic extension. We now review this material.

With the cardinal λ fixed, a partially ordered set $(\mathcal{P}, <)$ is said to be *standard* λ^+-*uniform* if $\mathcal{P} \subseteq \lambda^+ \times \mathcal{P}_\lambda(\lambda^+)$ (we refer here to subsets of λ^+ of size strictly less than λ), satisfying the following properties (where we take e.g. $p = (\alpha, u)$ and write $\mathrm{dom}\,(p)$ for u):

1. If $p \leq q$ then $\mathrm{dom}\, p \subseteq \mathrm{dom}\, q$.
2. For all $p, q, r \in \mathcal{P}$ with $p, q \leq r$ there is $r' \in \mathcal{P}$ so that $p, q \leq r' \leq r$ and $\mathrm{dom}\, r' = \mathrm{dom}\, p \cup \mathrm{dom}\, q$.
3. If $(p_i)_{i<\delta}$ is an increasing sequence of length less than λ, then it has a least upper bound q, with domain $\bigcup_{i<\delta} \mathrm{dom}\, p_i$; we will write $q = \bigcup_{i<\delta} p_i$, or more succinctly: $q = p_{<\delta}$.
4. For all $p \in \mathcal{P}$ and $\alpha < \lambda^+$ there exists a $q \in \mathcal{P}$ with $q \leq p$ and $\mathrm{dom}\, q = \mathrm{dom}\, p \cap \alpha$; furthermore, there is a unique maximal such q, for which we write $q = p\restriction\alpha$.
5. For limit ordinals δ, $p\restriction\delta = \bigcup_{\alpha<\delta} p\restriction\alpha$.
6. If $(p_i)_{i<\delta}$ is an increasing sequence of length less than λ, then $(\bigcup_{i<\delta} p_i)\restriction\alpha = \bigcup_{i<\delta}(p_i\restriction\alpha)$.
7. *(Indiscernibility)* If $p = (\alpha, v) \in \mathcal{P}$ and $h : v \to v' \subseteq \lambda^+$ is an order-isomorphism onto V' then $(\alpha, v') \in \mathcal{P}$. We write $h[p] = (\alpha, h[v])$. Moreover, if $q \leq p$ then $h[q] \leq h[p]$.
8. *(Amalgamation)* For every $p, q \in \mathcal{P}$ and $\alpha < \lambda^+$, if $p\restriction\alpha \leq q$ and $\mathrm{dom}\, p \cap \mathrm{dom}\, q = \mathrm{dom}\, p \cap \alpha$, then there exists $r \in \mathcal{P}$ so that $p, q \leq r$.

It is shown in [ShHL162] that under a diamond-like hypothesis, such partial orders admit reasonably generic objects. The precise formulation is given in A3.3 below.

A3.2 Density systems.

Let $\mathcal{P}$ be a standard λ^+-uniform partial order. For $\alpha < \lambda^+$, $\mathcal{P}_\alpha$ denotes the restriction of $\mathcal{P}$ to $p \in \mathcal{P}$ with domain contained in α. A subset G of $\mathcal{P}_\alpha$ is an *admissible ideal* (of $\mathcal{P}_\alpha$) if it is closed downward, is λ-directed (i.e. has upper bounds for all small subsets), and has no proper directed extension within $\mathcal{P}_\alpha$. For G an admissible ideal in $\mathcal{P}_\alpha$, $\mathcal{P}/G$ denotes the restriction of $\mathcal{P}$ to $\{p \in \mathcal{P} : p \restriction \alpha \in G\}$.

If G is an admissible ideal in $\mathcal{P}_\alpha$ and $\alpha < \beta < \lambda^+$, then an (α, β)-*density system* for G is a function D from pairs (u, v) in $P_\lambda(\lambda^+)$ with $u \subseteq v$ into subsets of $\mathcal{P}$ with the following properties:

(i) $D(u, v)$ is an upward-closed dense subset of $\{p \in \mathcal{P}/G : \mathrm{dom}\,(p) \subseteq v \cup \beta\}$;

(ii) For pairs $(u_1, v_1), (u_2, v_2)$ in the domain of D, if $u_1 \cap \beta = u_2 \cap \beta$ and $v_1 \cap \beta = v_2 \cap \beta$, and there is an order isomorphism from v_1 to v_2 carrying u_1 to u_2, then for any γ we have $(\gamma, v_1) \in D(u_1, v_1)$ iff $(\gamma, v_2) \in D(u_2, v_2)$.

An admissible ideal G' (of $\mathcal{P}_\gamma$) is said to *meet* the (α, β)-density system D for G if $\gamma \geq \alpha$, $G' \geq G$ and for each $u \in P_\lambda(\gamma)$ there is $v \in P_\lambda(\gamma)$ containing u such that G' meets $D(u, v)$.

A3.3 The genericity game.

Given a standard λ^+-uniform partial order $\mathcal{P}$, the *genericity* game for $\mathcal{P}$ is a game of length λ^+ played by Guelfs and Ghibellines, with Guelfs moving first. The Ghibellines build an increasing sequence of admissible ideals meeting density systems set by the Guelfs. Consider stage α. If α is a successor, we write α^- for the predecessor of α; if α is a limit, we let $\alpha^- = \alpha$. Now at stage α for every $\beta < \alpha$ an admissible ideal G_β in some $\mathcal{P}_{\beta'}$ is given, and one can check that there is a unique admissible ideal G_{α^-} in $\mathcal{P}_{\alpha^-}$ containing $\bigcup_{\beta<\alpha} G_{\beta'}$ (remember A 3.1(5)) [Lemma 1.3, ShHL 162]. The Guelfs now supply at most λ density systems D_i over G_{α^-} for (α, β_i) and also fix an element g_α in $\mathcal{P}/G^-_\alpha$. Let α' be minimal such that $g_\alpha \in \mathcal{P}_{\alpha'}$ and $\alpha' \geq \sup \beta_i$. The Ghibellines then build an admissible ideal $G_{\alpha'}$ for $\mathcal{P}_{\alpha'}$ containing G^-_α as well as g_α, and meeting all specified density systems, or forfeit the match; they let $G_{\alpha''} = G_{\alpha'} \cap \alpha''$ when $\alpha \leq \alpha'' < \alpha'$. The main result is that the Ghibellines can win with a little combinatorial help in predicting their opponents' plans.

For notational simplicity, we assume that G_δ is an $\aleph_2$-generic ideal on $\mathcal{AP}\restriction\delta$, when $\mathrm{cf}\,\delta = \aleph_2$ which is true on a club in any case.

A3.4 Dl_λ.

The combinatorial principle Dl_λ states that there are subsets Q_α of the power set of α for $\alpha < \lambda$ such that $|Q_\alpha| < \lambda$, and for any $A \subseteq \lambda$ the set $\{\alpha : A \cap \alpha \in Q_\alpha\}$ is stationary.

This follows from $\diamondsuit_\lambda$ or inaccessibility, obviously, and Kunen showed that for successors, Dl and $\diamondsuit$ are equivalent. In addition Dl_λ implies $\lambda^{<\lambda} = \lambda$.

A3.5 A general principle

Theorem

Assuming Dl_λ, the Ghibellines can win any standard λ^+-uniform $\mathcal{P}$-game.

This is Theorem 1.9 of [ShHL 162].

In our application we identify $\mathcal{AP}$ with a standard $\aleph_2^+$-uniform partial order via a certain coding. We first indicate a natural coding which is not quite the right one, then repair it.

First Try

An approximation $q = (\mathcal{A}, \underset{\sim}{\mathcal{F}}, \underset{\sim}{\varepsilon}, \underset{\sim}{\mathbf{k}},)$ will be identified with a pair (τ, u), where $u = \mathcal{A}$, and τ is the image of q under the canonical order-preserving map $h : \mathcal{A} \leftrightarrow \mathrm{otp}\,(\mathcal{A})$. One important point is that the first parameter τ comes from a fixed set T of size $2^{\aleph_1} = \aleph_2$; so if we enumerate T as $(\tau_\alpha)_{\alpha<\aleph_2}$ then we can code the pair (τ_α, u) by the pair (α, u). Under these successive identifications, $\mathcal{AP}$ becomes a standard $\aleph_2^+$-uniform partial order, as defined in §A3.1. Properties 1, 2, 4, 5, and 6 are clear, as is 7, in view of the uniformity in the iterated forcing $\mathbb{P}$, and properties 3, 8 were, in essence but not formally, stated in Claim 3.10.

The difficulty with this approach is that in this formalism, density systems cannot express nontrivial information: any generic ideal meets any density system, because for $q \le q'$ with $\mathrm{dom}\, q = \mathrm{dom}\, q'$, we will have $q = q'$; thus $D(u, u)$ will consist of all q with $\mathrm{dom}\, q = u$, for any density system D.

So to recode $\mathcal{AP}$ in a way that allows nontrivial density systems to be defined, we proceed as follows.

Second Try

Let $\iota : \aleph_2^+ \leftrightarrow \aleph_2^+ \times \aleph_2$ order preserving where $\aleph_2^+ \times \aleph_2$ is ordered lexicographically. Let $\pi : \aleph_2^+ \times \aleph_2 \longrightarrow \aleph_2^+$ be the projection on the first coordinate. First encode q by $\iota[q] = (\iota[\mathcal{A}], \ldots)$, then encode $\iota[q]$ by $(\tau, \pi[\mathcal{A}])$, where τ is defined much as in the first try – a description of the result of collapsing q into $\mathrm{otp}\, \pi[\mathcal{A}] \times \aleph_2$, after which τ is encoded by an ordinal label below $\aleph_2$. The point of this is that now the domain of q is the set $\pi[\mathcal{A}]$, and q has many extensions with the same domain. After this recoding, $\mathcal{AP}$ again becomes a $\aleph_2^+$-uniform partial ordering, as before. We will need some additional notation in connection with the indiscernibility condition. It will be convenient to view $\mathcal{AP}$ simultaneously from an encoded and a decoded point of view. One should now think of $q \in \mathcal{AP}$ as a quintuple

$(u, \mathcal{A}, \underset{\sim}{\mathcal{F}}, \varepsilon, \underset{\sim}{\mathbf{k}})$ with $\mathcal{A} \subseteq u \times \aleph_2$. If $h : u \leftrightarrow v$ is an order isomorphism, and q is an approximation with domain u, we extend h to a function h_* defined on $\mathcal{A}^q$ by letting it act as the identity on the second coordinate. Then $h[q]$ is the transform of q using h_*, and has domain v.

In order to obtain *least* upper bounds for increasing sequences, it is also necessary to allow some extra elements into $\mathcal{AP}$, by adding formal least upper bounds to increasing sequences of length $< \aleph_2$.

This provides the formal background for the discussion in §3. The actual construction should be thought of as a match in the genericity game for $\mathcal{AP}$, with the various assertions as to what may be accomplished corresponding to proposals by the Guelfs to meet certain density systems. To complete the argument it remains to specify these systems and to check that they are in fact density systems.

A3.6 The Major Density Systems

The main density systems under consideration were introduced implicitly in 3.11. Suppose that $\delta < \aleph_2$, $q \in \mathcal{AP}$ with $\delta \in \operatorname{dom} q \subseteq \aleph_2$, $q^*_\delta \leq q$, and $\underset{\sim}{z}$ is a $(\mathbb{P}\restriction \operatorname{dom} q)$-name. Define a density system $\underset{\sim}{D}^\delta_{q,\underset{\sim}{z}}(u, v)$ for $u \subseteq v \subseteq \aleph_3$ with $|v| \leq \aleph_1$ as follows. First, if $\operatorname{otp} u \leq \operatorname{otp} \operatorname{dom} q$ then let $\underset{\sim}{D}^\delta_{q,\underset{\sim}{z}}(u, v)$ degenerate to $\mathcal{AP}\restriction v$. Now suppose that $\operatorname{otp} u > \operatorname{otp} \operatorname{dom} q$ and that $h : \operatorname{dom} q \longrightarrow u$ is an order isormorphism from $\operatorname{dom} q$ to an initial segment of u. Let $q^* = h[q]$. Call an element r of $\mathcal{AP}$ a (u, v)-*witness* if:

1. $u \subseteq \operatorname{dom} r \subseteq v$;
2. $r \geq q^*$;
3. for some $\mathsf{p} \in \mathcal{P}\restriction\mathcal{A}^r$ with $\mathsf{p} \geq \mathsf{p}^\delta$, and some $(\mathbb{P}\restriction[\mathcal{A}^r \cap \delta])$-name $\underset{\sim}{x}$, $\underset{\sim}{F}_\delta(\underset{\sim}{x})$ is a $(\mathbb{P}\restriction[\mathcal{A}^r \cap \delta])$-name; and:
4. $\mathsf{p}' \Vdash_{\mathbb{P}\restriction\mathcal{A}^r}$ "$\{n : [\Gamma^1_{k^1_\delta(n)} \models R(\underset{\sim}{x}(n), \underset{\sim}{x}_\delta(n)) \iff \Gamma^2_{k^2_\delta(n)} \models \neg R(\underset{\sim}{F}_\delta(\underset{\sim}{x})(n), \underset{\sim}{z}(n))]\} \in \mathcal{F}^r$."

Let $D^\delta_{q,\underset{\sim}{z}}(u, v)$ be the set of $r \in \mathcal{AP}$ with $\operatorname{dom} r = v$ such that either r is a (u, v)-witness, or else there is no (u, v)-witness $r' \geq r$.

This definition has been arranged so that $D^\delta_{q,\underset{\sim}{z}}(u, v)$ is trivially dense. In §3 we wrote the argument as if no default condition had been used to guarantee density, so that the nonexistence of (u, v)-witnesses is called a "failure of density". Here we adjust the terminology to fit the style of [ShHL 162].

Now we return to the situation described in 3.12. We had $\mathbb{P}$-names $\underset{\sim}{F}$, $\underset{\sim}{k}^1$, $\underset{\sim}{k}^2$, and a condition $\mathsf{p} \in \mathbb{P}$, satisfying conditions (3,4) as stated there, and we considered the set $C = \{\delta < \aleph_3 : \operatorname{cof}(\delta) = \aleph_2,\ \delta \text{ is } (\underset{\sim}{F}, \underset{\sim}{k}^1, \underset{\sim}{k}^2, \mathsf{p})\text{-closed}\}$, and a stationary set S_C on which $\underset{\sim}{F}\restriction\delta$, p, ε_δ, k^1_δ were guessed by $\diamondsuit$. Then $\underset{\sim}{z} =: \underset{\sim}{F}(\underset{\sim}{x}_\delta)$ is a $(\mathbb{P}\restriction\mathcal{A}^q)$-name for some $q \in G$. Let $u = \operatorname{dom} q$, $q_0 = q\restriction\delta$. Now we consider the following condition used in 3.12:

(iv) For all $r \geq q$ in $\mathcal{AP}$ such that $r\restriction\delta \in G_\delta$, and $\underset{\sim}{x}$ a $(\mathbb{P}\restriction\mathcal{A}^{r\restriction\delta})$-name, with $\underset{\sim}{y} =: \underset{\sim}{F}(\underset{\sim}{x})$ a $(\mathbb{P}\restriction\mathcal{A}^{r\restriction\delta})$-name, we have:

$(*)_{\underset{\sim}{x},\underset{\sim}{y}}$ $\mathtt{p} \Vdash$ "The set $\{n : \Gamma^1_{\underset{\sim}{k}^1(n)} \models R(\underset{\sim}{x}(n), \underset{\sim}{x}_\delta(n))$ iff $\Gamma^2_{\underset{\sim}{k}^2(n)} \models R(\underset{\sim}{y}(n), \underset{\sim}{z}(n))\}$ is in $\underset{\sim}{\mathcal{F}}^r$".

We argued in 3.12 that we could confine ourselves to the case in which (iv) holds. We now go through this more carefully. Suppose on the contrary that we have $r \geq q$ in $\mathcal{AP}$ with $r\restriction\delta \in G_\delta$, and a $(\mathbb{P}\restriction\mathcal{A}^{r\restriction\delta})$-name $\underset{\sim}{x}$, so that $\underset{\sim}{y} =: \underset{\sim}{F}(\underset{\sim}{x})$ is a $(\mathbb{P}\restriction\mathcal{A}^{r\restriction\delta})$-name, and a condition $\mathtt{p}' \geq \mathtt{p}$, so that

$$\mathtt{p}' \Vdash \text{"The set } \{n : \Gamma^1_{\underset{\sim}{k}^1(n)} \models R(\underset{\sim}{x}(n), \underset{\sim}{x}_\delta(n)) \text{ iff } \Gamma^2_{\underset{\sim}{k}^2(n)} \models R(\underset{\sim}{y}(n), \underset{\sim}{z}(n))\} \text{ is not in } \underset{\sim}{\mathcal{F}}^r\text{"}.$$

Let $\alpha > \sup(\operatorname{dom} r)$, $u = \{\delta\} \cup \operatorname{dom} r \cup \{\sup \operatorname{dom} r\}$. Let $q^* \in G$, $q^* \geq r\restriction\delta, q$, and let π collapse u to $\operatorname{otp} u$. Set $D = D^{\pi(\delta)}_{\pi[q^*],\pi[\underset{\sim}{z}]}$. Fix $v \subseteq \alpha$, and $r' \in G_\alpha \cap \mathrm{D}(u,v)$. We can copy r via an order-isomorphism inside $\alpha \times \aleph_2$, fixing $r\restriction\delta$, so that the result can be amalgamated with r', to yield r'', which is then a (u,v)-witness above r'. Since $r' \in \mathrm{D}(u,v)$, this means that r' is itself a (u,v)-witness in G_α. As this is all that the construction in 3.12 was supposed to achieve, this case is covered by the discussion there.

3.7 Minor Density Systems

In the course of the argument in 3.12, we require two further density systems. In the course of that argument we introduced the set

$$S = \{\gamma \in S_C : \underset{\sim}{F}(\underset{\sim}{x}_\gamma) \text{ is a } [\mathbb{P}\restriction(\gamma+1)]\text{-name}\},$$

and argued that S is stationary. This led us to consider certain ordinals $\gamma < \delta$, with δ of cofinality $\aleph_2$, and an element $r_1 \in G_\delta$, at which point we claimed that we could produce a 1-1 order preserving function h with domain $\mathcal{A}^{r_1}$, equal to the identity on $\mathcal{A}^{r_1} \cap (\gamma+1)$, with $h(\min(\mathcal{A}^{r_1} \setminus (\gamma+1))) > \sup \mathcal{A}^{r_1}$, and $h[r_1] \in G_\delta$. More precisely, our claim was that this could be ensured by meeting suitable density systems.

For $\alpha < \aleph_2$, $q \in \mathcal{AP}\restriction\aleph_2$, define $\mathrm{D}^\alpha_q(u,v)$ as follows. If $(\{\alpha\} \cup \operatorname{otp} \operatorname{dom} q) \geq \operatorname{otp} u$ then let $\mathrm{D}^\alpha_q(u,v)$ degenerate. Otherwise, fix $k : (\{\alpha\} \cup \operatorname{dom} q) \longrightarrow u$ an order isomorphism onto an initial segment of u, and let $\beta = \inf(u \setminus \operatorname{range} k)$. Let $\mathrm{D}^\alpha_q(u,v)$ be the set of $r \in \mathcal{AP}$ with domain v such that $r\restriction v \setminus u$ contains the image of q under an order-preserving map h_0 which agrees with k below α and which carries $\inf(\mathcal{A}^q \setminus (\alpha \times \aleph_2))$ above β (i.e., above $(\beta, 0)$). The density condition corresponds to our ability to copy over part of q onto any set of unused ordinals in $(v \setminus \beta) \times \aleph_2$, recalling that $|\operatorname{dom} r| < \aleph_2$ for any $r \in \mathcal{AP}$, and then to perform an amalgamation.

For our intended application, suppose that γ, δ, r_1 are given as above, and let $u = \{\gamma\} \cup \operatorname{dom} r_1 \cup \{\sup \operatorname{dom} r_1)$. Let π be the canonical isomorphism of u with $\operatorname{otp} u$, and $\alpha = \pi(\gamma)$, $q = \pi[r_1]$. As G_δ meets D^α_q, we have $v \subseteq \delta$, and $r \in G_\delta \cap \mathrm{D}^\alpha_q(u, v)$. Then with $h = h_0 \circ \pi$, we have $h[r_1] \le r$, and our claim is verified.

Finally, a few lines later in the course of the same argument we mentioned that the claim proved in 3.14 can be construed as the verification that certain additional density systems are in fact dense, and that accordingly we may suppose that the condition r described there lies in G.

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