A Δ^2_2 Well-Order of the Reals And Incompactness of $L(Q^{MM})$

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Abstract

A forcing poset of size $2^{2^{\aleph_1}}$ which adds no new reals is described and shown to provide a Δ^2_2 definable well-order of the reals (in fact, any given relation of the reals may be so encoded in some generic extension). The encoding of this well-order is obtained by playing with products of Aronszajn trees: Some products are special while other are Suslin trees.

The paper also deals with the Magidor-Malitz logic: it is consistent that this logic is highly non compact.

Preface

This paper deals with three issues: the question of definable well-orders of the reals, the compactness of the Magidor-Malitz logic and the forcing techniques for specializing Aronszajn trees without addition of new reals.

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In the hope of attracting a wider readership, we have tried to make this paper as self contained as possible; in some cases we have reproved known results, or given informal descriptions to remind the reader of what he or she probably knows. We could not do so for the theorem that the iteration of $\mathbb{D}$ -complete proper forcing adds no reals, and the reader may wish to consult chapter V of Shelah [1982], or the new edition [1992]. Anyhow, we rely on this theorem only at one point.

The question of the existence of a definable well-order of the set of reals, $\mathbb{R}$, with all of its variants, is central in set theory. As a starting point for the particular question which is studied here, we take the theorem of Shelah and Woodin [1990] by which from the existence of a large cardinal (a supercompact and even much less) it follows that there is no well order of $\mathbb{R}$ in $L(\mathbb{R})$.

Assuming that there exists a cardinal which is simultaneously measurable and Woodin, Woodin [in preparation] has shown that: *If CH holds, then every* Σ^2_1 *set of reals is determined. Hence there is no* Σ^2_1 *well order of the reals.*

A Σ^2_i formula is a formula over the structure $\langle \mathbf{N}, \mathcal{P}(\mathbf{N}), \mathcal{P}(\mathcal{P}(\mathbf{N})), \in \ldots \rangle$ of type $\exists X_1 \subseteq \mathcal{P}(\mathbf{N}) \forall X_2 \subseteq \mathcal{P}(\mathbf{N}) \ldots \varphi(X_1 \ldots)$, where there are i alternations of quantifiers, and in φ all quantifications are over $\mathbf{N}$ and $\mathcal{P}(\mathbf{N})$. Equivalently, $\mathcal{P}(\mathbf{N})$ can be replaced by $\langle H = H(\omega_1), \in \rangle$ the collection of all hereditarily countable sets; this seems to be useful in applications. so a Σ^2_2 formula has the form $\exists X_1 \subseteq H \forall X_2 \subseteq H \varphi(X_1 \ldots)$, where φ is first order over H and the X_i's are predicates (subsets of H).

A natural question asked by Woodin is whether his theorem cited above could not be generalized to exclude Σ^2_2 well-order of $\mathbb{R}$: Perhaps CH and some large cardinal may imply that there is no Σ^2_2 well-order of $\mathbb{R}$. We give a negative answer by providing a forcing poset (of small size, $2^{2^{\aleph_0}}$) which adds no reals and gives generic extensions in which there exists a Σ^2_2 well-order of $\mathbb{R}$. Since supposedly any large cardinal retains its largeness after a "small" forcing extension, no large cardinal contradicts a Σ^2_2 well order of $\mathbb{R}$.

Specifically, we are going to prove the following Main Theorem.

Theorem A *Assume $\diamond_{\omega_1}$. Let $P(x)$ be a predicate (symbol). There is a (finite) sentence ψ in the language containing $P(x)$ with the Magidor-Malitz quantifiers, such that the following holds. Given any $\mathsf{P} \subseteq \omega_1$, (1) there is a model M of ψ, enriching $(\omega_1, <, \mathsf{P})$ such that $P^M = \mathsf{P}$, and (2) assuming $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ there is a forcing poset Q of size $\aleph_2$ satisfying the $\aleph_2$-c.c. and adding no reals such that in V^Q M is the single model of ψ (up to isomorphism).*

Recall that the Magidor-Malitz logic $L(Q^{MM})$ is obtained by adjoining to the regular first-order logic the quantifiers $Qxy\varphi(x,y)$ which is true in a structure of size $\aleph_1$ iff: there exists an uncountable subset of that structure's universe such that for any two distinct x and y in the set, $\varphi(x,y)$ holds. (See Magidor and Malitz [1977].)

Observe that if only CH is assumed in the ground model, but not $\diamond_{\omega_1}$, the theorem would still be aplicable since $\diamond_{\omega_1}$ can be obtained in such a case by a forcing which adds no reals and is of size $\aleph_1$. (See Jech [1978], Exercise 22.12.)

Let us see why Theorem A implies a Σ^2 well-order of $\mathbb{R}$ in the generic extension. Since CH is assumed, it is possible to find $\mathsf{P} \subseteq \omega_1$ which encodes in a natural way a well-order of $\mathbb{R}$ of type ω_1. For example, set $\mathsf{P} \subseteq \omega_1$ in such a way that $\mathsf{P} \cap [\omega\alpha, \omega\alpha + \omega)$, the intersection of P with the αth ω-block of ω_1, "is" a subset, r_α, of ω and so that $\langle r_\alpha \colon \alpha < \omega_1 \rangle$, is an enumeration of $\mathbb{R}$. Now use Theorem A to find a formula ψ and a model M of ψ (with $P^M = \mathsf{P}$) and a generic extension in which M is the unique model of ψ. In this generic extension, the relation $r_\alpha < r_\beta$, $\alpha < \beta$, can be defined by:

There is a model K of ψ where r_α appears in P^K before r_β does.

Now, for any formula φ in the Magidor-Malitz logic, the statement: "there is a model K of φ" is (equivalent to) a Σ^2_2 statement (see below), and hence the well-order of $\mathbb{R}$ is Δ^2_2 in the generic extension. (Since any Σ^2_2 linear order must be Δ^2_2.)

We can start with any relation $\mathsf{P} \subset \omega_1$ (not necessarily a well-order of the reals) and get by Theorem A a generic extension V^Q in which this relation is Δ^2_2.

To see the above remark, for any Magidor-Malitz formula φ, we encode the existence of $K \models \varphi$ as a Σ^2_2 statement concerning $H(\omega_1)$ thus:
There is a relation ε on $H(\omega_1)$ and a truth function which defines a model (H, ε) of enough set theory, in which ω_1^H is (isomorphic to) the real ω_1, and inside H there is a model K for the formula φ, such that: For any subformula $\delta(u, v)$ of φ with parameters in K, if $X \subseteq \omega_1$ is such that for any two distinct $a, b \in X$ $\delta(a, b)$ holds, then there is such an X in H as well.

Now this statement "is" Σ^2_2, and the model K of φ found in H is a real Magidor-Malitz model of φ, not only in the eyes of H.

The second issue of the paper, the "strong" incompactness of the Magidor-Malitz logic, is an obvious consequence of the fact that ψ has no non standard models.

The proof of the Main Theorem involves a construction of an ω_1 sequence of Suslin trees at the first stage (constructing the model M of ψ), and then an iteration of posets which specialize given Aronszajn trees at the second stage (making M the unique model of ψ). The main ingredient in the iteration is the definition of a new poset $\mathcal{S}(\mathsf{T})$ for specializing an Aronszajn tree T without addition of new reals.

For his well-known model of CH & *there are no Suslin trees* (SH), Jensen provides (in L) a poset which iteratively specializes each of the Aronszajn trees. Each step in this iteration (including the limit stages) is in fact a Suslin tree. Both the square and the diamond are judiciously used to construct this ω_2-sequence of Suslin trees. Since forcing with a Suslin tree adds no new reals, the generic extension satisfies CH. (See Devlin and Johnsbråten [1974].)

In Shelah [1982] (Chapter 5) this result is obtained in the general and more flexible setting of proper-forcing iterations which add no reals. In particular, a proper forcing which adds no reals and specializes a given Aronszajn tree is defined there.

The poset $\mathcal{S}(\mathsf{T})$ of our paper is simpler than the one in Shelah [1982] because it involves no closed unbounded subsets of ω_1, and so our paper could be profitably read by anyone who wants a (somewhat) simpler proof of Jensen's CH & SH.

The paper is organized as follows:
Section 1 gives preliminaries and sets our notation. Section 2 shows how to construct sequences of Suslin trees such that, at will, some products of the trees are Suslin while the others are special. Section 3 is a preservation theorem for countable support iteration of proper forcing: A Suslin tree cannot suddenly lost its Suslinity at limit stages of the iteration. Section 4 describes the poset which is used to specialize Aronszajn trees. Section 5 shows that the specializing posets of Section 4 can be iterated without adding reals. In Section 6 we start with a given family of Suslin trees and show how the iteration of the specializing posets obtains a model of ZFC in which this given family is the family of *all* Suslin trees; all other Suslin trees are killed. Sections 7 and 8 are the heart of the paper and the reader may want to look there first to get some motivation. In Section 8 a version of Theorem A is proved first which suffices to answer Woodin's question, and then the remaining details (by now easy) are given to complete the proof.

Concerning the related question for models where CH does not hold, let us report that:

1. Woodin obtained the following: Assume there is an inaccessible cardinal κ. Then there is a c.c.c forcing extension in which $\kappa = 2^{\aleph_0}$ (is weakly inaccessible) and there is a Δ^2_2-well ordering of the reals.

2. Extending the methods of this paper, Solovay obtained a forcing poset of size $2^{2^{\aleph_0}}$ such that the following holds in the extension:

(a) $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$,

(b) MA for σ-centered posets,

(c) There is a Δ^2_1-well ordering of the reals.

3. Motivated by this result of Solovay, Shelah obtained the following: If κ is an inaccessible cardinal and GCH holds on a cofinal segment of cardinals below κ, then there is an extension such that

   (a) $2^{\aleph_0} = \kappa$, cardinals and cofinalities are not changed,

   (b) MA,

   (c) There is a Δ^2_1-well ordering of the reals.

Theorem (A) was obtained by Shelah during his visit to Caltech in 1985 and he would like to thank H. Woodin for asking this question and R. Solovay for encouraging conversations. We also thank Solovay for some helpful suggestions which were incorporated here. The result of Section 6 (a model of ZFC with few Suslin trees) is due to Abraham and appeared in fact in Section 4 of Abraham and Shelah [1985]. (However, there the machinery of Jensen iteration of Suslin trees was used, while here the approach of proper forcing is used.) The poset $\mathcal{S}(\mathsf{T})$ for specializing an Aronszajn tree T was found by Abraham who proved that any Suslin tree S remains Suslin after the forcing, unless S is embeddable into T. As said above, $\mathcal{S}(\mathsf{T})$ is simpler than the corresponding poset P of Shelah [1982], but the closed unbounded set forcing involved in P is still necessary in order to make two Aronszajn trees isomorphic on a club.

1 Preliminaries

In this section we set our notations and remind the reader of some facts concerning trees and forcings. All of these appear with more details in the book of Jech [1978], or Todorčevič [1984] or in the monograph Devlin and Johansbråten [1974] which describes Jensen's results.

In saying that "T is a tree" we intend that the height of T is ω_1, each level T_α is countable ($\alpha < \omega_1$), and every node has $\aleph_0$ many (immediate) successors. We do not insist that the tree has a unique root.

For a node $t \in \mathsf{T}$ define its predecessor branch by

$$(\cdot, t) = \{s \in \mathsf{T} \mid s <_{\mathsf{T}} t\}.$$

Usually it is required for limit δ that $(\cdot, a) \neq (\cdot, b)$ for $a \neq b$ in T, but we allow branches with more than one least upper bound.

For a node $a \in \mathsf{T}$, *level*(a) is that ordinal α such that $a \in \mathsf{T}_\alpha$ (that is, the order-type of $(\cdot, a)$). We also say that a is of height α in this case. $\mathsf{T}\lceil\alpha$ is the tree consisting of all nodes of height $< \alpha$.

For a node $a \in \mathsf{T}$, $\mathsf{T}_a = \{x \in \mathsf{T} \mid a \leq_{\mathsf{T}} x\}$, is the tree consisting of all extensions of a in T.

A *branch* in a tree is a linearly ordered (usually downward closed) subset. An *antichain* is a pairwise incomparable subset of the tree. An Aronszajn tree is a tree with no uncountable branches. It is special if there is an order preserving map $f : \mathsf{T} \to \mathbb{Q}$, ($x <_{\mathsf{T}} y$ implies$f(x) < f(y)$). A *Suslin* tree is one with no uncountable antichain (and hence no uncountable chain as well). A Suslin tree has this property that any cofinal branch (in an extension of the universe) is in fact a generic branch. The reason being that for any dense open subset $D \subset \mathsf{T}$, for some α, $\mathsf{T}_\alpha \subset D$ (see Lemma 22.2 in Jech [1978]).

If $G \subset \mathsf{T}$ is a branch of length γ, then for $\alpha < \gamma$, G_α denotes $\mathsf{T}_\alpha \cap G$, and $G\lceil\alpha = G \cap (\mathsf{T}\lceil\alpha)$.

Product of trees: The product $\mathsf{T}^1 \times \mathsf{T}^2$ of two trees consists of all pairs $\langle a_1, a_2\rangle$, where for some α, $a_i \in \mathsf{T}^i_\alpha$. The pairs are ordered coordinatewise: $\langle a_1, a_2\rangle < \langle a_1', a_2'\rangle$ iff $a_i <_{\mathsf{T}^i} a_i'$ for both i's. The product of a finite number of trees is similarly defined.

When $\langle \mathsf{T}^{\xi} \mid \xi < \alpha \rangle$ is a sequence of trees, and $e = \langle \xi_1 \dots \xi_n \rangle$ is a sequence (or set) of indices, then the product of these n trees is denoted

$$\mathsf{T}^{(e)} = \times_{\xi \in e} \mathsf{T}^{\xi} = \mathsf{T}^{\xi_1} \times \dots \times \mathsf{T}^{\xi_n}.$$

This notation should not be confused with the one for their union:

$$\mathsf{T}^{e} = \bigcup \{ \mathsf{T}^{\xi} \mid \xi \in e \} = \mathsf{T}^{\xi_1} \cup \dots \cup \mathsf{T}^{\xi_n}.$$

The union of the trees is defined under the assumption that their domains are pairwise disjoint. (It is to simplify this definition that we drop the requirement for a unique root.)

A *derived* tree of T is formed by taking, for some $\alpha < \omega_1$, n distinct nodes, $a_1, \dots, a_n$, and forming the product $\mathsf{T}_{a_1} \times \mathsf{T}_{a_2} \times \cdots \mathsf{T}_{a_n}$.

The product of a Suslin tree with itself is never a Suslin tree. And the product of a special tree with any tree is again special.

In Devlin and Johansbråten [1974] Jensen constructs (using the diamond $\diamond$) a Suslin tree such that all of its derived trees are Suslin too. We will describe this construction in Section 3.1.

Let T be an Aronszajn tree (of *height* ω_1). A function $f : \mathsf{T} \to \mathbb{Q}$ is a *specialization* (of T) if $x <_{\mathsf{T}} y \Rightarrow f(x) <_{\mathsf{T}} f(y)$. When f is a partial function, it is called a *partial specializing* function on T.

${}^n\mathsf{T}_{\beta}$ denotes the set of all n-tuples $\bar{x} = \langle x_0, \dots, x_{n-1} \rangle$ where $x_i \in \mathsf{T}_{\beta}$ for all $i < n$. We also write $\bar{x} \in \mathsf{T}_{\beta}$ instead of $\bar{x} \in {}^n\mathsf{T}_{\beta}$. ${}^n\mathsf{T} = \cup \{ {}^n\mathsf{T}_{\beta} \mid \beta < \mathit{height}\mathsf{T} \}$. For $Y \subseteq^n \mathsf{T}$, $Y_{\gamma} = Y \cap^n \mathsf{T}_{\gamma}$. If $x \in \mathsf{T}_{\beta}$ and $\alpha \leq \beta$, then $x \lceil \alpha$ denotes the unique $y \leq_{\mathsf{T}} x$ with $y \in \mathsf{T}_{\alpha}$. Similarly, for $\bar{x} \in \mathsf{T}_{\beta}$, $\bar{x} \lceil \alpha \stackrel{def}{=} \langle x_0 \lceil \alpha, \dots, x_{n-1} \lceil \alpha \rangle \in \mathsf{T}_{\alpha}$. If $\alpha < \beta$, and $X \subseteq {}^n\mathsf{T}_{\beta}$, then $X \lceil \alpha \stackrel{def}{=} \{ \bar{x} \lceil \alpha \mid \bar{x} \in X \}$.

Also, if $h : {}^n\mathsf{T}_{\beta} \to \mathbb{Q}$ is a finite function, then for $\alpha < \beta$, if the projection taking $x \in^n T_{\beta}$ to $x \lceil \alpha$ is one to one, then $h \lceil \alpha : \mathsf{T}_{\alpha} \to \mathbb{Q}$ is the function h' defined by $h'(x \lceil \alpha) = h(x)$. And for a set, H, of finite functions, $H \lceil \alpha = \{ h \lceil \alpha \mid h \in H \}$.

We use similar notation for a branch, B, of T denoting by $B\lceil\alpha$ the subset of B consisting of those nodes of B of *height* $< \alpha$.

Sometimes, we think of $\bar{x} \in {}^n\mathsf{T}$ as a set rather than a sequence. For example, when we say that $\bar{x}_1$ and $\bar{x}_2$ are disjoint: in this case we refer to the range of the sequences, of course. More often, $\bar{x}$ refers to the sequence $\langle x_0, \dots, x_{n-1}\rangle$. For example, $\bar{x}_1 \leq_{\mathsf{T}} \bar{x}_2$ means that $\mathit{length}(\bar{x}_1) = \mathit{length}(\bar{x}_2) = n$, and for $i < n$, $x_{1i} \leq_{\mathsf{T}} x_{2i}$. We do not demand that $x_i \neq x_j$.

A set of n-tuples, $X \subseteq {}^n\mathsf{T}_\beta$, is said to be *dispersed* if for every finite $t \subseteq \mathsf{T}_\beta$ there is an n-tuple in X disjoint to t. The following Lemma is from Devlin and Johnsbråten [1974] (Lemma 7 in Chapter VI):

Lemma 1.1 *If T is an Aronszajn tree and $X \subseteq {}^n\mathsf{T}$ is uncountable and downward closed ($\bar{y} \leq_{\mathsf{T}} \bar{x} \in X \Rightarrow \bar{y} \in X$), then , for some $\beta < \omega_1$, there is an uncountable $Y \subseteq X$ such that:*

1. *For $\beta \leq \gamma_0 < \gamma_1 < \omega_1$, $Y_{\gamma_0} = Y_{\gamma_1}\lceil\gamma_0$.*
2. *$Y_\gamma = Y \cap {}^n\mathsf{T}_\gamma$ is dispersed for every $\beta \leq \gamma < \omega_1$ (Equivalently, Y_β is dispersed).*

2 Construction of Suslin trees

The diamond sequence, $\diamond$, on ω_1, enables the construction of Suslin trees with some degree of freedom concerning their products. For example, the construction of two Suslin trees A and B such that the product $\mathsf{A}\times\mathsf{B}$ is special; or the construction of three Suslin trees A, B and C such that $\mathsf{A} \times \mathsf{B} \times \mathsf{C}$ is special, but $\mathsf{A} \times \mathsf{B}$, $\mathsf{A} \times \mathsf{C}$ and $\mathsf{B} \times \mathsf{C}$ are Suslin trees. This freedom is demonstrated in this section by showing that, given any reasonable prescribed requirement on which products are Suslin and which are special, the diamond constructs a sequence $\langle \mathsf{S}(\zeta) \,|\, \zeta < \omega_1\rangle$ of trees satisfying this requirement. 'Reasonable' here means that no subproduct of a Suslin product is required to be special.

None of the *ideas* in this section is new, and we could have shortened our construction by referring to Devlin and Johnsbråten [1974], and leaving the details to the reader. We decided however to give a somewhat fuller presentation in the hope that some readers will find it useful. The constructions are presented gradually, so that for the more complex constructions we can concentrate on the main ideas and claim that some of the technical details are as before. In the following subsection we use the diamond to construct a Suslin tree such that all of its derived trees are Suslin as well. (Recall that a derived tree of T has the form $\mathsf{T}_{a_1} \times \ldots \times \mathsf{T}_{a_n}$ where $a_1, \ldots, a_n \in \mathsf{T}_\alpha$ are distinct members of the αth level of T for some $\alpha < \omega_1$.) Then we show the construction of two Suslin trees A and B such that $\mathsf{A} \times \mathsf{B}$ is special; and the last subsection gives the desired general construction.

For the rest of this section we assume a 'diamond' sequence $\langle S_\xi \,|\, \xi < \omega_i \rangle$ where $S_\xi \subseteq \xi$ and for every $X \subseteq \omega_1$, $\{\, \xi \mid X \cap \xi = S_\xi \}$ is stationary in ω_1.

2.1 A Suslin tree with all derived trees Suslin

Let us recall the construction of a Suslin tree A_α. The αth level of the Suslin tree A, is defined by induction on α. In order to be able to apply the diamond to A we wish to see A's universe as ω_1 and assume that the subtree $\mathsf{A}\lceil\alpha$ consists of the set of ordinals $\omega\alpha$. For $\alpha < \beta$ we shall require that the tree $\mathsf{A}\lceil\beta$ is an end-extension of $\mathsf{A}\lceil\alpha$ (the reader is asked to forgive us for using the notation $\mathsf{A}\lceil\beta$ even though the tree A itself has not yet been constructed).

At successor stages, the passage from $\mathsf{A}\lceil(\mu+1)$ to $\mathsf{A}\lceil(\mu+2)$, that is, the construction of $\mathsf{A}_{\mu+1}$, requires no special care: only that each node in A_μ has countably many extensions in $\mathsf{A}_{\mu+1}$.

At limit stage, $\delta < \omega_1$, first set $\mathsf{A}\lceil\delta = \bigcup_{\alpha<\delta} \mathsf{A}\lceil\alpha$, and then the δth level A_δ is obtained by defining (as follows) a countable set of branches, $\{\, b_i \mid i \in \omega\}$, and putting one point in A_δ above each b_i. The branch b_i is cofinal in $\mathsf{A}\lceil\delta$, and each node in $\mathsf{A}\lceil\delta$ is contained in at least one b_i.

If we only wish to construct a Suslin tree, then the diamond set $S_\delta \subseteq \mathsf{A}\lceil\delta$ is used as usual: Each node a in $\mathsf{A}\lceil\delta$ is first extended to some $x_0 \geq a$ in S_δ (if possible), and then, in ω steps, an increasing sequence, $x_0 < x_1 < \ldots$, is defined so that $level(x_i)$, $i < \omega_1$, is cofinal in δ. This sequence defines one of our countably many branches. We see now the need for the following statement to hold at every stage $\delta < \omega_1$.

For any $\zeta_0 < \zeta_1 < \delta$ *and* $a \in \mathsf{A}_{\zeta_0}$, *there is some* $b \in \mathsf{A}_{\zeta_1}$ *extending* a.

Now let us require a little more of A and ask that each of its derived trees is Suslin too (Devlin and Johansbråten [1974]). This variation is manifest in the construction of A_δ for limit δ, and is perhaps better described by means of a generic filter over a countable structure as follows.

Let $\mathcal{P} = \mathcal{P}(\mathsf{A}\lceil\delta)$ be the poset defined thus:

$$\mathcal{P} = \{\ \bar{a} | \textit{for some } n,\ \bar{a} = \langle a_0, \ldots, a_{n-1}\rangle \textit{ and, for some } \alpha < \delta, \textit{for all } 0 \leq i < n,\ a_i \in \mathsf{A}_\alpha\}$$

The *level* of $\bar{a} \in \mathcal{P}$ is the α such that $a_i \in \mathsf{A}_\alpha$, for all $i < length(\bar{a})$. A partial-order, $\bar{b}$ *extends* $\bar{a}$, is defined:
$\bar{b}$ *extends* $\bar{a}$ iff
level $(\bar{a}) \leq level(\bar{b})$, $length(\bar{a}) \leq length(\bar{b})$, and $\forall i < length(\bar{a})$, $a_i \leq b_i$ (in $\mathsf{A}\lceil\delta$).
It is not required for $\bar{a} \in \mathcal{P}$ to be one-to-one: $a_i = a_j$ is possible, although by genericity, they will split at some stage.

Now let M be a countable model (of a sufficient portion of set-theory) which includes $\mathcal{P}$ and $\mathsf{A}\lceil\delta$ and the diamond set S_δ; and let G be a $\mathcal{P}$-generic filter over M. Using suitably defined dense sets, it is easy to see that for each fixed $i < \omega$, $b_i = \{\, x \mid \text{for some } \bar{a} \in G,\ x = a_i\}$ is a branch in $\mathsf{A}\lceil\delta$ going all the way up to δ. The collection $\{\ b_i \mid i < \omega\}$ determines A_δ and this ends the definition of A.

Let $\mathsf{T} = \mathsf{A}_{t_0} \times \ldots \times \mathsf{A}_{t_{n-1}}$ be any derived tree of A, we will prove that T is Suslin. Let α be the level of $\langle t_0, \ldots, t_{n-1}\rangle$ in A (so $t_i \in \mathsf{A}_\alpha$ for all $i < n$). Let $E \subseteq \mathsf{T}$ be any dense open subset. By the diamond property, using some natural encoding of n-tuples of ordinals as ordinals, for some limit $\delta > \alpha$, $E \cap (\mathsf{T}\lceil\delta) = E \cap \delta = S_\delta$, and $E \cap \delta$ is dense open in $\mathsf{T}\lceil\delta$. We must prove that every $\bar{x} \in \mathsf{T}_\delta$ is in E, in order to be able to prove that an arbitrary antichain in T is countable. $\bar{x} \in \mathsf{T}_\delta$ has the form $\bar{x} = \langle x_0, \ldots, x_{n-1}\rangle$ where $x_k \in \mathsf{A}_{t_k}$. Since the t_k's are distinct, the x_k's give distinct branches of $\mathsf{A}\lceil\delta$. Recall the $\mathcal{P}(\mathsf{A}\lceil\delta)$ generic filter G (over M) used to define A_δ, and let $b_{i(k)}$ be the branch of $\mathsf{A}\lceil\delta$ which gave x_k. If for some $\bar{a} = \langle a_0, \ldots, a_{l-1}\rangle \in G$ the n-tuple $\langle a_{i(0)}, \ldots, a_{i(n-1)}\rangle$ is in the dense set E, then $\bar{x}$ which is above this n-tuple must be in E too.

The existence of such $\bar{a}$ in G is a consequence of the following density argument: Let D contains all those $\bar{a} \in \mathcal{P}(\mathsf{A}\lceil\delta)$ for which (1) $i(k) < \mathit{length}(\bar{a})$ for every $k < n$, and either the subsequence $s = \langle a_{i(0)}, \ldots, a_{i(k)}\rangle$ of $\bar{a}$ is not in T, or else $s \in E$. D is dense in $\mathcal{P}$ and $D \in M$, because S_δ is in M. So that $D \cap G \neq 0$, and any $\bar{a} \in D \cap G$ of height $> \alpha$ is as required.

2.2 The case of two trees

The construction in the previous section is combined now with the construction of a special Aronszajn tree to yield two Suslin trees A and B such that:

1. Each derived tree of A and of B is Suslin,

2. $\mathsf{A} \times \mathsf{B}$ is a special tree.

We commence by recalling the construction of a special Aronszajn tree A together with a strictly increasing $f : \mathsf{A} \to \mathbb{Q}$. In the inductive definition, $\mathsf{A}\lceil a$ and $f\lceil\alpha = f\lceil(\mathsf{A}\lceil\alpha)$, $\alpha < \omega_1$, are defined so that the following hold:

(1) For any $a \in \mathsf{A}\lceil\alpha$ *and rational* $\varepsilon > 0$, *and ordinal* $\tau < \alpha$ *such that* $\mathit{height}(a) < \tau$,

there is an extension $b > a$ *in* A_τ *such that* $0 < f(b) - f(a) < \varepsilon$.

This condition is needed at a limit stage α, if we don't want our cofinal branches to run out of rational numbers; it enables the assignment of $f(e) \in \mathbb{Q}$ for $e \in \mathsf{A}_\alpha$, but it requires some care to keep it true at all stages.

There is nothing very special at successor stages: Since we assume that each node has $\aleph_0$ many successors, condition (1) above may be achieved by assigning to these successors of e all the possible values of rational numbers $> f(e)$ (a forcing-like description of the successor stage is also possible—see below).

For a limit $\alpha < \omega_1$, it seems again convenient to formulate the construction of A_α and $f\lceil\mathsf{A}_\alpha$, in terms of a generic filter G over a countable structure M. So given $A\lceil\alpha$ and $f\lceil\alpha$, a countable poset $\mathcal{Q} = \mathcal{Q}(\mathsf{A}\lceil\alpha, f\lceil\alpha)$ is defined first.

Definition 2.1 Let $\mathcal{Q}$ be the collection of all pairs $(\bar{a}, \bar{q})$ of the form $\bar{a} = \langle a_0, \ldots, a_{n-1}\rangle$ $\bar{q} = \langle q(0), \ldots, q(n-1)\rangle$ such that:

1. $\bar{a}$ is an n-tuple in A_ξ for some $\xi < \alpha$.

2. $q(i) \in \mathbb{Q}$ and $f(a_i) < q(i)$ for all $i < n$.

Intuitively, $q(i)$ is going to be the value of $f(b)$ for $b \in \mathsf{A}_\alpha$ defined by the 'generic' branch $\{\ a_i \mid (\bar{a}, \bar{q}) \in G\}$. The order relation on $\mathcal{Q}$ is accordingly defined: $(\bar{a}_2, \bar{q}_2)$*extends*$(\bar{a}_1, \bar{q}_1)$ iff $\bar{a}_2$ *extends* $\bar{a}_1$ and $\bar{q}_1$ is an initial sequence of $\bar{q}_2$ (that is, $\mathit{length}(\bar{q}_1) \leq \mathit{length}(\bar{q}_2)$, and for $i < \mathit{length}(\bar{q}_1)$, $\bar{q}_1(i) = \bar{q}_2(i)$).

Now we turn to the construction of two Suslin trees A and B such that all the derived trees of A and B are Suslin and yet $\mathsf{A} \times \mathsf{B}$ is special. In this case both $\mathsf{A}\lceil\alpha$, $\mathsf{B}\lceil\alpha$ and the specializing function $f : (\mathsf{A}\lceil\alpha) \times (\mathsf{B}\lceil\alpha) \to \mathbb{Q}$ are simultaneously constructed by induction. There are three jobs to do at the limit αth stage: (i) to ensure that the cofinal branches of $\mathsf{A}\lceil\alpha$ and of its derived trees all pass through the diamond set S_α. (ii) To ensure the similar requirement for $\mathsf{B}\lceil\alpha$. (iii) To specialize $(\mathsf{A}\lceil\alpha + 1) \times (\mathsf{B}\lceil\alpha + 1)$. It turns out that it suffices to take care of (iii) in a natural way - and genericity will take care of the two other requirements, thereby ensuring that A and B and their derived trees are Suslin.

The inductive requirement (*1*) is needed here too, but in fact an even stronger requirement will be used:

(2) If $\bar{a}$, $\bar{b}$ are n and m tuples in $\mathsf{A}\lceil\alpha$ and $\mathsf{B}\lceil\alpha$, and for $\theta < \alpha$ $\bar{c}$ is an n-tuple in A_θ extending $\bar{a}$, and if $q: n \times m \to \mathbb{Q}$ is such that $\forall i, j\ f(a_i, b_j) < q(i, j)$, THEN there is an n-tuple $\bar{d}$ in B_θ, extending $\bar{b}$ such that $\forall i, j\ f(c_i, d_j) < q(i, j)$. (A similar requirement is made for $\bar{c}$ in B_θ.)

In fact, by taking smaller $q(i, j)$, it even follows that for any finite $D \subset B_\theta$ there is an n-tuple $\bar{d}$ in B_θ, extending $\bar{b}$ and disjoint to D, such that $\forall i, j\ f(c_i, d_j) = q(i, j)$. (A similar requirement is made for $\bar{c}$ in B_θ.) Again, we only describe the limit case, and leave the details of the successor case to the reader (take care of condition (*2*)). So assume $\alpha < \omega_1$ is a limit ordinal and $\mathsf{A}\lceil\alpha$ $(= \cup_{\mu<\alpha}\mathsf{A}\lceil\mu)$, $\mathsf{B}\lceil\alpha$ and $f\lceil\alpha$ are given.

Let $\mathbb{R} = \mathbb{R}((\mathsf{A}\lceil\alpha) \times (\mathsf{B}\lceil\alpha), f\lceil\alpha)$ be the poset defined in the following: $(\bar{a}, \bar{b}, \bar{q}) \in \mathbb{R}$ iff for some $\mu < \alpha$, $\bar{a}$ is an n-tuple in A_μ, $\bar{b}$ is an m-tuple in B_μ, and $\bar{q} : n \times m \to \mathbb{Q}$, are such that for all $0 \le i < n$, $0 \le j < m$, $f(a_i, b_j) < \bar{q}(i, j)$. Extension is defined naturally.

If M is now a chosen countable structure containing all the above, and the diamond S_α in particular, then pick an $\mathbb{R}$-generic filter, G, over M and define the αth levels A_α and B_α and extend f on $\mathsf{A}_\alpha \times \mathsf{B}_\alpha$ in the following way: For each i,

$u_i =$ *a node above the branch* $\{\ x \mid$ *For some* $(\bar{a}, \bar{b}, \bar{q}) \in G,\quad a_i = x\}$
$v_i =$ *a node above the branch* $\{\ y \mid$ *For some* $(\bar{a}, \bar{b}, \bar{q}) \in G,\quad b_i = y\}$
$f(u_i, v_j) = \bar{q}(i, j)$, *where* $(\bar{a}, \bar{b}, \bar{q}) \in G$ *for some* $\bar{a}, \bar{b}$

By condition (*2*), it is clear that this local forcing adds branches above every node, and that condition (*2*) continues to hold for $\alpha+1$. So the product of the trees A and B thus obtained is special; but why are A and B and each of their derived trees Suslin? To see that, we argue that if we restrict our attention to A, for example, then it is in fact the construction of a Suslin tree given in subsection 2.1 which describes A. For this aim we will define for each limit α a projection Π_{A} from the poset $\mathbb{R}$ used in the construction of $(\mathsf{A}\lceil\alpha+1)\times(\mathsf{B}\lceil\alpha+1)$ onto the poset $\mathcal{P}(\mathsf{A}\lceil\alpha)$ used in 2.1. Simply set $\Pi_{\mathsf{A}}(\bar{a},\bar{b},\bar{q})=\bar{a}$. We must check the following properties which ensure that the projection of the $\mathbb{R}$-generic filter is a $\mathcal{P}(\mathsf{A}\lceil\alpha)$-generic filter:

1. Π_{A} is order preserving: $x_0 \leq_{\mathbb{R}} x_1 \Rightarrow \Pi_{\mathsf{A}}(x_0) \leq_{\mathcal{P}} \Pi_{\mathsf{A}}(x_1)$.
2. Whenever $p\in\mathcal{P}$ extends $\Pi_{\mathsf{A}}(x_0)$, there is an extension x_1 of x_0 in $\mathbb{R}$ such that $\Pi_{\mathsf{A}}(x_1)=p$.

This is not difficult to prove by (*2*).

2.3 ω_1 many trees

In this section (waving our hands even harder) we extend the previous construction to ω_1 many Suslin trees with any reasonable requirement on which trees are Suslin and which are special.

Theorem 2.2 *Assume $\diamondsuit_{\omega_1}$. Let sp (for special) be a collection of non-empty finite subsets of ω_1 closed under supersets, and let su be those non-empty finite sets $e\subset\omega_1$ which are not in sp. Then there is a sequence of ω_1-trees $\langle \mathsf{A}^{\zeta}\,|\zeta<\omega_1\rangle$ such that for a finite set $e=\{\zeta_1,\dots,\zeta_n\}$*

1. *If $e\in\mathsf{sp}$, $\mathsf{A}^{(e)}\overset{Def}{=}\mathsf{A}^{\zeta_1}\times,\dots,\times\mathsf{A}^{\zeta_n}$ is special.*
2. *If $e\in\mathsf{su}$, $\mathsf{A}^{e}\overset{Def}{=}\mathsf{A}^{\zeta_1}\cup\ldots\cup\mathsf{A}^{\zeta_n}$ and all of its derived trees are Suslin.*

Proof: By induction on $\alpha<\omega_1$, the sequence $\{\ \mathsf{A}^{\zeta}\lceil\alpha+1\mid\zeta<\alpha\}$ is defined together with specializing functions $f_e\lceil\alpha+1$ for $e\in\mathsf{sp}$, $e\subseteq\alpha$. f_e is, of course, a specializing function from $\mathsf{A}^{(e)}$ into $\mathbb{Q}$. It is convenient to require that $f_e\lceil\alpha+1$ is only defined on the β levels of the product tree for $\beta>\max(e)$.

The definition of the trees requires some notations and preliminary definitions. Let $\alpha < \omega_1$ be any ordinal—successor or limit, and assume that $(\mathsf{A}^\zeta \restriction \alpha)$ for $\zeta < \alpha$, and $f_e \restriction \alpha$ for $e \subseteq \alpha$ in sp are given.

Let us define, for any finite $d \subseteq \alpha$,

$$\mathcal{P}(\mathsf{A}^d \restriction \alpha) = \times_{\xi \in d} \mathcal{P}(\mathsf{A}^\xi \restriction \alpha)$$

in the following. $a = \langle \overline{a}^\xi \mid \xi \in d \rangle \in \mathcal{P}(\mathsf{A}^d \restriction \alpha)$ if for some $\mu < \alpha$ for all $\xi \in d$, $\overline{a}^\xi$ is an n_ξ-tuple in A^ξ_μ enumerated as follows: $\overline{a}^\xi = \langle a^\xi_i \mid i \in I_\xi \rangle$ where $I_\xi = I_\xi(a) \subset \omega$ is a finite set of size n_ξ.

Extension is naturally defined in $\mathcal{P}(\mathsf{A}^d \restriction \alpha)$: b extends a iff for all $\xi \in d$, $I_\xi(a) \subseteq I_\xi(b)$ and for every $i \in I_\xi(a)$, $a^\xi_i < b^\xi_i$ in $\mathsf{A}^\xi \restriction \alpha$.

We need one more definition. For $a \in \mathcal{P}(\mathsf{A}^d \restriction \alpha)$, and $e \subseteq d$ with $e \in \mathsf{sp}$, let us say that q^e *bounds* a iff q^e is a function

$$q^e : \; I^{(e)} = \times_{\xi \in e} I_\xi(a) \to \mathbb{Q}$$

such that for every $\overline{i} = \langle i_\xi \mid \xi \in e \rangle \in I^{(e)}$,

$$f_e(\langle a^\xi_{i_\xi} \mid \xi \in e \rangle) < q^e(\overline{i}).$$

Now we can formulate the inductive property of the trees and functions at the α stage:

(3_α) *If* $a \in \mathcal{P}(\mathsf{A}^d \restriction \alpha)$, *where* $d \subseteq \alpha$ *is finite, and if* $\langle q^s \mid s \subseteq d, \; s \in \mathsf{sp} \rangle$ *is such that each* q^s *bounds* a, *then for every* $e \subseteq d$ *with* $e \in \mathsf{su}$, *and* b *extending* $a \restriction e$ *in* $\mathcal{P}(\mathsf{A}^e \restriction \alpha)$, *there is* $b_1 > a$ *in* $\mathcal{P}(\mathsf{A}^d \restriction \alpha)$ *such that each* q^s *bounds* b_1, *and* $b_1 \restriction e = b$.

Observe that this property makes sense for α's which are limit as well as for α's which are successor ordinals.

Let us now return to the inductive definition of the trees.

Case 1 α is a limit ordinal. In this case we first take the union of the trees and functions obtained so far. So for each $\xi < \alpha$, and $e \subseteq \alpha$ in sp:

$$\mathsf{A}^\xi \restriction \alpha = \bigcup_{\mu < \alpha} \mathsf{A}^\xi \restriction \mu, \;\; \textit{and} f_e \restriction \alpha = \bigcup_{\mu < \alpha} f_e \restriction \mu,$$

Then we add the α-th levels and extend f_e according to the following procedure.

A countable poset $\mathsf{R} = \mathsf{R}_\alpha$ is defined as a convenient way to express how the α-branches are added to each $(\mathsf{A}^\zeta \lceil \alpha)$, enabling the definition of A^ζ_α and of the extensions of the f_e's.

A condition in R gives finite information on the branches and the values of the appropriate f_e's. So a condition $r \in \mathsf{R}$ has two components: $r = \{a, \overline{q}\}$, where for some $\mu = \mu(a) < \alpha$, a gives information on the intersection of the (locally) generic α-branches with the μ level, and $\overline{q}$ tells us the future rational values on products of these branches. Formally, we require that for some finite $d = d(r) \subseteq \alpha$

(1) $a \in \mathcal{P}(\mathsf{A}^d \lceil \alpha)$,
(2) $\overline{q} = \langle q^e \mid e \subseteq d,\ e \in \mathsf{sp} \rangle$, and for each $e \subseteq d$ in sp, q^e bounds a.

In plain words, $r \in \mathsf{R}_\alpha$ has a finite domain $d \subseteq \alpha$ on which it speaks. For $\zeta \in d$, a^ζ_i is the intersection with A^ζ_μ, of the proposed ith branch added to $\mathsf{A}^\zeta \lceil \alpha$, and q^e gives information on how to specialize those trees required to be special. So if $e \subseteq d$ is in sp, then q^e gives rational upper bounds to the range of the specializing function f_e on the branches added to $\mathsf{A}^{(e)} \lceil \alpha$.

We write $d = d(r)$, $\bar{a} = \bar{a}(r)$, $\bar{q} = \bar{q}(r)$, $\mu = \mu(r)$ etc. to denote the components of $r \in \mathsf{R}$.

A countable M is chosen with R_α, S_α, the trees so far constructed and so on in M, and an R_α generic filter G over M is used to define the branches and the new values of f_e.

1. For $\zeta < \alpha$ and $i < \omega$, $u^\zeta_i = \{\, x \mid \textit{For some } r \in G,\ a^\zeta_i = x \}$, is the ith branch added to $\mathsf{A}^\zeta \lceil \alpha$. This determines A^ζ_α.

2. For $e \in \mathsf{sp}$, we define f_e on the αth level of $\mathsf{A}^{(e)}$, as follows. Any α-level node, w, of $\mathsf{A}^{(e)}$ has the form $\langle u^{\zeta_1}_{i_1}, \ldots, u^{\zeta_n}_{i_n} \rangle$ where $e = \{\, \zeta_1, \ldots, \zeta_n \}$; then $f_e(w) = q^e(i_1, \ldots, i_n)$, where q^e comes from G, (That is $q^e = \bar{q}(r)^e$ for some $r \in G$).

As evidenced by f_e, $\mathsf{A}^{(e)}$ becomes a special tree for $e \in \mathsf{sp}$. When $e \notin \mathsf{sp}$, A^e is Suslin and so are all of its derived trees. It is here that the assumption $e \notin \mathsf{sp} \Rightarrow$ for $e' \subseteq e$, $e' \notin \mathsf{sp}$ is used. We must prove that for $e \notin \mathsf{sp}$, the construction of A^e follows the specification described in subsection 2.1. To do that, observe that for $e \notin \mathsf{sp}$ the map Π taking r to $\langle \bar{a}^{\zeta}(r) \,|\, \zeta \in e \rangle$ is a projection of $\mathbb{R}$ onto $\mathcal{P}(\mathsf{A}^e \lceil \alpha)$.

Case 2 is for α a succesor ordinal. Put $\alpha = \rho + 1$. Not only the αth level has to be defined for all existing trees, but a new tree, A^{ρ}, and new functions must be introduced. The definition of the new functions f_e with $\rho \in e$ is somewhat facilitated by our assumption that these are only defined on the αth level.

3 Suslin-tree preservation by proper forcing

In this section it is shown that any Suslin tree S remains Suslin in a countable support iteration, if each single step of the iteration does not destroy the Suslin property of S. We assume our posets are separative: If q does not extend p then some extension of q is incompatible with p.

Let $\vec{\mathcal{P}} = \langle \mathcal{P}_\alpha \,|\, \alpha \leq \beta \rangle$ be a countable support iteration of length β (limit ordinal) of proper forcing posets; where $\mathcal{P}_{\alpha+1} = \mathcal{P}_\alpha * \mathcal{Q}_\alpha$ is a two step iteration: $\mathcal{P}_\alpha$ followed by $\mathcal{Q}_\alpha$. The following preservation theorem holds for $\vec{\mathcal{P}}$.

Theorem 3.1 *Let S be a Suslin tree of height ω_1; suppose for every $\beta' < \beta$, S remains Suslin in $V^{\mathcal{P}_{\beta'}}$. Then S remains Suslin in $V^{\mathcal{P}_\beta}$ as well.*

Proof: To show that every antichain of S is countable, it is enough to prove that:

For any dense open set $E \subseteq \mathsf{S}$ there is a level S_λ, $\lambda < \omega_1$, such that $\mathsf{S}_\lambda \subseteq E$.

So let $\underset{\sim}{E}$ be a $\mathcal{P}_\beta$ name of a dense open subset of S. Fix some countable elementary submodel, $M \prec H(\kappa)$, such that S, β, $\mathcal{P}_\beta$, $\underset{\sim}{E}$ etc. are in M, where κ is "big enough". ($H(\kappa)$ is the collection of all sets of cardinality hereditarily $< \kappa$. In fact, all that is needed is that M reflects enough of V to enable the following constructions and arguments to be carries out.)

- Let $\lambda = M \cap \omega_1$. λ is a countable ordinal.
- Let $\langle \beta(i) \,|\, i < \omega \rangle$ be an increasing ω-sequence of ordinals in $\beta \cap M$ and cofinal in $\beta \cap M$.
- Let $\{ b_n \mid n \in \omega \}$ be an enumeration of S_λ.

We will produce a condition $q \in \mathcal{P}_\beta$ (extending some given condition) such that for every $n \in \omega$, $q \Vdash b_n \in \underset{\sim}{E}$, and thus

$$q \Vdash \mathsf{S}_\lambda \subseteq \underset{\sim}{E}$$

First, a sequence $q_n \in \mathcal{P}_{\beta(n)}$, and $\underset{-}{p}_n$ is inductively constructed such that the following holds:

1. q_n is an M-generic condition for $\mathcal{P}_{\beta(n)}$ and $q_{n+1} \restriction \beta(n) = q_n$.
2. $\underset{-}{p}_n$ is a *name* in $V^{\mathcal{P}_{\beta(n)}}$, forced to be a condition in $\mathcal{P}_\beta \cap M$.
3. (a) $q_n \Vdash_{\mathcal{P}_{\beta(n)}} \underset{-}{p}_n \restriction \beta(n)$ *is in the canonical generic filter*, G_n.
   (b) $q_n \Vdash_{\mathcal{P}_{\beta(n)}} \underset{-}{p}_n$ *extends* $\underset{-}{p}_{n-1}$ *in* $\mathcal{P}_\beta$.
   (c) $q_n \Vdash_{\mathcal{P}_{\beta(n)}} (\underset{-}{p}_n \Vdash_{\mathcal{P}_\beta} b_n$ *is above some member of* $\underset{\sim}{E}$).

(Recall that the canonical generic filter G is defined so that $q \Vdash q \in G$ for every q).

Suppose for a moment that we do have such sequences, and this is how q is obtained: $q = \cup_{n<\omega} q_n$. Then $q \in \mathcal{P}_\beta$ extends each q_n, since $q_{n+1} \restriction \beta(n) = q_n$ is assumed for all n. We also have the following:

Claim 3.2 $q \Vdash_{\mathcal{P}_\beta} \mathsf{S}_\lambda \subseteq \underset{\sim}{E}$.

Proof: This is a consequence of 1-3 obtained as follows. Let $\underline{G}$ be the name of the $\mathcal{P}_\beta$ canonical generic filter. It is enough to show that for each n

(*) $\qquad q \Vdash_{\mathcal{P}_\beta} \underline{p}_n \in \underline{G},$

because then we use 3(c) to deduce that q forces b_n to be in $\underset{\sim}{E}$. To prove (*) we observe that

1. $q \Vdash_{\mathcal{P}_\beta} \underline{p}_n$ *extends* $\underline{p}_m$ *in* $\mathcal{P}_\beta$ *for* $m < n$, and
2. $q \Vdash_{\mathcal{P}_\beta} (\underline{p}_m \restriction \beta(m)) \in \underline{G}_m$ *for all* $m < \omega$.

Hence:

$$q \Vdash_{\mathcal{P}_\beta} (\underline{p}_m \restriction \beta(n)) \in \underline{G}_n \textit{for all } m < n.$$

From this it follows that for any q' extending q in $\mathcal{P}_\beta$, if q' determines $\underline{p}_m$, that is for some $p \in \mathcal{P}_\beta$, $q' \Vdash \underline{p}_m = p$, then $q' \Vdash p \restriction \beta(n) \in \underline{G}_n$, and hence q' extends $p \restriction \beta(n)$, for all n's, and thus q' extends p. Thus $q' \Vdash_{\mathcal{P}_\beta} \underline{p}_m \in \underline{G}$. This is so for an arbitrary extension of q which determines $\underline{p}_m$, and hence (*): $q \Vdash_{\mathcal{P}_\beta} \underline{p}_m \in \underline{G}$,

Return now to the construction of the sequences. Suppose $\underline{p}_n$ and q_n are constructed (or that we are about to start the construction).

In order to describe $\underline{p}_{n+1}$ and q_{n+1} (in that order), imagine a generic extension $V[G_n]$ of our universe V, where $G_n \subseteq \mathcal{P}_{\beta(n)}$ is a generic filter containing q_n. Then $M[G_n]$ can be formed; it is the G_n-interpretation of all $\mathcal{P}_{\beta(n)}$ names in M. Then $M[G_n] \prec H(\kappa)[G_n]$. S is still a Suslin tree in $V[G_n]$ by our assumption.

In $V[G_n]$, $\underline{p}_n$ is realized as a condition denoted p_n; $p_n \in \mathcal{P}_\beta \cap M$, and $p_n \restriction \beta(n) \in G_n$ by the inductive assumption in 3(a).

Since $\underset{\sim}{E}$ is forced to be dense in S, for any $s \in \mathsf{S}$ and $p \in \mathcal{P}_\beta$, there are $s \leq_{\mathsf{S}} s'$ and an extension p' of p in $\mathcal{P}_\beta$ such that

$$(**) \qquad p' \Vdash_{\mathcal{P}_\beta} s' \in \underset{\sim}{E}$$

Moreover, by genericity of G_n, we may require that

$$p \restriction \beta(n) \in G_n \Rightarrow p' \restriction \beta(n) \in G_n.$$

Thus, the set F of $s' \in \mathsf{S}$ for which there is $p' \in \mathcal{P}_\beta$ extending p_n with $p' \restriction \beta(n) \in G_n$ and satisfying $(**)$ is dense in S and is (defined) in $M[G_n]$.

Now S is a Suslin tree in $M[G_n]$, and hence every branch of $\mathsf{S} \restriction \lambda$ of length λ is $M[G_n]$ generic. (Recall $\lambda = \omega_1 \cap M$). Thus b_{n+1} (the $(n+1)$th node of S_λ) is above some node in F; and it is possible to pick p_{n+1} in $\mathcal{P}_\beta \cap M$ extending p_n with $p_{n+1} \restriction \beta(n) \in G_n$ and such that $p_{n+1} \Vdash_{\mathcal{P}_\beta} b_{n+1} \in \underset{\sim}{E}$.

This description of p_{n+1} made use of the $\mathcal{P}_{\beta(n)}$-generic filter G_n. Back in V, we define $\underset{\sim}{p}_{n+1}$ to be the *name* of that p_{n+1} in $V^{\mathcal{P}_{\beta(n)}}$ (and so, evidently, in $V^{\mathcal{P}_{\beta(n+1)}}$).

Next we define q_{n+1}. We demand the following from q_{n+1}:

1. q_{n+1} is an M-generic condition for $\mathcal{P}_{\beta(n+1)}$, and $q_{n+1} \restriction \beta(n) = q_n$.

2. $q_{n+1} \Vdash_{\mathcal{P}_{\beta(n+1)}} \underset{\sim}{p}_{n+1} \restriction \beta(n+1) \in G_{n+1}$.

The existence of q_{n+1} satisfying (1) and (2) is a general fact about proper forcing. It is a consequence of the following statement, which can be proved by induction on β_2:

Suppose $\beta_1 < \beta_2 \leq \beta$, and q_1 is an M-generic condition over $\mathcal{P}_{\beta_1}$, and $\underset{\sim}{p}$ is a name in $V^{\mathcal{P}_{\beta_1}}$ such that $q_1 \Vdash_{\mathcal{P}_{\beta_1}} \underset{\sim}{p} \in \mathcal{P}_{\beta_2} \cap M$ *and* $\underset{\sim}{p} \restriction \beta_1$ *is in the canonical* $\mathcal{P}_{\beta_1}$ *generic filter.* Then there is an M-generic condition over $\mathcal{P}_{\beta_2}$, q_2, such that $q_2 \restriction \beta_1 = q_1$, and $q_2 \Vdash_{\mathcal{P}_{\beta_2}} \underset{\sim}{p} \restriction \beta_2$ *is in the canonical* $\mathcal{P}_{\beta_2}$ *generic filter.*

4 How to specialize Aronszajn trees without adding reals

The forcing notions which turn a given Aronszajn tree into a special tree, naturally fall into two categories: those which use finite conditions and satisfy properties such as the c.c.c., and those which use infinite conditions and have nice closure properties.

In this section we describe how infinite conditions can be used to specialize an Aronszajn tree, without addition of new countable sets, and how to iterate such posets.

In a moment we will define the poset $\mathcal{S}(\mathsf{T})$ used to specialize an Aronszajn tree T. Meanwhile, let us see what are the problems with the direct approach, which takes the poset $\mathcal{S}_1$ of all specializing functions f defined on some downward closed countable subtree of the form $\mathsf{T}\lceil\alpha+1$. To see that this poset collapses ω_1 in forcing, look at the following dense open sets, defined for $n<\omega$.

$$D_n = \{\ f \in \mathcal{S}_1 \mid f\ 1 \textit{is defined on } \mathsf{T}\lceil\alpha+1, \textit{and for every } x \in \mathsf{T}_\alpha,\ f(x) \geq n\}$$

Clearly D_n is dense open. So for every $n<\omega$ there is an $\alpha<\omega_1$ such that some $f \in D_n$ defined on $\mathsf{T}\lceil\alpha+1$ is in the generic filter G. But if ω_1 is *not* collapsed, the generic filter must contain a condition which is simultaneously in every D_n, and this is a contradiction.

Thus, there must be some limitation on the growth of the generic specializing function. We may try the following poset: $\mathcal{S}_2$ consists of all specializing $f : \mathsf{T}\lceil\alpha+1 \to \mathbb{Q}$, $\alpha<\omega_1$, such that

$$\forall\alpha_0<\alpha\ \forall\bar{x}\in\mathsf{T}_{\alpha_0},\ \text{if } f(\bar{x})<\bar{q} \text{ then for some } \bar{y}\in\mathsf{T}_\alpha,\ \bar{x}<\bar{y} \text{ and } f(\bar{y})=\bar{q}.$$

Here, $\bar{x}$ is an n-tuple of nodes, $\bar{q}$ is an n-tuple of rational numbers, and $f(\bar{x})<\bar{q}$ is a shorthand for: $f(x_i)<q_i$ for all i's.

If we assume that every node in T has infinitely many successors, it is not difficult to see that any condition in $\mathcal{S}_2$ can be extended to any height. Therefore forcing with $\mathcal{S}_2$ specializes T. If T is a Suslin tree such that each derived tree of T is Suslin too, then $\mathcal{S}_2$ adds no new countable sets. If T is an arbitrary Aronszajn tree, however, $\mathcal{S}_2$ may collapse ω_1. Such is the case when T has the form $\mathsf{T}_1 \cup \mathsf{T}_2$, a disjoint union of T_1 with a copy T_2 of itself. Let $i : \mathsf{T}_1 \to \mathsf{T}_2$ be the map which takes a node in T_1 to its copy in T_2. Define

$$D_n = \left\{ f \in \mathcal{S}_2 | \begin{array}{l} dom(f) = \mathsf{T}\lceil\alpha + 1 \text{ and } \forall x \in \mathsf{T}_{1,\alpha} \\ f(x) > n \text{ or } f(i(x)) > n \end{array} \right\}$$

Again, D_n is seen to be dense open; and since the generic function contains a condition in D_n but there is no condition in the intersection of all the D_n's, ω_1 must be collapsed. This shows that the limitations one imposes on the growth of the generic specializing function must have a different character.

We are now going to define the poset $\mathcal{S} = \mathcal{S}(\mathsf{T})$ used to specialize a given Aronszajn tree T. A condition $p \in \mathcal{S}$ is a pair $p = (f, \Gamma)$ where f is a countable partial specialization of T, called an "approximation"; and Γ is an uncountable object, called a "promise". Its role is to ensure that $\mathcal{S}$ is a proper poset. Γ consists of "requirements", so we have to explain what these are first. Throughout, T is a fixed Aronszajn tree.

Definition 4.1 (1) We say that H is a *requirement* (of *height* $\gamma < \omega_1$) iff for some $n = n(H) < \omega$, H is a set of finite functions of the form $h : \mathsf{T}_\gamma \to \mathbb{Q}$, with $dom(h) \in {}^n\mathsf{T}_\gamma$.
(2) An *approximation* (on T) is a partial specializing function $f : \mathsf{T}\lceil(\alpha+1) \to \mathbb{Q}$; that is, an order-preserving function defined on $\bigcup_{\zeta\leq\alpha} \mathsf{T}_\zeta$ into the rationals. The countable ordinal α is called $last(f)$.

We say that a finite function $h : \mathsf{T}_\alpha \to \mathbb{Q}$ *bounds* f iff $\forall x \in dom(h)(f(x) < h(x))$. More generally, for $\beta \geq \alpha = last(f)$, $h : \mathsf{T}_\beta \to \mathbb{Q}$ bounds f iff $\forall x \in dom(h)(f(x\lceil\alpha) < h(x))$ (i.e., if $h\lceil\alpha$ is defined, then $h\lceil\alpha$ bounds f).
(3) An approximation f with $last(f) = \alpha$ is said to *fulfill* requirement H of

height α iff for every finite $t \subseteq \mathsf{T}_\alpha$ there is some $h \in H$ which bounds f and such that $dom(h)$ is disjoint to t.
(4) A *promise* Γ (for T) is a function $\langle \Gamma(\gamma) \,|\beta \leq \gamma < \omega_1 \rangle$ (β is denoted $\beta(\Gamma)$) such that

(a) $\Gamma(\gamma)$ is a countable collection of requirements of height γ. There is a fixed n such that $n = n(H)$ for all γ and $H \in \Gamma(\gamma)$.

(b) For $\gamma \geq \beta$, each $H \in \Gamma(\gamma)$ is dispersed. That is, for every finite $t \subseteq \mathsf{T}_\gamma$ for some $h \in H$, $t \cap dom(h) = \emptyset$.

(c) For every $\beta \leq \alpha_0 < \alpha_1 < \omega_1$,

$$\Gamma(\alpha_0) = \{ \; (X \lceil \alpha_0) \mid X \in \Gamma(\alpha_1) \}$$

(5) An approximation f fulfills promise Γ iff $last(f) \geq \beta(\Gamma)$, and f fulfills each requirement H in $\Gamma(last(f))$.

Definition 4.2 [of $\mathcal{S}(\mathsf{T})$] For any Aronszajn tree T define $\mathcal{S} = \mathcal{S}(\mathsf{T})$ by $p = (f, \Gamma) \in \mathcal{S}$ iff f is an approximation on T, Γ is a promise, and f fulfills Γ.

The partial order is naturally defined: $p_1 = (f_1, \Gamma_1)$*extends* $p_0 = (f_0, \Gamma_0)$ iff $f_0 \subseteq f_1$ and $\Gamma_0 \lceil (\omega_1 - last(f_1)) \subseteq \Gamma_1$. That is, any requirement of height $\gamma \geq last(f_1)$ in Γ_0 is also in Γ_1.

If $p = (f, \Gamma)$ is a condition in $\mathcal{S}$ we write $f = f(p)$, $\Gamma = \Gamma(p)$. In an abuse of notation, we write $last(p)$ for $last(f(p))$, and $p(x)$ instead of $f(x)$. We also call $last(p)$ 'the *height*' of p. (Recall that $f(p)$ is defined on $\mathsf{T} \lceil last(p) + 1$.)
Remark If our only aim is to obtain a model of CH & SH, it is enough to assume that $\Gamma(\gamma)$ is a singleton. The assumption that $\Gamma(\gamma)$ is a countable collection of requirements will be used in order to show that this forcing preserves certain Suslin trees.

Remark If $p = (f, \Gamma) \in \mathcal{S}$, γ is the *height* of p, and $g : \mathsf{T}\lceil\gamma + 1 \to \mathbb{Q}$ is a specializing function satisfying: $\forall x \in dom(f)\ (g(x) \leq f(x))$, then g fulfills the promise Γ which f fulfills.

This simple remark is used in the following way. Suppose that p_1 extends p_0; put $\mu_i = last(p_i)$, $f_i = f(p_i)$ for $i = 0, 1$.

Let δ be an order-preserving map of the set of positive rationals $\mathbb{Q}^+$ into $\mathbb{Q}^+$ such that $\delta(r) \leq r$ for all r. Then define, for any $x \in \mathsf{T}_\alpha$, where $\mu_0 < \alpha \leq \mu_1$,

$$g(x) = f_0(x\lceil\mu_0) + \delta(f_1(x) - f_0(x\lceil\mu_0)).$$

In words: g uses δ to compress f_1 on $\mathsf{T}\lceil\mu_1 + 1 \setminus \mathsf{T}\lceil\mu_0 + 1$.
Extend further g and, for $x \in \mathsf{T}\lceil\mu_0 + 1$, define $g(x) = f_0(x)$. Then $(g, \Gamma(p_1))$ is also an extension of p_0 of height μ_1.

Our next aim is to show that it is possible to extend conditions to any height, and to enlarge promises. Then we will show properness of $\mathcal{S}$. In the following subsection, $\mathcal{S}$ is shown to specialize only those trees it must specialize. Then, in the next subsection $\mathcal{S}$ is proved to satisfy the condition which allows to conclude that a countable support iteration of such posets adds no new countable sets.

Lemma 4.3 (The extension lemma) *If $p \in \mathcal{S}$ and $last(p) < \mu < \omega_1$, then there is an extension q of p in $\mathcal{S}$ with $\mu = last(q)$, and such that $\Gamma(q) = \Gamma(p)$. Moreover, if $h : \mathsf{T}_\mu \to \mathbb{Q}$ is finite and bounds p, then h bounds an extension q of p of height μ.*

Proof: The 'moreover' clause of the Lemma is, in fact, a direct consequence of the first part and the Remark above. Indeed, if h bounds p as in the Lemma, pick first *any* extension p_1 of p with $\mu = last(p_1)$, and then correct p_1 as follows to obtain q.

Put $\mu_0 = last(p)$, $f = f(p)$. For some $d > 0$, $\forall x \in dom(h)$

$$h(x) > f(x\lceil\mu_0) + d.$$

Let δ be an order-preserving map of $\mathbb{Q}^+$ into the interval $(0, d)$ such that $\delta(x) < x$ for all x. Now use the Remark to correct p_1 and to obtain an extension q of p which satisfy for every $x \in \mathsf{T}\lceil((\mu+1)-\mu_0)$, $q(x)-p(x\lceil\mu_0) < d$. Hence h bounds q.

The proof of the first part of the Extension Lemma is done by induction on μ. Since the proof is quite easy, only the outline is given.

CASE I $\mu = \mu_0 + 1$ is a successor ordinal. By the inductive assumption, $last(p) = \mu_0$ can be assumed, and we have to extend $f = f(p)$ on T_{μ_0+1}, fulfilling all the requirements in $\Gamma(\mu)$ $(\Gamma = \Gamma(p))$. Given any requirement $H \in \Gamma(\mu)$, we know that $H\lceil\mu_0 = H_0 \in \Gamma(\mu_0)$ is fulfilled by f. So, H_0 contains an infinite pairwise disjoint set of functions h which bound f. This allows plenty of time to extend f, in ω steps, and to keep the promise Γ at the level μ.

CASE II μ is a limit ordinal. Pick an increasing sequence of ordinals $\mu_i, i < \omega$, cofinal in μ. We are going to define an increasing sequence $p_i \in \mathcal{S}$ (beginning with $p_0 = p$) and finite $h_i : \mathsf{T}_\mu \to \mathbb{Q}$ which bound p_i, by induction on $i < \omega$. Then we will set $q = (f, \Gamma)$, by $f = \bigcup\{\ f(p_i) | i < \omega\} \cup \bigcup\{\ h_i | i < \omega\}$, and $\Gamma = \Gamma(p)$. $last(p_i) = \mu_i$, and the passage from p_i to p_{i+1} uses the inductive assumption for μ_{i+1}. The role of the h_i's is not only to ensure that f is bounded on the μ branches determined by T_μ, but also to ensure that the promise made in $\Gamma(p) = \Gamma$, namely $\Gamma(\mu)$, is kept by $h = \bigcup_{i<\omega} h_i$. Each requirement $H \in \Gamma(\mu)$ must appear infinitely often in a list of missions, and at each step, $i < \omega$, of the definition, h_{i+1} takes care of one more $h \in H$, so that finally an infinite pairwise disjoint subset of H consists of functions which bound f. It is here that we use the assumption that $\Gamma(\mu_i) = \Gamma(\mu)\lceil\mu_i$, i.e., that $\Gamma(\mu_i) = \{\ (H\lceil\mu_i) \mid H \in \Gamma(\mu)\}$. Next we show that promised can be added.

Let $p = (f, \Gamma) \in \mathcal{S}$ be a condition of height μ, and let Ψ be any promise.

We say that p 'includes' Ψ iff for all γ such that $\mu \leq \gamma < \omega_1$

$$\Psi(\gamma) \subseteq \Gamma(\gamma).$$

That is, any requirement $H \in \Psi(\gamma)$ is already in $\Gamma(\gamma)$. If p includes Ψ then, obviously p fulfills Ψ. Otherwise, it is not always possible to extend p to fulfill Ψ. However, if the following simple condition holds, then this can be done.

Lemma 4.4 *[Addition of promises] Let $p \in \mathcal{S}$, put $\mu = last(p)$. Let Ψ be a promise with $\mu < \beta = \beta(\Psi)$. Suppose for some finite $g : \mathsf{T}_\mu \to \mathbb{Q}$ (called a* basis *for Ψ), g bounds $f(p)$ and*

$$\forall \gamma \geq \beta,\ \forall H \in \Psi(\gamma),\ \forall h \in H\ (h \lceil \mu = g),$$

then there is an extension p_1 of p in $\mathcal{S}$ of height β which includes Ψ.

Proof: This is an easy application of the Extension Lemma. Put $f = f(p)$, then for some rational $d > 0$, $\forall x \in dom(g)\ g(x) > f(x) + d$.

Now every $H \in \Psi(\beta)$ is a dispersed collection of functions h with $h \lceil \mu = g$. Let p_1 be any extension of p of *height* β; set $f_1 = f(p_1)$. The desired extension of p will be obtained by correcting f_1 so as to fulfill $\Psi(\beta)$ and then to add Ψ. This is done as follows.

Let δ be an order-preserving map of the positive rationals into the rational interval $(0, d)$, such that $\delta(r) < r$ for every r. Define now for $x \in T_\alpha,\ \mu < \alpha \leq \beta$: $f_2(x) = f(x \lceil \mu) + \delta(f_1(x) - f(x \lceil \mu))$. Then $f_2 \cup f$ fulfills each $H \in \Psi(\beta)$, and thus gives the desired extension.

The properness of $\mathcal{S}$ is not so easy to prove, and it is here that the need for the promises appears. Given an elementary countable substructure $M \prec H(\kappa)$, such that $\mathcal{S} \in M$, and given a condition $p_0 \in M$, we have to find an "M-generic" condition q extending p_0. In fact, we will find q with a stronger property which implies that no new reals are added: for every dense open set $D \subseteq \mathcal{S}$ in M, $q \in D$.

As in the definition of q in the Extension Lemma (the limit case), here too an increasing sequence $p_i \in \mathcal{S} \cap M$ of conditions and finite functions $h_i : \mathsf{T}_\mu \to \mathbb{Q}$ are defined; where $\mu = \omega_1 \cap M$. But now we are faced with an extra mission in defining p_{i+1}: to put p_{i+1} in D, the i-th dense open subset of $\mathcal{S}$ in M (in some enumeration of the countable M). The problem with this mission is that perhaps whenever r extends p_i is in D, then h_i does not bound r.

To show that this bad event never happens, requires the following main Lemma.

Lemma 4.5 *Let T be an Aronszajn tree. Let $M \prec H(\kappa)$ be a countable elementary substructure, where κ is some big enough cardinal; $\mathsf{T}, \mathcal{S} = \mathcal{S}(\mathsf{T}) \in M$. Let $p \in M$ be a condition in $\mathcal{S}$, $\mu = \omega_1 \cap M$ and $h : \mathsf{T}_\mu \to \mathbb{Q}$ be a finite function which bounds p. Let $D \subseteq \mathcal{S}$, $D \in M$ be dense open. Then there is an extension of p, $r \in D \cap M$, such that h bounds r.*

Proof: Assume for the sake of a contradiction that this is not so, and let T, M, p, h etc. be a counterexample. Let $\mu_0 = \mathit{last}\ p$; $\bar{x} = \mathit{domain}\ (h)$ enumerated in some way; so $\bar{x} \in {}^n\mathsf{T}_\mu$, $\bar{x} = \langle x_0, \ldots, x_{n-1} \rangle$. Put $\bar{q} = h(\bar{x})$; that is, $q_i = h(x_i)$. Denote $\bar{v} = \bar{x} \lceil \mu_0$; then $\bar{v} \in {}^n\mathsf{T}_{\mu_0}$, and we may assume $v_i \neq v_j$ for $i \neq j$ (or else, extend p above the splittings of $\bar{x}$). In M:

If $r \in D \cap M$ extends p, then h does not bound r.

Put $g_0 = h \lceil \mu_0$. Then $g_0 \in M$. Say that a finite function $g : \mathsf{T}_\gamma \to \mathbb{Q}$ is *bad* iff

1. $\mu_0 \leq \gamma < \omega_1$, and $g \lceil \mu_0 = g_0$.
2. Whenever $r \in D$ extends p and $\gamma \geq \mathit{last}(r)$, g does not bound r.

In other words, g is bad if it mimics $h \lceil \gamma$, but it may live on other n-tuples of T. Of course, $h \lceil \gamma$ itself is bad for any γ with $\mu_0 \leq \gamma < \mu$. It follows that, in M and hence in $H(\kappa)$, there are uncountably many bad g's. Indeed, if there were only countably many bad functions, there would be a bound γ, in M, for $\{ \mathit{height}(g) \mid g \text{ is bad}\}$; and as $\gamma < \mu$, $h\lceil\gamma$ would not be bad.

Observe that if g is bad and $\mu_0 \leq \gamma_0 < \mathit{height}(g)$, then $g\lceil\gamma_0$ is bad too.

Now put

$$B = \{ \mathit{dom}(g) \mid g \text{ is bad}\}.$$

Then B is uncountable and closed downwards (above μ_0) subset of $\bigcup_{\mu_0\leq\gamma<\omega_1} {}^n\mathsf{T}_\gamma$. As T is an Aronszajn tree, Lemma 1.1 implies that for some $\beta > \mu_0$ and some $B^0 \subseteq \mathsf{B}$, if we put $B^0_\gamma = B^0 \cap {}^n\mathsf{T}_\gamma$, then

1. For $\beta \leq \gamma_0 < \gamma_1 < \omega_1$, $B^0_{\gamma_0} = B^0_{\gamma_1}\lceil\gamma_0$, and
2. B^0_β (and thus every B^0_γ, $\beta < \gamma$) is dispersed.

We may find B^0 in M, since only parameters in M were mentioned in its definition. For $\beta \leq \gamma < \omega_1$, let $\Psi(\gamma)$ consists of $H_\gamma = \{ g | g \text{ is bad and } \mathit{dom}(g) \in B^0_\gamma\}$. By Lemma 4.4 (Addition of promises), there is an extension p_o of p in M of height β which includes Ψ. That is, if $\Gamma_0 = \Gamma(p_0)$, then for every $\gamma \geq \mathit{last}(p_0) = \beta$, $H_\gamma \in \Gamma_0(\gamma)$.

Now let $r \in \mathcal{S}$ be *any* condition extending p_0 and in D. Let $\gamma = \mathit{last}(r)$. Since r fulfills Γ_0, for some $g \in H_\gamma$, g bounds r. But this contradicts the fact that g is bad.

4.1 Specialization, while Safeguarding Suslin trees

Suppose that we care about a Suslin tree S, and wish to specialize an Aronszajn tree T while keeping S Suslin. Obviously, this is not always possible: for example if S *is* T, or if they contain isomorphic uncountable subtrees. We will show in this section that, if T remains Aronszajn even after the addition of a cofinal branch to S, then the poset $\mathcal{S}(\mathsf{T})$ specializes T while keeping S Suslin.

Theorem 4.6 *Let* S *be a Suslin tree, and* T *be an Aronszajn tree such that* $\|\mathsf{T} \textit{ is Aronszajn}\|^{\mathsf{S}} = 1$. *Then* $\|\mathsf{S} \textit{ is Suslin}\|^{\mathcal{S}(\mathsf{T})} = 1$.

Proof: The forcing poset $\mathcal{S} = \mathcal{S}(\mathsf{T})$ was defined in the previous subsection and shown there to be proper. To prove the theorem we let $\underset{\sim}{D}$ be a name in $\mathcal{S}$ forcing of a dense open subset of the tree S. We will find a condition $p \in \mathcal{S}$ (extending an arbitrarily given condition in $\mathcal{S}$) such that for some $\mu < \omega_1$, $p \Vdash_{\mathcal{S}} \mathsf{S}_\mu \subseteq \underset{\sim}{D}$. This is enough to show that $\mathcal{S}$ does not destroy the Susliness of S. The framework for the construction of p is similar to the one for showing the properness of $\mathcal{S}$, and the following Lemma suffices for the proof of the Theorem.

Lemma 4.7 *Let S and T be as in the Theorem. Let $\underset{\sim}{D}$ be a name in $\mathcal{S} = \mathcal{S}(T)$ forcing of a dense open subset of S. Let $M \prec H(\kappa)$ be a countable elementary substructure, containing $\mathsf{T}, \mathsf{S}, \underset{\sim}{D}$, and let $p_0 \in \mathcal{S} \cap M$ be a condition. Let $\mu = \omega_1 \cap M$, and $h_0 : \mathsf{T}_\mu \to \mathbb{Q}$ be a finite function which bounds p_0. For any $b \in \mathsf{S}_\mu$ there is an extension $p \in \mathcal{S} \cap M$ of p_0 such that h_0 bounds p, and $p \Vdash_{\mathcal{S}} b \in \underset{\sim}{D}$.*

Proof: Assume that this Lemma does not hold. Let M, p_0, h_0 etc. be a counterexample. Put $\mu_0 = \mathit{last}(p_0)$, and $g_0 = h_0 \lceil \mu_0$.

The Suslin tree S is a c.c.c. forcing notion which adds no new countable sets. We are going to define first a *name* $\underset{\sim}{B}$, in S forcing, of an uncountable tree of 'bad' functions, and derive a promise Γ out of this $\underset{\sim}{B}$, a promise which, when adjoined to p_0, will give the desired contradiction.

Forcing with S, extend $b \in \mathsf{S}_\mu$ to a (generic) branch G of S, and let $V[G]$ be the extension of the universe V thus obtained. We have:

$$M[G] \prec H(\kappa)[G] = H(\kappa)^{V[G]}.$$

In $V[G]$, and hence in $M[G]$, T is still an Aronszajn tree by the assumption of the Theorem. The following definition is carried out in $V[G]$, but all its parameters are in $M[G]$:

Definition 4.8 A finite function $h : \mathsf{T}_\gamma \to \mathbb{Q}$ is bad iff:
1. $\mu_0 \le \gamma < \omega_1$, and $h\lceil\mu_0 = g_0$.
2. Whenever $p \in \mathcal{S}$ extends p_0 and $\gamma \ge lastp$ and $G_\gamma = e$, if $p \Vdash_{\mathcal{S}} e \in \underset{\sim}{D}$ then h does not dominate p. (Recall that G_γ is the unique node in $G \cap \mathsf{S}_\gamma$.)

For any $\mu_0 \le \gamma < \mu$, $h_0\lceil\gamma$ is bad. (If not, by elementarity of $M[G]$, there is, in M, an extension p of p_0, of height γ and such that $h_0\lceil\gamma$ bounds p and $p \Vdash_{\mathcal{S}} (G_\gamma) \in \underset{\sim}{D}$. But then, as $b > G_\gamma$, $p \Vdash_{\mathcal{S}} b \in \underset{\sim}{D}$, in contradiction to our assumption.) Hence the set of bad functions is uncountable.

Obviously, if h, of *height* γ, is bad and $\mu_0 \le \gamma' < \gamma$, then $h\lceil\gamma'$ is bad too.

We know how to find (in $M[G]$) an ordinal $\mu_0 \le \beta < \omega_1$, and a collection $B(\gamma)$, $\beta \le \gamma < \omega_1$, such that $B(\gamma)$ is a set of bad functions of height γ, and
1. For $\beta \le \gamma_0 < \gamma_1 < \omega_1$, $B(\gamma_0) = B(\gamma_1)\lceil\gamma_0$,
2. $B(\beta)$ is dispersed. (See Definition 4 (4)(b), and Lemma 1.1.)

Let $\underset{\sim}{B} \in M$ be a name of B in V^{S}, and let $b_0 < b$ be a condition in S which forces these properties of B. In particular, b_0 forces "all functions in $\underset{\sim}{B}(\gamma)$ are bad".

Now, back in V, we define the promise Γ. For every countable $\gamma \ge \beta$, $\Gamma(\gamma)$ is the collection of all requirements H of height γ such that $\|H = \underset{\sim}{B}(\gamma)\|^{\mathsf{S}} > 0$. Again $\Gamma \in M$. Since S is a c.c.c. poset, $\Gamma(\gamma)$ is countable, and since S adds no new countable sets, $\Gamma(\gamma)$ is non-empty (some condition in S above b_0 'describes' $\underset{\sim}{B}(\gamma)$) and $\Gamma(\gamma)$ is countable.

Since g_0 is a basis of Γ, and g_0 bounds p_0, there is an extension p_1 of p_0, in $\mathcal{S} \cap M$, which includes Γ. (See the Addition of Promises Lemma 4.4.)

Next, find a node $d \in \mathsf{S}$, with $b_0 < d$, such that $p_2 \Vdash_{\mathcal{S}} d \in \underset{\sim}{D}$ for some extension p_2 of p_1 with $p_2 \in \mathcal{S} \cap M$. This is possible since $\underset{\sim}{D}$ is assumed to be a name in $V^{\mathcal{S}}$ such that $\|\underset{\sim}{D}$ is dense in $\mathsf{S}\|^{\mathcal{S}} = 1$.

Let $\gamma = last(p_2)$, and let $d_1 > d$ be a node in S which forces "$H = \underset{\sim}{B}(\gamma)$" for some requirement H of height γ. Then $H \in \Gamma(\gamma)$ and so some $h \in H$

bounds p_2 (as p_2 fulfills Γ). But $d_1 \Vdash_S h$ *is bad*, contradicts $p_2 \Vdash_{\mathcal{S}} d \in \underset{\sim}{D}$.

5 α-properness and ω_1-D-completeness of $\mathcal{S}$

In chapter V of Shelah [1982] (Sections 3,5 and 6) the notions of α-properness and $\mathbb{D}$ -completeness are defined, and an $< \omega_1$-proper, simple $\mathbb{D}$ -complete forcing which specializes an Aronszajn tree is described. Section 7 there shows that the iteration of such forcings adds no reals. In chapters VII and VIII different notions of chain conditions are introduced: the $\aleph_2$-e.c.c and the $\aleph_2$-p.i.c. Any of them can be used to show that our iterations satisfy the $\aleph_2$-c.c. (The second is particularly useful if $2^{\aleph_1} > \aleph_2$). We shall review here these definitions, but will not give proofs for the preservation theorems which may be found in the Proper Forcing book.

To use the theory of proper forcings which add no reals we will show that (1) the specializing poset $\mathcal{S} = \mathcal{S}(\mathsf{T})$ is α-proper for every $\alpha < \omega_1$, and that (2) for some simple ω_1-completeness system $\mathbb{D}$, $\mathcal{S}$ is $\mathbb{D}$ -complete.

By "a tower of *length* $\alpha + 1$ of substructures of $H(\lambda)$" we mean here a sequence $\bar{N} = \langle N_i \,|\, i \leq \alpha \rangle$ of countable $N_i \prec H(\lambda)$ such that

1. $\bar{N}$ is continuously increasing. ($N_\delta = \bigcup_{i<\delta} N_i$, for limit $\delta \leq \alpha$).
2. $\langle N_j \,|\, j \leq i \rangle \in N_{i+1}$.

Definition 5.1 P is α-proper ($\alpha < \omega_1$), iff for every large enough λ and tower $\langle N_i \,|\, i \leq \alpha \rangle$ of countable substructures of $H(\lambda)$ of *length* $\alpha+1$, if $P \in N_0$ and $p \in P \cap N_0$, then there is an extension q of p in P such that q is an (N_i, P)-generic condition *for every* $i \leq \alpha$.

Theorem 5.2 *$\mathcal{S}$ is α-proper for every $\alpha < \omega_1$.*

We only *indicate* the proof since there is not much to say which was not said for the case $\alpha = 1$. The proof is by induction on α. The case of a successor ordinal is an obvious application of the inductive assumption and the properness of $\mathcal{S}$. In case α is a limit ordinal, given a tower $\langle N_i \,|i \leq \alpha\rangle$ as in the definition, pick an increasing ω-sequence $i_n < \alpha$, $n < \omega$, converging to α. We will define an increasing sequence $p_n \in N_{i_n+1}$ such that p_n is (N_j, P)-generic condition for every $j \leq i_n$. The inductive assumption and the assumption that $\langle N_k \,|k \leq i_n\rangle \in N_{i_n+1}$ are used to get p_n in the elementary substructure N_{i_n+1}. We must be careful so that $f \stackrel{\text{def}}{=} \bigcup_{n<\omega} f(p_n)$ is bounded on every branch of $\mathsf{T}\lceil\alpha$ determined by points in T_α, and that f fulfills every requirement in $\Gamma(\alpha)$ for $\Gamma = \Gamma(p_n)$, $n < \omega$. But we learned how to do it when proving properness of $\mathcal{S}$.

The $\mathbb{D}$ -completeness of $\mathcal{S}$ is equally simple, if only the definition is clear. Let us review it on the informal level first.

Think on the difference between the poset $\mathcal{P}$, for adding a new subset to ω_1 with countable conditions on the one hand, and the poset T, a Suslin tree, on the other hand. Both posets add no new countable sets, but while posets like $\mathcal{P}$ can be iterated without adding reals, an iteration of Suslin trees can add a new real (see Jensen and Johansbråten [1974]).

Pick a countable $M \prec H(\lambda)$ with $\mathcal{P}, \mathsf{T} \in M$ and look for $\mathcal{P}$-generic and T-generic filters $G_{\mathcal{P}}$ and G_{T} over M which have an upper bound in $\mathcal{P}$, and in T (this is what it takes to show "no new countable sets are added"). While $G_{\mathcal{P}}$ can be defined, in a sense, from within M; the definition of G_{T} requires knowing T_α. To clearly see this difference, let $\Pi : M \to \overline{M}$ be the transitive collapse of M onto the transitive structure $\overline{M}$. If we only have $\overline{M}$ at hand (and a countable enumeration of $\overline{M}$) then we can define a $\mathcal{P}$-generic filter over $\overline{M}$, and any such filter has an upper bound in $\mathcal{P}$. However, for T the situation is radically different: even though any branch of $\mathsf{T}\cap M$ is $\overline{M}$-generic, there is no way to know which branches have an upper bound in T, unless T_α is given to us.

For the poset $\mathcal{S}(\mathsf{T})$ (T now an Aronszajn tree) the situation is subtly in between $\mathcal{P}$ and T: It seems that we need to know T_α (and more) to define

M-generic filters over $\mathcal{S}$, but in fact this is less crucial: there is room for some errors. Let us make this more precise in the following. Recall the properness proof, and suppose that $M \prec H(\lambda)$ and the collapse $\Pi : M \to \overline{M}$ are given. We seek to find a generic filter G over $\overline{M}$ such that $\Pi^{-1}G$ has an upper bound in $\mathcal{S}$. Besides $\overline{M}$, the only parameters of importance were T_μ $(\mu = \omega_1 \cap M)$ and the function Γ which assigns to each $p \in \mathcal{S} \cap M$, the countable set $\Gamma(p)(\mu)$ of requirements (in the sense of T'_μ) of height μ. Suppose that not the real T_μ and Γ are given, but just a countable set of cofinal branches of $\mathsf{T} \cap \overline{M}$ (called T'_μ) and any function Γ' such that $\Gamma'(p)$ is a countable collection of requirements of height μ, and for every $p \in \mathcal{S} \cap M$ and $\beta \in \omega_1 \cap M$, $\Gamma'(p)(\beta) = \{ X \lceil \beta \mid X \in \Gamma'(p)\}$. Then the increasing, $\overline{M}$-generic sequence of conditions, $p_i \in \mathcal{S} \cap \overline{M}$, $i < \omega$, could be defined to give a filter G. Of course, if T'_μ and Γ' are arbitrary, then we cannot be sure that Π^{-1} of the filter G thus obtained has an upper bound in $\mathcal{S}$. *However*, the following observation comes to our rescue: given a countable collection $\{ \langle \mathsf{T}^i_\mu, \Gamma^i \rangle \mid i < \omega\}$, it is possible to find a generic G which is good for *every* $\langle \mathsf{T}^i_\mu, \Gamma^i \rangle$. 'Good' in the sense that if some $\langle \mathsf{T}^i_\mu, \Gamma^i \rangle$ were the real thing, then G would have an upper bound in the external $\mathcal{S}$. This is the essence of the notion of simple $\mathbb{D}$ -completeness. For completeness, we give now the definition from chapter V of Shelah's Proper Forcing [1982]. The reader can then see that $\mathcal{S}$ is indeed $\mathbb{D}$ -complete for an ω-completeness simple system. The theory developed there shows that the countable support iteration of α-proper $(\alpha < \omega_1)$ and simple $\mathbb{D}$ -complete posets adds no new countable sets.

Definition of $\mathbb{D}$ -completeness. (See Definitions 5.2, 5.3 and 5.5 in Shelah [1982]). For any structure N, let $\pi : N \to \bar{N}$ denote the Mostowski collapse of N to a transitive structure. When enough set-theory is present in $\bar{N}$, the forcing relation can be defined in $\bar{N}$. So, given a poset $\bar{P} \in \bar{N}$, if $\bar{N}$ is countable, an $\bar{N}$-generic filter over $\bar{P}$ can be found, and the generic extension $\bar{N}[G]$ can be formed. We let

$Gen(\bar{N}, \bar{P}, \bar{p}) = \{G \subset \bar{P} \mid G \textit{ is } \bar{N} - \textit{generic filter over } \bar{P}, \textit{ and } \bar{p} \in G\}.$

The function $\mathbb{D}$ is called an $\aleph_1$-completeness system iff
For every countable transitive model $\bar{N}$ *(of enough set-theory) and* $\bar{p} \in \bar{P} \in \bar{N}$, $\mathbb{D}(\bar{N}, \bar{P}, \bar{p})$ *is a family of subsets of* $Gen(\bar{N}, \bar{P}, \bar{p})$ *such that every intersection of countably many sets in that family is non-empty.*

Thus if $G \in A \in \mathbb{D}(\bar{N}, \bar{P}, \bar{p})$ then G is an $\bar{N}$-generic filter over $\bar{P}$ with $\bar{p} \in G$, and if $A^i \in \mathbb{D}(\bar{N}, \bar{P}, \bar{p})$ then $\cap_{i<\omega} A^i$ is non-empty.

Given a completeness system $\mathbb{D}$, we say that the poset P is $\mathbb{D}$-complete iff for some large enough κ the following holds: For every $N \prec H(\kappa)$, with $\bar{P} \in N$ and for every $p \in P$, let $\pi : N \to \bar{N}$ be the transitive collapse of N; put $\bar{P} = \pi(P)$, $\bar{p} = \pi(p)$. There is some $A \in \mathbb{D}(\bar{N}, \bar{P}, \bar{p})$ such that for every $G \in A$:
$\pi^{-1}(G) = \{\pi^{-1}(g) \mid g \in G\}$ contains an upper bound in P.

Finally, let us say that the completeness system $\mathbb{D}$ is *simple* iff it is given by a formula $\psi(G, \bar{P}, \bar{p}, x)$ in the following way:
$\mathbb{D}(\bar{N}, \bar{P}, \bar{p}) = \{A_x \mid x \subset \bar{N}\}$, where
$A_x = \{G \in Gen(\bar{N}, \bar{P}, \bar{p}) \mid \langle \bar{N} \cup \mathcal{P}(\bar{N}), \in\rangle \models \psi(G, \bar{P}, \bar{p}, x)\}$.

In our case, the parameter x describes T_α and the function $p \longmapsto \Gamma(p)(\alpha)$, where $\alpha = \omega_1^{\bar{N}}$.

As for the $\aleph_2$-chain condition of $\mathcal{S}$, it follows from CH by the obvious remark that if two conditions have the same specializing function (but different promises) then they are compatible. In Chapter 8 of Shelah [1982] the notion of $\aleph_2$-p.i.c ($\aleph_2$ proper isomorphism condition) is defined, and it is shown that countable support iteration of length ω_2 of such posets satisfies the $\aleph_2$-c.c. if CH is assumed. Our posets $\mathcal{S}$ clearly satisfy the $\aleph_2$-p.i.c. and hence that result may be applied to conclude that the $\aleph_2$ chain condition holds for the iteration.

6 Models with few Suslin trees

Suppose S and all the derived trees of S are Suslin trees. We shall find now a generic extension in which the only Suslin trees are S and its derived trees, and such that no new countable sets are added by this extension. The extension is obtained as an iteration of length $2^{\aleph_1} = \aleph_2$ of posets of type $\mathcal{S}(\mathsf{T})$ described as follows.

By the result of Section 3, we know that if S and its derived trees remain Suslin at each stage of the iteration, then this also holds for the final limit of the iteration. We know that no new countable sets are added by the iteration of $\mathcal{S}(\mathsf{T})$ forcings, and that the $\aleph_2$-chain condition holds. The definition of the iteration is such that if, for some $\alpha < \omega_2$, $\mathcal{P}_\alpha$ is defined, then $\mathcal{P}_{\alpha+1}$ is obtained as an iteration $\mathcal{P}_\alpha * \mathcal{Q}_\alpha$ where (in $V^{\mathcal{P}_\alpha}$) $\mathcal{Q}_\alpha$ has the form $\mathcal{S}(\mathsf{T})$ for an 'appropriate' Aronszajn tree (T is appropriate if for any derived tree S^1 of S, $\|\mathsf{T}$ is Aronszajn$\|^{\mathsf{S}^1} = 1$). Then, by Theorem 4.6, $\|$all derived trees of S are Suslin $\|^{\mathcal{S}(T)} = 1$.

When care is taken of all appropriate T as above, the final extension $V[G]$ satisfies

1. S and its derived trees are Suslin.

2. For any Aronszajn tree T, either T is special, or for some derived tree S^1 of S, $\|\mathsf{T}$ is not Aronszajn$\|^{\mathsf{S}^1} > \mathbf{0}$.

The latter possibility implies, as the following Lemma shows, that T contains a club-isomorphic copy of a derived tree of S.

Lemma 6.1 *Assume that* S *and its derived trees are all Suslin. Suppose* T *is an Aronszajn tree, and* $\|\mathsf{T}$ *is not Aronszajn*$\|^{\mathsf{S}^1} = 1$ *for a derived tree* S^1 *of* S. *Assume further that* S^1 *is of least dimension with this property. Then* S^1 *is embeddable on a club set into* T.

Proof: Let us fix first some notation. S^1 has the form $\mathsf{S}_{a_1} \times \ldots \times \mathsf{S}_{a_n}$ for some n-tuple $\langle a_1, \ldots, a_n \rangle$ of distinct elements of S_γ for some $\gamma < \omega_1$. n is called the dimension of S^1. Now, when we say that $e \in \mathsf{S}^1$ is of the form $e = \langle e_1, \ldots, e_n \rangle$ it is assumed that $e_i > a_i$ in S.

For this Lemma, we assume that any node of limit hight in T is determined by its predeccessors. Let b be a name in V^{S^1} such that

$$\|b \text{ is a cofinal branch in } \mathsf{T}\|^{\mathsf{S}^1} = 1.$$

For any $e_1 \in \mathsf{S}^1$ and $\alpha < \omega_1$ there is an extension e_2 of e_1 which determines $b_\alpha = b \cap \mathsf{T}_\alpha$. That is, for some $x \in \mathsf{T}_\alpha$, $e_2 \Vdash_{\mathsf{S}^1} x \in b$. The set D_α of all conditions in S^1 which thus determine b_α is a dense open set in S^1. Since S^1 is Suslin, there is a club set $E \subseteq \omega_1$ of limit ordinals such that for $\alpha \in E$, if $e \in (\mathsf{S}^1)_\alpha$ then $e \in D_\beta$ for all $\beta < \alpha$. That is, e determines b_β for all $\beta < \alpha$. But then e must determine b_α as well, since there is a *single* node in T_α above all those determined b_α's. So, we have that for $\alpha \in E$, $(\mathsf{S}^1)_\alpha \subseteq D_\alpha$; hence for every $e \in (\mathsf{S}^1)_\alpha$ there is some $f(e) \in \mathsf{T}_\alpha$ with $e \Vdash_{\mathsf{S}^1} b_\alpha = f(e)$. As we will see, f is an embedding of S^1 on some club into T. Clearly f is an order preserving map of $\mathsf{S}^1 \restriction E$ into $\mathsf{T} \restriction E$. We will find a club set $D \subseteq E$ such that, on $\mathsf{S}^1 \restriction D$, f is one-to-one. The basic observation is that, as T is Aronszajn (and any node in S^1 has extensions to every higher level), every $e \in \mathsf{S}^1 \restriction E$ has two extensions, e_1 and e_2 such that $f(e_1) \neq f(e_2)$. The following is a slight strengthening of this, which is obtained from the minimality of the dimension n of S^1.

Claim: For every $e \in \mathsf{S}^1$, and for every set of indices $h \subset \{1, \ldots, n\}$ (strict inclusion) there are two extensions, $e' = \langle e'_1, \ldots, e'_n \rangle$ and $e'' = \langle e''_1, \ldots, e''_n \rangle$ of e, such that $f(e') \neq f(e'')$; and $e'_i = e''_i$ for $i \in h$.

Proof: Suppose this is not so, and for some $h = \{h(1), \ldots, h(k)\} \subseteq \{1, \ldots, n\}$ with $k < n$, for some $e = \langle e_1, \ldots, e_n \rangle$ for every two extensions e' and e'' of e with the same restriction on h, $f(e') = f(e'')$. This means that restricted to S^1_e, the function f actually depends on $\mathsf{S}^2 = \mathsf{S}_{e_{h(1)}} \times \ldots \times \mathsf{S}_{e_{h(k)}}$.

This enables us to define a name, in S^2 forcing, of a branch in T. But the dimension k of S^2 contradicts the minimality of n.

Now the proof of the Lemma can be concluded by showing that the embedding f defined above is one-to-one on a club set. This follows from the Claim since not only S^1 but any other derived tree of S is Suslin. Take for example a countable elementary substructure, M, of some H_κ, and put $\delta = M \cap \omega_1$. We claim that if $e^1 \neq e^2$ are in $(\mathsf{S}^1)_\delta$ then $f(e^1 \neq f(e^2)$. Confusing sequences with sets, put $e = e^1 \cup e^2$; then for some k with $n < k \leq 2n$, e is a k-tuple. e 'is' in fact an M-generic branch of a derived tree of S of dimension k. What the Claim implies is that for a dense open set of k-tuples of the form $e' \cup e''$ in that derived tree, $f(e') \neq f(e'')$. Since e is in that dense set, $f(e') \neq f(e'')$.

The generalization of our discussion to any collection of trees poses no problems. Suppose we are given a collection $\mathcal{U}$ of Suslin trees such that if $\mathsf{S} \in \mathcal{U}$ then all derived trees of S are Suslin. Assume: $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Then iterate $\mathcal{S}(\mathsf{T})$ posets, just as before, so that all trees in $\mathcal{U}$ and their derived trees remain Suslin. We know that this is possible, for any Aronszajn tree T, unless $\|\mathsf{T}$ is not Aronszajn$\|^{\mathsf{S}^1} > 0$ for some S^1 which is a derived tree of a tree in $\mathcal{U}$. In such a case, we know that T must contain a restriction to a club set of a derived tree of some $\mathsf{S} \in \mathsf{T}$.

7 The uniqueness of simple primal Suslin sequences

This section sets the preliminaries needed to prove the main theorem: the notions of *simple* and *primal* sequences of Suslin trees are defined, and the uniqueness of such sequences is proved. Using this material and the machinery developed to construct Suslin trees and to specialize them at will, the Encoding Theorem will be easily demonstrated in the following section.

We will deal here not only with ω_1-sequences of Aronszajn trees, but also with I-sequences, $\mathcal{T} = \langle \mathsf{T}^\zeta \,|\, \zeta \in I \rangle$, of Aronszajn trees, where I is an ω_1-like set of indices.

A linear order $(I, <)$ is said to be ω_1-like iff it is uncountable but all proper initial segments are countable. In this paper we need a slightly stronger version, and add to these requirements of ω_1-like that any point has a successor, and that a first element exists.

We say that $a \in I$ is a 'limit' point if it is not a successor (so the first element is a limit). A point $a \in I$ is said to be 'even' iff it is a limit or it has the form $\delta + n$, where δ is a limit and $n < \omega$ is an even integer. Similarly 'odd' points of I are defined.

We will call the members of I 'indices', since this is how they will be used. In some cases, I is or is isomorphic to ω_1, but in general an ω_1-like order need not to be well-founded. Indeed, an important point of our argument is that, in some universe, the Magidor-Malitz quantifiers can force I to be well-founded.

Let $\mathcal{T} = \langle \mathsf{T}^\zeta \,|\, \zeta \in I \rangle$ be a sequence of Aronszajn trees. Recall that for $d \in [I]^{<\omega} - \{\emptyset\}$ (a finite non empty subset of I), $\mathsf{T}^d = \bigcup_{\zeta \in d} \mathsf{T}^\zeta$ is the disjoint union of the Aronszajn trees with indices in d. A derived tree of T^d is thus a product of derived trees of the T^ζ's. One of these derived trees is $\mathsf{T}^(d) = \times_{\zeta \in d} \mathsf{T}^\zeta$.

Let su be a collection of non-empty finite subsets of I which is closed under subsets, and let $\mathsf{sp} = [I]^{<\omega} - \{\emptyset\} - \mathsf{su}$ be the complement of su. (sp is closed under supersets.) We say then that $(\mathsf{su}, \mathsf{sp})$ is a pattern (over I)

Definition 7.1 We say that the I-sequence $\mathcal{T}$ of Aronszajn trees has the pattern $(\mathsf{su}, \mathsf{sp})$ if

1. For $d \in \mathsf{su}$, every derived tree of T^d is Suslin.
2. For $d \in \mathsf{sp}$, $\mathsf{T}^{(d)}$ is a special tree.

Definition 7.2

1. A collection $\mathcal{U}$ of Suslin trees is *primal* iff all derived trees of trees in $\mathcal{U}$ are Suslin, and for any Suslin tree A there exists some $\mathsf{S} \in \mathcal{U}$ such that a derived tree of S is club-embeddable into A.

2. The sequence $\mathcal{T}$ with Suslin-special pattern $(\mathsf{su}, \mathsf{sp})$ is called primal iff the collection $\mathcal{U} = \{\ \mathsf{T}^d \mid d \in \mathsf{su}\}$ is primal.

We may summarize our results obtained so far in the following:

Theorem 7.3 1. *Assume* $\diamond_{\omega_1}$. *Given any pattern* $(\mathsf{su}, \mathsf{sp})$ *over* ω_1*, there exists an* ω_1*-sequence of Aronszajn trees* $\mathcal{T} = \langle \mathsf{T}^\zeta \,|\zeta \in \omega_1 \rangle$ *with this pattern. (Section 2.3.)*
2. *Assume* $2^{\aleph_0} = \aleph_1$ *and* $2^{\aleph_1} = \aleph_2$*. Let* $\mathcal{U}$ *be a collection of Suslin trees such that for* $\mathsf{S} \in \mathcal{U}$ *all derived trees of* S *are Suslin as well. There is then an* $\aleph_2$*-c.c. generic extension which adds no new countable sets, and in which* $\mathcal{U}$ *is a primal collection of Suslin trees. (Section 6.)*

7.1 Simple Patterns

Let I be an ω_1-like order. We will have to refer to quadruples $\zeta_1 < \zeta_2 < \zeta_3 < \zeta_4$ of indices given in increasing order in I, with some simple properties called types. For example, $\bar{\zeta} = \langle \zeta_1, \ldots, \zeta_4 \rangle$ is of type $\langle$odd, even, odd, even$\rangle$ if ζ_1 and ζ_3 are odd, and ζ_2, ζ_4 are even. Similar notations are obvious.

Now we define when the pattern pair $(\mathsf{su}, \mathsf{sp})$ is said to be *simple*: If su contains all tuples in the columns Suslin, and sp contains all tuples in the column Special, in the following table. In case $I = \omega_1$, for each limit ordinal δ, pick a canonical well-order E_δ of ω of order-type δ. By standard encoding, we assume that $E_\delta \subseteq \omega$.

Suslin	Special
All triples, pairs, and singletons	All quintuples
$\langle$even, even, even, even$\rangle$	
$\langle$odd, odd, odd, odd$\rangle$	
$\langle$odd, even, odd, even$\rangle$	$\langle$even, odd, even, odd$\rangle$
$\langle$odd, even, even, odd$\rangle$	$\langle$odd, even, even, even$\rangle$
$\langle$limit δ, even, odd, odd$\rangle$	$\langle$non-limit even, even, odd, odd$\rangle$
$\langle\zeta,\ \zeta+1$, odd, odd$\rangle$	$\langle\alpha,\ \beta$,odd, odd$\rangle$where α and β have different parity, and $\alpha+1<\beta$.
$\langle\delta+1,\ \delta+3,\ \delta+4,\ \delta+(5+i)\rangle$ where $i\in E_\delta$, δ a limit	$\langle\delta+1,\ \delta+3,\ \delta+4,\ \delta+(5+i)\rangle$ where δ is limit and $i\notin E_\delta$.

In the table we used the sets $E_\delta\subseteq\omega$; these are required only in case I is well-ordered.

It should be checked that if $d\subseteq I$ appears in the Suslin column of the table, and $e\subseteq I$ appears in the special column, then $e\not\subseteq d$. The Suslin tuples are closed under subsets, and even after closing the the special tuples under supersets, disjoint sets are obtained.

The following theorem explains the use of simple sequences (sequences of trees with simple patterns). In fact, as the reader may find out, there are other notions of simplicity which can be used to derive the conclusion of the Theorem.

Theorem 7.4 (Unique Pattern) *Suppose that $\mathcal{T}=\langle \mathsf{T}^\zeta\,|\zeta\in\omega_1\rangle$, and $\mathcal{A}=\{\ \mathsf{A}^\zeta\mid\zeta\in I\}$ are ω_1 and ω_1-like sequences of Suslin trees with simple Suslin-special patterns. IF $\mathcal{T}$ is primal, then I is isomorphic to ω_1, and $\mathcal{T}$ and $\mathcal{A}$ have the* same *Suslin-special pattern.*

Proof: We are going to define an order isomorphism $d: I\to\omega_1$, and show that for every $\zeta\in I$, A^ζ contains a club-embedding of a derived tree of $\mathsf{T}^{d(\zeta)}$. This suffices to derive the equality of the patterns of $\mathcal{T}$ and $\mathcal{A}$. We shall use the following easy observations: If A is special and B is any tree, then

$\mathsf{A}\times\mathsf{B}$ is special too. If $h : \mathsf{A} \to \mathsf{B}$ is a club embedding of the tree A into B, then

1. B is special $\Rightarrow$ A is special (on a club set of levels and hence on all levels. See Devlin and Johnsbråten [1974]).

2. A is special $\Rightarrow$ B is not Suslin.

Let $(\mathsf{su}_1, \mathsf{sp}_1)$, $(\mathsf{su}_2, \mathsf{sp}_2)$ be the patterns of $\mathcal{T}$ and $\mathcal{A}$ respectively. Since $\mathcal{T}$ is assumed to be primal, for every Suslin tree A^ζ there is $d = d(\xi) \in \mathsf{su}_1$ such that A^ξ contains a club image of a derived tree of T^d. Our aim is to prove that

$d(\xi)$is a singleton, and d establishes an isomorphism of Ionto ω_1.

This will be achieved in the following steps.

(a) *There is no quintuple $\xi_1, \ldots, \xi_5$ such that, for all indices i, j, $d(\xi_i) = d(\xi_j)$.* Suppose, for the sake of a contradiction, that for some $d \in \mathsf{su}_1$, $d = d(\xi_i)$ for five indices $\xi_1, \ldots, \xi_5$. Then there are club embeddings, h_i, from derived trees of T^d into A^{ξ_i}, $1 \le i \le 5$. We may combine these embeddings into a club embedding of a derived tree of T^d into $\times_{1\le i\le 5}\mathsf{A}^{\xi_i}$. But since the product of a quintuple of trees is a special tree, this derived tree of T^d cannot be Suslin.

(b) *There are not uncountably many ξ's with $|d(\xi)| > 1$.* To see this, suppose the contrary, and let an uncountable set $X \subseteq I$ be such that for $\xi \in X$, $d(\xi)$ contains more than one element. We may assume that the finite sets $d(\xi), \xi \in X$, form a Δ-system, and that either all members of X are odd or all are even. Hence, for all quadruples $d \subset X$, A^d is Suslin. We will get the contradiction be considering the two possibilities for the Δ-system. If the core of the system is all of $d(\xi)$, i.e., $d = d(\xi_1) = d(\xi_2)$ for $\xi_1, \xi_2 \in X$, then a contradiction to (a) is obtained.

If the core of the system is strictly included in $d(\xi)$, for $\xi \in X$, then for a quadruple $\xi_1, \dots, \xi_4$ in X, $d = \bigcup_{1 \le i \le 4} d(\xi_i)$ contains ≥ 5 indices. Now $\times_{i \in d} \mathsf{T}^i$ is a special tree, and has a derived tree which is club embeddable into $\times_{1 \le i \le 4} \mathsf{A}^{\xi_i}$ which is a Suslin tree. This is clearly impossible since a derived tree of a special tree is special.

Now that we have proved that on a co-countable set $d(\xi)$ is a singleton, we proceed to show that $d(\xi)$ is a singleton for every ξ.

(c) *For every* $\xi \in I$, $|d(\xi)| = 1$. This will enable us to change notation and write $d(\xi) \in \omega_1$ (instead of $d(\xi) \subseteq \omega_1$). Assume, for some ξ_1, $|d(\xi_1)| \ge 2$. Suppose, for example, that ξ_1 is even. We can find (in the co-countable set of (b)) even indices ξ_2, ξ_3, ξ_4 such that $c = \bigcup_{1 \le i \le 4} d(\xi_i)$ contains ≥ 5 indices. Since $e = \{\, \xi_1, \dots, \xi_4\} \in \mathsf{su}_2$, all derived trees of A^e are Suslin, and in particular $B = \times_{i \in e} A^i$ is Suslin. On the other hand, there is an embedding of a derived tree of T^c on a club into B, but this is impossible as any such derived tree of T^c is special (as $| \, c \, | \ge 5$).

(d) *d is one-to-one.* Suppose that $\xi_1 < \xi_2$ but $d(\xi_1) = d(\xi_2)$. Consider the four possibilities for (ξ_1, ξ_2). (1) both are even, (2) both are odd, (3) ξ_1 is even and ξ_2 is odd, (4) ξ_1 is odd and ξ_2 is even.

In each case, it is possible to find $\zeta_1,\ \zeta_2$ such that $e = \{\xi_1, \xi_2\} \cup \{\zeta_1, \zeta_2\} \in \mathsf{sp}_2$. (For example, if both ξ_1 and ξ_2 are even, find odd $\zeta_1,\ \zeta_2$ such that $\xi_1 < \zeta_1 < \xi_2 < \zeta_2$.) Yet $t = \{\ d(\xi) \mid \xi \in e\}$ contains at most 3 indices, and a club embedding of a derived tree of T^t into $\times_{\xi \in e} \mathsf{A}^\xi$ is obtained. This contradicts the fact that the first tree is Suslin and the latter is special.

At this stage we don't know yet that d is order preserving; but as d is one-to-one from an ω_1-like order into an ω_1-like order, for any α, if $\beta > \alpha$ is sufficiently large, then $d(\beta) > d(\alpha)$. This simple remark is used below.

(e) ξ *is even if and only if* $d(\xi)$ *is even.* Let us first check this: Could it be that there are both uncountably many even ξ's with $d(\xi)$ odd, and uncountably many odd ξ's with $d(\xi)$ even? No, because in such a case (using the remark made above) we will find a quadruple $\langle \xi_1, \ldots, \xi_4 \rangle$ of type $\langle$even, odd, even, odd$\rangle$ with d-image of type $\langle$odd, even, odd, even$\rangle$. But this is impossible since the first type is in the special column, and the second in the Suslin column of the simplicity table.

It follows now that there are not uncountably many odd ξ's for which $d(\xi)$ is even. For otherwise, using our result above, on a co-countable set of even ξ's, $d(\xi)$ is even. Then a quadruple of type $\langle$even, odd, even, odd$\rangle$ has d-image of type $\langle$even, even, even, even$\rangle$. Again this is impossible. Similarly, there are no uncountable many even ξ's with $d(\xi)$ odd.

So there can be at most countably many changes of parity. In fact even if a single odd ξ_1 is with even $d(\xi_1)$, we get a contradiction by finding even ξ_2, ξ_3, ξ_4 so that $\langle \xi_1 \ldots \xi_4 \rangle$ is of type $\langle$odd, even, even, even$\rangle$, but its d-image is of type $\langle$even, even, even, even$\rangle$. Likewise, there is no even ξ with odd $d(\xi)$.

(f) d *is order preserving.* Since d is one-to-one, it is order preserving on an uncountable set. First we prove that if $\xi_1 < \xi_2$ is of type $\langle$even, odd$\rangle$ or of type $\langle$odd, even$\rangle$, then $d(\xi_1) < d(\xi_2)$. Assume this is not the case, and $d(\xi_1) > d(\xi_2)$. Then find ξ_3, ξ_4 such that $\langle \xi_1, \ldots, \xi_4 \rangle$ is of type $t_1 =\langle$even, odd, even, odd$\rangle$ or of type $t_2 =\langle$odd, even, even, odd$\rangle$ and such that only ξ_1 and ξ_2 change places.

Then, since the first two coordinates change their place, the d-image of this quadruple is (respectively) of type t_2 or t_1. But since t_1 and t_2 are in different columns, a contradiction is derived.

To see now that d is order preserving on *any* pair, pick $\xi_1 < \xi_2$ with the same parity. Then, since ξ_1+1 is of the other parity, $d(\xi_1) < d(\xi_1+1) < d(\xi_2)$.

(g) *δ is limit iff $d(\delta)$ is limit.*
This follows from the fact that ⟨limit, even, odd, odd⟩ is of type Suslin, while ⟨even but not limit, even, odd, odd⟩ is of type special.

(h) *For every ξ, $d(\xi+1) = d(\xi)+1$.* This is a consequence of the assumption that $\langle \xi, \xi+1, \text{odd, odd}\rangle$ is of type Suslin, while $\langle d(\xi), d(\xi+1), \text{ odd, odd}\rangle$ is of type special in case $d(\xi) + 1 < d(\xi + 1)$.

(i) *d is onto ω_1.* Since d preserves the order, I is well-ordered as well. Now, since for limit $\delta \in I$, $d(\delta)$ is limit, d maps the block $[\delta, \delta + \omega)$ onto the block $[d(\delta), d(\delta) + \omega)$; thus the order-type of δ is the same as the order-type of $d(\delta)$ (use E_δ). Hence $\delta = d(\delta)$ and d is onto.

So we have concluded that d is the identity, and thence the Unique Pattern Theorem.

8 The encoding scheme

We now have all the ingredients for the main result—to show how to encode subsets of ω_1 with simple patterns of Suslin sequences. The Magidor-Malitz quantifiers provide a concise way to describe our result.

Recall that the quantified formula $Qx, y\ \varphi(x, y)$ holds in a structure M if there is a set $\mathsf{A} \subseteq M$ of cardinality $\aleph_1$, and for any two distinct $x, y \in \mathsf{A}$, M satisfies $\varphi(x, y)$. (Magidor and Malitz [1977])

Let us see what can be stated in this language. We may say that the set A (a unary predicate) is uncountable, simply by stating $Qx, y(\mathsf{A}(x))$. To say that a linear order relation $\prec$ is ω_1-like we just say that it is uncountable, and any initial segment $\{\, x \mid x \prec y\}$ is countable (and the obvious first-order properties).

Let us accept a slightly freer notion of ω_1-trees: that of ω_1-like trees. These are trees with set of levels, not ω_1, but ω_1-like. The predecessors of a node in an ω_1-like tree form a countable chain which is not necessarily well-ordered. Since an ω_1-like order embedds ω_1, any ω_1-like tree contains an ω_1 tree.

The notions of ω_1-like Suslin trees, and ω_1-like special trees can be defined and characterized in the Magidor-Malitz logic. There is a sentence σ (in the language containing a binary relation $<$) such that $\mathsf{T} \models \sigma$ iff $(\mathsf{T}, <)$ is an ω_1-like Suslin tree. σ will simply state that T is an uncountable tree (with obvious properties), and there is no uncountable set of pairwise incomparable nodes in T.

Going one more step, we describe now a sentence, φ, which holds true only in simple ω_1-like sequences of ω_1-like Suslin trees $\langle \mathsf{T}^\zeta \,|\zeta \in I\rangle$ where I is an ω_1-like ordering $\prec$. For this we may have to introduce a one-place predicate symbol I, and a two-place predicate $\mathsf{T}(a, i)$ which, for a particular $i \in I$, describes the tree T^i. There is need also for a function which specializes those products which must be specialized. Giving more information on how φ looks may annoy the reader who can find these details for herself; so we stop and state our theorem.

Theorem 8.1 (Encoding Theorem) *There is a sentence ψ in the Magidor-Malitz logic which contains, besides the symbols $\prec$ etc. described above, a one place predicate $P(x)$ such that the following holds: Assuming $\diamond_{\omega_1}$, for any $X \subseteq \omega_1$,*

1. *There is a model $M \models \psi$ for which $I^M = \omega_1$ and $P^N = X$.*

2. *Assume $2^{\aleph_1} = \aleph_2$. There is a generic extension of the universe which adds no new countable sets and collapses no cardinals, and such that in this extension: If N is any model satisfying ψ, then I^N has order-type*

ω_1, and (identifying I^N with ω_1)

$$P^N = X$$

Proof: Let us first remark that the assumptions $\diamond_{\omega_1}$ and $2^{\aleph_1} = \aleph_2$ are not crucial, since these assumptions can be obtained with a forcing of size $2^{2^{\aleph_1}}$.

The sentence ψ is the simple-sequence sentence φ partially described above with the addition of

$$\forall \zeta \in I(\langle 3, 5, 6, \zeta \rangle \in \mathsf{su} \text{ iff } P(\zeta))$$

Given $X \subseteq \omega_1$ (assume that X contains only ordinals > 6), let $\langle \mathsf{su}, \mathsf{sp} \rangle$ be a simple pattern such that $X = \{ \zeta \in \omega_1 \mid \langle 3, 5, 6, \zeta \rangle \in \mathsf{su} \}$. Then use $\diamondsuit_{\omega_1}$ to construct an ω_1-sequence $\mathcal{T} = \langle \mathsf{T}^\zeta \,|\zeta \in \omega_1 \rangle$ of Suslin trees which has the pattern $\langle \mathsf{su}, \mathsf{sp} \rangle$. This takes care of *1.*

The generic extension is a countable support iteration of specializing posets which keeps every derived tree of $\{ \mathsf{T}^d \mid d \in \mathsf{su} \}$ Suslin but specializes any Aronszajn tree, when possible. As we showed (Section 6), an iteration of *length* $\aleph_2$ suffices to make $\mathcal{T}$ a primal sequence: Any Aronszajn tree A in the extension is either special or it contains a club-embedding of a derived tree of some T^d, $d \in \mathsf{su}$. Now we have the required uniqueness. Let N be a model of ψ. Then $I = I^N$ is an ω_1-like ordering, and for each $i \in I$, a Suslin tree A^i can be reconstituted from the set of a's for which $\mathsf{T}(a, i)$ holds in N. The sequence $\mathcal{A} = \langle \mathsf{A}^i \,|i \in I \rangle$ is simple and with pattern $\langle \mathsf{su}^N, \mathsf{sp}^N \rangle$. The Unique Pattern Theorem 7.4 can now be applied to $\mathcal{T}$ and $\mathcal{A}$ to yields $I^N \approx \omega_1$. And $\mathsf{su} = \mathsf{su}^N$. That is, the two sequences have the same pattern. From this it follows that $X = P^N$.

8.1 The complete proof of Theorem A

To obtain the Δ^2_2 encoding of any subset of $\mathbb{R}$, Theorem 8.1 is sufficient. However, the statement of Theorem A in the Preface is neater because it provides a categorical sentence, while Theorem 8.1 only establishes the uniqueness of P^N. The sentence ψ described above contains predicates and function symbols other than P, and they too must be encoded by the unique pattern sequence of trees.

To prove Theorem A, we "catch our tail" in the following way. Not only the tree sequence $\langle \mathsf{T}^\xi \lceil \alpha + 1 \mid \xi \in \alpha \rangle$ is constructed inductively, but so is the simple Suslin-special pattern itself. More precisely, at the limit α stage, we encode the countable structure so far defined (the trees and P etc.) as a subset of the ordinal-interval $(\alpha, \alpha + \omega)$, and we put $\langle 3, 5, 6, \zeta \rangle \in \mathsf{su}$ for $\xi \in (\alpha, \alpha + \omega)$ so that it encodes that countable structure. The categorical sentence ψ' tells us this fact as well. The proof continues just as before: any model M for ψ' determine a Suslin-special pattern which must be the unique such simple pattern, but it determines in turn M, and hence the uniqueness of M.

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