BOREL WHITEHEAD GROUPS

SAHARON SHELAH

Institute of Mathematics
The Hebrew University
Jerusalem, Israel

Rutgers University
Mathematics Department
New Brunswick, NJ USA

MSRI
Berkeley, CA USA

ABSTRACT. We investigate the Whiteheadness of Borel abelian groups ($\aleph_1$-free, without loss of generality as otherwise this is trivial). We show that CH (and even WCH) implies any such abelian group is free, and always $\aleph_2$-free.

1998 Mathematics Subject Classification 03C60, 03E15, 20K20.
Key words and phrases. Abelian groups, Whitehead groups, Borel sets, Borel Abelian groups.

I would like to thank Alice Leonhardt for the beautiful typing.
First Typed - 98/Mar/4
Latest Revision - 99/May/17
§1, §2 done, Fall '89
Publication no. 402

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

§0 Introduction

It is independent of set theory whether every Whitehead group is free [Sh 44]. The problem is called Whitehead's problem. In addition, Whitehead's problem is independent of set theory even under the continuum hypothesis [Sh:98]. An interesting problem suggested by Dave Marker is the Borel version of Whitehead's problem, namely,
Question: Is every Whitehead group coded by a Borel set free? (For a precise definition of a Borel code, see below.) In the present paper, we will give a partial answer to this question.

0.1 Definition. 1) We say that $\bar\psi = \langle \psi_0, \psi_1 \rangle$ is a code for a Borel abelian group if:

- (a) $\psi_0(\dots,\dots)$ codes a Borel equivalence relation $E = E^{\bar\psi}$ on a subset $B_* = B_*^{\bar\psi}$ of ${}^\omega 2$ so $[\psi_0(\eta,\eta) \leftrightarrow \eta \in B_*]$ and $[\psi_0(\eta,\nu) \rightarrow \eta \in B_* \ \& \ \nu \in B_*]$, the group will have a set of elements $B = B_*^{\bar\psi}/E^{\bar\psi}$
- (b) $\psi_1 = \psi_1(x,y,z)$ codes a Borel set of triples from ${}^\omega 2$ such that $\{(x/E^{\bar\psi}, y/E^{\bar\psi}, z/E^{\bar\psi}) : \psi_1(x,y,z)\}$ is the graph of a function from $B \times B$ to B such that $(B,+)$ is an abelian group.

2) We say Borel$^+$ if (b) is replaced by:

- $(b)'$ ψ_1 codes a Borel function from $B_* \times B_*$ to B_* which respects $E^{\bar\psi}$, the function is called $+$ and $(B,+)$ is an abelian group (well, we should denote the function which $+$ induces from $(B_*/E^{\bar\psi}) \times (B_*/E^{\bar\psi})$ into $B_*/E^{\bar\psi}$ by e.g. $+_{E^{\bar\psi}}$, but are not strict).

3) We let $B^{\bar\psi} = B_{\bar\psi} = (B,+)$ be the group coded by $\bar\psi$; abusing notation we may write B for $B_{\bar\psi}$.
4) An abelian group B is called Borel if it has a Borel code similarly "Borel$^+$".

Clearly
0.2 Observation: The set of codes for Borel abelian groups is Π^1_2.

An interesting problem suggested by Dave Marker is the Borel version of Whitehead's problem: namely
0.3 Question: Is every Borel$^+$ Whitehead group free?

In this paper we will give a partial answer to this question, even for the "Borel" (without +) version. We will show that every Borel Whitehead group is $\aleph_2$-free. In particular, the continuum hypothesis implies that every Borel Whitehead group is free. This latter result provides a contrast to the author's proof ([Sh:98]) that it is consistent with CH that there is a Whitehead group of cardinality $\aleph_1$ which is not free.

We refer the reader to [EM] for the necessary background material on abelian groups.

Suppose B is an uncountable $\aleph_1$-free abelian group. Let $S_0 = \{G \subset B : |G| = \aleph_0$ and B/G is not $\aleph_1$-free$\}$. It is well known that if B is not $\aleph_2$-free, then S_0 is stationary. We will argue that the converse is true for Borel abelian groups and the answer is quite absolute. Lastly, we deal with weakening Borel to Souslin.

0.4 Question: If B is an $\aleph_2$-free Borel abelian group, what can be the n in the analysis of a nonfree $\aleph_2$-free abelian subgroup of B from [Sh 161] (or see [EM] or [Sh 523])?

We thank Todd Eisworth and the referee for corrections.

§1 On $\aleph_2$-freeness

1.1 Hypothesis. Let B be an $\aleph_1$-free Borel abelian group. Let $\bar{\psi}$ be a Borel code for B.

Let $S_B = S_{\bar{\psi}} = \{K \subseteq B : K$ is a countable subgroup and B/K is not $\aleph_1$-free$\}$.

1.2 Lemma. *1) If S_B is stationary, then B is not $\aleph_2$-free.*
2) Moreover, there is an increasing continuous sequence $\langle G_i : i < \omega_1 \rangle$ of countable subgroups of B such that G_{i+1}/G_i is not free for each $i < \omega_1$.

Remark. On such proof in model theory see [Sh 43, §2], [BKM78] and [Sch85].

Proof. We work in a universe $V \models ZFC$. Force with $\mathbf{P} = \{p : p$ is a function from some $\alpha < \omega_1$ to ${}^\omega 2\}$. Let $G \subseteq \mathbf{P}$ be V-generic and let $V[G]$ denote the generic extension.

Since $\mathbf{P}$ is $\aleph_1$-closed, forcing with $\mathbf{P}$ adds no new reals. Thus $\bar{\psi}$ still codes B in the generic extension, i.e. $B_{\bar{\psi}}^{V[G]} = B_{\bar{\psi}}^V$. Forcing with $\mathbf{P}$ also adds no new countable subsets of B hence "B is $\aleph_1$-free" holds in V iff it holds in $V[G]$. Similarly if $K \subset B$ is countable, then "B/K is $\aleph_1$-free" holds in V iff it holds in $V[G]$. Thus, $S_{\bar{\psi}}^V = S_{\bar{\psi}}^{V[G]}$. Moreover, since $\mathbf{P}$ is proper, $S_{\bar{\psi}}$ remains stationary (see [Sh:f, Ch.III]).

Since $V[G] \models CH$, we can write

$$B = \bigcup_{\alpha < \omega_1} B_\alpha,$$

where $\bar{B} = \langle B_\alpha : \alpha < \omega_1 \rangle$ is an increasing continuous chain of countable pure subgroups. Let $S = \{\alpha < \omega_1 : B/B_\alpha$ is not $\aleph_1$-free$\}$. Since $S_{\bar{\psi}}$ is stationary (as a subset of $[B]^{\aleph_0}$), clearly S is a stationary subset of ω_1. So $V[G] \models$ "B is not free".

By Pontryagin's criteria for each $\alpha \in S$ there are $n_\alpha \in \omega$ and $a_0^\alpha, \dots, a_{n_\alpha}^\alpha$ such that

$$PC(B_\alpha \cup \{a_0^\alpha, \dots, a_{n_\alpha}^\alpha\})/B_\alpha$$

is not free, where $PC(X) = PC(X, B)$ is the pure closure of the subgroup of B which X generates. We choose n_α minimal with this property.

Work in $V[G]$. Let κ be a regular cardinal such that $\mathscr{H}(\kappa)$ satisfies enough axioms of set theory to handle all of our arguments, and let $<^*$ be a well ordering of $\mathscr{H}(\kappa)$. Let $N \preceq (\mathscr{H}(\kappa), \in, <^*)$ be countable such that $\bar{\psi}, S, \langle B_\alpha : \alpha < \omega_1 \rangle$ and $\langle \langle a_0^\alpha, \dots, a_{n_\alpha}^\alpha \rangle : \alpha < \omega_1 \rangle$ belong to N.

The model N has been built in $V[G]$, but since forcing with $\mathbf{P}$ adds no new reals, there is a transitive model $N_0 \in V$ isomorphic to N and let h be an isomorphism from N onto N_0. Clearly h maps $\bar{\psi}$ to $\bar{\psi}$. From now on we work in V. Hence $\mathscr{H}(\kappa)$ below is different from the one above.

We build an increasing continuous elementary chain $\langle N_\alpha : \alpha < \omega_1 \rangle$, choosing N_α by induction on α, each N_α countable as follows. Note the N_α's are neither necessarily transitive nor even well founded.

Let $\Gamma = \Gamma_\alpha = \{\varphi(v) : N_\alpha \models$ "$\{\delta \in h(S) : \varphi(\delta)\}$ is stationary" and $\varphi \in \Phi_\alpha\}$ where Φ_α is the set of first order formulas with parameters from N_α in the vocabulary $\{\in, <^*\}$ and with the only free variable v. Let $\leq_{\Gamma_\alpha}$ be the following partial order of $\Gamma_\alpha : \theta \leq_{\Gamma_\alpha} \varphi$ iff $N_\alpha \models$ "$(\forall x)[\varphi(x) \to \theta(x)]$". Let t_α be a subset of Γ_α such that:

(a) t_α is downward closed, i.e. if $\theta \leq_{\Gamma_\alpha} \varphi$ and $\varphi \in t_\alpha$ then $\theta \in t_\alpha$

(b) t_α is directed

(c) for some countable $M_\alpha \prec (\mathscr{H}(\kappa), \in, <^*)$ to which N_α belongs, if $\Gamma \in M_\alpha, \Gamma \subseteq \Gamma_\alpha$ is a dense subset of Γ_α then $t_\alpha \cap \Gamma \neq \emptyset$.

Clearly by the density if $\varphi \in \Gamma_\alpha$ and $\theta \in \Phi_\alpha$, then $\varphi \wedge \theta \in \Gamma_\alpha$ or $\varphi \wedge \neg\theta \in \Gamma_\alpha$. Thus, t_α is a complete type over N_α. Since N_α has definable Skolem functions (as $<^*$ was a well ordering), we can let $N_{\alpha+1}$ be the Skolem hull of $N_\alpha \cup \{b_\alpha\}$ where $N_\alpha \prec N_{\alpha+1}, b_\alpha \in N_{\alpha+1}$ realizes t_α.

We claim that $N_{\alpha+1}$ has no "new natural numbers", i.e. if $N_{\alpha+1} \models$ "c is a natural numbers" then $c \in N_\alpha$. Why? As $c \in N_{\alpha+1}$ clearly for some $f \in N_\alpha$ we have $N_\alpha \models$ "f is a function with domain ω_1, the countable ordinals" and $N_{\alpha+1} \models$ "$f(b_\alpha) = c$". Let

$$\begin{aligned}\mathscr{D}_f = \big\{\varphi(v) \in \Gamma_\alpha : &N_\alpha \models \text{“}(\forall x)(\varphi(x) \to f(x) \text{ is not a natural number})\text{”} \\ &\text{or for some } d \in N_\alpha \text{ we have} \\ &N_\alpha \models \text{“}(\forall x)(\varphi(x) \to f(x) = d)\text{”}\big\}.\end{aligned}$$

It is easy to check that $\mathscr{D}_f$ is a subset of Γ_α, it belongs to M_α and it is a dense subset of Γ_α; hence $t_\alpha \cap \mathscr{D}_f \neq \emptyset$. Let $\varphi(x) \in \mathscr{D}_f \cap t_\alpha$, so $N_{\alpha+1} \models \varphi[b_\alpha]$, and by the definition of $\mathscr{D}_f$ we get the desired conclusion.

If $N_\alpha \models$ "b is a countable ordinal" then $N_{\alpha+1} \models$ "$b < b_\alpha$ & b_α is a countable ordinal". Also $N_{\alpha+1} \models$ "$b_\alpha \in h(S)$".

We claim that b_α is the least ordinal of $N_{\alpha+1} \backslash N_\alpha$ in the sense of $N_{\alpha+1}$. Assume $N_{\alpha+1} \models$ "c is a countable ordinal, $c < b_\alpha$" so for some $f \in N_\alpha$ we have $N_\alpha \models$ "$f : \omega_1 \to \omega_1$ is a function" and $N_{\alpha+1} \models$ "$c = f(b_\alpha)$", $N_{\alpha+1} \models$ "$f(b_\alpha) < b_\alpha$". Then $N_\alpha \models$ "$\{\beta \in h(S) : f(\beta) < \beta\}$ is a stationary subset of ω_1". Let $\mathscr{D} = \{\varphi(v) \in \Gamma_\alpha : N_\alpha \models$ "$(\forall v)(\varphi(v) \to v$ is a countable ordinal)" and $N_\alpha \models$ "$(\exists \gamma < \omega_1)(\forall v)(\varphi(v) \to f(v) = \gamma) \vee (\forall v)(\varphi(v) \to f(v) \geq v)$"$\}$. By Fodor's lemma (which N_α satisfies) $\mathscr{D}$ is a dense subset of Γ_α and clearly $\mathscr{D} \in M_\alpha$. Since t_α is sufficiently generic, there is a $\gamma \in N_\alpha$ such that $N_{\alpha+1} \models$ "$f(b_\alpha) = \gamma$".

Now N_α is not necessarily wellfounded but it has standard ω and without loss of generality $N_\alpha \models$ "$a \subseteq \omega$" implies $a = \{n < \omega : N_\alpha \models$ "$n \in a$"$\}$ so as $h(\bar\psi) = \bar\psi$ clearly $N_\alpha \models$ "$x/E^{\bar\psi} \in B$" $\Rightarrow x/E^{\bar\psi} \in B$, and $N_\alpha \models$ "$x, y, z \in B_*, x/E^{\bar\psi} + y/E^{\bar\psi} = z/E^{\bar\psi}$" $\Rightarrow x/E^{\bar\psi} + y/E^{\bar\psi} = z/E^{\bar\psi}$. Also if $N_\alpha \models$ "$x/E^{\bar\psi}, y/E^{\bar\psi}$ are distinct members of B, i.e. $\neg x E^{\bar\psi} y$", then $x/E^{\bar\psi} \neq y/E^{\bar\psi}$.

For each $\alpha < \omega_1$, if $N_\alpha \models$ "$b < \omega_1$", let B^α_b be the group $(h(\bar B))_b$ as interpreted in N_α, i.e. N_α thinks that B^α_b is the b-th group in the increasing chain $h(\bar B)$. Clearly $B^\alpha_b \subseteq B$ if $E^{\bar\psi}$ is the equality, otherwise let $\mathbf{j}^\alpha_b$ map $(x/E^{\bar\psi})^{N_\alpha}$ to $x/E^{\bar\psi}$, so $\mathbf{j}^\alpha_b$ embeds B^α_b into B; let this image be called G^α_b. Also in N_α there is a bijection between B^α_b and ω. If $\gamma > \alpha$, since $N_\alpha \preceq N_\gamma$ have the same natural numbers, clearly $B_b^\alpha = B_b^\gamma$ when $E^{\bar\psi}$ is equality or $\mathbf{j}_b^\alpha = \mathbf{j}_b^\gamma$ and $G_b^\alpha = G_b^\gamma$ in the general case. In particular, $G_{b_\alpha}^{\alpha+1}$ is the union of $\{G_b^\alpha : N_\alpha \models \text{“}b < \omega_1\text{”}\}$.

For $\alpha < \omega_1$, let $G_\alpha = G_{b_\alpha}^{\alpha+1}$ and let $(h(\langle\langle b_\ell^\alpha : \ell \leq n_\alpha\rangle : \alpha \in S\rangle))(b_\alpha) \in N_{\alpha+1}$ be $\langle (a_\ell^{b_\alpha}/E^{\bar\psi})^{N_\alpha} : \ell \leq m_\alpha\rangle$, so $N_{\alpha+1}$ thinks that $\langle a_\ell^{b_\alpha}/E^{\bar\psi} : \ell \leq m_\alpha\rangle$ witness that $h(B)/B_{b_\alpha}^{\alpha+1}$ is not free. Clearly $a_0^{b_\alpha}/E^{\bar\psi}, \dots, a_{m_\alpha}^{b_\alpha}/E^{\bar\psi} \in G_{\alpha+1}$ and

$$PC(G_\alpha \cup \{a_0^{b_\alpha}/E^{\bar\psi}, \dots, a_{m_\alpha}^{b_\alpha}/E^{\bar\psi}\})/G_\alpha$$

is not free. So $G_{\alpha+1}/G_\alpha$ is not free. Let $G = \bigcup\limits_{\alpha<\omega_1} G_\alpha$. Then G is not free. But G is a subgroup of B, thus B is not $\aleph_2$-free. $\square_{1.2}$

Remark. Instead of the forcing we could directly build the N_α's but we have to deal with stationary subsets of $[{}^\omega 2]^{\aleph_0}$ instead of ω_1.

1.3 Corollary. If B is an $\aleph_1$-free Borel abelian group, <u>then</u> B is $\aleph_2$-free if and only if $\{K \subseteq B : |K| = \aleph_0$ and B/K is $\aleph_1$-free$\}$ is not stationary.

<u>1.4 Fact</u>: If $2^{\aleph_0} < 2^{\aleph_1}$ then every Borel Whitehead group B is $\aleph_2$-free.

Proof. By [DvSh 65] (or see [EM]) as $2^{\aleph_0} < 2^{\aleph_1}$ we have: if G be a Whitehead group of cardinality $\aleph_1$ (hence is $\aleph_1$-free) and $G = \bigcup\limits_{\alpha<\omega_1} G_\alpha$ is such that $\langle G_\alpha : \alpha < \omega_1\rangle$ is an increasing continuous chain of countable subgroups, then $\{\alpha : G_{\alpha+1}/G_\alpha$ is not free$\}$ does not contain a closed unbounded set (see [EM, Ch.XII,1.8]). Thus, if B is not $\aleph_2$-free, then the subgroup G constructed in the proof of lemma 1.2 is not Whitehead. Since being Whitehead is a hereditary property (see [EM]), B is not Whitehead.

$\square_{1.4}$

The lemma shows that

1.5 Conclusion. For Borel abelian groups $B^{\bar\psi}$, “$B^{\bar\psi}$ is $\aleph_2$-free” is absolute (in fact it is a $\sum_1^1$ property of $\bar\psi$).

Proof. The formula will just say that there is a model of a suitable fragment of ZFC (e.g. ZC) with standard ω to which $\bar\psi$ belongs and it satisfies “$B^{\bar\psi}$ is $\aleph_2$-free”. $\square_{1.5}$

§2 On $\aleph_2$-free Whitehead

2.1 Theorem. *If B is a Borel Whitehead group, then B is $\aleph_2$-free.*

2.2 Conclusion: (CH or just $2^{\aleph_0} < 2^{\aleph_1}$) Every Whitehead Borel abelian group is free.

Before we prove we quote [Sh 44, Definition 3.1].

2.3 Definition. 1) If L is a subset of the $\aleph_1$-free abelian group, $G, PC(L,G)$ is the smallest pure subgroup of G which contains L. Note that if H is a pure subgroup of $G, L \subseteq H$ then $PC(L,G) = PC(L,H)$. We omit G if it is clear.
2) If H is a subgroup of G, L a finite subset of $G, a \in G$, then the statement $\pi(a,L,H,G)$ means that: $PC(H \cup L) = PC(H) \oplus PC(L)$ but for no $b \in PC(H \cup L \cup \{a\})$ do we have $PC(H \cup L \cup \{a\}) = PC(H) \oplus PC(L \cup \{b\})$.

Proof. Assume B is not $\aleph_2$-free. We repeat the proof of Lemma 1.2. So in $V^{\mathbf{P}}, B$ is a non-free $\aleph_1$-free abelian group of cardinality $\aleph_1$. Hence by [Sh 44, p.250,3.1(3)], B satisfies possibility I or possibility II where we have chosen $\bar B = \langle B_\alpha : \alpha < \omega_1\rangle$ increasing continuous with B_α a countable pure subgroup, $B = \bigcup_{\alpha<\omega_1} B_\alpha$; the possibilities are explained below. The proof splits into the two cases.

Possibility I: By [Sh 44, p.250].

So we can find (still in $V^{\mathbf{P}}$) an ordinal $\delta < \omega_1$ and $a^\ell_i \in B$ for $i < \omega_1, \ell < n_i$ such that

- (A) $\{a^i_\ell + B_\delta : i < \omega_1, \ell \le n_i\}$ is independent in B/B_δ
- (B) $\pi(a^\ell_{n_i}, L_i, B_\delta, B)$ where $L_i = \{a^i_\ell : \ell < n_i\}$.

This situation does not survive well under the process and the proof of Lemma 1.2 but after some analysis a revised version will.

Without loss of generality $n_i = n(*) = n^*$ (by the pigeon hole principle). Let $N \prec (\mathscr{H}(\chi), \in, <^*)$ be countable such that $B_\delta, B, \langle B_\alpha : \alpha < \omega_1\rangle, \langle\langle a^i_0, \dots, a^i_{n_i}\rangle : i < \omega_1\rangle$ belong to N. We can find $M \in V, M \cong N$; without loss of generality M is transitive (so $M \models$ "n is a natural number" iff n is a natural number). We now work in V.

Let $\mathfrak{B} \prec (\mathscr{H}(\chi), \in, <^*)$ be countable, $M \in \mathfrak{B}$, note that $\mathscr{H}(\chi)^{\mathfrak{B}} \neq \mathscr{H}(\chi)$ and $\mathscr{H}(\chi)^V = \mathscr{H}(\chi) \neq \mathscr{H}(\chi)^{V^{\mathbf{P}}}$. Let Φ_M be the set of first order formulas $\varphi(v)$ in the vocabulary $\{\in, <^*\}$ and parameters from M and the only free variable v. Now we imitate the proof of [Sh 202]. Let $\Gamma = \{\varphi(v) \in \Phi_M : M \models$ "$\{\alpha < \omega_1 : \varphi(\alpha)\}$ is uncountable$\}$" (equivalently Γ is $\{a \subseteq \omega_1 : |a| = \aleph_1\}^M$). We can find $\langle t_\eta(v) : \eta \in {}^\omega 2\rangle$ such that:

- (a) each $t_\eta(v)$ a suitable generic subset of Γ, i.e. Γ is ordered by $\varphi_1(v) \le \varphi_2(v)$ if $M \models (\forall v)(\varphi_2(v) \to \varphi_1(v))$ so $t_\eta(v)$ is directed, downward closed and is not disjoint to any dense subset of Γ from $\mathfrak{B}$

(b) for $k < \omega, \eta_0, \dots, \eta_{k-1} \in {}^\omega 2$ which are pairwise distinct $\langle t_{\eta_0}(v), \dots, t_{\eta_{k-1}}(v)\rangle$ is generic too (for Γ^k), i.e. if $\mathscr{D} \in \mathfrak{B}$ is a dense subset of Γ^k then $\prod_{\ell<k} t_{\eta_\ell}(v)$ is not disjoint to $\mathscr{D}$.

(See explanation in the end of the proof of case II).
So for each $\eta, t_\eta(v)$ is a complete type over M hence we can find $M_\eta, M \prec M_\eta, M_\eta$ the Skolem hull of $M \cup \{y_\eta\}$ such that y_η realizes $t_\eta(v)$ in M_η. So $M_\eta \models$ "y_η a countable ordinal". Without loss of generality if $M_\eta \models$ "$\rho \in {}^\omega 2$" then $\rho \in {}^\omega 2$ and $\rho(n) = i \Leftrightarrow M_\eta \models$ "$\rho(n) = i$" when $n < \omega, i < 2$.
Let $h : N \to M$ be the isomorphism from N onto M (so $h \in V^{\mathbf{P}}$). We still use B_δ! As $\bar{a} = \langle\langle a^i_\ell : \ell \le n^*\rangle : i < \omega_1\rangle \in N$ we can look at $\bar{a}$ and $h(\bar{a})$ as a two-place function (with variables written as superscript and subscript). So we can let $a^\eta_\ell(\ell \le n^*, \eta \in {}^\omega 2)$ be reals such that: $M_\eta \models$ "$h(\bar{a})^{y_\eta}_\ell = a^\eta_\ell$". By absoluteness $a^\ell_\eta \in B$ (more exactly $a^\ell_\eta \in B_* = B^{\bar{\psi}}_*, a^\ell_n/E^{\bar{\psi}} \in B$) and $\pi(a^\eta_{n^*}, \langle a^\eta_\ell : \ell < n^*\rangle, B_\delta, B)$. If we can prove that $\langle a^\eta_\ell : \eta \in {}^\omega 2, \ell \le n^*\rangle$ is independent over $B_\delta (= h(B_\delta))$, then the proof of [Sh:98, 3.3] finish our case: proving B is not Whitehead group. But independence is just a demand on every finite subset. So it is enough to prove

$\otimes$ if $k < \omega, \eta_0, \dots, \eta_{k-1} \in {}^\omega 2$ are distinct, then $\{a^{\eta_m}_\ell : \ell \le n^*, m < k\}$ is independent over B_δ.

We prove this by induction on k. For $k = 0$ this is vacuous, for $k = 1$ it is part of the properties of each $\langle a^\eta_\ell : \ell \le n^*\rangle$. So let us prove it for $k + 1$. Remember that $\langle t_{\eta_0}(v), \dots, t_{\eta_k}(v)\rangle$ (more exactly $\prod_{\ell \le k} t_{\eta_\ell}(v)$) is a generic subset of Γ^k.

Assume the desired conclusion fails. So by absoluteness we can find $\varphi_\ell(v) \in t_{\eta_\ell}(v)$ and $s^m_\ell \in \mathbb{Z}$ for $m \le k, \ell \le n^*$ such that:

$\oplus$ if $t'_{\eta_m}(v) \subseteq \Gamma$ is generic over $\mathfrak{B}$ for $m \le k$, moreover $\langle t'_{\eta_m}(v) : m \le k\rangle$ is a generic subset of Γ^{k+1} over $\mathfrak{B}$ and $\varphi_m(v) \in t'_{\eta_m}(v)$, <u>then</u> (defining M'_{η_m} by $t'_{\eta_m}(v)$ and $a^{\eta_m}_\ell$ as before) $\sum_{\substack{\ell \le n^* \\ m \le k}} s^m_\ell a^{\eta_m}_\ell = t \in B_\delta$.

Clearly for $m \le k$ we have $M \models$ "$\{v : \varphi_m(v) \wedge v$ a countable ordinal$\}$ has order type ω_1" and without loss of generality also $M \models$ "$\{v : M \models$ "$\neg\varphi_m(v) \wedge v$ a countable ordinal"$\}$ has order type ω_1".

So in M there are $g_0, \dots, g_k \in M$ such that: $M \models$ "g_i is a permutation of ω_1, for $i \le k$ we have $(\forall v)(\varphi_0(v) \leftrightarrow \varphi_0(g_i(v))$ and $g_0(v), g_1(v), \dots, g_k(v)$ are pairwise distinct". Let for $m \le k, t^i_{\eta_0}(v) = \{\varphi(v) \in \Gamma : \varphi(g_i(v)) \in t_{\eta_0}(v)\}$. Let in $M_{\eta_0}, y^i_{\eta_0} = [g_i(y_{\eta_0})]^{M_{\eta_0}}, a^{\eta_0,i}_\ell = [h(\bar{a})^{(y^i_{\eta_0})}_\ell]^{M_{\eta_0}}$. Now $y^i_{\eta_0}$ realizes $t^i_{\eta_0}(v)$ and M_{η_0} is also the Skolem hull of $M \cup \{y^i_{\eta_0}\}$ and $\langle t^i_{\eta_0}(v), t_{\eta_1}(v), \dots, t_{\eta_k}(v)\rangle \subseteq \Gamma^{k+1}$ is generic over $\mathfrak{B}$ and $\varphi_0(v) \in t^i_{\eta_0}(v), \varphi_1(v) \in t_{\eta_1}(v), \dots, \varphi_k(v) \in t_{\eta_k}(v)$. Hence for each $i \le k$ in B we have $\sum_{\ell \le n^*} s^0_\ell a^{\eta_0,i}_\ell + \sum_{\substack{0<m\le k \\ \ell \le n^*}} s^m_\ell a^{\eta_m}_\ell = t \in B_\delta$.

By linear algebra $\{a^{\eta_0,i}_\ell : i \le k, \ell \le n^*\}$ is not independent (actually, $i = 0, 1$ suffices - just subtract the equations). By absoluteness this holds in M_{η_0}. But the formula saying this is false holds in $(\mathscr{H}(\chi), \in, <^*)$ hence in N, hence in M, hence in M_η (it speaks on $\bar{a}, B, B_\delta$), contradiction. So $\oplus$ fails hence $\otimes$ holds so (as said before $\otimes$) we have finished Possibility I.

Possibility II of [Sh 44, p.250]: In this case we have "not possibility I" but $S = \{\delta < \omega_1 : \delta$ a limit ordinal and there are a^δ_ℓ for $\ell \le n_\delta$ such that $\pi(a^\delta_{\eta_\delta}, \{a^\delta_\ell : \ell < n_\delta\}, B_\delta, B)\}$ is stationary; all in $V^{\mathbf{P}}$. Now without loss of generality we can find $\langle \alpha^\delta_n : n < \omega\rangle$ such that: $\alpha^\delta_n < \alpha^\delta_{n+1}, \delta = \bigcup\limits_{n<\omega} \alpha^\delta_n$, and there are $y^\delta_m \in B_{\delta+1}, t^\delta_m \in B_{\alpha^\delta_n+1}$ and $s^\delta_{m,\ell} \in \mathbb{Z}$, (for $\ell < n_\delta$) such that:

$\boxtimes(*)_0$ $y^\delta_0 = a^\delta_{n_\delta}$ and

$(*)_2$ $s^\delta_{m,n_\delta} y^\delta_{m+1} = \sum\limits_{\ell<n_\delta} s^\delta_{m,\ell} a^\delta_\ell + y^\delta_m + t^\delta_m$

$(*)_3$ $s^\delta_{m,n_\delta} > 1$, morever if s is a proper divisor of s^δ_{m,n_δ} (e.g. 1) then sy^δ_{m+1,n_δ} is not in $B_\delta + \langle\{a^\delta_i : \ell < n_\delta\} \cup \{y^\delta_m\}\rangle_B$

$(*)_4$ if $\alpha \in \delta\backslash\{\alpha^\delta_n : n < \omega\}$ then $PC_B(B_{\alpha+1} \cup \{a^\delta_0, \dots, a^\delta_{n_\delta}\}) = PC_B(B_\alpha \cup \{a^\delta_0, \dots, a^\delta_{n_\delta}\}) + B_{\alpha+1}$

[why? known, or see later.]

Without loss of generality $\delta \in S \Rightarrow n_\delta = n^*$. So as in the proof of Lemma 1.2 we can choose countable $N \prec (\mathscr{H}(\chi), \in, <^*)$ such that $\bar{a} = \left\langle\langle a^\delta_\ell : \ell \le n^*\rangle : \delta \in S\right\rangle, \bar{\alpha} = \left\langle\langle\alpha^\delta_n : n < \omega\rangle : \delta \in S\right\rangle, \left\langle(\langle s^\delta_{m,\ell} : \ell \le n^*\rangle, y^\delta_m, t^\delta_m)_{m<\omega} : \delta \in S\right\rangle$ belong to N, then define M and choose $\mathfrak{B}$ as before. We let this time $\Gamma = \Gamma_M$ be as in the proof of Lemma 1.2, that is $\{\varphi(v) : M \models$ "$\{\delta \in S : \varphi(\delta)\}$ stationary$\}$". Now we work in V. We can find $\langle t_\eta(v) : \eta \in {}^\omega 2\rangle$ such that:

(a) each $t_\eta(v) \subseteq \Gamma$ is generic over $\mathfrak{B}$ as before hence
(b) for $k < \omega$ and pairwise distinct $\eta_0, \dots, \eta_{k-1} \in {}^\omega 2, \langle t_{\eta_0}, \dots, t_{\eta_{k-1}}\rangle$ is generic for Γ^k over $\mathfrak{B}$
(c) letting M_η, y_η be such that: $M \prec M_\eta, M_\eta$ the Skolem hull of $M_\eta \cup \{y_\eta\}, y_\eta$ realizes $t_\eta(v)$ in M_η we have
    (i) $M_\eta \models$ "y_η is a countable ordinal $\in S$"
    (ii) $M \models$ "a is a countable ordinal" $\Rightarrow M_\eta \models$ "$a < y_\eta$"
    (iii) if $y \in M_\eta$ satisfies (i) + (ii) then $M_\eta \models$ "$y_\eta \le y$".

So looking at $h : N \to M$ the isomorphism, then $\alpha^\eta_n =: [h(\bar{\alpha})]^{y_\eta}_n$ for $n < \omega$ satisfies:

$$M_\eta \models \text{"}\alpha^\eta_n \text{ a countable ordinal"}$$

$$M_\eta \models \text{"}\alpha^\eta_n < \alpha^\eta_{n+1} < y_\eta\text{"}$$

$$M_\eta \models \text{"the set}\{[h(\bar{\alpha})]^{y_\eta}_n : n < \omega\} \text{ is unbounded below } y_\eta\text{"}$$

hence $\{\alpha^\eta_n : n < \omega\} \subseteq M$ is unbounded among the countable ordinals of M.
Now by easy manipulation (see proof below):

(c) if $\eta_1 \neq \eta_2 \in {}^\omega 2$ then $\{\alpha^{\eta_1}_n : n < \omega\} \cap \{\alpha^{\eta_2}_n : n < \omega\}$ is finite.

(We can be lazy here demanding just that no $\{\alpha^\eta_n : n < \omega\}$ is included in the union of a finite set with the union of finitely many sets of the form $\{\alpha^\nu_n : n < \omega\}$ where $\nu \in {}^\omega 2 \backslash \{\eta\}$, which follows from pairwise generic, and one has to do slightly more abelian group theory work below).

Now we can let $a^\eta_\ell = [(h(\bar a))^{y_\eta}_\ell]^{M_\eta}$. By linear algebra we get the independence of $\{a^\eta_\ell : \eta \in {}^\omega 2 \text{ and } \ell \leq n^*\}$ over $A = B \cap M$ i.e. $\{a/E^\psi : a \in B_* \cap M\}$ hence a contradiction to our being in possibility II (or directly get $\otimes$ in the proof in the case possibility I holds).
An alternative is the following:
We are assuming that in $V^{\mathbf{P}}$, possibility I fails. So also in V, letting $A = M \cap B^{\bar\psi}$ the following set is countable: $K[A] =: \{\langle a_\ell : \ell \leq n\rangle : n < \omega, a_\ell \in B, \langle a_\ell : \ell \leq n\rangle$ independent over A in B and $\pi(a_n, \{a_\ell : \ell < n\}, A, B)\}$ (see proof later).
For each such $\bar a = \langle a_\ell : \ell \leq n\rangle$ we can look at a relevant type it realizes over A

$$t(\bar a, A) = \Big\{(\exists y)(sy = \sum_{\ell \leq n} s_\ell x_\ell) : B \models (\exists y)(sy = \sum s_\ell a_\ell), \quad s, s_\ell \text{ integers}\Big\}$$

so $\{t(\bar a, A) : \bar a \in K[A]\}$ is countable. But for the $\eta \in {}^\omega 2$ the types $t(\langle a^\eta_\ell : \ell < n_\eta\rangle, A)$ are pairwise distinct, contradiction, so actually case II never occurs.

We still have some debts in the treatment of possibility II.
<u>Why do clauses (b) and (c) hold</u>? For each n we let

$$\Gamma_{M,n} = \Big\{\varphi(v) : \begin{array}{ll} (i) & \varphi(v) \text{ is a first order formula with parameters from } M \\ (ii) & \text{for some } \beta^*_\ell \in M \cap \omega_1 \text{ for } \ell < n \text{ we have} \\ & M \models \text{``}(\forall v)(\varphi(v) \rightarrow v \in h(S)) \ \& \ \bigwedge_{\ell < n} (h(\bar a))^v_\ell = \beta^*_\ell) \\ (iii) & M \models \text{``}(\forall \beta < \omega_1)(\exists^{\text{stat}} v < \aleph_1)[(\varphi(v) \ \& \ \beta < (h(\bar a))^v_n)]\text{''} \end{array}\Big\}.$$

Now note:

$\otimes_0$ $\Gamma_{M,n} \subseteq \Gamma_M$

$\otimes_1$ if $\varphi(v) \in \Gamma_M$ and $n < \omega$ then for some $m \in [n, \omega)$ and $\beta_\ell \in M \cap \omega_1$ for $\ell < m$ we have "$\varphi(v) \ \& \ \bigwedge_{\ell < m}$ "$(h(\bar a))^v_\ell = \beta_\ell$" belongs to $\Gamma_{M,m}$

$\otimes_2$ if $\varphi(v) \in \Gamma_{M,n}$ and $\beta \in M \cap \omega_1$ then $\varphi'(v) = \varphi(v) \ \& \ \beta < (h(\bar a))^v_n$ belongs to $\Gamma_{M,n}$.

Now let $\langle \mathscr{D}_n : n < \omega \rangle$ be the family of dense open subsets of Γ_M which belong to $\mathfrak{B}$. We choose by induction on n, $\langle \varphi_\eta(v) : \eta \in {}^n 2 \rangle, k_\eta < \omega$ such that:

(α) $\varphi_n(v) \in \Gamma_{M,k_\eta}$

(β) $\varphi_\eta(v) \in \mathscr{D}_\ell$ if $\ell < \ell g(\eta)$

(γ) $\varphi_\eta(v) \leq_\Gamma \varphi_{\eta \char`\^ \langle i \rangle}(v)$ for $i = 0, 1$

(δ) if $\eta_0 \neq \eta_1 \in {}^n 2, \eta_i \vartriangleleft \nu_i \in {}^{n+1}2$ for $i = 0, 1$ and $k_{\eta_0} \leq k < k_{\nu_0}$ and $M \models (\forall v)(\varphi_{\nu_0}(v) \rightarrow (h(\bar{a}))^v_k = \beta)$ then $M \models (\forall v)[\varphi_{\nu_1}(v) \rightarrow \bigwedge\limits_{\ell < k_{\nu_1}} (h(\bar{a}))^v_\ell \neq \beta]$.

There is no problem to do it and $t_\eta(v) = \{\varphi(v) \in \Gamma_M : \varphi(v) \leq_{\Gamma_M} \varphi_{\eta \restriction n}(v)$ for some $n < \omega\}$ for $\eta \in {}^\omega 2$ are as required.

Why does $\boxtimes$ hold?

For $\delta \in S$ let $w_\delta = \{\alpha < \delta : PC_B(B_{\alpha+1} \cup \{a^\delta_0, \ldots, a^\delta_{n_\alpha}\})$ is not equal to $PC_B(B_\alpha \cup \{a^\delta_0, \ldots, a^\delta_{n,\alpha}\}) + B_{\alpha+1} \subseteq B\}$.

Let $S' = \{\delta \in S : (\forall \alpha < \delta)(|w_\delta \cap \alpha| < \aleph_0)\}$, if S' is stationary we get $\boxtimes$, otherwise $S \backslash S'$ is stationary, and for $\delta \in S \backslash S'$ let $\alpha_\delta =$ Min$\{\alpha : w_\delta \cap \alpha$ is infinite$\}$. By Fodor's lemma for some $\alpha(*) < \omega_1, S'' = \{\delta \in S \backslash S' : \alpha_\delta = \alpha(*)\}$ is stationary hence uncountable and we can get possibility I, contradiction. $\square_{2.1}$

§3 Refinements

We may wonder if we can weaken the demand "Borel".

3.1 Definition. 1) We say $\bar{\psi}$ is a code for a Souslin abelian group if in Definition 0.1 we weaken the demand on ψ_0, ψ_1 to being a $\sum_1^1$ relation.
2) A model M of a fragment of ZFC is essentially transitive if:

- (a) if $M \models$ "x is an ordinal" and $(\{y : y <^M x\}, \in^M)$ is well ordered then x is an ordinal and $M \models$ "$y \in x$" $\Leftrightarrow y \in x$
- (b) if α is an ordinal, $(\{y : y <^M \alpha\}, \in^M)$ is well ordered and $M \models$ "α an ordinal, $\mathrm{rk}(x) = \alpha$", then $M \models$ "$y \in x$" $\Leftrightarrow y \in x$.

3) For M essentially transitive with standard ω such that $\bar{\psi} \in M$ let B^M is $B^{\bar{\psi}}$ as interpreted in M and $\mathrm{trans}(M) = \{x \in M : x \text{ as in (b) of part (2)}\}$.

3.2 Fact. 1) "$\bar{\psi}$ codes a Souslin abelian group" in a Π_2^1 property.
2) If M is a model of a suitable fragment of set theory (comprehension is enough), <u>then</u> M is isomorphic to an essentially transitive model.
3) If M is an essentially transitive model with standard ω of a suitable fragment of ZFC and $\bar{\psi} \in$ trans(M), (note $\bar{\psi}$ is really a pair of subsets of $\mathscr{H}(\aleph_0)$), <u>then</u> as $B^M = (B^{\bar{\psi}})^M \subseteq \mathrm{trans}(M)$ there is a homomorphism $\mathbf{j}_M$ from B^M into $B = B^{\bar{\psi}}$ such that $M \models$ "$t = x/E^{\bar{\psi}}$" implies $\mathbf{j}_M(t) = x/E^{\bar{\psi}}$.
4) If $M \prec N$ are as in (3), then $\mathbf{j}_M \subseteq \mathbf{j}_N$.

Proof. Straightforward.

3.3 Claim. *1) In 1.2, 2.1 we can assume that $B = B^{\bar{\psi}}$ is only Souslin.*
2) If $B = B^{\bar{\psi}}$ is not $\aleph_2$-free, <u>then</u> case I of [Sh 44](3.1) holds, moreover the conclusion of case I in the proof of 2.1 holds.

Remark. If only ψ_1 is Souslin, i.e. is $\sum_1^1$, just repeat the proofs.

Proof. For both we imitate the proof of 2.1.

In both possibilities, for each $\eta \in {}^\omega 2$, let G_η be the group which $\bar{\psi}$ defines in M_η, (the M_η's chosen as there). So $\mathbf{j}_{M_\eta}$ is a homomorphism from G_η into B. However, $\mathbf{j}_M \subseteq \mathbf{j}_{M_\eta}$ and $\mathbf{j}_M$ is one to one (noting that h, the unique isomorphism from N onto M, is the identity on $({}^\omega 2) \cap N$, hence on $B_* \cap N$, and also $B^V = B^{V^{\mathbf{P}}}$). Let $B' = \mathrm{Rang}(\mathbf{j}_M)$. Now in defining $\pi(x, L, B', B)$ we can add that we cannot find $L' \cup \{x'\} \subseteq PC(B'_\delta \cup L \cup \{x\})$ such that $\pi(x', L', B', B)$ and $|L'| < |L|$, i.e. the n is minimal. As B is $\aleph_1$-free, this implies that $\mathbf{j}_M \restriction PC(B' \cup \{a_\ell^n : \ell \leq n^*\}, B)^{M_\eta}$ is one to one and by easy algebraic argument, we can get, for 2.1, non-Whiteheadness and for 1.2, non $\aleph_2$-freeness. $\square_{3.3}$

3.4 Fact. 1) "$B^{\bar\psi}$ is non-$\aleph_2$-free" is a $\sum_1^1$-property of $\bar\psi$, assuming $B^{\bar\psi}$ is a $\aleph_1$-free Souslin abelian group.
2) "$\bar\psi$ codes a $\aleph_1$-free Souslin abelian group" is a Π_2^1-property of $\bar\psi$.

Proof. Just check.

REFERENCES.

[BKM78] Jon Barwise, Matt Kaufmann, and Michael Makkai. Stationary logic. *Annals of Mathematical Logic*, **13**:171–224, 1978.

[DvSh 65] Keith J. Devlin and Saharon Shelah. A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$. *Israel Journal of Mathematics*, **29**:239–247, 1978.

[EM] Paul C. Eklof and Alan Mekler. *Almost free modules: Set theoretic methods*, volume 46 of *North–Holland Mathematical Library*. North–Holland Publishing Co., Amsterdam, 1990.

[Sch85] J. Schmerl. Transfer theorems and their application to logics. In J.Barwise and S.Feferman, editors, *Model Theoretic Logics*, pages 177–209. Springer-Verlag, 1985.

[Sh 44] Saharon Shelah. Infinite abelian groups, Whitehead problem and some constructions. *Israel Journal of Mathematics*, **18**:243–256, 1974.

[Sh 43] Saharon Shelah. Generalized quantifiers and compact logic. *Transactions of the American Mathematical Society*, **204**:342–364, 1975.

[Sh:98] Saharon Shelah. Whitehead groups may not be free, even assuming CH. II. *Israel Journal of Mathematics*, **35**:257–285, 1980.

[Sh 202] Saharon Shelah. On co-κ-Souslin relations. *Israel Journal of Mathematics*, **47**:139–153, 1984.

[Sh 161] Saharon Shelah. Incompactness in regular cardinals. *Notre Dame Journal of Formal Logic*, **26**:195–228, 1985.

[Sh 523] Saharon Shelah. Existence of Almost Free Abelian groups and reflection of stationary set. *Mathematica Japonica*, **45**:1–14, 1997.

[Sh:f] Saharon Shelah. *Proper and improper forcing.* Perspectives in Mathematical Logic. Springer, 1998.