On Taylor's problem

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By extending finite theorems Erdős and Rado proved that for every infinite cardinal κ there is a κ-chromatic triangle-free graph [3]. In later work they were able to add the condition that the graph itself be of cardinal κ [4]. The next stage, eliminating 4-circuits, turned out to be different, as it was shown by Erdős and Hajnal [1] that every uncountably chromatic graph contains a 4-circuit. In fact, every finite bipartite graph must be contained, but odd circuits can be omitted up to a certain length. This solved the problem "which finite graphs must be contained in every κ-chromatic graph" for every $\kappa > \omega$. The next result was given by Erdős, Hajnal, and Shelah [2], namely, every uncountably chromatic graph contains all odd circuits from some length onward. They, as well as Taylor, asked the following problem. If κ, λ are uncountable cardinals and X is a κ-chromatic graph, is there a λ-chromatic graph Y such that every finite subgraph of Y appears as a subgraph of X. In [2] the following much stronger conjecture was posed. If X is uncountably chromatic, then for some n it contains all finite subgraphs of the so-called n-shift graph. This conjecture was, however, disproved in [5].

Here we give some results on Taylor's conjecture when the additional hypotheses $|X| = \kappa$, $|Y| = \lambda$ are imposed.

We describe some (countably many) classes $\mathcal{K}^{n,e}$ of finite graphs and prove that if $\lambda^{\aleph_0} = \lambda$ then every λ^+-chromatic graph of cardinal λ^+ contains, for some n, e, all members of $\mathcal{K}^{n,e}$ as subgraphs. On the other hand, it is consistent for every regular infinite cardinal κ that there is a κ^+-chromatic graph on κ^+ that contains finite subgraphs only from $\mathcal{K}^{n,e}$. We get, therefore, some models of set theory, where the finite subraphs of graphs with $|X| = \mathrm{Chr}(X) = \kappa^+$ for regular uncountable cardinals κ are described.

We notice that in [6] all countable graphs are described which appear in every graph with uncountable coloring number.

Notation. $\overline{x}$ will denote a finite string of ordinals. $\overline{x} < \overline{y}$ means that $\max(\overline{x}) < \min(\overline{y})$.

Definition. Assume that $1 \le n < \omega$, $e : \{1, 2, \ldots, 2n\} \to \{0, 1\}$ is a function with $|f^{-1}(0)| = n$. We are going to define the structures in $\mathcal{K}^{n,e}$ as follows. They will be of the form $H = (V, <, U, X, h_1, \ldots, h_n)$ where $(V, <)$ is a finite linearly ordered set, $U \subseteq V$, X is a graph on U, $h_i : U \to V$ satisfy $h_1(x) < \cdots < h_n(x) = x$ for $x \in U$. The elements in $\mathcal{K}_0^{n,e}$ are those isomorphic to $(V, <, U, X, h_1, \ldots, h_n)$ where $V = \{1, 2, \ldots, n\}$, $<$ is the natural ordering, $U = \{n\}$, $X = \emptyset$, $h_i(n) = i$ $(1 \le i \le n)$.

If $H = (V, <, U, X, h_1, \ldots, h_n)$ is a structure of the above form, and $x \in V$, we form the edgeless amalgamation $H' = H +_x H$ as follows. Put $H' = H +_x H = (V', <', U', X', h'_1, \ldots, h'_n)$ where $(V', <')$ has the $<'$-*ordered* decomposition $V' = W \cup V_0 \cup V_1$, if we put $V'_i = W \cup V_i$ for $i < 2$ then the structures

$$\left(V'_i, <' |V'_i, U' \cap V'_i, h'_1|V'_i, \ldots, h'_n|V'_i\right)$$

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are both isomorphic to H for $i < 2$ and $\min(V_i)$ correspond to x under the isomorphisms.

If $H = (V, <, U, X, h_1, \ldots, h_n)$ is a structure of the above form, and $x \in U$, we also form the one-edge amalgamation $H' = H *_x H$ as follows. Enumerate in increasing order $e^{-1}(0)$ as $\{a_1, \ldots, a_n\}$ and $e^{-1}(1)$ as $\{b_1, \ldots, b_n\}$. Put $H' = (V', <', U', X', h'_1, \ldots, h'_n)$ where $(V', <')$ has the *ordered* decomposition $V' = V_0 \cup V_1 \cup \cdots \cup V_{2n}$; $H'|(V_0 \cup \bigcup\{V_i : e(i) = \varepsilon\})$ are isomorphic to H ($\varepsilon = 0, 1$) if x_0, x_1 are the points corresponding to x, then $h'_i(x_0) = \min(V_{a_i})$, $h'_i(x_1) = \min(V_{b_i})$, and the only extra edge in X' is $\{x_0, x_1\}$.

We then put

$$\begin{aligned}\mathcal{K}^{n,e}_{t+1} = &\big\{H +_x H : H = (V, <, U, \ldots) \in \mathcal{K}^{n,e}_t, x \in V\big\}\\ &\cup \big\{H *_y H : H = (V, <, U, \ldots) \in \mathcal{K}^{n,e}_t, y \in U\big\},\end{aligned}$$

and finally $\mathcal{K}^{n,e} = \bigcup\{\mathcal{K}^{n,e}_t : t < \omega\}$.

Theorem 1. *If $|G| = \mathrm{Chr}(G) = \lambda^+$, $\lambda^{\aleph_0} = \lambda$, then, for some n, e, G contains every graph in $\mathcal{K}^{n,e}$ as subgraph.*

We start with some technical observations.

Lemma 1. *If $t_n : \lambda^+ \to \lambda^+$ are functions ($n < \omega$), then there is a λ-coloring $F : \lambda^+ \to \lambda$ such that for $F(\alpha) = F(\beta)$, i, $j < \omega$, $\alpha < t_i(\beta) < t_j(\alpha)$ may not hold.*

Proof. As $\lambda^{\aleph_0} = \lambda$, it suffices to show this for two functions $t_0(\alpha)$, $t_1(\alpha)$, with $t_1(\alpha) > \alpha$. We prove the stronger statement that there is a function $F : \lambda^+ \to [\lambda]^\lambda$ such that if $\alpha < t_0(\beta) < t_1(\alpha)$ then $F(\alpha) \cap F(\beta) = \emptyset$. Let $\langle N_\xi : \xi < \lambda^+\rangle$ be a continuous, increasing sequence of elementary submodels of $\langle \lambda^+; <, t_0, t_1, \ldots\rangle$ with $\gamma_\xi = N_\xi \cap \lambda^+ < \lambda^+$. $C = \{\gamma_\xi : \xi < \lambda^+\}$ is closed, unbounded. We define $F|\gamma_\xi$ by transfinite recursion on ξ. If $F|\gamma_\xi$ is given, and β has $t_0(\beta) < \gamma_\xi \le \beta < \gamma_{\xi+1}$, by elementarity $\tau = \sup\{t_1(\alpha) : \alpha < t_0(\beta)\} < \gamma_\xi$, and there is a β' with $t_0(\beta') = t_0(\beta)$, $\tau < \beta' < \gamma_\xi$. Put $H(\beta) = F(\beta')$, otherwise, i.e., when $\gamma_\xi \le t_0(\beta)$, put $H(\beta) = \lambda$. To get $F|[\gamma_\xi, \gamma_{\xi+1})$, we disjointize $\{H(\beta) : \gamma_\xi \le \beta < \gamma_{\xi+1}\}$, i.e., find $F(\beta) \subseteq H(\beta)$ of cardinal λ such that $F(\beta_0) \cap F(\beta_1) = \emptyset$ for $\beta_0 \ne \beta_1$. We show that this F works. Assume that $F(\alpha)$, $F(\beta)$ are not disjoint. By induction we can assume that either α or β is between γ_ξ and $\gamma_{\xi+1}$. By the disjointization process some of them must be smaller than γ_ξ. If $\beta < \gamma_\xi \le \alpha < \gamma_{\xi+1}$ then $t_0(\beta) < \gamma_\xi$ as N_ξ is an elementary submodel, so $t_0(\beta) < \alpha$. Assume now that $\alpha < \gamma_\xi \le \beta < \gamma_{\xi+1}$. Our construction then selected a β' with $t_0(\beta') = t_0(\beta)$ and $F(\beta) \subseteq H(\beta) = F(\beta')$ which is, by the inductive hypothesis, disjoint from $F(\alpha)$. □

Lemma 2. *If $C = \{\delta_\xi : \xi < \lambda^+\}$ is a club then there is a function $K : [\lambda^+]^{\aleph_0} \to \lambda$ such that if $K(A) = K(B)$ and $A \cap [\delta_\xi, \delta_{\xi+1}) \ne \emptyset$ and $B \cap [\delta_\xi, \delta_{\xi+1}) \ne \emptyset$ for some $\xi < \lambda^+$ then $A \cap \delta_{\xi+1} = B \cap \delta_{\xi+1} =$ and so $A \cap B$ is an initial segment both in A and B.*

Proof. Fix for every $\beta < \lambda^+$ an into function $F_\beta : \alpha \to \lambda$ such that for $\beta_0 < \beta_1 < \beta_2$, $F_{\beta_1}(\beta_0) \ne F_{\beta_2}(\beta_1)$ holds. This can be done by a straightforward inductive construction.

If $A \in [\lambda^+]^{\aleph_0}$ put $X(A) = \{\xi : A \cap [\delta_\xi, \delta_{\xi+1}) \ne \emptyset\}$. Let $\mathrm{tp}(X(A)) = \eta$. Enumerate $X(A)$ as $\{\tau^A_\theta : \theta < \eta\}$. Let $K(A)$ be a function with domain η, at $\theta < \eta$, if $\tau^A_\theta = \xi$, let

$$K(A)(\theta) = \big\langle \{F_{\tau^A_\theta}(\tau^A_{\theta'}) : \theta' < \theta\}, \{F_{\delta_{\xi+1}}(y) : y \in A \cap \delta_{\xi+1}\}\big\rangle.$$

Assume now that $K(A) = K(B)$, $\xi \in X(A) \cap X(B)$. If $\xi = \tau^A_\theta = \tau^B_{\theta'}$ then $\theta = \theta'$ by the properties of F above. The second part of the definition of $K(A)$ gives that $A \cap \delta_{\xi+1} = B \cap \delta_{\xi+1}$. □

Proof of Theorem 1. We first show that one can assume that G is λ^+-chromatic on every closed unbounded set.

Lemma 3. *There is a function $f : \lambda^+ \to \lambda^+$ such that if $C \subseteq \lambda^+$ is a closed unbounded set then $\bigcup\{[\alpha, f(\alpha)] : \alpha \in C\}$ is λ^+-chromatic.*

Proof. Assume that the statement of the Lemma fails. Put $f_0(\alpha) = \alpha$, for $n < \omega$ let C_n witness that $f_n : \lambda^+ \to \lambda^+$ is not good and $f_{n+1}(\alpha) = \min(C_n - (\alpha + 1))$. As, by assumption, $\bigcup\{[\alpha, f_n(\alpha)] : \alpha \in C_n, n < \omega\}$ is $\leq \lambda$-chromatic, there is a

$$\gamma \notin \bigcup\Big\{[\alpha, f_n(\alpha)] : \alpha \in C_n, n < \omega\Big\}, \gamma > \min\Big(\bigcap\Big\{C_n : n < \omega\Big\}\Big).$$

Clearly, $\gamma \notin C_n$ $(n < \omega)$, and if now $\alpha_n = \max(\gamma \cap C_n)$, then $\alpha_n < \gamma$, and $\alpha_{n+1} < \alpha_n$ $(n < \omega)$, a contradiction. □

By slightly re-ordering λ^+ we can state Lemma 3 as follows. If $C \subseteq \lambda^+$ is a closed unbounded set, then $S(C) = \bigcup\{[\lambda\alpha, \lambda(\alpha + 1)) : \alpha \in C\}$ is λ^+-chromatic. Put, for $\tau < \lambda$, $C \subseteq \lambda^+$ a club set, $S_\tau(C) = \bigcup\{\lambda\alpha + \tau : \alpha \in C\}$. If, for every $\tau < \lambda$ there is some closed unbounded C_τ that $S_\tau(C_\tau)$ is λ-chromatic, then for $C = \bigcap\{C_\tau : \tau < \lambda\}$, $S(C)$ is the union of at most λ graphs, each $\leq \lambda$-chromatic, a contradiction.

There is, therefore, a $\tau < \lambda$ such that $S_\tau(C)$ is λ^+-chromatic whenever C is a closed unbounded set. Mapping $\lambda\alpha+\tau$ to α we get a graph on λ^+, order-isomorphic to a subgraph of the original graph which is λ^+-chromatic on every closed unbounded set. From now on we assume that our original graph G has this property.

We are going to build a model $M = \langle \lambda^+; <, \lambda, G, \ldots\rangle$ by adding countably many new functions.

For n, e as in the Definition, φ a first order formula, let $G^{n,e}_\varphi$ be the following graph. The vertex set is $V_\varphi = \{\langle \overline{x}_0, \ldots, \overline{x}_n\rangle : \overline{x}_0 < \cdots < \overline{x}_n, M \models \varphi(\overline{x}_0, \ldots, \overline{x}_n)\}$ and $\langle \overline{x}_0, \ldots, \overline{x}_n\rangle$, $\langle \overline{y}_0, \ldots, \overline{y}_n\rangle$ are joined, if $\overline{x}_0 = \overline{y}_0$, $\{\overline{x}_1, \ldots, \overline{x}_n\}$, $\{\overline{y}_1, \ldots, \overline{y}_n\}$ interlace by e, and finally $\{\min(\overline{x}_n), \min(\overline{y}_n)\} \in G$. We introduce a new quantifier $Q^{n,e}$ with $Q^{n,e}\varphi$ meaning that the above graph, $G^{n,e}_\varphi$ is λ^+-chromatic. If, however, $\mathrm{Chr}(G^{n,e}_\varphi) \leq \lambda$, we add a good λ-coloring to M. We also assume that M is endowed with Skolem functions.

Lemma 4. *There exist n, e and $\alpha_1 < \cdots < \alpha_n < \lambda^+$ such that $t(\alpha_i) < \alpha_{i+1}$ if $t : \lambda^+ \to \lambda^+$ is a function in M and if $\overline{x}_0 \subseteq \alpha_1$, $\overline{x}_i \subseteq [\alpha_i, \alpha_{i+1})$ $(1 \leq i < n)$, $\overline{x}_n \subseteq [\alpha_n, \lambda^+)$, $\min(\overline{x}_i) = \alpha_i$, and φ is a formula, $M \models \varphi(\overline{x}_0, \ldots, \overline{x}_n)$, then $M \models Q^{n,e}\varphi$.*

Proof. Assume that the statement of the Lemma does not hold, i.e., for every n, e, $\alpha_1, \ldots, \alpha_n$ there exist $\overline{x}_0, \ldots, \overline{x}_n$ contradicting it.

Let, for $\alpha < \lambda^+$, $B_\alpha \subseteq \lambda^+$ be a countable set such that $\alpha \in B_\alpha$, and if n, e, $\alpha_1, \ldots, \alpha_n \in B_\alpha$ are given , then a counter-example as above is found with $\overline{x}_0, \ldots, \overline{x}_n \subseteq B_\alpha$. We require that B_α be Skolem-closed. Let B^+_α be the ordinal closure of B_α, $B^+_\alpha =$

$\{\gamma(\alpha,\xi) : \xi \le \xi_\alpha\}$ be the increasing enumeration, $\alpha = \gamma(\alpha,\tau_\alpha)$. Let $\{M_\xi : \xi < \lambda^+\}$ be a continuous, increasing chain of elementary submodels of M such that $\delta_\xi = M_\xi \cap \lambda^+ < \lambda^+$. Clearly, $C = \{\delta_\xi : \xi < \lambda^+\}$ is a closed, unbounded set. We take a coloring of the sets $\{B^+_\alpha : \alpha < \lambda^+\}$ by λ colors that satisfies Lemma 2, if α, β get the same color then the structures $(B^+_\alpha; B_\alpha, M)$ and $(B^+_\beta; B_\beta, M)$ are isomorphic and we also require that if $\overline{x}_0,\dots,\overline{x}_n \subseteq B_\alpha$ and $\overline{y}_0,\dots,\overline{y}_n \subseteq B_\beta$ are in the same positions, i.e., are mapped onto each other by the order isomorphism between B_α and B_β and $(\overline{x}_0,\dots,\overline{x}_n)$ is colored by the λ-coloring of $G^{n,e}_\varphi$, then $(\overline{y}_0,\dots,\overline{y}_n)$ is also colored and gets the same color. All this is possible, as $\lambda^{\aleph_0} = \lambda$. We also assume that our coloring satisfies Lemma 1 with some functions $\{t_n : n < \omega\}$ that $B^+_\alpha = \{t_n(\alpha) : n < \omega\}$.

As G is λ^+-chromatic on C, there are $\alpha < \beta$, both in C, joined in G, getting the same color. By our conditions, $B^+_\alpha \cap B^+_\beta$ is initial segment in both, and beyond that they do not even intersect into the same complementary interval of C. As our structures are isomorphic, this holds for B_α, B_β, as well.

We now let $B^+_\alpha = \bigcup\{B^+_\alpha(i) : i < i_\alpha\}$, $B^+_\beta = \bigcup\{B^+_\beta(i) : i < i_\beta\}$ be the ordered decompositions given by the following equivalence relations. For x, $y \in B^+_\alpha$, $x \le y$, $x \sim y$ if either $[x,y] \cap B^+_\beta = \emptyset$ or $[x,y] \cap B^+_\beta \supseteq [x,y] \cap B^+_\alpha$. Similarly for B^+_β. By Lemma 2, $B^+_\alpha \cap B^+_\beta = B^+_\alpha(0) = B^+_\beta(0)$.

Lemma 5. *i_α, i_β are finite.*

Proof. Otherwise, as B^+_α, B^+_β are ordinal closed, $\gamma = \min\bigl(B^+_\alpha(\omega)\bigr) = \min\bigl(B^+_\beta(\omega)\bigr)$ is in $B^+_\alpha \cap B^+_\beta$, so $\gamma \in B^+_\alpha(0)$, a contradiction. □

Enumerate

$$\Bigl\{\xi \le \xi_\alpha : \text{there is a } 0 < i < \omega \text{ such that either } \gamma(\alpha,\xi) = \min\bigl(B^+_\alpha(i)\bigr) \text{ or } \gamma(\beta,\xi) = \min\bigl(B^+_\beta(i)\bigr)\Bigr\} \cup \Bigl\{\tau_\alpha\Bigr\}$$

as $\xi_1 < \xi_2 < \dots < \xi_n$. By Lemma 1, if $\alpha < \beta$, $\alpha = \min(B^+_\alpha(i_\alpha - 1))$, $\beta = \min(B^+_\beta(i_\beta - 1))$, $B^+_\alpha(i_\alpha - 1) < B^+_\beta(i_\beta - 1)$. So $\xi_n = \tau_\alpha$. We let $\alpha_i = \gamma(\alpha,\xi_i)$, $\beta_i = \gamma(\beta,\xi_i)$. If $\alpha_i = \min(B^+_\alpha(j))$ then $\alpha_i \in B_\alpha$ and, by isomorphism, $\beta_i \in B_\beta$. We show that for every $i < n$, $t \in M$, $t(\alpha_i) < \alpha_{i+1}$ and $t(\beta_i) < \beta_{i+1}$. As $(B^+_\alpha; B_\alpha, M)$ and $(B^+_\beta; B_\beta, M)$ are isomorphic, for every i it suffices to show this either for α_i or for β_i. For $i = n-1$ this follows from the fact that α (as well as β) is from C. If $i < n$ then either α_{i-1} and α_i are separated by an element of B^+_β or vice versa. Assume the former. Then, by Lemma 2, α_{i-1} and α_i are in different intervals of C so necessarily $t(\alpha_{i-1}) < \alpha_i$ holds.

Let e be the interlacing type of $\{\alpha_i : 1 \le i \le n\}$, $\{\beta_i : 1 \le i \le n\}$. By our indirect assumption, there are a formula φ, $\overline{x}_i$, $\overline{y}_i$ $(0 \le i \le n)$ in the same position in B_α, B_β such that $\overline{x}_i \subseteq [\alpha_i, \alpha_{i+1})$, $\overline{y}_i \subseteq [\beta_i, \beta_{i+1})$ etc, and $M \models \varphi(\overline{x}_0,\dots,\overline{x}_n) \wedge \varphi(\overline{y}_0,\dots,\overline{y}_n)$ and $(\overline{x}_0,\dots,\overline{x}_n)$, $(\overline{y}_0,\dots,\overline{y}_n)$ are joined in $G^{n,e}_\varphi$, but they get the same color in the good coloring of $G^{n,e}_\varphi$, a contradiction which proves Lemma 4. □

Now fix n, e, and $\alpha_1 < \cdots < \alpha_n < \lambda^+$ as in Lemma 4. We call a formula φ *dense* if there exist $\overline{x}_0 \subseteq \alpha_1$, $\overline{x}_i \subseteq [\alpha_i, \alpha_{i+1})$ $(1 \le i < n)$, $\overline{x}_n \subseteq [\alpha_n, \lambda^+)$, $\min(\overline{x}_i) = \alpha_i$ such that $M \models \varphi(\overline{x}_0, \ldots, \overline{x}_n)$. If $H = (V, <, U, X, h_1, \ldots, h_n) \in \mathcal{K}^{n,e}$, $V = \{0, 1, \ldots, s\}$, a *φ-rich copy of H* is some string $(\overline{y}_0, \ldots, \overline{y}_s)$ such that $\overline{y}_0 < \cdots < \overline{y}_s$, if $\{i, j\} \in X$ then $\{\min(\overline{y}_i), \min(\overline{y}_j)\} \in G$ and for every $v \in U$, $M \models \varphi(\overline{y}_0, \overline{y}_{h_1(v)}, \ldots, \overline{y}_{h_n(v)})$.

Lemma 5. *For every $H \in \mathcal{K}^{n,e}$ if φ is dense there is a φ-rich copy of H in G.*

Lemma 6. *For every $H = (V, <, U, X, h_1, \ldots, h_n) \in \mathcal{K}^{n,e}$, $q \in U$, if φ is dense, there is a φ-rich copy $(\overline{y}_0, \ldots, \overline{y}_s)$ of H such that $\min(\overline{y}_{h_i(q)}) = \alpha_i$ for $1 \le i \le n$.*

We notice that Lemma 5 obviously concludes the proof of Theorem 1 and Lemma 6 clearly implies Lemma 5. Also, they trivially hold for $H \in \mathcal{K}^{n,e}_0$. We prove these two Lemmas simultaneously.

Claim 1. *If Lemma 5 holds for some H then Lemma 6 holds for H, as well.*

Proof. Assume that Lemma 5 holds for $H = (V, <, U, X, h_1, \ldots, h_n) \in \mathcal{K}^{n,e}$ and for any dense φ but Lemma 6 fails for a certain $q \in U$ and a dense φ. This statement can be written as a formula $\theta(\alpha_1, \ldots, \alpha_n)$. As φ is dense, $M \models \varphi(\overline{x}_0, \ldots, \overline{x}_n)$ for some appropriate strings, so also $M \models \psi(\overline{x}_0, \ldots, \overline{x}_n)$ where $\psi = \varphi \wedge \theta(\min(\overline{x}_1), \ldots, \min(\overline{x}_n))$. As ψ is dense, by Lemma 5 there is a ψ-rich copy $(\overline{y}_0, \ldots, \overline{y}_s)$ of H but then $M \models \theta(\min(\overline{y}_{h_1(q)}), \ldots, \min(\overline{y}_{h_n(q)}))$, a contradiction. □

Claim 2. *If Lemma 6 holds for $H = (V, <, U, X, h_1, \ldots, h_n)$ and $x \in V$ then Lemma 5 holds for $H' = H +_x H$.*

Proof. Select $q \in U$ such that $x = h_i(q)$ for some $1 \le i \le n$. By Lemma 6, there is a φ-rich copy $(\overline{y}_0, \ldots, \overline{y}_s)$ of H such that $\min(\overline{y}_{h_i(q)}) = \alpha_i$ for $1 \le i \le n$. As $\alpha_i > t(\alpha_{i-1})$ holds for every function t in the skolemized structure M there are φ-rich copies of H which agree with this below x but their x elements are arbitrarily high. We can, therefore, get a φ-rich copy of H'. □

Claim 3. *If Lemma 6 holds for $H = (V, <, U, X, h_1, \ldots, h_n)$ and $y \in U$ then Lemma 5 holds for $H' = H *_y H$.*

Proof. Let $(\overline{y}_0, \ldots, \overline{y}_s)$ be a φ-rich copy of H such that $\min(\overline{y}_{h_i(q)}) = \alpha_i$ for $1 \le i \le n$. The elements in the $(\overline{y}_0, \ldots, \overline{y}_s)$ string can be redistributed as $(\overline{x}_0, \ldots, \overline{x}_n)$ such that $\min(\overline{x}_i) = \alpha_i$ and then the fact that they form a φ-rich copy of H can be written as $M \models \psi(\overline{x}_0, \ldots, \overline{x}_n)$ for some formula ψ. As ψ is dense, by Lemma 4, $M \models Q^{n,e}\psi$ holds, so there are two strings, $(\overline{x}_0, \ldots, \overline{x}_n)$ and $(\overline{x}'_0, \ldots, \overline{x}'_n)$ both satisfying ψ, interlacing by e, and $\{\min(\overline{x}_n), \min(\overline{x}'_n)\} \in G$. This, however, gives a φ-rich copy of H'. □

Theorem 2. *If n, e are as in the Definition, λ is an infinite cardinal, $\lambda^{<\lambda} = \lambda$, then there exists a λ^+-c.c., $< \lambda$-closed poset $Q = Q_{n,e,\lambda}$ which adds a λ^+-chromatic graph of cardinal λ^+ all whose finite subgraphs are subgraphs of some element of $\mathcal{K}^{n,e}$.*

Proof. Put $q = (V, U, X, h_1, \ldots, h_n) \in Q$ if $V \in [\lambda^+]^{<\lambda}$, $U \subseteq V$, $X \subseteq [V]^2$, every h_i is a function $U \to V$ with $h_1(x) < \cdots < h_n(x) = x$ for $x \in U$ and every finite substructure of $(q,<)$ is a substructure of some element of $\mathcal{K}^{n,e}$. Order Q as follows. $q' = (V',U',X',h'_1,\dots,h'_n) \le q = (V,U,X,h_1,\dots,h_n)$ iff $V' \supseteq V$, $U = U' \cap V$, $X = X' \cap [V]^2$, $h'_i \supseteq h_i$ $(1 \le i \le n)$. Clearly, $(Q,\le)$ is $<\lambda$-closed.

Lemma 7. *$(Q,\le)$ is λ^+-c.c.*

Proof. By the usual Δ-system arguments it suffices to show that if the conditions $q^i = (V \cup V^i, U^i, X^i, h^i_1,\dots,h^i_n)$ are order isomorphic $(i<2)$, $V < V^0 < V^1$ then they are compatible. A finite subset of $V \cup V^0 \cup V^1$ can be included into some $s \cup s_0 \cup s_1$ where s_0 and s_1 are mapped onto each other by the isomorphism between q_0 and q_1. By condition, $q|s \cup s_0$ is a substructure of some structure $H \in \mathcal{K}^{n,e}$. But then $q|s \cup s_0 \cup s_1$ is a substructure of an edgeless amalgamation of H. □

If $G \subseteq Q$ is generic then $Y = \bigcup\{X : (V,U,X,\dots) \in G\}$ is a graph on a subset of λ^+ all whose finite subgraphs are subgraphs of some member of $\mathcal{K}^{n,e}$. The following Lemma clearly concludes the proof of the Theorem.

Lemma 8. $\mathrm{Chr}(Y) = \lambda^+$.

Proof. Assume, toward a contradiction, that 1 forces that $f : \lambda^+ \to \lambda$ is a good coloring of Y. Let $M_1 \prec M_2 \prec \cdots \prec M_n$ be elementary submodels of $(H((2^\lambda)^+); Q, f, \dots,)$ with $\lambda \subseteq M_0$, $[M_i]^{<\lambda} \subseteq M_i$. Put $\delta_i = M_i \cap \lambda^+ < \lambda^+$. Notice that $\mathrm{cf}(\delta_i) = \lambda$. Let $p' = (V',U',X',h'_1,\dots,h'_n)$ where $V = \{\delta_1,\dots,\delta_n\}$, $U = \{\delta_n\}$, $X = \emptyset$, $h_i(\delta_n) = \delta_i$. Choose $p = (V,U,X,h_1,\dots,h_n) \le p'$ forcing $f(\delta_n) = \xi$ for some $\xi < \lambda$. Let $\psi_n(\pi, x_1,\dots,x_n)$ be the following formula. π is an order isomorphism $V \to \lambda^+$, $\pi(\delta_i) = x_i$ and $\pi(p)$ forces that $f(x_n) = \xi$. Let $\delta_{n+1} = \lambda$. For $0 \le i < n$ define $\psi_i(\pi, x_1,\dots,x_i)$ meaning that $\pi : V \cap \delta_{i+1} \to \lambda^+$ is order preserving and there are arbitrarily large $x_{i+1} < \lambda^+$ and $\pi' \supseteq \pi$ such that $\psi_{i+1}(\pi', x_1,\dots,x_{i+1})$ holds.

Claim 4. *$\psi_i(\mathrm{id}|V \cap \delta_{i+1}, \delta_1,\dots,\delta_i)$ for $0 \le i \le n$.*

Proof. This is obvious for $i = n$. If $\psi_i(\mathrm{id}|V \cap \delta_{i+1}, \delta_1,\dots,\delta_i)$ fails, then, by definition, there would be a bound for the possible x_{i+1} values for which $\psi_{i+1}(\pi', \delta_1,\dots,\delta_i, x_{i+1})$ holds for some $\pi' \supseteq \mathrm{id}|V \cap \delta_{i+1}$. But then this bound is smaller than δ_{i+1} so ψ_{i+1} fails, too. □

Returning to the proof of Lemma 8, we define the following function t. Let $\{a_1,\dots,a_n\}$, $\{b_1,\dots,b_n\}$ be as in the definition of the one-edge amalgamation. Put, for $1 \le i \le n$, $t(i) = j$ iff $b_{j-1} < a_i < b_j$ where $b_0 = 0$, $b_{n+1} = 2n+1$. Set $\pi_0 = \mathrm{id}|V \cap \delta_1$. We know that $\psi_0(\pi_0)$ holds. By induction on $1 \le i \le n$ select π_i in such a way that $\pi_{i+1} \supseteq \pi_i$, if we let $\pi_i(\delta_i) = \delta'_i$ then $\psi_i(\pi_i, \delta'_1,\dots,\delta'_i)$ holds and $\sup(V \cap \delta_{t(i)}) < \delta'_i$ and $\mathrm{Ran}(\pi_i) < \delta_{t(i)}$. This is possible as $M_1,\dots,M_n$ are elementary submodels. Finally, $\pi_n(p)$ is a condition interlacing with p by e and it forces that $f(\delta'_n) = \xi$. Now if we take the union of them plus the edge $\{\delta_n, \delta'_n\}$ then an argument as in Lemma 7 shows that we get a condition which forces a contradiction. □

Theorem 3. *If GCH holds there is a cardinal, cofinality, and GCH preserving (class) notion of forcing in which for every n, e, and regular $\lambda \ge \omega$ there is a λ^+-chromatic graph on λ^+ all whose finite subgraphs are subgraphs of some elements of $\mathcal{K}^{n,e}$.*

Proof. For $\lambda \geq \omega$ regular let Q_λ be the product of $Q_{n,e,\lambda}$ of Theorem 2 with finite supports if $\lambda = \omega$, and complete supports otherwise. Notice that Q_λ is a λ^+-c.c. notion of forcing of cardinal λ^+. For λ singular let Q_λ be the trivial forcing.

Our notion of forcing is the Easton-support limit of the Q_λ's, i.e., the direct limit of P_α where $P_{\alpha+1} = P_\alpha \oplus Q_\alpha$ with Q_α defined in the ground model. For α limit, $p \in P_\alpha$ iff $p(\beta) \in Q_\beta$ for all $\beta < \alpha$, and $|\mathrm{Dom}(p) \cap \kappa| < \kappa$ for $\kappa \leq \alpha$ regular.

Given n, e, and λ as in the statement of the Theorem, the extended model can be thought as the generic extension of some model first with $Q_{n,e,\lambda}$ then with P_λ which is of cardinal λ so it cannot change the chromatic number of a graph from λ^+ to λ.

Assume that the cofinality of some ordinal α collapses to a regular λ. P splits as $P_\lambda \oplus Q_\lambda \oplus R$ where R is $\leq \lambda$-closed, $|P_\lambda| \leq \lambda$ and Q_λ is λ^+-c.c., so in fact the λ^+-c.c. $P_{\lambda+1}$ changes the cofinality of α which is impossible. This also implies that no cardinals are collapsed.

If τ is regular, all subsets of τ are added by the τ^+-c.c. $P_{\tau+1}$ of cardinal τ^+ so 2^τ remains τ^+. If τ is singular we must bound $\tau^{\mathrm{cf}(\tau)}$. The sets of size $\mathrm{cf}(\tau)$ are added by $P_{\mathrm{cf}(\tau)+1}$ so we can bound the new value of $\tau^{\mathrm{cf}(\tau)}$ by $\tau^{\mathrm{cf}(\tau)^+} = \tau^+$. □

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