UNIVERSAL THEORIES CATEGORICAL IN POWER AND κ-GENERATED MODELS

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Abstract. We investigate a notion called *uniqueness in power* κ that is akin to categoricity in power κ, but is based on the cardinality of the generating sets of models instead of on the cardinality of their universes. The notion is quite useful for formulating categoricity-like questions regarding powers below the cardinality of a theory. We prove, for (uncountable) universal theories T, that if T is κ-unique for one uncountable κ, then it is κ-unique for every uncountable κ; in particular, it is categorical in powers greater than the cardinality of T.

It is well known that the notion of categoricity in power exhibits certain irregularities in "small" cardinals, even when applied to such simple theories as universal Horn theories. For example, a countable universal Horn theory categorical in one uncountable power is necessarily categorical in all uncountable powers, by Morley's theorem, but it need not be countably categorical.

Tarski suggested that, for universal Horn theories T, this irregularity might be overcome by replacing the notion of categoricity in power by that of freeness in power. T is *free in power* κ, or κ*-free*, if it has a model of power κ and if all such models are free, in the general algebraic sense of the word, over the class of all models of T. T is a *free theory* if each of its models is free over the class of all models of T. It is trivial to check that, for $\kappa > |T|$, categoricity and freeness in power κ are the same thing. For $\kappa \leq |T|$ they are not the same thing. For example, the (equationally axiomatizable) theory of vector spaces over the rationals is an example of a free theory, categorical in every uncountable power, that is ω-free, but not ω-categorical. Tarski formulated the following problem: Is a universal Horn theory that is free in one infinite power necessarily free in all infinite powers? Is it a free theory?

Baldwin, Lachlan, and McKenzie, in Baldwin-Lachlan [1973], and Palyutin, in Abakumov-Palyutin-Shishmarev-Taitslin [1973], proved that a countable ω-categorical universal Horn theory is ω_1-categorical, and hence categorical in all infinite powers. Thus, it is free in all infinite powers. Givant [1979] showed that a countable ω-free, but not ω-categorical, universal Horn theory is also ω_1-categorical, and in fact it is a free theory. Further, he proved that a universal Horn theory, of any cardinality κ, that is κ-free, but not κ-categorical, is necessarily a free theory.

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Independently, Baldwin-Lachlan, Givant, and Palyutin all found examples of countable ω_1-categorical universal Horn theories that are not ω-free, and of countable ω_1- and ω-categorical universal Horn theories that are not free theories.

Thus, Tarski's implicit problem remains: find a notion akin to categoricity in power that is regular for universal Horn theories, i.e., if it holds in one infinite power, then it holds in every infinite power.

One of the difficulties with the notions of categoricity in power and freeness in power is that they are defined in terms of the cardinality of the universes of models instead of the cardinality of the generating sets. This causes difficulties when trying to work with powers $< |T|$.

Let's call a model $\mathfrak{A}$ *strictly κ-generated* if κ is the minimum of the cardinalities of generating sets of $\mathfrak{A}$. We define a theory T to be *κ-unique* if it has, up to isomorphisms, exactly one strictly κ-generated model. For cardinals $\kappa > |T|$, the notions of κ-categoricity, κ-freeness, and κ-uniqueness coincide (in the case when T is universal Horn). When $\kappa = |T|$, we have

$$\kappa\text{-}categoricity \quad \Rightarrow \quad \kappa\text{-}freeness \quad \Rightarrow \quad \kappa\text{-}uniquenes \quad ,$$

but none of the reverse implications hold.

Givant [1979], p. 24, asked, for universal Horn theories T, whether κ-uniqueness is the regular notion that Tarski was looking for, i.e., (1) Does κ-uniqueness for one infinite κ imply it for all infinite κ? For countable T, he answered this question affirmatively by showing that ω-uniqueness is equivalent to categoricity in uncountable powers. For uncountable T, he provided a partial affirmative answer by showing that categoricity in power $> |T|$ implies κ-uniqueness for all infinite κ, and is, in turn, implied by κ-uniqueness when $\kappa = |T|$ and κ is regular. However, the problem whether κ-uniqueness implies categoricity in powers $> |T|$ when $\omega \leq \kappa < |T|$, or when $\omega < \kappa = |T|$ and κ is singular, was left open.

In this paper we shall prove the following:

Theorem. *A universal theory T that is κ-unique for some $\kappa > \omega$ is κ-unique for every $\kappa > \omega$. In particular, it is categorical in powers $> |T|$.*

It follows from the previous remarks that a universal Horn theory T which is κ-unique for some $\kappa > \omega$ is κ-unique for every $\kappa \geq \omega$. Thus, the only part of (1) that still remains open is the case when T is uncountable and ω-unique. A more general formulation of this open problem is the following:

Problem. *Is an ω-unique universal theory T necessarily categorical in powers $>$ $|T|$? In particular, is a countable ω-unique (or ω-categorical) universal theory ω_1-categorical?*

An example due to Palyutin, in Abakumov-Palyutin-Shishmarev-Taitslin [1973], shows that a countable universal theory categorical in uncountable powers need not be ω-unique. In fact, in Palyutin's example the finitely generated models are all finite, and there are countably many non-isomorphic, strictly ω-generated models. Thus, for universal theories, κ-uniqueness for some $\kappa > \omega$ does not imply ω-uniqueness.

To prove our theorem, we shall show that, under the given hypotheses, the theory of the infinite models of T is complete, superstable, and unidimensional, and that all sufficiently large models are a-saturated. Thus, we shall make use of some of the notions and results of stability theory that are developed in Shelah [1990] (see also Shelah [1978]). We will assume that the reader is acquainted with the elements of model theory and with such basic notions from stability theory as superstability, a-saturatedness, strong type, regular type, and Morley sequence. We begin by reviewing some notation and terminology, and then proving a few elementary lemmas.

The letters m and n shall denote finite cardinals, and κ and λ infinite cardinals. The cardinality of a set U is denoted by $|U|$. The set-theoretic difference of A and B is denoted by $A - B$. If ϑ is a function, and $\bar{a} = \langle a_0, \dots, a_{n-1}\rangle$ is a sequence of elements in the domain of ϑ, then $\vartheta(\bar{a})$ denotes the sequence $\langle \vartheta(a_0), \dots, \vartheta(a_{n-1})\rangle$. We denote the restriction of ϑ to a subset X of its domain by $\vartheta{\restriction}X$, and a similar notation is employed for the restriction of a relation. A sequence $\langle X_\xi : \xi < \lambda\rangle$ of sets is *increasing* if $X_\xi \subseteq X_\eta$ for $\xi < \eta < \lambda$, and *continuous* if $X_\delta = \bigcup_{\xi<\delta} X_\xi$ for limit ordinals $\delta < \lambda$.

We use German letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ to denote models, and the corresponding Roman letters $A, B, C, \ldots$ to denote their respective universes. If $\tau(x_0, \ldots, x_{n-1})$ is a term in a (fixed) language L for $\mathfrak{A}$, and if and $\bar{a}$ an n-termed sequence of elements in A, symbolically $\bar{a} \in {}^nA$, then the value of τ at $\bar{a}$ in $\mathfrak{A}$ is denoted by $\tau^{\mathfrak{A}}[\bar{a}]$, or simply by $\tau[\bar{a}]$. A similar notation is used for formulas. Suppose $\bar{a} \in {}^nA$ and $X \subseteq A$. The type of $\bar{a}$ over X (in $\mathfrak{A}$), i.e. the set of formulas in the language of $\langle \mathfrak{A}\,,\, x\,\rangle_{x\in X}$ that are satisfied by $\bar{a}$ in the latter model, is denoted by $tp^{\mathfrak{A}}(\bar{a}, X)$, or simply by $tp(\bar{a}, X)$, when no confusion can arise. The strong type of $\bar{a}$ over X (in $\mathfrak{A}$), i.e., the set of formulas in the language of $\langle \mathfrak{A}\,,\, b\,\rangle_{b\in A}$ that are almost over X and that are satisfied by $\bar{a}$, is denoted by $stp^{\mathfrak{A}}(\bar{a}, X)$, or simply by $stp(\bar{a}, X)$. If $p(\bar{x})$ is a strong type and $E \subseteq A$ is a base for p in $\mathfrak{A}$, then $p{\restriction}^*E$ denotes the set of formulas in p that are almost over E.

We write $\mathfrak{A} \subseteq \mathfrak{B}$ to express that $\mathfrak{A}$ is a submodel of $\mathfrak{B}$. The submodel of $\mathfrak{A}$ generated by a set $X \subseteq A$ is denoted by $\mathfrak{Sg}^{\mathfrak{A}}(X)$, or simply by $\mathfrak{Sg}(X)$, and its universe by $Sg(X)$. A model is *μ-generated* if it is generated by a set of cardinality μ, and *strictly μ-generated* if it is μ-generated, but not ν-generated for any $\nu < \mu$. Every model $\mathfrak{A}$ has a generating set of minimal cardinality, and hence is strictly μ-generated for some (finite or infinite) μ. If X is a generating set of $\mathfrak{A}$ of minimal cardinality, and Y is any other generating set of $\mathfrak{A}$, then there must be a subset Z of Y of power at most $|X| + \omega$ such that Z generates X, and hence also $\mathfrak{A}$. A set $X \subseteq A$ is *irredundant* (in $\mathfrak{A}$) if, for every $Y \subsetneqq X$ we have $\mathfrak{Sg}(Y) \neq \mathfrak{Sg}(X)$.

A model $\mathfrak{A}$ is an *n-submodel* of $\mathfrak{B}$, and $\mathfrak{B}$ an *n-extension* of $\mathfrak{A}$, in symbols $\mathfrak{A} \preccurlyeq_n \mathfrak{B}$, if for every Σ_n-formula $\varphi(x_0, \ldots, x_{k-1})$ in the language of $\mathfrak{A}$, and every $\bar{a} \in {}^kA$, we have $\mathfrak{A} \models \varphi[\bar{a}]$ iff $\mathfrak{B} \models \varphi[\bar{a}]$. A 0-submodel of $\mathfrak{B}$ is just a submodel in the usual sense of the word, and an elementary submodel—in symbols, $\mathfrak{A} \preccurlyeq \mathfrak{B}$—is just an n-submodel for each n. We write $\mathfrak{A} \prec \mathfrak{B}$ to express the fact that $\mathfrak{A}$ is a *proper* elementary submodel of $\mathfrak{B}$, i.e., $\mathfrak{A} \preccurlyeq \mathfrak{B}$ and $\mathfrak{A} \neq \mathfrak{B}$. A theory is *model complete* if, for any two models $\mathfrak{A}$ and $\mathfrak{B}$, we have $\mathfrak{A} \preccurlyeq \mathfrak{B}$ iff $\mathfrak{A} \subseteq \mathfrak{B}$. For any theory T, we denote by T_∞ the theory of the infinite models of T.

We shall always use the phrase *dense ordering* to mean a non-empty dense linear ordering without endpoints. It is well-known that the theory of such orderings admits elimination of quantifiers. Hence, in any such ordering $\mathfrak{U} = \langle U\,,\,<\rangle$, if $\bar{a}$ and $\bar{b}$ are two sequences in ${}^nU$ that are *atomically equivalent*, i.e., that satisfy the same atomic formulas, then they are *elementarily equivalent*, i.e., they satisfy the same elementary formulas.

Fix a linear ordering $\mathfrak{V} = \langle V\, , <\rangle$, and set $W = {}^{\omega\geq}V = \bigcup_{\mu\leq\omega} {}^{\mu}V$. Let $\lessdot$ be the (proper) initial segment relation on W, and, for each $\mu \leq \omega$, let P_μ be the set of elements in W with domain μ, i.e., $P_\mu = {}^{\mu}V$. There is a natural lexicographic ordering, $<$, on W induced by the ordering of V: $f < g$ iff either $f \lessdot g$ or else there is a natural number n in the domain of both f and g such that $f\restriction n = g\restriction n$ and $f(n) < g(n)$ in $\mathfrak{V}$. Take h to be the binary function on W such that, for any f, g in W, $h(f, g)$ is the greatest common initial segment of f and g. We shall call the structure $\mathfrak{W} = \langle W\, , <\, , \lessdot\, , P_\mu\, , h\,\rangle_{\mu\leq\omega}$ the *full tree structure over* $\mathfrak{V}$ *with* $\omega+1$ *levels*, or, for short, the *full tree over* $\mathfrak{V}$. Any substructure of the full tree over $\mathfrak{V}$ that is *downward closed*, i.e., closed under initial segments, is called a *tree over* $\mathfrak{V}$. A *tree* is any structure isomorphic to a tree structure over some ordering. A tree $\mathfrak{U}$ over a dense order $\mathfrak{V}$ is itself called *dense* if: (i) for every f in U with finite domain, say n, the set of immediate successors of f in U, i.e., the set of extensions of f in U with domain $n + 1$, is densely ordered by $<$ in $\mathfrak{U}$, or, put a different way, $\{g(n) : f \lessdot g\}$ is dense under the ordering inherited from $\mathfrak{V}$; (ii) every element in U with a finite domain is an initial segment of an element in U with domain ω. Just as with dense orderings, the theory of the class of dense trees admits elimination of quantifiers.

A model $\mathfrak{A}$ is κ-*homogeneous* if, for every cardinal $\mu < \kappa$ and every pair $\bar{a}, \bar{b} \in {}^{\mu}A$ of elementarily equivalent sequences, there is an automorphism ϑ of $\mathfrak{A}$ taking $\bar{a}$ to $\bar{b}$. It is well-known that, for regular cardinals κ, any model has κ-homogenous elementary extensions (usually of large cardinality). It follows from our remarks above that, if $\mathfrak{U}$ is a κ-homogeneous dense ordering or tree, and if $\bar{a}, \bar{b} \in {}^{\mu}U$ are atomically equivalent (where $\mu < \kappa$), then there is an automorphism of $\mathfrak{U}$ taking $\bar{a}$ to $\bar{b}$. A model $\mathfrak{A}$ is *weakly* ω-*homogenous* provided that, for every n, every pair of sequences $\bar{a}, \bar{a}' \in {}^{n}A$ that are elementarily equivalent, and every $b \in A$, there is a $b' \in A$ such that $\bar{a}\hat{\ }\langle b\rangle$ and $\bar{a}'\hat{\ }\langle b'\rangle$ are elementarily equivalent. It is well known that a countable, weakly ω-homogenous model is ω-homogeneous. Dense orderings and dense trees are always weakly ω-homogeneous.

We turn, now, to some notions from stability theory. A model $\mathfrak{A}$ is *a-saturated* if, for any strong type p (consistent with the theory of $\langle \mathfrak{A}\, , a\,\rangle_{a\in A}$), if $E \subseteq A$ is a finite base for p, then $p\restriction^{*}E$ is realized in $\mathfrak{A}$. We say that $\mathfrak{A}$ is *a-saturated in* $\mathfrak{B}$, in symbols $\mathfrak{A} \preccurlyeq_a \mathfrak{B}$, provided that $\mathfrak{A} \preccurlyeq \mathfrak{B}$ and that, for any strong type p of $\mathfrak{B}$ based on a finite subset E of A, if $p\restriction^{*}E$ is realized in $\mathfrak{B}$, then it is realized in $\mathfrak{A}$. A model $\mathfrak{A}$ is a-prime over a set X if (i) $X \subseteq A$ and $\mathfrak{A}$ is a-saturated; (ii) whenever $\mathfrak{B}$ is an a-saturated model elementarily equivalent to $\mathfrak{A}$ and such that $X \subseteq B$, then there is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$ that leaves the elements of X fixed. (Here, we assume that $\mathfrak{A}$ and $\mathfrak{B}$ are elementary substructures of some monster model). As is shown in Shelah [1990], Chapter IV, Theorem 4.18, for complete, superstable theories, the a-prime model over X exists and is unique, up to isomorphic copies over X. We shall denote it by $\mathfrak{Pr}_a(X)$. Let $\mathfrak{A}$ be a model and B, C subsets of A with $C \subseteq B$. A type $p(\bar{x})$ over B *splits over* C if there are $\bar{b}, \bar{c}$ from B such that $tp(\bar{b}, C) = tp(\bar{c}, C)$, and there is a formula $\varphi(\bar{x}, \bar{y})$ over C such that $\varphi(\bar{x}, \bar{b})$ and $\neg\varphi(\bar{x}, \bar{c})$ are both in p.

We now fix a universal theory T in a language L of arbitrary cardinality. In what follows let L' be an expansion of L with built-in Skolem functions, and T' any Skolem theory in L' that extends T. Without loss of generality, we may assume that, for every term $\tau(x_0, ..., x_{n-1})$ of L' there is a function symbol f of L' of rank

n such that the equation $f(x_0, ..., x_{n-1}) = \tau$ holds in T'.

For every infinite ordering $\mathfrak{U} = \langle U , \leq \rangle$ and (complete) Ehrenfeucht-Mostowski set Φ of formulas of L' compatible with T', there is a model $\mathfrak{M}' = EM(\mathfrak{U}, \Phi)$ of T'—called a (*standard*) *Ehrenfeucht-Mostowski model of* T'—such that $\mathfrak{M}'$ is generated by U (in particular, $U \subseteq M'$), and U is a set of Φ-indiscernibles in $\mathfrak{M}'$ with respect to the ordering of $\mathfrak{U}$, i.e., if $\varphi(x_0, ..., x_{n-1}) \in \Phi$, and if $\bar{a} \in {}^nU$ satisfies $a_i < a_j$ for $i < j < n$, then $\mathfrak{M}' \models \varphi[\bar{a}]$.

Shelah [1990], Chapter VII, Theorem 3.6, establishes the existence of certain generalizations of Ehrenfeucht-Mostowski models. Suppose that T is, e.g., nonsuperstable, and that the sequence $\langle \varphi_n(\bar{x}, \bar{y}) : n < \omega \rangle$ of formulas witnesses this nonsuperstability (see, e.g., Shelah [1990], Chapter II, Theorems 3.9 and 3.14). Then there is a *generalized* Ehrenfeucht-Mostowski set Φ from which we can construct, for every tree $\mathfrak{U}$, a *generalized* Ehrenfeucht-Mostowski model $\mathfrak{M}' = EM(\mathfrak{U}, \Phi)$ of T'. In other words, given a tree $\mathfrak{U}$, there a model $\mathfrak{M}'$ of T' generated by a sequence $\langle \bar{a}_u : u \in U \rangle$ of Φ-indiscernibles with respect to the atomic formulas of $\mathfrak{U}$; in particular, if $\bar{w}$ and $\bar{v}$ in ${}^nU$ are atomically equivalent in $\mathfrak{U}$, then $\langle \bar{a}_{w_0}, \dots, \bar{a}_{w_{n-1}} \rangle$ and $\langle \bar{a}_{v_0}, \dots, \bar{a}_{v_{n-1}} \rangle$ are elementarily equivalent in $\mathfrak{M}'$. Moreover, for w in $P_n^{\mathfrak{M}'}$ and v in $P_\omega^{\mathfrak{M}'}$, we have that $\mathfrak{M}' \models \varphi_n[\bar{a}_w, \bar{a}_v]$ iff $w \lessdot v$. In general, the sequences $\bar{a}_u$ may be of length greater than 1. However, to simplify notation we shall act as if they all have length 1, and in fact, we shall identify $\bar{a}_u$ with u. Thus, we shall assume that $\mathfrak{M}'$ is generated by U and that two sequences from U which are atomically equivalent in $\mathfrak{U}$ are elementarily equivalent in $\mathfrak{M}'$.

Here are some well-known facts about Ehrenfeucht-Mostowski models. Let $\mathfrak{U}, \mathfrak{V}$ be infinite orderings or trees, and $\mathfrak{M}' = EM(\mathfrak{U}, \Phi)$, $\mathfrak{N}' = EM(\mathfrak{V}, \Phi)$. (In the case of trees, we must assume that $\mathfrak{M}'$ and $\mathfrak{N}'$ exist.)

Fact 1. *If ϑ is any embedding of $\mathfrak{V}$ into $\mathfrak{U}$, then the canonical extension of ϑ to N' is an elementary embedding of $\mathfrak{N}'$ into $\mathfrak{M}'$.*

In particular,

Fact 2. *Any automorphism of $\mathfrak{U}$ extends to an automorphism of $\mathfrak{M}'$.*

Fact 3. *If $\mathfrak{V} \subseteq \mathfrak{U}$, then $\mathfrak{Sg}^{\mathfrak{M}'}(V) \preccurlyeq \mathfrak{M}'$, and $\mathfrak{Sg}^{\mathfrak{M}'}(V)$ is isomorphic to $\mathfrak{N}'$ via a canonical isomorphism that is the identity on V.*

Fact 4. *If $\mathfrak{U}$ is a linear order, then U is irredundant in $\mathfrak{M}'$. If $\mathfrak{U}$ is a tree, then for any element f and subset X of U, if f is not an initial segment of any element of X, then f is not generated by X in $\mathfrak{M}'$.*

We shall always denote the reduct of $\mathfrak{M}'$ to L by $\mathfrak{M}$, in symbols, $\mathfrak{M} = \mathfrak{M}' \restriction L$. Facts 1, 2, and 4 transfer automatically from $\mathfrak{M}'$ to $\mathfrak{M}$. However, $\mathfrak{M}$ is usually not generated by U. We shall therefore formulate versions of the above facts that apply to $\mathfrak{Sg}^{\mathfrak{M}}(U)$, and, more generally, to a collection of models that lie between $\mathfrak{Sg}^{\mathfrak{M}}(U)$ and $\mathfrak{Sg}^{\mathfrak{M}'}(U) = \mathfrak{M}'$.

Definition. *Suppose $\mathfrak{M}' = EM(\mathfrak{U}, \Phi)$, and $K \subseteq L' - L$.*

(i) *For each $f \in K$, let R_f be the relation corresponding to $f^{\mathfrak{M}'}$, i.e., $R_f[a_0, \dots , a_n]$ iff $f^{\mathfrak{M}}[a_0, \dots , a_{n-1}] = a_n$. Set $\mathfrak{M}_K = \langle \mathfrak{M}, R_f \rangle_{f \in K}$.*

(ii) *For each $V \subseteq U$ set*

$$K{*}V = V \cup \{f^{\mathfrak{M}'}[\bar{a}] : f \in K, \bar{a} \text{ from } V\} \quad .$$

Thus, the elements of $K*V$ are just the elements of V, together with the elements of $\mathfrak{M}'$ that can be obtained from sequences of V by single applications of f in K.

Lemma 1. *Let $\mathfrak{U}, \mathfrak{V}$ be dense orderings or dense trees, $\mathfrak{M}' = EM(\mathfrak{U}, \Phi)$, $\mathfrak{N}' = EM(\mathfrak{V}, \Phi)$, and $K \subseteq L' - L$. (Thus, in the case of dense trees, we postulate that generalized Ehrenfeucht-Mostowski models for Φ exist.)*

- (i) *U is a set of indiscernibles in $\mathfrak{Sg}^{\mathfrak{M}}(K*U)$ for the atomic formulas of Φ (with respect to the atomic formulas of $\mathfrak{U}$); in particular, if $\bar{a}$ and $\bar{b}$ are two atomically equivalent sequences in $\mathfrak{U}$, then they satisfy the same atomic formulas in $\mathfrak{Sg}^{\mathfrak{M}}(K*U)$.*
- (ii) *U is a set of indiscernibles in $\mathfrak{Sg}^{\mathfrak{M}}(K*U)$ for some complete Ehrenfeucht-Mostowski set compatible with T, i.e., if $\bar{a}$ and $\bar{b}$ are two atomically equivalent sequences in $\mathfrak{U}$, then they are elementarily equivalent in $\mathfrak{Sg}^{\mathfrak{M}}(K*U)$.*
- (iii) *If ϑ embeds $\mathfrak{V}$ into $\mathfrak{U}$, then the canonical extension of ϑ to $Sg^{\mathfrak{N}}(K*V)$ elementarily embeds $\mathfrak{Sg}^{\mathfrak{N}}(K*V)$ into $\mathfrak{Sg}^{\mathfrak{M}}(K*U)$.*

In particular:

- (iv) *Every automorphism of $\mathfrak{U}$ extends canonically to an automorphism of $\mathfrak{Sg}^{\mathfrak{M}}(K*U)$.*
- (v) *If $\mathfrak{V} \subseteq \mathfrak{U}$, then $\mathfrak{Sg}^{\mathfrak{M}}(K*V) \preccurlyeq \mathfrak{Sg}^{\mathfrak{M}}(K*U)$.*

The lemma continues to hold if we replace "$\mathfrak{M}$" and "$\mathfrak{N}$" everywhere by "$\mathfrak{M}_K$" and "$\mathfrak{N}_K$" respectively.

Proof. Since satisfaction of atomic formulas is preserved under submodels, part (i) is trivial. Part (i) directly implies (iv), and part (v) implies (iii). Thus, we only have to prove (ii) and (v). We begin with (v).

First, assume that $\mathfrak{U}$ is ω-homogeneous. We easily check that the Tarski-Vaught criterion holds. Here are the details. Suppose $\bar{a}$ is an n-termed sequence from $\mathfrak{Sg}(K*V)$ satisfying $\exists x\varphi(x, \bar{y})$ in $\mathfrak{Sg}(K*U)$; say, b is an element of $\mathfrak{Sg}(K*U)$ such that $\langle b\rangle\hat{\ }\bar{a}$ satisfies $\varphi(x, \bar{y})$. Let $\bar{c}$ and $\bar{d}$ be sequences of elements from $\mathfrak{V}$ and $\mathfrak{U}$ that generate $\bar{a}$ and b respectively. Thus, there are $f_0, \ldots, f_{k-1}$ and $g_0, \ldots, g_{l-1}$ in K, and a term $\sigma(\bar{z}, \bar{w})$ and terms $\tau_0(\bar{u}, \bar{v}), \ldots, \tau_{n-1}(\bar{u}, \bar{v})$ in L such that (adding dummy variables to simplify notation)

$$a_i = \tau_i\big[f_0[\bar{c}], \ldots, f_{k-1}[\bar{c}], \bar{c}\big] \quad \text{for } i < n, \text{ and} \quad b = \sigma\big[g_0[\bar{d}], \ldots, g_{l-1}[\bar{d}], \bar{d}\big] \quad .$$

Hence, the sequence

$$\big\langle\, \sigma\big[g_0[\bar{d}], \ldots, g_{l-1}[\bar{d}], \bar{d}\,\big], \tau_0\big[f_0[\bar{c}], \ldots, f_{k-1}[\bar{c}], \bar{c}\big], \ldots, \tau_{n-1}\big[f_0[\bar{c}], \ldots, f_{k-1}[\bar{c}], \bar{c}\big] \,\big\rangle$$

satisfies φ in $\mathfrak{Sg}(K*U)$. Since $\mathfrak{V}$ is dense, there is a sequence $\bar{e}$ from $\mathfrak{V}$ such that $\bar{c}\hat{\ }\bar{d}$ is atomically equivalent to $\bar{c}\hat{\ }\bar{e}$ in $\mathfrak{U}$. By ω-homogeneity there is an automorphism of $\mathfrak{U}$ taking $\bar{c}\hat{\ }\bar{d}$ to $\bar{c}\hat{\ }\bar{e}$. In view of (iv), this automorphism extends to an automorphism of $\mathfrak{Sg}(K*U)$. Thus,

$$\big\langle\, \sigma\big[g_0[\bar{e}], \ldots, g_{l-1}[\bar{e}], \bar{e}\,\big], \tau_0\big[f_0[\bar{c}], \ldots, f_{k-1}[\bar{c}], \bar{c}\big], \ldots, \tau_{n-1}\big[f_0[\bar{c}], \ldots, f_{k-1}[\bar{c}], \bar{c}\big] \,\big\rangle$$

satisfies φ in $\mathfrak{Sg}(K*U)$. Since $\sigma[g_0[\bar{e}], \ldots, g_{l-1}[\bar{e}], \bar{e}]$ is in $\mathfrak{Sg}(K*V)$, this shows that the Tarski-Vaught criterion is satisfied. Hence, $\mathfrak{Sg}(K*V) \preccurlyeq \mathfrak{Sg}(K*U)$.

Now let $\mathfrak{U}$ be an arbitrary dense ordering or dense tree. Take $\mathfrak{W}$ to be an ω-homogeneous extension of $\mathfrak{U}$, and set $\mathfrak{P}' = EM(\mathfrak{W}, \Phi)$. Thus, $\mathfrak{M}' \preccurlyeq \mathfrak{P}'$, and hence $\mathfrak{M} \preccurlyeq \mathfrak{P}$, by Fact 3. By the case just treated we have

$$\mathfrak{Sg}^{\mathfrak{M}}(K*U) = \mathfrak{Sg}^{\mathfrak{P}}(K*U) \preccurlyeq \mathfrak{Sg}^{\mathfrak{P}}(K*W)$$

and

$$\mathfrak{Sg}^{\mathfrak{M}}(K*V) = \mathfrak{Sg}^{\mathfrak{N}}(K*V) \preccurlyeq \mathfrak{Sg}^{\mathfrak{N}}(K*W) \quad .$$

Hence, $\mathfrak{Sg}^{\mathfrak{M}}(K*V) \preccurlyeq \mathfrak{Sg}^{\mathfrak{M}}(K*U)$, as was to be shown. This proves (v).

For (ii), suppose $\bar{a}, \bar{b} \in {}^nU$ are atomically equivalent in $\mathfrak{U}$. Let $\mathfrak{W}$ be an ω-homogeneous elementary extension of $\mathfrak{U}$ that includes the elements of $\bar{a}$ and $\bar{b}$, and set $\mathfrak{P}' = EM(\mathfrak{W}, \Phi)$. By Fact 3 we may assume that $\mathfrak{M}' \preccurlyeq \mathfrak{P}'$, and hence that $\mathfrak{M} \preccurlyeq \mathfrak{P}$. Since $\mathfrak{W}$ is ω-homogeneous, there is an automorphism of $\mathfrak{W}$ taking $\bar{a}$ to $\bar{b}$. By Fact 2, this extends to an automorphism ϑ of $\mathfrak{P}'$. Now ϑ maps W one-one onto itself and preserves all the operations of $\mathfrak{P}'$. In particular, an appropriate restriction is an automorphism of $\mathfrak{Sg}^{\mathfrak{P}}(K*W)$. Thus, $\bar{a}$ and $\bar{b}$ are elementarily equivalent in $\mathfrak{Sg}^{\mathfrak{P}}(K*W)$. By part (v) they remain elementarily equivalent in $\mathfrak{Sg}^{\mathfrak{P}}(K*U) = \mathfrak{Sg}^{\mathfrak{M}}(K*U)$.

The above proof obviously remains valid if we replace (implicit and explicit occurrences of) "$\mathfrak{M}$" and "$\mathfrak{N}$" everywhere by "$\mathfrak{M}_K$" and "$\mathfrak{N}_K$" respectively. □

Lemma 2. *Suppose $\mathfrak{U}$ has power κ, $\mathfrak{M}' = EM(\mathfrak{U}, \Phi)$, and $K \subseteq L'$ has power at most κ. Then $\mathfrak{Sg}^{\mathfrak{M}}(K*U)$ is strictly κ-generated.*

Proof. Since $|U| = \kappa \le |K|$, we have $|K*U| = \kappa$. Thus, $\mathfrak{Sg}(K*U)$ is κ-generated. Suppose now, for contradiction, that $\mathfrak{Sg}(K*U)$ is μ-generated for some (finite or infinite) $\mu < \kappa$. A standard argument then gives us a set $V \subseteq U$, of power μ when $\omega \le \mu$, and finite when $\mu < \omega$, such that $K*V$ generates $\mathfrak{Sg}(K*U)$. In particular, $K*V$ generates U in $\mathfrak{M}$. But then V generates U in $\mathfrak{M}'$, which is impossible, by Fact 4. Indeed, either U is irredundant in $\mathfrak{M}'$ or else a cardinality argument gives us an f in U that is not an initial segment of any element of V. □

Main Hypothesis. *Throughout the remainder of the paper we fix an uncountable cardinal κ and assume that T is a κ-unique universal theory in a language L of arbitrary cardinality.*

As before, L' will be an arbitrary expansion of L with built-in Skolem functions, T' a Skolem theory in L' extending T, and Φ a standard or generalized Ehrenfeucht-Mostowski set compatible with T'. Again, recall that if $\mathfrak{M}' = EM(\mathfrak{U}, \Phi)$, then we set $\mathfrak{M} = \mathfrak{M}' \restriction L$. Our first goal is to show that T_∞—the theory of the infinite models of T—is complete.

Lemma 3. *Let $\mathfrak{U}$ be a dense ordering of power κ, and $\mathfrak{M}' = EM(\mathfrak{U}, \Phi)$. Then for every n and every countable set $H \subseteq L' - L$, there is countable set K, with $H \subseteq K \subseteq L' - L$, such that*

$$\mathfrak{Sg}^{\mathfrak{M}}(K*U) \preccurlyeq_n \mathfrak{M} \quad .$$

The same is true if $\mathfrak{U}$ is a dense tree, provided that $\mathfrak{M}'$ exists.

Proof. The proof is by induction on n, for all H at once. The case $n = 0$ is trivial: take $K = H$. Assume, now, that the lemma holds for a given $n \ge 0$ and for all countable sets $H \subseteq L' - L$. Suppose, for contradiction, that H_0 is a countable subset of $L' - L$ such that,

(1) For every countable K with $H_0 \subseteq K \subseteq L' - L$ we have

$$\mathfrak{Sg}^{\mathfrak{M}}(K * U) \not\preccurlyeq_{n+1} \mathfrak{M} \quad .$$

We construct an increasing sequence $\langle H_\xi : \xi < \omega_1 \rangle$ of countable subsets of $L' - L$ such that, setting

$$\mathfrak{A}_\xi = \mathfrak{Sg}^{\mathfrak{M}}(H_\xi * U)\,,$$

we have

(2) $\mathfrak{A}_\xi \preccurlyeq_n \mathfrak{M}$ and $\mathfrak{A}_\xi \not\preccurlyeq_{n+1} \mathfrak{A}_{\xi+1}$ for all $\xi < \omega_1$.

The set H_0 is given. Suppose H_ξ has been constructed. Since $\mathfrak{A}_\xi \not\preccurlyeq_{n+1} \mathfrak{M}$, by (1), there is a Π_n-formula $\varphi_\xi(x_0, \dots, x_{p-1}, y_0, \dots, y_{q-1})$ in L, appropriate terms $\tau_0, \dots, \tau_{q-1}$ in L and function symbols $f_0, \dots, f_{r-1}$ in H_ξ, and a sequence $\bar{a} = \langle a_0, \dots, a_{s-1} \rangle$ from U such that (adding dummy variables to simplify notation)

(3) $\langle f_0[\bar{a}], \dots, f_{r-1}[\bar{a}], a_0, \dots, a_{s-1}\rangle$ satisfies $\exists x_0 \dots \exists x_{p-1} \varphi_\xi(x_0, \dots, x_{p-1}, \tau_0, \dots, \tau_{q-1})$ in $\mathfrak{M}$, but not in $\mathfrak{A}_\xi$.

(Here we are using the fact that $H_\xi * U$ generates $\mathfrak{A}_\xi$.) Thus, there are Skolem functions $g_0, \dots, g_{p-1}$ in L' such that

(4) $\langle g_0[\bar{a}], \dots, g_{p-1}[\bar{a}], f_0[\bar{a}], \dots, f_{r-1}[\bar{a}], a_0, \dots, a_{s-1} \rangle$ satisfies the Π_n-formula $\varphi_\xi(x_0, \dots, x_{p-1}, \tau_0, \dots, \tau_{q-1})$ in $\mathfrak{M}$.

By the induction hypothesis there is a countable $H_{\xi+1}$ with

$$H_\xi \cup \{g_0, \dots, g_{p-1}\} \subseteq H_{\xi+1} \subseteq L' - L$$

and such that, setting $\mathfrak{A}_{\xi+1} = \mathfrak{Sg}(H_{\xi+1} * U)$, we have $\mathfrak{A}_{\xi+1} \preccurlyeq_n \mathfrak{M}$. However, we put witnesses in $\mathfrak{A}_{\xi+1}$ to insure that $\mathfrak{A}_\xi \not\preccurlyeq_{n+1} \mathfrak{A}_{\xi+1}$. At limit stages δ take $H_\delta = \bigcup_{\xi<\delta} H_\xi$. This completes the construction.

Set

$$G = \textstyle\bigcup_{\xi<\omega_1} H_\xi \quad \text{and} \quad \mathfrak{B} = \mathfrak{Sg}^{\mathfrak{M}}(G * U) \quad .$$

Clearly, $\mathfrak{B} = \bigcup_{\xi<\omega_1} \mathfrak{A}_\xi$. Therefore, from (2) and our construction we obtain

(5) $\mathfrak{A}_\xi \preccurlyeq_n \mathfrak{B} \preccurlyeq_n \mathfrak{M}$ and $\mathfrak{A}_\xi \not\preccurlyeq_{n+1} \mathfrak{B}$ for every $\xi < \omega_1$.

Next, we prove:

(6) For every $\xi < \omega_1$ and every countable, dense $X \subseteq U$ we have

$$\mathfrak{Sg}(H_\xi * X) \not\preccurlyeq_{n+1} \mathfrak{B} \quad .$$

Indeed, fix $\xi < \omega_1$ and select a sequence $\bar{a}$ so that (3) and (4) hold. Given $X \subseteq U$, choose a sequence $\bar{b}$ from X that is atomically equivalent to $\bar{a}$ in $\mathfrak{U}$. This is possible because X is dense. By indiscernibility in $\mathfrak{M}'$, and (4),

$$\langle g_0[\bar{b}], \dots, g_{p-1}[\bar{b}], f_0[\bar{b}], \dots, f_{r-1}[\bar{b}], b_0, \dots, b_{s-1} \rangle$$

satisfies $\varphi_\xi(x_0,\dots,x_{p-1},\tau_0,\dots,\tau_{q-1})$ in $\mathfrak{M}$, and hence also in $\mathfrak{B}$, by (5). Suppose, for contradiction, that $\mathfrak{Sg}(H_\xi * X) \preccurlyeq_{n+1} \mathfrak{B}$. Then

(7) $\langle f_0[\bar b],\dots,f_{r-1}[\bar b],b_0,\dots,b_{s-1}\rangle$ satisfies $\exists x_0\dots\exists x_{p-1}\varphi_\xi(x_0,\dots,x_{p-1},\tau_0,\dots,\tau_{q-1})$ in $\mathfrak{Sg}(H_\xi * X)$.

Now we have $\mathfrak{Sg}(H_\xi * X) \preccurlyeq \mathfrak{A}_\xi$, by Lemma 1(v). Therefore, (7) holds with "$\mathfrak{Sg}(H_\xi * X)$" replaced by "$\mathfrak{A}_\xi$". Since $\bar a$ and $\bar b$ are elementarily equivalent in the expansion $\mathfrak{Sg}^{\mathfrak{M}_{H_\xi}}(H_\xi * U)$ of $\mathfrak{A}_\xi$, by Lemma 1(ii), we conclude that $\langle f_0[\bar a],\dots,f_{r-1}[\bar a],a_0,\dots,a_{s-1}\rangle$ satisfies

$$\exists x_0\dots\exists x_{p-1}\varphi_\xi(x_0,\dots,x_{p-1},\tau_0,\dots,\tau_{q-1})$$

in $\mathfrak{A}_\xi$. But this contradicts (3); so the proof of (6) is completed.

We now work towards a contradiction of (6) by producing a $\xi < \omega_1$ and a countable, dense set $X \subseteq U$ such that

(8) $\mathfrak{Sg}(H_\xi * X) \preccurlyeq \mathfrak{B}$.

Since $\mathfrak{B} = \mathfrak{Sg}^{\mathfrak{M}}(G * U)$ and $\mathfrak{Sg}^{\mathfrak{M}}(U)$ are both strictly κ-generated, by Lemma 2, they are isomorphic, by κ-uniqueness. Thus, there is a set $V \subseteq B$ and a dense ordering $\sqsubseteq$ on V such that $\mathfrak{V} = \langle V , \sqsubseteq \rangle$ and $\mathfrak{U}$ are isomorphic, V generates $\mathfrak{B}$, and V is a set of indiscernibles in $\mathfrak{B}$ with respect to atomic formulas (under the ordering $\sqsubseteq$).

We define increasing sequences, $\langle X_n : n\in\omega \rangle$ and $\langle Y_n : n\in\omega\rangle$, of countable, dense subsets of $\mathfrak{U}$ and $\mathfrak{V}$ respectively, and an increasing sequence $\langle \rho_n : n\in\omega\rangle$ of countable ordinals, such that

(9) $\mathfrak{Sg}^{\mathfrak{B}}(H_{\rho_n} * X_n) \subseteq \mathfrak{Sg}^{\mathfrak{B}}(Y_n) \subseteq \mathfrak{Sg}^{\mathfrak{B}}(H_{\rho_{n+1}} * X_{n+1})$.

Indeed, set $\rho_0 = 0$ and take X_0 to be an arbitrary countable, dense subset of U. Since H_0 is countable, so is $H_0 * X_0$. Therefore, there is a countable $Y_0 \subseteq V$ that generates $H_0 * X_0$ in $\mathfrak{B}$. By throwing in extra elements, we may assume that Y_0 is dense.

Now suppose that ρ_n , X_n and Y_n have been defined. Since Y_n is countable and $\mathfrak{B} = \bigcup_{\xi<\omega_1}\mathfrak{A}_\xi$, there is a $\rho_{n+1} < \omega_1$ such that $Y_n \subseteq A_{\rho_{n+1}}$. Without loss of generality we may take $\rho_{n+1} > \rho_n$. Since $H_{\rho_{n+1}} * U$ generates $\mathfrak{A}_{\rho_{n+1}}$, there is a countable subset $X_{n+1} \subseteq U$ such that $H_{\rho_{n+1}} * X_{n+1}$ generates Y_n in $\mathfrak{B}$. Of course we may choose X_{n+1} so that it is dense and includes X_n . As before, we now choose a countable, dense $Y_{n+1} \subseteq V$ that generates X_{n+1} in $\mathfrak{B}$ and includes Y_n . This completes the construction.

Set

$$\rho = \sup\{\rho_n : n\in\omega\} \quad , \quad X = \textstyle\bigcup_{n\in\omega} X_n \quad , \quad Y = \bigcup_{n\in\omega} Y_n \quad .$$

Then $\rho < \omega_1$ and $H_\rho = \bigcup_{n\in\omega} H_{\rho_n}$, by definition of the sequence $\langle H_\xi : \xi < \omega_1 \rangle$. Also, $\langle X , \le \rangle$ and $\langle Y , \sqsubseteq \rangle$ are countable, dense submodels of $\mathfrak{U}$ and $\mathfrak{V}$ respectively. By (5), (9), and Lemma 1(v) we have

$$\mathfrak{Sg}^{\mathfrak{M}}(H_\rho * X) = \mathfrak{Sg}^{\mathfrak{B}}(H_\rho * X) = \textstyle\bigcup_{n<\omega}\mathfrak{Sg}^{\mathfrak{B}}(H_{\rho_n} * X_n) = \bigcup_{n<\omega}\mathfrak{Sg}^{\mathfrak{B}}(Y_n) = \mathfrak{Sg}^{\mathfrak{B}}(Y)$$

and

$$\mathfrak{Sg}^{\mathfrak{B}}(Y) = \mathfrak{Sg}^{\mathfrak{M}}(Y) \preccurlyeq \mathfrak{Sg}^{\mathfrak{M}}(V) = \mathfrak{Sg}^{\mathfrak{B}}(V) = \mathfrak{B} \quad .$$

Thus, we have constructed a countable $\xi < \omega_1$ and a countable, dense $X \subseteq U$ such that (8) holds. This contradicts (6). $\square$

Lemma 4. *For every countable $H \subseteq L' - L$, there is a countable K, with $H \subseteq K \subseteq L' - L$, such that, whenever $\mathfrak{V}$ is a dense ordering and $\mathfrak{N}' = EM(\mathfrak{V}, \Phi)$, we have*

$$\mathfrak{Sg}^{\mathfrak{N}}(K * V) \preccurlyeq \mathfrak{N} \quad .$$

The same is true when $\mathfrak{V}$ is a dense tree, provided that $EM(\mathfrak{U}, \Phi)$ exists for dense trees $\mathfrak{U}$.

Proof. Let $\mathfrak{U}$ be a dense ordering of power κ, and set $\mathfrak{M}' = EM(\mathfrak{U}, \Phi)$. We use Lemma 3 to define an increasing sequence $\langle H_n : n < \omega \rangle$ of countable subsets of $L' - L$ such that

(1) $\mathfrak{Sg}^{\mathfrak{M}}(H_n * U) \preccurlyeq_n \mathfrak{M}$.

Indeed, we set $H_0 = H$, and given H_n satisfying (1), we apply Lemma 3 to obtain H_{n+1}. Setting $K = \bigcup_{n \in \omega} H_n$, we easily check that

(2) $\mathfrak{Sg}^{\mathfrak{M}}(K * U) \preccurlyeq \mathfrak{M}$.

Now suppose that $\mathfrak{V}$ is any dense ordering or tree, and that $\mathfrak{N}' = EM(\mathfrak{V}, \Phi)$. We first treat the case when $\mathfrak{V}$ is a submodel of $\mathfrak{U}$. By Fact 3 we may assume, without loss of generality, that $\mathfrak{N}' \preccurlyeq \mathfrak{M}'$. Hence,

(3) $\mathfrak{N} \preccurlyeq \mathfrak{M}$.

In view of (3) and Lemma 1(v), we get

(4) $\mathfrak{Sg}^{\mathfrak{N}}(K * V) = \mathfrak{Sg}^{\mathfrak{M}}(K * V) \preccurlyeq \mathfrak{Sg}^{\mathfrak{M}}(K * U)$.

Combining (2)–(4) gives

(5) $\mathfrak{Sg}^{\mathfrak{N}}(K * V) \preccurlyeq \mathfrak{N}$.

For the case when $\mathfrak{V}$ is not a submodel of $\mathfrak{U}$, take a countable, dense submodel $\mathfrak{W}$ of $\mathfrak{V}$. Set $\mathfrak{P}' = EM(\mathfrak{W}, \Phi)$. Again, we may assume that $\mathfrak{P}' \preccurlyeq \mathfrak{N}'$, so

(6) $\mathfrak{P} \preccurlyeq \mathfrak{N}$,

(7) $\mathfrak{Sg}^{\mathfrak{P}}(K * W) = \mathfrak{Sg}^{\mathfrak{N}}(K * W) \preccurlyeq \mathfrak{Sg}^{\mathfrak{N}}(K * V)$.

Furthermore, there is an embedding of $\mathfrak{W}$ into $\mathfrak{U}$, and this embedding extends to an elementary embedding ϑ of $\mathfrak{P}'$ into $\mathfrak{M}'$. Hence, as usual,

(8) ϑ induces an elementary embedding of $\mathfrak{P}$ into $\mathfrak{M}$,

(9) ϑ induces an elementary embedding of $\mathfrak{Sg}^{\mathfrak{P}}(K * W)$ into $\mathfrak{Sg}^{\mathfrak{M}}(K * U)$.

Using (2) and (6)–(9), we readily verify that (5) holds. Here are the details. Let $\varphi(\bar{x})$ be a formula of L, and $\bar{a}$ a sequence of appropriate length from $\mathfrak{Sg}^{\mathfrak{N}}(K * V)$. Then there is a finite sequence $\bar{v}$ from V that generates $\bar{a}$ (with the help of K). Choose a $\bar{v}'$ in W that is atomically equivalent to $\bar{v}$, and let $\bar{a}'$ be the sequence obtained from $\bar{v}'$ in the same way that $\bar{a}$ is obtained from $\bar{v}$. Then $\bar{a}$ and $\bar{a}'$ are elementarily equivalent in $\mathfrak{N}'$—and therefore also in $\mathfrak{N}$—and in $\mathfrak{Sg}^{\mathfrak{N}}(K{*}V)$ by Fact 3 and Lemma 1(ii). Hence,

$$\begin{array}{llll} \mathfrak{N} \models \varphi[\bar a] & \text{iff} & \mathfrak{N} \models \varphi[\bar a'] & \\ & \text{iff} & \mathfrak{P} \models \varphi[\bar a'] & \text{by (6)}, \\ & \text{iff} & \mathfrak{M} \models \varphi[\vartheta(\bar a')] & \text{by (8)}, \\ & \text{iff} & \mathfrak{Sg}^{\mathfrak{M}}(K*U) \models \varphi[\vartheta(\bar a')] & \text{by (2)}, \\ & \text{iff} & \mathfrak{Sg}^{\mathfrak{P}}(K*W) \models \varphi[\bar a'] & \text{by (9)}, \\ & \text{iff} & \mathfrak{Sg}^{\mathfrak{N}}(K*V) \models \varphi[\bar a'] & \text{by (7)}, \\ & \text{iff} & \mathfrak{Sg}^{\mathfrak{N}}(K*V) \models \varphi[\bar a] \quad . & \end{array}$$

□

Theorem 5. T_∞ *is complete.*

Proof. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two infinite models of T. For $i = 1, 2$, let T_i be the theory of $\mathfrak{B}_i$, let L' be an expansion of the language of T_i with built-in Skolem functions, T_i' a Skolem theory in L' extending T_i, and Φ_i a standard Ehrenfeucht-Mostowski set in L' compatible with T_i'. Let $\mathfrak{U}$ be a dense ordering of power κ, and set $\mathfrak{M}_i' = EM(\mathfrak{U}, \Phi_i)$ and $\mathfrak{M}_i = \mathfrak{M}_i' \restriction L$. By Lemma 4 there is a countable $K_i \subseteq L' - L$ such that

(1) $\quad \mathfrak{Sg}^{\mathfrak{M}_i}(K_i{*}U) \preccurlyeq \mathfrak{M}_i \quad .$

Now $\mathfrak{Sg}^{\mathfrak{M}_1}(K_1 * U)$ and $\mathfrak{Sg}^{\mathfrak{M}_2}(K_2 * U)$ are strictly κ-generated models of T, by Lemma 2. Therefore, they are isomorphic, by the κ-uniqueness of T. From this and (1) we conclude that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are elementarily equivalent. But $\mathfrak{M}_i'$, and hence also $\mathfrak{M}_i$, is a model of the theory of $\mathfrak{B}_i$. In consequence, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are elementarily equivalent, as was to be shown. □

Our next goal is to prove:

Theorem 6. T_∞ *is superstable.*

Proof. Set $U = (\kappa+1) \times \mathbb{Q}$ and $\mathfrak{U} = \langle\, U\, , <\rangle$, where $<$ is the lexicographic ordering on U. We define a substructure $\mathfrak{V}$ of the full tree over U by specifying its universe. It is the smallest set V satisfying the following conditions: all finite levels of the full tree are included in V, i.e., $\bigcup_{n\in\omega} {}^nU \subseteq V$; all eventually constant functions from ${}^\omega U$ are in V; for each limit ordinal $\delta < \omega_1$ we choose a strictly increasing function f_δ in ${}^\omega\omega_1$ such that $\sup\{f_\delta(n) : n \in \omega\} = \delta$, and we put into V a copy of f_δ from ${}^\omega V$ called g_δ and determined by: $g_\delta(n) = \langle\, f_\delta(n)\, , 0\,\rangle$ for each n.

(1) For every countable set $W \subseteq U$, the set $V \cap {}^{\omega\geq}W$ is also countable.

Indeed, the finite levels, ${}^nW$, and the set of eventually constant functions of ${}^\omega W$ are all countable. Moreover, the domain of W, i.e., $\{\delta : \langle\, \delta\, , q\,\rangle \in W \text{ for some } q \in \mathbb{Q}\}$, is countable, so the set of g_δ in ${}^\omega W$, with $\delta < \omega_1$, is also countable. This proves (1).

Let Φ be a standard Ehrenfeucht-Mostowski set compatible with T' (our Skolem extension of T), and set $\mathfrak{M}' = EM(\mathfrak{U}, \Phi)$. Assume, for contradiction, that T

is not superstable, and let $\langle \varphi_n(x,y) : n < \omega\rangle$ be a sequence of formulas witnessing this nonsuperstability. Then, as mentioned in the preliminaries, there is an Ehrenfeucht-Mostowski set Ψ compatible with T' such that the generalized Ehrenfeucht-Mostowski model $\mathfrak{N}' = EM(\mathfrak{V},\Psi)$ exists, and, for u from $P_n^{\mathfrak{V}}$ and v from $P_\omega^{\mathfrak{V}}$, we have $\mathfrak{N}' \models \varphi_n[u,v]$ iff $u \lessdot v$. Set $\mathfrak{M} = \mathfrak{M}'{\restriction}L$ and $\mathfrak{N} = \mathfrak{N}'{\restriction}L$.

By Lemma 4, there are countable sets $J, K \subseteq L' - L$ such that

(2) $\quad \mathfrak{Sg}(J{*}U) \preccurlyeq \mathfrak{M} \quad$ and $\quad \mathfrak{Sg}(K{*}V) \preccurlyeq \mathfrak{N}$.

By Lemma 2, $\mathfrak{Sg}(J*U)$ and $\mathfrak{Sg}(K*V)$ are both strictly κ-generated. Hence, κ-uniqueness implies that they are isomorphic. Let ϑ be such an isomorphism.

We now define two strictly increasing, continuous sequences, $\langle X_\alpha : \alpha < \omega_1\rangle$ and $\langle Y_\alpha : \alpha < \omega_1\rangle$, of countable, dense subsets of U and V respectively, each Y_α being downward closed, i.e., closed under initial segments. (i) Let $X_0 = \{\kappa\} \times \mathbb{Q}$ and take Y_0 to be any countable, dense, downward closed subset of V. Suppose, now, that X_α and Y_α have been defined. Since both of these sets are countable, by assumption, so are $J{*}X_\alpha$ and $K{*}Y_\alpha$. For any a in $J{*}U$ there is obviously a finite subset $C_a \subseteq V$ such that $K{*}C_a$ generates $\vartheta(a)$ (in $\mathfrak{Sg}(K{*}V)$). (ii) For each a in $J{*}X_\alpha$, put the elements of C_a into $Y_{\alpha+1}$. Since $J{*}X_\alpha$ is countable, this adds only countably many elements to $Y_{\alpha+1}$. (iii) Similarly, for every b in $K{*}Y_\alpha$, choose a finite set $D_b \subseteq U$ such that $J{*}D_b$ generates $\vartheta^{-1}(b)$ (in $\mathfrak{Sg}(J{*}U)$), and put each element of D_b into $X_{\alpha+1}$. (iv) For each $\langle \xi\,,\, q\rangle$ in X_α, put all of $\{\xi\} \times \mathbb{Q}$ into $X_{\alpha+1}$. (v) Put all of ${}^{\omega\geq}(\alpha \times \mathbb{Q}) \cap V$ into $Y_{\alpha+1}$. The latter set remains countable, by (1). (vi) If necessary, add countably many more elements to $X_{\alpha+1}$ and $Y_{\alpha+1}$ to insure that these sets are dense, that $X_\alpha \subset X_{\alpha+1}$ and $Y_\alpha \subset Y_{\alpha+1}$, and that $Y_{\alpha+1}$ is downward closed. (vii) For limit ordinals δ, set $X_\delta = \bigcup_{\alpha<\delta} X_\alpha$ and $Y_\delta = \bigcup_{\alpha<\delta} Y_\alpha$.

Setting

(3) $\quad \mathfrak{A}_\alpha = \mathfrak{Sg}(J{*}X_\alpha)$ and $\mathfrak{B}_\alpha = \mathfrak{Sg}(K{*}Y_\alpha)$ for every $\alpha < \omega_1$,

we see from conditions (ii) and (iii) that, for each $\alpha < \omega_1$, we have

$$\vartheta(A_\alpha) \subseteq B_{\alpha+1} \quad \text{and} \quad \vartheta^{-1}(B_\alpha) \subseteq A_{\alpha+1} \quad .$$

From this it easily follows that

(4) For each limit ordinal $\delta < \omega_1$, the (appropriate restriction of the) mapping ϑ is an isomorphism of $\mathfrak{A}_\delta$ onto $\mathfrak{B}_\delta$.

Set

$$E = \left\{\delta < \omega_1 : \delta \text{ is a limit ordinal and } {}^{\omega\geq}(\omega_1 \times \mathbb{Q}) \cap Y_\delta = \left[\textstyle\bigcup_{\beta<\delta} {}^{\omega\geq}(\beta \times \mathbb{Q})\right] \cap V\right\}$$

and

$$E' = \left\{\delta \in E : \delta = \sup(\delta \cap E) = \sup\{\alpha \in E : \alpha < \delta\}\right\} \quad .$$

(5) E and E' are closed, unbounded sets.

To see that E is unbounded, observe, first of all, that, by conditions (v) and (vii), we have

(6) $\quad {}^{\omega\geq}(\omega_1 \times \mathbb{Q}) \cap Y_\delta \supseteq \left[\bigcup_{\beta<\delta} {}^{\omega\geq}(\beta \times \mathbb{Q})\right] \cap V$ for every limit ordinal $\delta < \omega_1$.

Therefore, we need only establish the reverse inclusion for unboundedly many limit ordinals. Since a v in ${}^{\omega\geq}(\omega_1 \times \mathbb{Q}) \cap V$ must have a countable range, there is an ordinal $\gamma_v < \omega_1$ such that v is in ${}^{\omega\geq}(\gamma_v \times \mathbb{Q})$. Let α_0 be an arbitrary countable ordinal. Since Y_{α_0} is countable, the supremum of the γ_v, over all v in Y_{α_0}, is countable. Hence, we can find a countable $\alpha_1 > \alpha_0$ such that

$$ {}^{\omega\geq}(\omega_1 \times \mathbb{Q}) \cap Y_{\alpha_0} \subseteq {}^{\omega\geq}(\alpha_1 \times \mathbb{Q}) \quad . $$

Continuing in this fashion, we obtain a strictly increasing sequence $\langle \alpha_n : n < \omega \rangle$ of countable ordinals such that

$$ {}^{\omega\geq}(\omega_1 \times \mathbb{Q}) \cap Y_{\alpha_n} \subseteq {}^{\omega\geq}(\alpha_{n+1} \times \mathbb{Q}) \quad . $$

Set $\delta = \sup\{\alpha_n : n < \omega\}$. Then

$$ {}^{\omega\geq}(\omega_1 \times \mathbb{Q}) \cap Y_\delta \subseteq \left[\bigcup_{\beta<\delta} {}^{\omega\geq}(\beta \times \mathbb{Q})\right] \cap V \quad . $$

In view of (6), we see that equality actually holds in the preceding line. This shows that E is unbounded. The rest of the proof of (5) is easy, so we leave it to the reader.

(7) For every δ in E' and every α in $E \cap \delta$, the type $tp(g_\delta, B_\delta)$ splits over B_α .

To prove (7), recall that g_δ is not in ${}^{\omega}(\beta \times \mathbb{Q})$ for any $\beta < \delta$. We thus see, from the definition of E, that g_δ is not in Y_δ. Hence, by Fact 4, g_δ is not in B_δ.

The function g_δ is defined in terms of a strictly increasing, ordinal-valued function f_δ whose supremum is δ. Since $\alpha < \delta$, there must be an $n < \omega$ such that $f_\delta(n) > \alpha$. Let h_1 and h_2 be the extensions of $g_\delta \restriction n$ to $n+1$ determined by

$$ h_1(n) = g_\delta(n) = \langle\, f_\delta(n)\,,\, 0\,\rangle \qquad \text{and} \qquad h_2(n) = \langle\, f_\delta(n)\,,\, 1\,\rangle \quad . $$

Notice that $h_1 = g_\delta \restriction (n+1)$. Clearly, h_1 and h_2 are in Y_δ, by conditions (v) and (vii). Because α is in E, and $f_\delta(n) > \alpha$, we can apply the definition of E to conclude that h_1 and h_2 are not in Y_α, and hence are not initial segments of any elements in Y_α. Therefore, they are not in B_α, by Fact 4. They realize the same type over B_α, by tree indiscernibility. Moreover, $\mathfrak{N} \models \varphi_{n+1}[h_1, g_\delta]$ and $\mathfrak{N} \models \neg\varphi_{n+1}[h_2, g_\delta]$, since φ_{n+1} codes the initial segment relation between elements in $P^{\mathfrak{B}}_{n+1}$ and $P^{\mathfrak{B}}_{\omega}$. Thus, both $\varphi_{n+1}(h_1, \bar{x})$ and $\neg\varphi_{n+1}(h_2, \bar{x})$ are in $tp(g_\delta, B_\delta)$. This completes the proof of (7).

We now work towards a contradiction to (7).

(8) For every δ in E' and every a in $Sg(J{*}U) - A_\delta$, there is an α in $E \cap \delta$ such that $tp(a, A_\delta)$ does not split over A_α .

To see that (8) contradicts (7), take any δ in E', and set $a = \vartheta^{-1}(g_\delta)$. Since g_δ is in $Sg(K{*}V) - B_\delta$, we get that a is in $Sg(J{*}U) - A_\delta$, by (4). Hence, there is an α in $E \cap \delta$ such that $tp(a, A_\delta)$ does not split over A_α, by (8). But then, applying ϑ to a, $tp(a, A_\delta)$, and A_δ, we get that $tp(g_\delta, B_\delta)$ does not split over B_α, by (4). This contradicts (7).

To prove (8), fix a δ in E' and an a in $Sg(J{*}U) - A_\delta$. Then there is a sequence $\bar{u} = \langle u_0, \dots, u_{r-1}\rangle$ from U, function symbols $f_0, \dots f_{n-1}$ in J, and a term σ from L, such that (adding dummy variables)

(9) $a = \sigma\big[f_0[\bar{u}], \dots, f_{n-1}[\bar{u}], \bar{u}\big]$.

Each u_i has the form $u_i = \langle\, \gamma_i\,,\, q_i\,\rangle$ for a unique $\gamma_i \leq \kappa$ and q_i in $\mathbb{Q}$. The set

$$\Gamma_i = \big\{\beta \leq \kappa : \{\beta\} \times \mathbb{Q} \subseteq X_\delta \text{ and } \gamma_i \leq \beta\big\}$$

is nonempty, since it contains κ, by condition (i). We shall denote the smallest element of this set by γ_i'. Notice that $u_i \in X_\delta$ iff $\gamma_i' = \gamma_i$, by condition (iv). Now, by definition, $\{\gamma_i'\} \times \mathbb{Q} \subseteq X_\delta$ for each $i < r$. Since δ is in E', there is, for each $i < r$, a β in $E \cap \delta$ such that $\{\gamma_i'\} \times \mathbb{Q} \subseteq X_\beta$. Take α to be the maximum of these β. Then α is in $E \cap \delta$ and $\{\gamma_i'\} \times \mathbb{Q} \subseteq X_\alpha$ for each $i < r$.

To verify that $tp(a, A_\delta)$ does not split over A_α, let $\psi(x, \bar{y})$ be a formula of L, and let $\bar{b}$ and $\bar{c}$ be sequences of equal length from A_δ, such that $\psi(x, \bar{b})$ and $\neg\psi(x, \bar{c})$ are in $tp(a, A_\delta)$, i.e.,

(10) $\mathfrak{M} \models \psi[a, \bar{b}] \wedge \neg\psi[a, \bar{c}]$.

We must prove that

(11) $tp(\bar{b}, A_\alpha) \neq tp(\bar{c}, A_\alpha)$.

Since $\bar{b}$ and $\bar{c}$ come from A_δ, there is a finite sequence $\bar{v}$ from X_δ that generates $\bar{b}$ and $\bar{c}$ with the help of some elements of J. Using the ordinals γ_i', we shall construct a sequence $\bar{u}' = \langle u_0', \dots, u_{r-1}'\rangle$ in X_α such that, for each i, we have $u_i' = u_i$ iff u_i is in X_δ, and

(12) $u_i' < u_k'$ iff $u_i < u_k$ and $u_i' = u_k'$ iff $u_i = u_k$,

$u_i' < v_j$ iff $u_i < v_j$ and $u_i' = v_j$ iff $u_i = v_j$.

for all appropriate j, k.

The construction of $\bar{u}'$ is not difficult: we shall set $u_i' = \langle\, \gamma_i'\,,\, q_i'\,\rangle$, where q_i' is chosen from $\mathbb{Q}$. Since $\{\gamma_i'\} \times \mathbb{Q} \subseteq X_\alpha$, this assures that u_i' will be in X_α. If γ_i is in Γ_i, then $\gamma_i' = \gamma_i$. In this case, take $q_i' = q_i$, so that $u_i' = u_i$. Suppose, now, that $v_j = \langle\, \beta_j\,,\, r_j\,\rangle$. If $u_i < v_j$, then we have either $\gamma_i < \beta_j$, or else $\gamma_i = \beta_j$ and $q_i < r_j$. In the first case, observe that $\beta_j \in \Gamma_i$, since v_j is in X_δ (here we are using again condition (iv)). Therefore, $\gamma_i' \leq \beta_j$. If $\gamma_i' < \beta_j$, then we may choose q_i' however we wish. If $\gamma_i' = \beta_j$, then we must choose $q_i' < r_j$. This is possible since $\{\gamma_i'\} \times \mathbb{Q} \subseteq X_\delta$. In the second case, when $\gamma_i = \beta_j$, we have γ_i in X_δ, so $u_i' = u_i$. In both cases we obtain $u_i' < v_j$. The other cases in (12) are treated similarly. Notice, however, that we must choose q_i' so that (12) holds for all appropriate j and k at once. To do this, we use the density of $\mathbb{Q}$.

From (12) we conclude that $\bar{u}'{}^\frown\bar{v}$ and $\bar{u}{}^\frown\bar{v}$ are atomically equivalent in $\mathfrak{U}$. Because U is a set of order indiscernibles in $\mathfrak{M}'$, it follows that $\bar{u}'{}^\frown\bar{v}$ and $\bar{u}{}^\frown\bar{v}$ are elementarily equivalent in $\mathfrak{M}'$. Recalling (9), set $a' = \sigma\big[f_0[\bar{u}'], \dots, f_{n-1}[\bar{u}'], \bar{u}'\big]$, and notice that a' is in A_α, since $\bar{u}'$ is from X_α.

Since $\bar{u}'$ generates a' in the same way that $\bar{u}$ generates a, and since $\bar{v}$ generates $\bar{b}$ and $\bar{c}$, we conclude that $a'\hat{\ }\bar{b}\hat{\ }\bar{c}$ and $a\hat{\ }\bar{b}\hat{\ }\bar{c}$ are elementarily equivalent in $\mathfrak{M}'$. In particular, from (10) we get that

$$\mathfrak{M} \models \psi[a', \bar{b}] \wedge \neg\psi[a', \bar{c}] \quad .$$

Thus, $\psi(a', \bar{y})$ is in $tp(\bar{b}, A_\alpha)$, while $\neg\psi(a', \bar{y})$ is in $tp(\bar{c}, A_\alpha)$. This proves (11), and hence (8). $\Box$

As is well-known, the fact that T_∞ is superstable (and hence stable) implies that order indiscernibles are totally indiscernible. Thus, if $\mathfrak{M}' = EM(\mathfrak{U}, \Phi)$, then any two one-one sequences in ${}^nU$ satisfy the same formulas in $\mathfrak{M}$ (but not necessarily in $\mathfrak{M}'$). It follows that Lemma 1 (with $K = \varnothing$) goes through without any references to order. Thus, if $\mathfrak{M}'$ and $\mathfrak{N}'$ are as in Lemma 1, then any injection ϑ of V into U extends canonically to an elementary embedding of $\mathfrak{Sg}^{\mathfrak{N}}(V)$ into $\mathfrak{Sg}^{\mathfrak{M}}(U)$, and this extension is an isomorphism iff ϑ is onto. In particular, the isomorphism type of a subalgebra (of some very large model) generated by an infinite set of indiscernibles with respect to Φ is uniquely determined by the cardinality of the set of indiscernibles. To give a precise definition of such algebras, we fix a very large dense ordering $\mathfrak{Z}$ (as large as we will need for any argument in this paper), set $\mathfrak{C} = EM(\mathfrak{Z}, \Phi)$, and let $\langle Z_\lambda : \lambda \text{ an infinite cardinal } \leq |Z|\rangle$ be an increasing sequence of subsets of Z such that $|Z_\lambda| = \lambda$.

Definition 7. *Let $\mathfrak{F}_\lambda = \mathfrak{Sg}^{\mathfrak{C}}(Z_\lambda)$. The set Z_λ is referred to as a Φ-basis of $\mathfrak{F}_\lambda$*

The algebras $\mathfrak{F}_\lambda$, for $\lambda \geq \omega$, have many properties in common with free algebras. For example, if U and V are Φ-bases of $\mathfrak{F}_\lambda$ and $\mathfrak{F}_\mu$ respectively, and if ϑ maps V one-one into U then the canonical extension of ϑ maps $\mathfrak{F}_\mu$ elementarily into $\mathfrak{F}_\lambda$. This is just a reformulation of what was said above. For another example, notice that $\mathfrak{F}_\lambda$ is strictly
λ-generated, by Lemma 2. It follows that *any* infinite Φ-basis of $\mathfrak{F}_\lambda$ must have cardinality λ; otherwise we would have $\mathfrak{F}_\mu \cong \mathfrak{F}_\lambda$ for some $\mu \neq \lambda$, forcing the strictly μ-generated model $\mathfrak{F}_\mu$ to be λ-generated and visa-versa.

For the case when T_∞ is superstable, we readily extend Lemma 1(v) to cover $\preccurlyeq_a$:

Lemma 8. *Let $\mathfrak{U}, \mathfrak{V}$ be dense orderings, $\mathfrak{M}' = EM(\mathfrak{U}, \Phi)$, and $K \subseteq L' - L$. If $\mathfrak{V} \subseteq \mathfrak{U}$, then*

$$\mathfrak{Sg}^{\mathfrak{M}}(K{*}V) \preccurlyeq_a \mathfrak{Sg}^{\mathfrak{M}}(K{*}U) \quad .$$

The lemma continues to hold if we replace "$\mathfrak{M}$" by "$\mathfrak{M}_K$".

Proof. We begin with the case when $\mathfrak{U}$ is ω_1-homogeneous. Let p be a strong type of $\mathfrak{M}$ based on a finite subset E of $Sg(K{*}V)$, and suppose that $p\restriction^{*}E$ is realized in $\mathfrak{Sg}(K{*}U)$ by $\bar{b}$. Let $\bar{v} = \langle v_0, \ldots, v_{m-1}\rangle$ be a finite sequence in $\mathfrak{V}$ that generates E in $\mathfrak{M}$, with the help of finitely many $f \in K$, and let $\bar{u} = \langle u_0, \ldots, u_{n-1}\rangle$ be a finite sequence from $\mathfrak{U}$ that generates $\bar{b}$. Because of the density of $\mathfrak{V}$, we can certainly find a sequence $\bar{u}' = \langle u_0', \ldots, u_{n-1}'\rangle$ in $\mathfrak{V}$ such that $\bar{v}\hat{\ }\bar{u}$ and $\bar{v}\hat{\ }\bar{u}'$ are atomically equivalent in $\mathfrak{U}$. Using again the density of $\mathfrak{V}$, is not difficult to construct an ω-sequence $\bar{w}$ in V such that $\bar{v}\hat{\ }\bar{w}\hat{\ }\bar{u}$ and $\bar{v}\hat{\ }\bar{w}\hat{\ }\bar{u}'$ are atomically equivalent and the range, X, of $\bar{v}\hat{\ }\bar{w}$ is dense. Assume, for the moment, that this done. Since $\mathfrak{U}$ is ω_1-homogeneous, there is an automorphism of $\mathfrak{U}$ taking $\bar{v}\hat{\ }\bar{w}\hat{\ }\bar{u}$ to $\bar{v}\hat{\ }\bar{w}\hat{\ }\bar{u}'$. Extend it to an automorphism ϑ

of $\mathfrak{M}'$. Then ϑ also induces an automorphism of $\mathfrak{Sg}(K*U)$, and it is the identity mapping on $Sg(K*X)$. Because X is dense, $\mathfrak{Sg}(K*X)$ is an elementary submodel of $\mathfrak{Sg}(K*V)$, and therefore p is stationary over this submodel. Since ϑ fixes an elementary submodel over which p is stationary, we see that ϑ must map $p\restriction^*E$ to itself. Thus, $\vartheta(\bar{b})$ realizes $p\restriction^*E$ in $\mathfrak{Sg}(K*U)$. But $\mathfrak{Sg}(K*V)$ is an elementary substructure of $\mathfrak{Sg}(K*U)$, by Lemma 1(v), and all the formulas of $p\restriction^*E$ are over $\mathfrak{Sg}(K*V)$. Thus, $\vartheta(\bar{b})$ must realize $p\restriction^*E$ in $\mathfrak{Sg}(K*V)$.

Now suppose that $\mathfrak{U}$ is arbitrary. Let $\mathfrak{W}$ be an ω_1-homogeneous extension of $\mathfrak{U}$, and set $\mathfrak{N}' = EM(\mathfrak{W}, \Phi)$. Then by the previous case,

$$\mathfrak{Sg}^{\mathfrak{M}}(K*V) = \mathfrak{Sg}^{\mathfrak{N}}(K*V) \preccurlyeq_a \mathfrak{Sg}^{\mathfrak{N}}(K*W)$$

and

$$\mathfrak{Sg}^{\mathfrak{M}}(K*U) = \mathfrak{Sg}^{\mathfrak{N}}(K*U) \preccurlyeq_a \mathfrak{Sg}^{\mathfrak{N}}(K*W) \quad .$$

Therefore, $\mathfrak{Sg}^{\mathfrak{M}}(K*V) \preccurlyeq_a \mathfrak{Sg}^{\mathfrak{M}}(K*U)$.

Returning to $\bar{w}$, a simple example should suffice to indicate how it is to be constructed. Suppose, for example, that

$$v_i < u_0 < u_1 < u_2 < u_3 < v_j \quad ,$$

and that no other elements of the sequences $\bar{v}$ and $\bar{u}$ are in the interval between v_i and v_j. Of course,

$$v_i < u_0' < u_1' < u_2' < u_3' < v_j \quad .$$

If $u_0' \leq u_0$, or else if $u_3' \geq u_3$, then include in the range of $\bar{w}$ a dense set between v_i and u_0', or else between u_3' and v_j, respectively. Now consider the case when $u_0 < u_0' < u_3' < u_3$. If $u_1' \leq u_1$, or else if $u_2' \geq u_2$ then include in the range of $\bar{w}$ a dense set between u_0' and u_1', or else between u_2' and u_3', respectively. There remains the case when $u_1 < u_1' < u_2' < u_2$. In this case, include in the range of $\bar{w}$ a dense set between u_1' and u_2'. $\square$

The next lemma is the analogue of Lemma 3 for the notion $\preccurlyeq_a$.

Lemma 9. *Let $\mathfrak{U}$ be a dense ordering of power κ, and $\mathfrak{M}' = EM(\mathfrak{U}, \Phi)$. Then for every countable $H \subseteq L' - L$ there is a countable K with $H \subseteq K \subseteq L' - L$ such that*

$$\mathfrak{Sg}^{\mathfrak{M}}(K*U) \preccurlyeq_a \mathfrak{M} \quad .$$

Proof. The proof is quite similar to the proof of Lemma 3. Suppose, for contradiction, that H is a counterexample to the assertion of the lemma. Analogously to the proof of Lemma 3, we construct an increasing, continuous sequence $\langle H_\xi : \xi < \omega_1 \rangle$ of countable subsets of $L' - L$ such that

(1) $\mathfrak{Sg}(H_\xi * U) \preccurlyeq \mathfrak{Sg}(H_{\xi+1} * U) \preccurlyeq \mathfrak{M}$, but $\mathfrak{Sg}(H_\xi * U) \not\preccurlyeq_a \mathfrak{Sg}(H_{\xi+1} * U)$.

Indeed, by Lemma 4 we can take H_0 to be a countable extension of H such that $\mathfrak{Sg}(H_0 * U) \preccurlyeq \mathfrak{M}$. Suppose, now, that H_ξ has been defined, and that $\mathfrak{Sg}(H_\xi * U) \preccurlyeq \mathfrak{M}$. By the assumption on H, $\mathfrak{Sg}(H_\xi * U)$ is not a-saturated in $\mathfrak{M}$. Thus, there is a strong type p_ξ based on a finite subset E_ξ of $Sg(H_\xi * U)$, such that $p_\xi\restriction^* E_\xi$ is realized in $\mathfrak{M}$—say by $\bar{a}_\xi$—but $p_\xi\restriction^* E_\xi$ is not realized in $\mathfrak{Sg}(H_\xi * U)$. Since $\bar{a}_\xi$ is generated by U in $\mathfrak{M}'$, there is a finite set $F \subseteq L' - L$ such that $F * U$ generates $\bar{a}_x i$ in $\mathfrak{M}$. Take $H_{\xi+1}$ to be a countable extension of $H_\xi \cup F$ in $L' - L$ such that $\mathfrak{Sg}(H_{\xi+1} * U) \preccurlyeq \mathfrak{M}$. Then $\mathfrak{Sg}(H_\xi * U) \preccurlyeq \mathfrak{Sg}(H_{\xi+1} * U)$. By construction, $\bar{a}_\xi$ realizes $p_\xi \restriction {}^* E_\xi$ in $\mathfrak{Sg}(H_{\xi+1} * U)$, so $\mathfrak{Sg}(H_\xi * U) \not\preccurlyeq_a \mathfrak{Sg}(H_{\xi+1} * U)$. This completes the verification of (1).

Set $G = \bigcup_{\xi<\omega_1} H_\xi$ and $\mathfrak{B} = \bigcup_{\xi<\omega_1} \mathfrak{Sg}(H_\xi * U) = \mathfrak{Sg}(G * U)$.

(2) For every $\xi < \omega_1$ and every countably infinite $X \subseteq U$ we have $\mathfrak{Sg}(H_\xi * X) \not\preccurlyeq_a \mathfrak{B}$.

Indeed, let W be a countable, dense subset of U containing X and containing finite subsets U_0 and U_1 of U such that $H_\xi * U_0$ generates E_ξ and $G * U_1$ generates $\bar{a}_\xi$ in $\mathfrak{M}$. Then, by Lemma 1(v),

(3) $\mathfrak{Sg}(G * W) \preccurlyeq \mathfrak{Sg}(G * U) = \mathfrak{B}$.

Therefore, to prove (2) it suffices to show that $\mathfrak{Sg}(H_\xi * X) \not\preccurlyeq_a \mathfrak{Sg}(G * W)$.

Since $p_\xi \restriction {}^* E_\xi$ is realized by $\bar{a}_\xi$ in $\mathfrak{Sg}(G * U)$, by construction, we see from (3) and the definition of W that it must also be realized by $\bar{a}_\xi$ in $\mathfrak{Sg}(G * W)$. Also,

(4) $\mathfrak{Sg}(H_\xi * W) \preccurlyeq \mathfrak{Sg}(H_\xi * U)$,

by Lemma 1(v). Since $p_\xi \restriction {}^* E_\xi$ is not realized in $\mathfrak{Sg}(H_\xi * U)$, by construction, we see from (4) that it is not realized in $\mathfrak{Sg}(H_\xi * W)$.

We now transfer this situation to a strong type over $\mathfrak{Sg}(H_\xi * X)$. Because $\mathfrak{W}$ is countable and dense, and X is infinite, there is an automorphism of $\mathfrak{W}$ taking U_0 into X (this is the point of introducing W to replace U). Extend it to an automorphism ϑ of $\mathfrak{N}' = \mathfrak{Sg}^{\mathfrak{M}'}(W)$. Then appropriate restrictions of ϑ are automorphisms of $\mathfrak{Sg}(G * W)$ and $\mathfrak{Sg}(H_\xi * W)$. Therefore,

(5) $\vartheta(p_\xi) \restriction^* \vartheta(E_\xi)$ is realized in $\mathfrak{Sg}(G * W)$ (by $\vartheta(\bar{a}_\xi)$), but not in $\mathfrak{Sg}(H_\xi * W)$.

Suppose, now, for contradiction, that $\mathfrak{Sg}(H_\xi * X) \preccurlyeq_a \mathfrak{Sg}(G * W)$. Then

(6) $\vartheta(p_\xi) \restriction^* \vartheta(E_\xi)$ is realized in $\mathfrak{Sg}(H_\xi * X)$.

On the other hand, this supposition also gives

(7) $\mathfrak{Sg}(H_\xi * X) \preccurlyeq \mathfrak{Sg}(G * W)$.

Since $\mathfrak{Sg}(H_\xi * U) \preccurlyeq \mathfrak{Sg}(G * U)$, by construction, we get from (3) and (4) that

(8) $\mathfrak{Sg}(H_\xi * W) \preccurlyeq \mathfrak{Sg}(G * W)$.

Combining (7) and (8), we arrive at $\mathfrak{Sg}(H_\xi * X) \preccurlyeq \mathfrak{Sg}(H_\xi * W)$. In view of (6), this forces $\vartheta(p_\xi) \restriction^* \vartheta(E_\xi)$ to be realized in $\mathfrak{Sg}(H_\xi * W)$, which contradicts (5). This proves (2)

We now construct a $\xi < \omega_1$ and a countable $X \subseteq U$ that contradict (2). Since $\mathfrak{B}$ is strictly κ-generated, by Lemma 2, it is isomorphic to $\mathfrak{F}_\kappa$. Thus, there is a set V of generators of $\mathfrak{B}$, of size κ, that is totally indiscernible in $\mathfrak{B}$. Exactly as in the proof of Lemma 3, we construct increasing sequences, $\langle X_n : n \in \omega \rangle$ and $\langle Y_n : n \in \omega \rangle$, of countable subsets of U and V respectively, and an increasing sequence $\langle \rho_n : n < \omega \rangle$ of countable ordinals, such that

$$\mathfrak{Sg}(H_{\rho_n} * X_n) \subseteq \mathfrak{Sg}(Y_n) \subseteq \mathfrak{Sg}(H_{\rho_{n+1}} * X_{n+1})$$

for every $n < \omega$. Set

$$\xi = \sup\{\rho_n : n < \omega\} \quad , \quad X = \bigcup_{n\in\omega} X_n \quad , \quad \text{and} \quad Y = \bigcup_{n<\omega} Y_n \quad .$$

Then

$$\mathfrak{Sg}(H_\xi * X) = \textstyle\bigcup_{n<\omega} \mathfrak{Sg}(H_{\rho_n} * X_n) = \bigcup_{n<\omega} \mathfrak{Sg}(Y_n) = \mathfrak{Sg}(Y) \quad .$$

Since $\mathfrak{Sg}(Y) \preccurlyeq_a \mathfrak{B}$ by Lemma 8 (with $K = \varnothing$) and the total indiscernibility of Y and V, we conclude that $\mathfrak{Sg}^{\mathfrak{M}}(H_\xi * X) \preccurlyeq_a \mathfrak{B}$. This is just the desired contradiction to (2). □

Lemma 10. *For every countable $H \subseteq L' - L$, there is a countable K, with $H \subseteq K \subseteq L' - L$, such that, whenever $\mathfrak{V}$ is a dense ordering and $\mathfrak{N}' = EM(\mathfrak{V}, \Phi)$, we have*

$$\mathfrak{Sg}^{\mathfrak{N}}(K * V) \preccurlyeq_a \mathfrak{N} \quad .$$

Proof. The proof is almost identical to the proof of the second part of Lemma 4, with "$\preccurlyeq$" replaced everywhere by "$\preccurlyeq_a$". One uses Lemma 9 in place of the first part of the proof of Lemma 4, and Lemma 8 in place of Lemma 1(v) and Facts 1 and 3. Notice, for example, that, taking $K = L' - L$, we get essentially Fact 3 with "$\preccurlyeq$" replaced by "$\preccurlyeq_a$". We leave the details to the reader. □

Lemma 11. *$\mathfrak{F}_\lambda$ is a-saturated for every $\lambda \geq \omega$.*

Proof. By Shelah [1987], the proof of Theorem 2.1 on pp. 285–287, T has an Ehrenfeucht-Mostowski model $\mathfrak{N}' = EM(\mathfrak{W}, \Delta)$ of power $> \max\{(2^{|T|})^+, \kappa\}$ such that $\mathfrak{N} = \mathfrak{N}' \restriction L$ is a-saturated. By Lemma 10 there is a countable $K \subseteq L' - L$ such that $\mathfrak{Sg}^{\mathfrak{N}}(K * W) \preccurlyeq_a \mathfrak{N}$. Let $\mathfrak{U}$ be a dense submodel of $\mathfrak{W}$ of power κ. Then $\mathfrak{Sg}^{\mathfrak{N}}(K * U) \preccurlyeq_a \mathfrak{Sg}^{\mathfrak{N}}(K * W)$, by Lemma 8. Thus, we immediately see that $\mathfrak{Sg}^{\mathfrak{N}}(K * U)$, too, is a-saturated. Since it is strictly κ-generated, by Lemma 2, it is isomorphic to $\mathfrak{F}_\kappa$, by κ-uniqueness. Thus, $\mathfrak{F}_\kappa$ is a-saturated.

Next, we turn to $\mathfrak{F}_\omega$. Recall that $\mathfrak{F}_\kappa$ has a Φ-basis Z_κ that extends a Φ-basis Z_ω of $\mathfrak{F}_\omega$. Hence, $\mathfrak{F}_\omega \preccurlyeq \mathfrak{F}_\kappa$ and even $\mathfrak{F}_\omega \preccurlyeq_a \mathfrak{F}_\kappa$, by Lemmas 1(v) and 8 (with $K = \varnothing$) and the total indiscernibility of Z_κ and Z_ω. Since $\mathfrak{F}_\kappa$ is a-saturated, by the previous lemma, so is $\mathfrak{F}_\omega$.

Now consider any $\lambda > \omega$. To show that $\mathfrak{F}_\lambda$ is a-saturated, consider an arbitrary strong typed p based on a finite subset E of F_λ. Let V a denumerably infinite subset of the Φ-basis Z_λ of $\mathfrak{F}_\lambda$ that contains a set of generators for E. Since $\mathfrak{Sg}(V)$ is isomorphic to $\mathfrak{F}_\omega$, and the latter is a-saturated, $p \restriction^* E$ is realized in $\mathfrak{Sg}(V)$. But $\mathfrak{Sg}(V) \preccurlyeq \mathfrak{F}_\lambda$, so $p \restriction^* E$ is also realized in $\mathfrak{F}_\lambda$. □

Lemma 12. *There is a cardinal λ such that every model of T of power $> \lambda$ is a-saturated.*

Proof. Let λ be the Hanf number for omitting types in languages of cardinality at most $|T|$. Suppose, for contradiction, that there is a model $\mathfrak{A}$ of T of power $> \max\{\lambda, \kappa\}$ that is not a-saturated. Then there is a strong type $p = p\restriction^* E$ (over $\mathfrak{A}$) based on a finite subset E of A such that p is omitted by $\mathfrak{A}$. Since there are at most $|T|$ many inequivalent formulas in p, by Shelah [1990], Chapter III, Lemma 2.2(2), we may assume that p is a (possibly incomplete) *type*—as opposed to a strong type—over some subset B of A of cardinality at most $|T|$.

Let L' be a language with built-in Skolem functions that extends L and that includes constants for the elements of B. Let T' be a Skolem theory in L' that extends the theory of $\langle \mathfrak{A}\,,\, b\rangle_{b\in B}$. By Morley's Omitting Types Theorem and the choice of λ, there is an Ehrenfeucht-Mostowski model $\mathfrak{N}'$ of T' over a dense ordering $\mathfrak{U}$ of indiscernibles of cardinality $\geq \lambda$ such that $\mathfrak{N}'$ omits the type p (see Morley [1965], Theorem 3.1, or Chang-Keisler [1973], Exercise 7.2.4). Let $\mathfrak{V}$ be a dense submodel of $\mathfrak{U}$ of power κ, and set $\mathfrak{M}' = \mathfrak{Sg}^{\mathfrak{N}'}(V)$. Then $\mathfrak{M}'$ is the corresponding Ehrenfeucht-Mostowski model over $\mathfrak{V}$, and $\mathfrak{M}' \preccurlyeq \mathfrak{N}'$, by Fact 3. Obviously, $\mathfrak{M}'$, and hence also $\mathfrak{M}$, omits p.

Let $H \subseteq L' - L$ be a finite set containing all the individual constant symbols for elements of E. By Lemma 4, there is a countable K with $H \subseteq K \subseteq L' - L$ such that $\mathfrak{Sg}(K * V) \preccurlyeq \mathfrak{M}$. Thus, E is a subset of the universe of $\mathfrak{Sg}(K*V)$. Since p, as a strong type, is based on a subset of the model $\mathfrak{Sg}(K*V)$, every formula φ of p is equivalent to a formula φ' with parameters from $Sg(K*V)$, by Shelah *ibid.*, Lemma 2.15(1). Let p' be the set of these formulas φ'. Then p' is a strong type based on E that is equivalent to p. Because $\mathfrak{M}$ omits p, it also omits p'. Therefore, the elementary substructure $\mathfrak{Sg}(K*V)$ must omit p'. This shows that $\mathfrak{Sg}(K*V)$ is not a-saturated. But that is impossible: since $\mathfrak{Sg}(K*V)$ is strictly κ-generated (by Lemma 2), it is isomorphic to $\mathfrak{F}_\kappa$, and we have seen that $\mathfrak{F}_\kappa$ is a-saturated. □

Lemma 13. *$\mathfrak{F}_\lambda$ is a-prime over any Φ-basis, for $\lambda \geq \omega$. Consequently, $\mathfrak{F}_\omega$ is also a-prime over $\varnothing$.*

Proof. Let U be a Φ-basis of $\mathfrak{F}_\lambda$. Then U generates any given sequence $\bar{a}$ from $\mathfrak{F}_\lambda$, so $tp(\bar{a}, U)$ is clearly atomic. In particular, $\mathfrak{F}_\lambda$ is a-atomic over U. Moreover, $\mathfrak{F}_\lambda$ is also a-saturated, by Lemma 11, and there are obviously no Morley sequences in $\mathfrak{F}_\lambda$ over U. By Shelah's second characterization theorem for a-prime models, $\mathfrak{F}_\lambda$ is a-prime over U (see Shelah [1990], Chapter IV, Definition 4.3 and Theorem 4.18).

Now an a-prime model over a countable set is also a-prime over $\varnothing$ (see *ibid.*). Hence, $\mathfrak{F}_\omega$ is also a-prime over $\varnothing$. □

We now work inside of some monster model. Let $p_0, \dots, p_{n-1}$ be pairwise orthogonal regular types, all based, say, on a finite set E. By realizing the strong type of E over $\varnothing$ in $\mathfrak{Pr}_a(\varnothing)$, and then passing to automorphic copies of $p_0, \dots, p_{n-1}$, we may assume that E is a subset of the universe of $\mathfrak{Pr}_a(\varnothing)$. For each $i < n$, let I_i be an infinite Morley sequence built from $p_i\restriction^* E$.

Lemma 14. *$\mathfrak{Pr}_a(E \cup \bigcup_{i<n} I_i)$ is strictly λ-generated, where $\lambda = |\bigcup_{i<n} I_i|$.*

Proof. The proof is by induction on λ. If $\lambda = \omega$, then, as mentioned in the preceding lemma, $\mathfrak{Pr}_a(E \cup \bigcup_{i<n} I_i)$ is a-prime over $\varnothing$. But, we just saw that $\mathfrak{F}_\omega$ is a-prime

over $\varnothing$. Hence, these two models are isomorphic, by the uniqueness of a-prime models. Since $\mathfrak{F}_\omega$ is strictly ω-generated, by Lemma 2, so is $\mathfrak{Pr}_a(E \cup \bigcup_{i<n} I_i)$.

Now suppose that the lemma is true for all infinite $\mu < \lambda$. Represent $\bigcup_{i<n} I_i$ as a strictly increasing sequence $\langle \bar{a}_\alpha : \alpha < \lambda \rangle$ with the property that, for each $i < n$, the sequence contains infinitely many members of I_i. For each infinite $\beta < \lambda$, set $\mathfrak{B}_\beta = \mathfrak{Pr}_a(E \cup \{\bar{a}_\alpha : \alpha < \beta\})$. We may arrange this so that $\mathfrak{B}_\beta \preccurlyeq \mathfrak{B}_\eta$ whenever $\beta \leq \eta$, and, at limit stages δ, that $\mathfrak{B}_\delta = \bigcup_{\beta<\delta} \mathfrak{B}_\beta$. By the induction hypothesis, each $\mathfrak{B}_\beta$ is $|\beta|$-generated, since $\omega \leq |\beta| < \lambda$. Therefore, $\mathfrak{Pr}_a(E \cup \bigcup_{i<n} I_i) = \bigcup_{\beta<\lambda} \mathfrak{B}_\beta$ is generated by a set of cardinality at most $\lambda \cdot \lambda = \lambda$.

To see that no set of smaller cardinality can generate it, we proceed by contradiction. Let X be an infinite set of power $\nu < \lambda$ that generates $\mathfrak{Pr}_a(E \cup \bigcup_{i<n} I_i)$. Since $\mathfrak{Pr}_a(E \cup \bigcup_{i<n} I_i)$ is a-atomic over $E \cup \bigcup_{i<n} I_i$, for each finite sequence $\bar{b}$ from X there is a finite subset $I_{\bar{b}} \subseteq \bigcup_{i<n} I_i$ such that

$$stp(\bar{b}, E \cup I_{\bar{b}}) \vdash stp(\bar{b}, E \cup \textstyle\bigcup_{i<n} I_i) \quad .$$

Set $J = \bigcup\{I_{\bar{b}} : \bar{b}$ is a finite sequence from $X\}$. Then J has power at most ν and

(1) $\quad stp(\bar{b}, E \cup J) \vdash stp(\bar{b}, E \cup \bigcup_{i<n} I_i)$ for every (finite) sequence $\bar{b}$ from X .

Since $\nu < \lambda$, there is a sequence $\bar{a} = \langle a_0, \dots, a_{k-1} \rangle$ that is in $\bigcup_{i<n} I_i$, but not in J. Thus, $\bar{a}$ is independent over $E \cup J$. This is readily seen to contradict (1). For example, because X generates $\mathfrak{Pr}_a(E \cup \bigcup_{i<n} I_i)$, there is a sequence $\bar{b}$ from X and terms $\sigma_i(\bar{x})$ of L, for $i < k$, such that $\sigma_i[\bar{b}] = a_i$. Now

$$stp(\bar{b}, E \cup J) \vdash \bigwedge_{i<k} \sigma_i(\bar{x}) = a_i \quad ,$$

by (1), so there is a formula $\varphi(\bar{x})$ in $L^*(E \cup J)$ (the set of formulas almost over $E \cup J$) such that

$$\vdash \varphi(\bar{x}) \to \bigwedge_{i<k} \sigma_i(\bar{x}) = a_i \quad .$$

Thus, $\bar{a}$ is definable, over $L^*(E \cup J)$, since, e.g.,

$$\vdash y = a_i \leftrightarrow \exists \bar{x}(\varphi(\bar{x}) \wedge \sigma_i(\bar{x}) = y) \qquad \text{for each } i < k \quad .$$

But then $\bar{a}$ is in the algebraic closure of $E \cup J$, so it cannot be independent over this set. We have reached our contradiction. $\square$

Theorem 15. *T is categorical in every cardinality $> |T|$.*

Proof. We begin by proving that T must be unidimensional. Indeed, suppose for contradiction that there are two orthogonal regular types, p_1 and p_2, over a finite subset E of $\mathfrak{Pr}_a(\varnothing)$. Working inside a monster model, for $i = 1, 2$, let I_i be a Morley sequence of cardinality κ built from $p_i \restriction^* E$. Then $\mathfrak{Pr}_a(E \cup I_1)$ and $\mathfrak{Pr}_a(E \cup I_1 \cup I_2)$ are both strictly κ-generated models, by the previous lemma, and hence isomorphic, by κ-uniqueness. But this is impossible: the first model has a single dimension of cardinality κ, the p_1-dimension, and all the other dimensions are ω; the second model has exactly two dimensions of cardinality κ, the p_1- and the p_2-dimensions. Thus, T can only have one regular type, up to equivalence. Let's denote this type by p.

Now let λ ($\geq |T|$) be so large that all the models of T of power $> \lambda$ are a-saturated (see Lemma 12). Then T is λ^+-categorical. Indeed, if $\mathfrak{M}$ and $\mathfrak{N}$ are models of power λ^+, then they both are a-saturated, by choice of λ, and have p-dimension λ^+, by unidimensionality. Hence, they must be isomorphic, by Shelah [1990], Chapter V, Theorem 2.10. This shows that T is categorical in some power $> |T|$. By the results of Shelah [1974], it is categorical in every power $> |T|$. $\square$

Theorem 16. *T is λ-unique for every uncountable λ.*

Proof. If T is definitionally equivalent to a countable theory, and is categorical in every infinite power, i.e., if T is a totally categorical theory, then the theorem is obvious. Suppose, now, that it is not totally categorical. We shall use Theorems 1–3, including the proof of Theorem 2, from Laskowski [1988]. According to these, T_∞ must have a minimal prime model $\mathfrak{M}_0$. Moreover, there is a type p based on M_0 with the following properties:

(1) If $\mathfrak{N}$ is any elementary extension of $\mathfrak{M}_0$, then any two maximal Morley sequences in $\mathfrak{N}$ built from $p{\restriction}M_0$ must have the same cardinality. This is called the p-dimension of $\mathfrak{N}$.

(2) If $\mathfrak{N}$ and $\mathfrak{N}'$ are elementary extensions of $\mathfrak{M}_0$, with the same p-dimension, then there is an isomorphism of $\mathfrak{N}$ onto $\mathfrak{N}'$ that fixes M_0 .

(3) If $\mathfrak{M}_0 \preccurlyeq \mathfrak{N} \prec \mathfrak{N}'$, then $p{\restriction}N$ is realized in $\mathfrak{N}'$.

To simplify notation, we shall assume that p is a 1-type.

Since T_∞ is a universal-existential theory categorical in power, it is model complete, by Lindström's theorem (see, e.g., Chang-Keisler [1973], Theorem 3.1.12).

Fix $\lambda > \omega$, and let $\mathfrak{M}$ be a strictly λ-generated model of T, say X is a generating set of power λ. Our goal is to prove that $\mathfrak{M}$ has p-dimension λ. Let W be a countably infinite subset of X. Then $\mathfrak{Sg}(W) \preccurlyeq \mathfrak{M}$, by model completeness. Without loss of generality, we may suppose that $\mathfrak{M}_0$ is an elementary submodel of $\mathfrak{Sg}(W)$. We now define a strictly increasing, continuous sequence, $\langle Y_\xi : \xi < \lambda\rangle$, of subsets of X of size $< \lambda$, such that

(4) $\mathfrak{M}_0 \preccurlyeq \mathfrak{Sg}(Y_\xi) \prec \mathfrak{Sg}(Y_{\xi+1}) \preccurlyeq \mathfrak{M}$ for $\xi < \lambda$.

Indeed, take $Y_0 = W$. If Y_ξ has been defined, then $\mathfrak{Sg}(Y_\xi) \neq \mathfrak{M}$, since $\mathfrak{M}$ is strictly λ-generated and Y_ξ has power $< \lambda$. Hence, there is an element u_ξ in X that is not generated by Y_ξ. We set $Y_{\xi+1} = Y_\xi \cup \{u_\xi\}$. Property (4) follows by our choice of Y_0 and u_ξ, for each ξ, and by model completeness.

For each $\xi < \lambda$, the type $p \restriction Sg(Y_\xi)$ is realized in $\mathfrak{Sg}(Y_{\xi+1})$, by (3) and (4). Therefore, using our strictly increasing chain $\langle \mathfrak{Sg}(Y_\xi) : \xi < \lambda\rangle$ of elementary substructures of $\mathfrak{M}$, we can build from $p{\restriction}M_0$ a Morley sequence in $\mathfrak{M}$ of length at least λ. Thus, $\mathfrak{M}$ has p-dimension at least λ.

Suppose now that I is a maximal Morley sequence in $\mathfrak{M}$ built from $p{\restriction}M_0$. Then $\mathfrak{Sg}(M_0 \cup I) = \mathfrak{M}$. For otherwise we would have $\mathfrak{Sg}(M_0 \cup I) \prec \mathfrak{M}$, by model completeness. Hence, we could realize $p{\restriction}Sg(M_0 \cup I)$ in $\mathfrak{M}$, by (3), and thus extend I to a larger Morley sequence in $\mathfrak{M}$ built from $p{\restriction}M_0$, contradicting its maximality. Since X also generates $\mathfrak{M}$, and has cardinality $\lambda > \omega$, there is a subset J of I of cardinality $\leq \lambda$ such that $M_0 \cup J$ generates X, and hence also $\mathfrak{M}$. In particular, $M_0 \cup J$ generates I. But then $J = I$, since any element in $I - J$ would be independent over $M_0 \cup J$, and hence could not be generated by this set. We conclude that $|I| \leq \lambda$. In other words, $\mathfrak{M}$ has p-dimension at most λ, and hence exactly λ.

We have shown:

(6) Any strictly λ-generated model of T extending $\mathfrak{M}_0$ has p-dimension λ .

Now let $\mathfrak{M}$ and $\mathfrak{N}$ be any two strictly λ-generated models of T. By passing to isomorphic copies, we may assume that $\mathfrak{M}_0$ is an elementary substructure of each. Hence, by (2) and (6), $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic over M_0. This shows that T is λ-unique. □

By the preceding theorem, the noncountably generated models of T are, up to isomorphisms, precisely the structures $\mathfrak{F}_\lambda$, for $\lambda > \omega$.

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