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Pointwise compact and stable sets of measurable functions

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Introduction In a series of papers culminating in [Ta84], M.Talagrand, the second author and others investigated at length the properties and structure of pointwise compact sets of measurable functions. A number of problems, interesting in themselves and important for the theory of Pettis integration, were solved subject to various special axioms. It was left unclear just how far the special axioms were necessary. In particular, several results depended on the fact that it is consistent to suppose that every countable relatively pointwise compact set of Lebesgue measurable functions is 'stable' in Talagrand's sense; the point being that stable sets are known to have a variety of properties not shared by all pointwise compact sets. In the present paper we present a model of set theory in which there is a countable relatively pointwise compact set of Lebesgue measurable functions which is not stable, and discuss the significance of this model in relation to the original questions. A feature of our model which may be of independent interest is the following: in it, there is a closed negligible set $Q \subseteq [0,1]^2$ such that whenever $D \subseteq [0,1]$ has *outer* measure 1 then

$$Q^{-1}[D] = \{x : \exists\ y \in D, (x,y) \in Q\}$$

has *inner* measure 1 (see 2G below).

1. The model We embark immediately on the central ideas of this paper, setting out a construction of a partially ordered set which forces a fairly technical proposition in measure theory (1S below); the relevance of this proposition to pointwise compact sets will be discussed in §2. The construction is complex, and rather than give it in a single stretch we develop it cumulatively in 1E, 1I, 1Q below; it is to be understood that each notation introduced in these paragraphs, as well as those in the definitions 1A, 1K, 1L, is to stand for the remainder of the section. After each part of the construction we give lemmas which can be dealt with in terms of the construction so far, even if their motivation is unlikely to be immediately clear.

When we come to results involving Forcing, we will try to follow the methods of [Ku80]; in particular, in a p.o.set, '$p \leq q$' will always mean that p is a stronger condition than q.

1A Definition If $\mathcal{A}$ is any family of sets not containing $\emptyset$, set

$$\mathrm{dp}(\mathcal{A}) = \min\{\#(I) : I \cap A \neq \emptyset\ \forall\ A \in \mathcal{A}\}.$$

Observe that $\mathrm{dp}(\mathcal{A}) = 0$ iff $\mathcal{A} = \emptyset$ and that $\mathrm{dp}(\mathcal{A} \cup \mathcal{B})$ is at most the cardinal sum of $\mathrm{dp}(\mathcal{A})$ and $\mathrm{dp}(\mathcal{B})$. (Of course much more can be said.)

1B Lemma Suppose that $n, l, k \in \mathbb{N}$, with n, l not less than 2, and that ϵ is such that $0 < \epsilon \leq \frac{1}{2}$ and $l\epsilon^k \geq (k+2)\ln n$. Then there is a set $W \subseteq n \times n$ (we identify n with the set of its predecessors) such that $\#(W) \leq \epsilon n^2$ and whenever $I \in [n]^l$ and $J_0, \ldots, J_{l-1} \in [n]^{\leq k}$ are disjoint, there are $i \in I$, $j < l$ such that $\{i\} \times J_j \subseteq W$.

proof If $k = 0$ this is trivial; suppose that $k > 0$. Set $\Omega = \mathcal{P}(n \times n)$. Give Ω a probability for which the events $(i,j) \in W$, as (i,j) runs over $n \times n$, are independent with probability ϵ. If $W \in \Omega$ is a random set, then

$$\Pr(\#(W) \leq \epsilon n^2) > \tfrac{1}{4}$$

because $\epsilon \leq \frac{1}{2}$ and $\#(W)$ has the binomial distribution $B(n^2, \epsilon)$. On the other hand, if $J \in [n]^{\leq k}$ and $i < n$, $\Pr(\{i\} \times J \subseteq W) \geq \epsilon^k$. So if $I \in [n]^l$ and $J_0, \ldots, J_{l-1}$ are disjoint members of $[n]^{\leq k}$,

$$\Pr(\{i\} \times J_j \not\subseteq W \ \ \forall\ i \in I,\ j < l) \leq (1-\epsilon^k)^{l^2} \\ \leq \exp(-l^2\epsilon^k).$$

Accordingly

$$\begin{aligned}\Pr(\exists\ I \in [n]^l,\ \text{disjoint}\ J_0, \dots, J_{l-1} \in [n]^{\leq k}\ \text{such that}\ \{i\} \times J_j \not\subseteq W \ \ \forall\ i \in I,\ j < l) \\ \leq \#([n]^l)\#([n]^{\leq k})^l \exp(-l^2\epsilon^k) \\ \leq n^l n^{kl} \exp(-l^2\epsilon^k) \\ = \exp((k+1)l\ln n - l^2\epsilon^k) \leq \frac{1}{4}\end{aligned}$$

because

$$l^2\epsilon^k - (k+1)l\ln n \geq l\ln n \geq 2\ln 2.$$

There must therefore be some $W \in \Omega$ of the type required.

Remark Compare the discussion of cliques in random graphs in [Sp87], pp. 18-20.

1C Lemma Let m and l be strictly positive integers and $\mathcal{A}$ a non-empty family of non-empty sets. Let $\mathbb{T}$ be the family of non-empty sets $\mathcal{T} \subseteq \mathcal{A}^m$. For $\mathcal{T} \in \mathbb{T}$ write $\mathcal{T}^* = \{\mathbf{t}{\restriction}j : \mathbf{t} \in \mathcal{T},\ j \leq m\} \subseteq \bigcup_{j \leq m} \mathcal{A}^j$. For $\mathcal{T}$, $\mathcal{T}_0 \in \mathbb{T}$ say that $\mathcal{T} \preccurlyeq \mathcal{T}_0$ if $\mathcal{T} \subseteq \mathcal{T}_0$ and

$$\mathrm{dp}(\{u : \mathbf{t}^\frown u \in \mathcal{T}^*\}) \geq \mathrm{dp}(\{u : \mathbf{t}^\frown u \in \mathcal{T}_0^*\})/2l$$

for every $\mathbf{t} \in \mathcal{T}^* \setminus \mathcal{T}$. Fix $\mathcal{T}_0 \in \mathbb{T}$ and a cover $\langle \mathcal{S}_i \rangle_{i<2l}$ of $\mathcal{T}_0$. Then there is a $\mathcal{T} \preccurlyeq \mathcal{T}_0$ such that $\mathcal{T} \subseteq \mathcal{S}_i$ for some $i < 2l$.

[**Notation:** In this context we use ordinary italics, 'u', for members of $\mathcal{A}$, and bold letters, '$\mathbf{t}$', for finite sequences of members of $\mathcal{A}$.]

proof For $\mathbf{t} \in \mathcal{T}_0^* \setminus \mathcal{T}_0$ set

$$\alpha_{\mathbf{t}} = \mathrm{dp}(\{u : \mathbf{t}^\frown u \in \mathcal{T}_0^*\})/2l > 0.$$

For $i < 2l$ define $\langle \mathcal{S}_i^{(j)} \rangle_{j \leq m}$ by setting $\mathcal{S}_i^{(m)} = \mathcal{S}_i \cap \mathcal{T}_0$,

$$\mathcal{S}_i^{(j)} = \{\mathbf{t} : \mathbf{t} \in \mathcal{A}^j \cap \mathcal{T}_0^*,\ \mathrm{dp}(\{u : \mathbf{t}^\frown u \in \mathcal{S}_i^{(j+1)}\}) \geq \alpha_{\mathbf{t}}\}$$

for $j < m$. An easy downwards induction (using the fact that dp is subadditive) shows that $\mathcal{T}_0^* \cap \mathcal{A}^j = \bigcup_{i<2l} \mathcal{S}_i^{(j)}$ for every $j \leq m$. In particular, there is some $i < 2l$ such that $\emptyset \in \mathcal{S}_i^{(0)}$. Now define $\mathcal{T}$ by

$$\mathcal{T} = \{\mathbf{t} : \mathbf{t} \in \mathcal{A}^m,\ \mathbf{t}{\restriction}j \in \mathcal{S}_i^{(j)} \ \ \forall\ j \leq m\} \subseteq \mathcal{T}_0 \cap \mathcal{S}_i,$$

and see that $\mathcal{T} \preccurlyeq \mathcal{T}_0$, as required.

1D Corollary Let n, l, k and W be as in Lemma 1B. Take $r \leq k$, let Z be the cartesian product n^r and set

$$\tilde{W} = \{(i,z) : i < n,\ z \in Z,\ (i, z(j)) \in W \ \ \forall\ j < r\}.$$

Let m, $\mathcal{A}$, $\mathbb{T}$ and $\preccurlyeq$ be as in Lemma 1C, and take $\mathcal{T}_0 \in \mathbb{T}$, $H : \mathcal{T}_0 \to n$ any function. Then

either there are $i < n$, $\mathcal{T} \preccurlyeq \mathcal{T}_0$ such that $H(\mathbf{t}) = i$ for every $\mathbf{t} \in \mathcal{T}$

or there is a $J \in [n]^{\leq rl}$ such that for every $z \in (n \setminus J)^r$ there is a $\mathcal{T} \preccurlyeq \mathcal{T}_0$ such that $(H(\mathbf{t}), z) \in \tilde{W}$ for every $\mathbf{t} \in \mathcal{T}$.

proof Set

$$A = \{z : z \in Z,\ \exists\ \mathcal{T} \preccurlyeq \mathcal{T}_0 \text{ such that } (H(\mathbf{t}), z) \in \tilde{W} \ \ \forall\ \mathbf{t} \in \mathcal{T}\}.$$

If $A \supseteq (n \setminus J)^r$ for some $J \in [n]^{\leq rl}$, we have the second alternative; suppose otherwise. Then we can find $z_0, \dots, z_{l-1} \in Z \setminus A$ such that the sets $J_j = \{z_j(i) : i < r\}$ are all disjoint. Each J_j belongs to $[n]^{\leq k}$, so by the choice of W,

$$I = \{i : \{i\} \times J_j \not\subseteq W \ \ \forall\ j < l\}$$

has cardinal less than l. Now observe that if $\mathbf{t} \in \mathcal{T}_0$ then either $H(\mathbf{t}) \in I$ or $(H(\mathbf{t}), z_j) \in \tilde{W}$ for some $j < l$. So we have a cover of $\mathcal{T}_0$ by the sets

$$\mathcal{S}_i = \{\mathbf{t} : H(\mathbf{t}) = i\} \text{ for } i \in I,$$
$$\mathcal{S}'_j = \{\mathbf{t} : (H(\mathbf{t}), z_j) \in \tilde{W}\} \text{ for } j < l.$$

By Lemma 1C, there is a $\mathcal{T} \preccurlyeq \mathcal{T}_0$ such that either $\mathcal{T} \subseteq \mathcal{S}_i$ for some $i \in I$ or $\mathcal{T} \subseteq \mathcal{S}'_j$ for some $j < l$. But we cannot have $\mathcal{T} \subseteq \mathcal{S}'_j$, because $z_j \notin A$; so $\mathcal{T} \subseteq \mathcal{S}_i$ for some i, and we have the first alternative.

Remark 1C-1D are of course elementary, but their significance is bound to be obscure; they will be used in 1R below. An essential feature of 1C is the fact that the denominator '$2l$' in the definition of $\preccurlyeq$ is independent of the size of $\mathcal{A}$.

1E Construction: part 1 (a) Take a sequence $\langle n_k \rangle_{k\in\mathbb{N}}$ of integers increasing so fast that
(i) $n_0 \geq 4$;
(ii) $n_k > 2^{k+1}$;
(iii) writing $\tilde{c}_l = \prod_{i<l} 2^{n_i}$, then
$\ln(2^{-k-1}n_k) \geq 2^l(k+1)(\tilde{c}_l^{l+1}\ln 2 + \tilde{c}_l^{l-1}\ln n_{l-1})$ for $1 \leq l \leq k$;
(iv) $\ln n_k \geq (2^{k+1})^k(k+2)$;
(v) writing $\lceil\alpha\rceil$ for the least integer greater than or equal to α,
$(k+1)\ln(2\lceil(\ln n_k)^2\rceil) \leq 2^{-k}\lceil\ln(2^{-k-1}n_k)\rceil$;
(vi) $2^k k\lceil(\ln n_k)^2\rceil(\prod_{i<k} 2^{n_i})^{k+1} \leq n_k$;
(vii) $\ln(2^{-k-1}n_k) \geq (k+1)\ln(2\tilde{c}_k^{k+1} + 2k)$
for every $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, let V_k be the cartesian product $\prod_{i<k} n_i$.

(b) For each $k \in \mathbb{N}$, let T_k be the set of those subsets t of V_k expressible as $t = \prod_{i<k} C_i(t)$ where $C_i(t) \subseteq n_i$ and $\#(C_i(t)) \geq (1 - 2^{-i-1})n_i$ for each $i < k$. Set $T = \bigcup_{k\in\mathbb{N}} T_k$, and for $t \in T$ say that $\mathrm{rank}(t) = k$ if $t \in T_k$. For t, $t' \in T$ say that $t \leq t'$ if $\mathrm{rank}(t) \leq \mathrm{rank}(t')$ and $C_i(t) = C_i(t')$ for every $i < \mathrm{rank}(t)$. Then T is a finitely-branching tree of height ω in which the T_k are the levels and 'rank' is the rank function. For $t \in T$ write $T^{(t)}$ for the subtree $\{t' : t' \leq t \text{ or } t \leq t'\}$, $\mathrm{suc}(t)$ for $\{t' : t \leq t', \mathrm{rank}(t') = \mathrm{rank}(t) + 1\}$.

(c) For $k \in \mathbb{N}$, set
$$\gamma_k = (k+1)/\ln(\lceil 2^{-k-1}n_k\rceil);$$
$2^{-k-1}n_k > 1$ by (a)(ii) above. For $t \in T$ define $d_t : \mathcal{P}T \to \mathbb{R} \cup \{-\infty\}$ by writing
$$d_t(S) = \gamma_{\mathrm{rank}(t)}\ln(\mathrm{dp}(\{C : t \times C \in S\}))$$
for every $S \subseteq T$, allowing $d_t(S) = -\infty$ if $S \cap \mathrm{suc}(t) = \emptyset$. Observe that $d_t(T) \geq k+1$ whenever $\mathrm{rank}(t) = k$ (because
$$\mathrm{dp}(\{C : C \subseteq n_k, \#(C) \geq (1 - 2^{-k-1})n_k\}) \geq \lceil 2^{-k-1}n_k\rceil.)$$

(d) Let $\mathbb{Q}$ be the set of subtrees $q \subseteq T$ such that
$q \neq \emptyset$;
if $t \leq t' \in q$ then $t \in q$;
if $t \in q$ then $q \cap \mathrm{suc}(t) \neq \emptyset$;
writing $\delta_k(q) = \min\{d_t(q) : t \in q \cap T_k\}$ for $k \in \mathbb{N}$, $\lim_{k\to\infty}\delta_k(q) = \infty$.
Observe that $\delta_k(T) \geq k+1$ for each $k \in \mathbb{N}$, so that $T \in \mathbb{Q}$ and $\mathbb{Q} \neq \emptyset$.

(e) For q, $q' \in \mathbb{Q}$ say that $q \leq q'$ if $q \subseteq q'$. Then $(\mathbb{Q}, \leq, T)$ is a p.o.set (that is, a pre-ordered set with a top element, as in [Ku80]). Observe that if $t \in q \in \mathbb{Q}$ then $q \cap T^{(t)} \in \mathbb{Q}$ and $q \cap T^{(t)} \leq q$.

(f) For q, $q' \in \mathbb{Q}$ and $k \in \mathbb{N}$ say that $q \leq_k q'$ if $q \leq q'$ and $q \cap T_k = q' \cap T_k$ and
$$d_t(q) \geq \min(k, d_t(q')) - 2^{-k}$$
for every $t \in q$. Note that $\leq_k$ is not transitive unless $k = 0$.

Remarks Of course the point of the sequence $\langle n_k \rangle_{k\in\mathbb{N}}$ on which the rest of this construction will depend is that it increases 'as fast as we need it to'. The exact list given in (a) above is of no significance and will be used only as a list of clues to the (elementary) arguments below which depend on the rapidly-increasing nature of the sequence. This is why we have made no attempt to make the list as elegant or as short as possible.

Three elements may be distinguished within the construction of $\mathbb{Q}$. First, it is a p.o.set of rapidly branching trees; that is, if $t \in q \in \mathbb{Q}$, $q \cap \mathrm{suc}(t)$ is large compared with $T_{\mathrm{rank}(t)}$, except for t of small rank. This is the basis of most of the (laborious but routine) work down to 1P below. Second, there is a natural $\mathbb{Q}$-name for a subset of $X = \prod_{k\in\mathbb{N}} n_k$ of large measure; a generic filter in $\mathbb{Q}$ leads to a branch of T and hence to the Ψ of 1Q(d). Third, the use of dp in the definition of 'rapidly branching' ((c)-(d) above) is what makes possible the side-step in the last part of the proof of 1R.

1F Lemma $\mathbb{Q}$ is proper.

proof This is a special case of Proposition 1.18 in [Sh326]. (In fact, the arguments of 1G-1H below show that $\mathbb{Q}$ satisfies Axiom A, and is therefore proper; see [Ba84], 2.4.)

1G Lemma Let $k \in \mathbb{N}$ and let ζ be an ordinal. Suppose that A is a set with $\#(A) \leq \exp(2^{-k}/\gamma_j) - 1$ for every $j \geq k$, and that τ is a $\mathbb{Q}$-name for a member of A. Let Δ be a $\mathbb{Q}$-name for a countable subset of ζ. Then for every $q \in \mathbb{Q}$ there are a $q' \leq_k q$, a function $H : T_k \to A$ and a countable (ground-model) set $D \subseteq \zeta$ such that

$$q' \cap T^{(t)} \Vdash_{\mathbb{Q}} \tau = H(t) \;\; \forall \; t \in q' \cap T_k,$$
$$q' \Vdash_{\mathbb{Q}} \Delta \subseteq D.$$

proof (a) Set $m = \#(A)$. The point is that if $j \geq k$ and $t \in T_j$ and $\langle S_i \rangle_{i \leq m}$ is a family of subsets of T, then

$$\begin{aligned} d_t(\bigcup_{i \leq m} S_i) &= \gamma_j \ln(\mathrm{dp}(\bigcup_{i \leq m} \{C : t \times C \in S_i\})) \\ &\leq \gamma_j \ln(\sum_{i \leq m} \mathrm{dp}(\{C : t \times C \in S_i\})) \\ &\leq \gamma_j \ln((m+1) \max_{i \leq m} \mathrm{dp}(\{C : t \times C \in S_i\})) \\ &= \gamma_j \ln(m+1) + \max_{i \leq m} \gamma_j \ln(\mathrm{dp}(\{C : t \times C \in S_i\})) \\ &\leq 2^{-k} + \max_{i \leq m} d_t(S_i). \end{aligned}$$

(b) For each $a \in A$, let S_a be the set

$$\{t : t \in q, \mathrm{rank}(t) \geq k, \exists \; p \in \mathbb{Q}, \, D \in [\zeta]^{\leq \omega},$$
$$p \leq_k q \cap T^{(t)}, \, p \Vdash_{\mathbb{Q}} \tau = a \;\; \& \;\; \Delta \subseteq D\}.$$

If $t \in q \setminus S_a$ and $\mathrm{rank}(t) \geq k$, then $d_t(S_a) < \min(k, d_t(q)) - 2^{-k}$. For if $S_a \cap \mathrm{suc}(t) = \emptyset$, $d_t(S_a) = -\infty$. While if $S_a \cap \mathrm{suc}(t) \neq \emptyset$, then for each $s \in \mathrm{suc}(t) \cap S_a$ we can find $p_s \in \mathbb{Q}$ and $D_s \in [\zeta]^{\leq \omega}$ such that $p_s \leq_k q \cap T^{(s)}$, $p_s \Vdash_{\mathbb{Q}} \tau = a$ and $p_s \Vdash_{\mathbb{Q}} \Delta \subseteq D_s$. If we now set

$$p = \bigcup_{s \in \mathrm{suc}(t) \cap S_a} p_s, \;\; D = \bigcup_{s \in \mathrm{suc}(t) \cap S_a} D_s,$$

then $p \subseteq q \cap T^{(t)}$ and $p \Vdash_{\mathbb{Q}} \tau = a$ and $p \Vdash_{\mathbb{Q}} \Delta \subseteq D$. Because $t \notin S_a$, $p \not\leq_k q \cap T^{(t)}$ and there must be an $s \in p$ such that $d_s(p) < \min(k, d_s(q \cap T^{(t)})) - 2^{-k}$; evidently $s = t$ and

$$d_t(S_a) < \min(k, d_t(q)) - 2^{-k},$$

as claimed.

(c) Suppose, if possible, that there is a $t_0 \in q \cap T_k \setminus \bigcup_{a \in A} S_a$. Set

$$p = \{t : t \in q \cap T^{(t_0)}, \, t' \notin \bigcup_{a \in A} S_a \;\; \forall \; t' \leq t\}.$$

Then p is a subtree of T. For every $t \in p$ with $t \geq t_0$,

$$d_t(q) \leq \max(\{d_t(p)\} \cup \{d_t(S_a) : a \in A\}) + 2^{-k}$$

because $\#(A) = m$. But $d_t(S_a) < d_t(q) - 2^{-k}$ for every $a \in A$, by (b) above, so $d_t(p) \geq d_t(q) - 2^{-k}$ (and $p \cap \mathrm{suc}(t) \neq \emptyset$). This shows both that p has no maximal elements and that $\delta_i(p) \geq \delta_i(q) - 2^{-k}$ for every $i \geq k$, so that $p \in \mathbb{Q}$. Because $\mathbb{Q}$ is proper, we can find a $p' \leq p$ and a countable $D \subseteq \zeta$ such that $p' \Vdash \Delta \subseteq D$ ([Sh82], p. 81, III.1.16). Next, there are $p'' \leq p'$, $a \in A$ such that $p'' \Vdash_{\mathbb{Q}} \tau = a$. Let $j \in \mathbb{N}$ be such that $\delta_i(p'') \geq k$ whenever $i \geq j$, and take $t \in p''$ such that $\mathrm{rank}(t) \geq \max(k, j)$. Then $p'' \cap T^{(t)}$ witnesses that $t \in S_a$; which is impossible.

(d) Accordingly we have for every $t \in q \cap T_k$ an $H(t) \in A$, a countable set D_t and a $p_t \in \mathbb{Q}$ such that

$$p_t \Vdash_{\mathbb{Q}} \tau = H(t) \;\; \& \;\; \Delta \subseteq D_t,$$
$$p_t \leq_k q \cap T^{(t)}.$$

Set $q' = \bigcup_{t \in q \cap T_k} p_t$, $D = \bigcup_{t \in q \cap T_k} D_t$; then $q' \leq_k q$, D is countable, $q' \Vdash_{\mathbb{Q}} \Delta \subseteq D$ and $q' \cap T^{(t)} \Vdash_{\mathbb{Q}} \tau = H(t)$ for every $t \in q \cap T_k$.

1H Lemma Let $\langle q_k \rangle_{k \in \mathbb{N}}$ be a sequence in $\mathbb{Q}$ such that $q_{2k+2} \leq_{k+1} q_{2k+1} \leq_k q_{2k}$ for every $k \in \mathbb{N}$. Then $\hat{q} = \bigcap_{k \in \mathbb{N}} q_k$ belongs to $\mathbb{Q}$ and is accordingly a lower bound for $\{q_k : k \in \mathbb{N}\}$ in $\mathbb{Q}$; also $\hat{q} \cap T_{k+1} = q_{2k+1} \cap T_{k+1}$ for each $k \in \mathbb{N}$.

proof Because each q_k is a finitely-branching subtree of T with no maximal elements, so is $\hat{q}$, and $d_t(\hat{q}) = \lim_{k \to \infty} d_t(q_k)$ for every $t \in \hat{q}$. Moreover, if $t \in \hat{q}$ and $k \leq l \in \mathbb{N}$,

$$d_t(q_{2l}) \geq \min(k, d_t(q_{2k})) - 3.2^{-k} + 3.2^{-l},$$

(induce on l, using the definition of $\leq_l$), so we have

$$\delta_i(\hat{q}) = \lim_{l \to \infty} \delta_i(q_{2l}) \geq \min(k, \delta_i(q_{2k})) - 3.2^{-k}$$

for every i, $k \in \mathbb{N}$; consequently $\lim_{i \to \infty} \delta_i(\hat{q}) = \infty$ and $\hat{q} \in \mathbb{Q}$. Now if $k \in \mathbb{N}$ and $2k+1 \leq l$, $q_{l+1} \cap T_{k+1} = q_l \cap T_{k+1}$, so $\hat{q} \cap T_{k+1} = q_{2k+1} \cap T_{k+1}$.

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1I Construction: part 2 Let κ be the cardinal $\mathfrak{c}^+$ (evaluated in the ground model).

(a) Let $(\langle \mathbb{P}_\xi \rangle_{\xi \leq \kappa}, \langle \mathbb{Q}_\xi \rangle_{\xi < \kappa})$ be a countable-support iteration of p.o.sets, as in [Ku80], chap. 8, such that each $\mathbb{Q}_\xi$ is a $\mathbb{P}_\xi$-name for a p.o.set with the same definition, interpreted in $V^{\mathbb{P}_\xi}$, as the p.o.set $\mathbb{Q}$ of 1E. (Note that T is absolute, and so, in effect, is $\langle d_t \rangle_{t \in T}$, because each d_t is determined by its values on the finite set $\mathcal{P}(\text{suc}(t))$; so that the difference between $\mathbb{Q}$ and $\mathbb{Q}_\xi$ subsists in the power of $\mathbb{P}_\xi$ to add new subsets of T. Also each $\mathbb{Q}_\xi$ is 'full' in Kunen's sense.) Write $\mathbb{P} = \mathbb{P}_\kappa$.

(b) If $\zeta \leq \kappa$, $K \in [\zeta]^{<\omega}$, $k \in \mathbb{N}$ and p, p' belong to $\mathbb{P}_\zeta$, say that $p \leq_{K,k} p'$ if $p \leq p'$ and

$$p \restriction \xi \Vdash_{\mathbb{P}_\xi} p(\xi) \leq_k p'(\xi) \ \forall \ \xi \in K,$$

taking $\leq_k$ here to be a $\mathbb{P}_\xi$-name for the relation on $\mathbb{Q}_\xi$ corresponding to the relation $\leq_k$ on $\mathbb{Q}$ as defined in 1E(f). Of course $\leq_{K,k}$ is not transitive unless $K = \emptyset$ or $k = 0$.

1J Lemma (a) $\mathbb{P}_\zeta$ is proper for every $\zeta \leq \kappa$.

(b) If $\xi < \kappa$, $\zeta \leq \kappa$ then $\mathbb{P}_{\xi+\zeta}$ may be identified with a dense subset of the iteration $\mathbb{P}_\xi * \mathbb{P}'_\zeta$, where $\mathbb{P}'_\zeta$ is a $\mathbb{P}_\xi$-name with the same definition, interpreted in $V^{\mathbb{P}_\xi}$, as the definition of $\mathbb{P}_\zeta$ in V.

(c) For every $\zeta < \kappa$,

$$\mathbb{1}_{\mathbb{P}_\zeta} \Vdash_{\mathbb{P}_\zeta} 2^\omega < \kappa.$$

(d) If $\zeta \leq \kappa$ has uncountable cofinality, A is a (ground-model) set, $\dot{f}$ is a $\mathbb{P}_\zeta$-name for a sequence in A and $p \in \mathbb{P}_\zeta$, then we can find $\xi < \zeta$, $p' \leq p$ and a $\mathbb{P}_\xi$-name $\dot{g}$ such that

$$p' \Vdash_{\mathbb{P}_\zeta} \dot{f} = \dot{g}.$$

(e) If A is a (ground-model) set and $\dot{f}$ is a $\mathbb{P}$-name for a sequence in A, then we can find a $\xi < \kappa$ and a $\mathbb{P}_\xi$-name $\dot{g}$ such that

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \dot{f} = \dot{g}.$$

proof (a) This is just because $\mathbb{Q}$ is proper, as noted in 1F; see [Sh82], p. 90, Theorem III.3.2.

(b) This now follows by induction on ζ. The inductive step to a successor ordinal is trivial, because if we can think of $\mathbb{P}_{\xi+\zeta}$ as dense in $\mathbb{P}_\xi * \mathbb{P}'_\zeta$ then we can identify $\mathbb{Q}_{\xi+\zeta}$ with $\mathbb{Q}'_\zeta$. As for the inductive step to limit ζ, any member of $\mathbb{P}_{\xi+\zeta}$ can be regarded as (p, p') where $p \in \mathbb{P}_\xi$ and p' is a $\mathbb{P}_\xi$-name for a member of $\mathbb{P}'_\zeta$. On the other hand, given $(p, p') \in \mathbb{P}_\xi * \mathbb{P}'_\zeta$, we have a $\mathbb{P}_\xi$-name $\dot{J}$ for the support of p' which in $V^{\mathbb{P}_\xi}$ is a countable subset of ζ. But because $\mathbb{P}_\xi$ is proper there are a $p_1 \leq p$ and a countable ground-model set $I \subseteq \zeta$ such that $p_1 \Vdash_{\mathbb{P}_\xi} \dot{J} \subseteq I$ ([Sh82], p. 81, III.1.16). Now (p_1, p') can be re-interpreted as a member of $\mathbb{P}_{\xi+\zeta}$ stronger than (p, p'). Thus $\mathbb{P}_{\xi+\zeta}$ is dense in $\mathbb{P}_\xi * \mathbb{P}'_\zeta$, as claimed.

(c) [Sh82], p. 96, III.4.1.

(d) [Sh82], p. 171, V.4.4.

(e) By [Sh82], p. 96, III.4.1, $\mathbb{P}$ satisfies the κ-c.c.; because κ is regular, (d) gices the result.

1K Definition Let $\zeta \leq \kappa$, $p \in \mathbb{P}_\zeta$.

(a) Define $\mathbf{U}(p)$, $\langle p^{(\mathbf{u})}\rangle_{\mathbf{u}\in\mathbf{U}(p)}$ as follows. A finite function $\mathbf{u}\subseteq\zeta\times T$ belongs to $\mathbf{U}(p)$ if
either $\mathbf{u}=\emptyset$, in which case $p^{(\mathbf{u})}=p$,
or $\mathbf{u}=\mathbf{v}\cup\{(\xi,t)\}$ where $\mathbf{v}\in\mathbf{U}(p)$, $\operatorname{dom}(\mathbf{v})\subseteq\xi<\zeta$, and

$$p^{(\mathbf{v})}\restriction\xi\Vdash_{\mathbb{P}_\xi} t\in p^{(\mathbf{v})}(\xi),$$

in which case $p^{(\mathbf{u})}$ is defined by writing

$$p^{(\mathbf{u})}(\eta)=p^{(\mathbf{v})}(\eta)\ \ \forall\ \ \eta\in\zeta\setminus\{\xi\},$$
$$p^{(\mathbf{u})}(\xi)=p^{(\mathbf{v})}(\xi)\cap T^{(t)}.$$

(b) Observe that if $\mathbf{u}\in\mathbf{U}(p)$ then $p^{(\mathbf{u})}(\xi)=p(\xi)$ for $\xi\in\zeta\setminus\operatorname{dom}(\mathbf{u})$, $p^{(\mathbf{u})}(\xi)=p(\xi)\cap T^{(\mathbf{u}(\xi))}$ if $\xi\in\operatorname{dom}(\mathbf{u})$; $\mathbf{U}(p)$ is just the set of finite functions $\mathbf{u}$ for which these formulae define such a $p^{(\mathbf{u})}\in\mathbb{P}_\zeta$. Of course $p^{(\mathbf{u})}\le p$ for every $\mathbf{u}\in\mathbf{U}(p)$.

(c) Note that if $\xi\le\zeta$, $p\in\mathbb{P}_\zeta$, $\mathbf{u}\in\mathbf{U}(p)$ then $\mathbf{u}\restriction\xi\in\mathbf{U}(p\restriction\xi)$ and $(p\restriction\xi)^{(\mathbf{u}\restriction\xi)}=p^{(\mathbf{u})}\restriction\xi$.

(d) If $p\in\mathbb{P}_\zeta$, $\mathbf{u}\in\mathbf{U}(p)$ and $\mathbf{v}\subseteq\zeta\times T$ is a finite function such that $\operatorname{dom}(\mathbf{u})\subseteq\operatorname{dom}(\mathbf{v})$ and $\mathbf{u}(\xi)\le\mathbf{v}(\xi)$ in T for every $\xi\in\operatorname{dom}(\mathbf{u})$, then $\mathbf{v}\in\mathbf{U}(p)$ iff $\mathbf{v}\in\mathbf{U}(p^{(\mathbf{u})})$, and in this case $p^{(\mathbf{v})}=(p^{(\mathbf{u})})^{(\mathbf{v})}$ (induce on $\#(\mathbf{v})$).

(e) We shall mostly be using not the whole of $\mathbf{U}(p)$ but the sets $\mathbf{U}(p;K,k)=\mathbf{U}(p)\cap T_k^K$ for $K\in[\zeta]^{<\omega}$, $k\in\mathbb{N}$, writing T_k^K for the set of functions from K to T_k.

1L Definition For $\zeta\le\kappa$, $K\in[\zeta]^{<\omega}$, $k\in\mathbb{N}$ and $p\in\mathbb{P}_\zeta$, say that p is **(K,k)-fixed** if for every $\eta\in K$, $\mathbf{u}\in\mathbf{U}(p;K\cap\eta,k)$ there is a (ground-model) set $A\subseteq T_k$ such that

$$p^{(\mathbf{u})}\restriction\eta\Vdash_{\mathbb{P}_\eta} p(\eta)\cap T_k=A.$$

Equivalently, p is (K,k)-fixed if $\mathbf{U}(p;K,k)\supseteq\mathbf{U}(p_1;K,k)$ for every $p_1\le p$.

1M Lemma Suppose $\zeta\le\kappa$, $K\in[\zeta]^{<\omega}$, $k\ge1$ and that A is a finite set with $2^{c^m}a^{c^{m-1}}\le\exp(2^{-k}/\gamma_i)$ for every $i\ge k$, where $c=\#(T_k)$, $m=\#(K)$ and $a=\#(A)$. Let τ be a $\mathbb{P}_\zeta$-name for a member of A, and Δ a $\mathbb{P}_\zeta$-name for a countable subset of κ. Then for any $p\in\mathbb{P}_\zeta$ there are $p_1\le_{K,k}p$, a function $H:T_k^K\to A$ and a countable (ground-model) set $D\subseteq\kappa$ such that

$$p_1\text{ is }(K,k)\text{-fixed},$$
$$p_1^{(\mathbf{u})}\Vdash_{\mathbb{P}_\zeta}\tau=H(\mathbf{u})\ \ \forall\ \ \mathbf{u}\in\mathbf{U}(p_1;K,k),$$
$$p_1\Vdash_{\mathbb{P}_\zeta}\Delta\subseteq D.$$

proof Induce on $m=\#(K)$. If $m=0$ we may take any $a\in A$, $p_1'\le p$ such that $p_1'\Vdash\tau=a$, and (again using [Sh82], III.1.16, this time based on 1Ja) a countable D and a $p_1\le p_1'$ such that $p_1\Vdash_{\mathbb{P}_\zeta}\Delta\subseteq D$; now set $H(\emptyset)=a$.

For the inductive step to $\#(K)=m\ge1$, let ξ be $\max K$. As explained in 1Jb, $\mathbb{P}_\zeta$ may be regarded as a dense subset of $\mathbb{P}_{\xi+1}*\mathbb{P}'$; arguing momentarily in $V^{\mathbb{P}_{\xi+1}}$ we can find a $\mathbb{P}_{\xi+1}$-name $\hat r_0$ for a member of $\mathbb{P}'$, a $\mathbb{P}_{\xi+1}$-name τ' for a member of A and a $\mathbb{P}_{\xi+1}$-name Δ' for a countable set such that

$$(p\restriction\xi+1,\hat r_0)\le p\text{ in }\mathbb{P}_{\xi+1}*\mathbb{P}',$$
$$(p\restriction\xi+1,\hat r_0)\Vdash_{\mathbb{P}_{\xi+1}*\mathbb{P}'}\tau'=\tau,$$
$$(p\restriction\xi+1,\hat r_0)\Vdash_{\mathbb{P}_{\xi+1}*\mathbb{P}'}\Delta\subseteq\Delta'.$$

Now let Δ_0' be a $\mathbb{P}_{\xi+1}$-name for a countable subset of $\zeta\setminus(\xi+1)$ such that

$$\mathbb{1}_{\mathbb{P}_{\xi+1}}\Vdash_{\mathbb{P}_{\xi+1}}\operatorname{supp}(\hat r_0)=\Delta_0'.$$

Because $\#(A)=a<2^{c^m}a^{c^{m-1}}\le\exp(2^{-k}/\gamma_i)$ for every $i\ge k$, we can use Lemma 1G in $V^{\mathbb{P}_\xi}$ to find $\tilde H$, $\tilde q$, $\tilde\Delta$ such that

$$\tilde H\text{ is a }\mathbb{P}_\xi\text{-name for a function from }T_k\text{ to }A,$$
$$\tilde\Delta\text{ is a }\mathbb{P}_\xi\text{-name for a countable subset of }\kappa,$$
$$\tilde q\in\mathbb{Q}_\xi,$$
$$p\restriction\xi\Vdash_{\mathbb{P}_\xi}\tilde q\le_k p(\xi),$$
$$p\restriction\xi\Vdash_{\mathbb{P}_\xi}\big(\tilde q\cap T^{(t)}\Vdash_{\mathbb{Q}_\xi}\tau'=\tilde H(t)\ \ \forall\ \ t\in\tilde q\cap T_k\big),$$
$$p\restriction\xi\Vdash_{\mathbb{P}_\xi}\big(\tilde q\Vdash_{\mathbb{Q}_\xi}\Delta'\cup\Delta_0'\subseteq\tilde\Delta\big).$$

Now consider the pair $(\tilde H,\tilde q\cap T_k)$. This can be regarded as a $\mathbb{P}_\xi$-name for a member of $A_1=A^{T_k}\times\mathcal{P}T_k$, and $a_1=\#(A_1)=2^ca^c$, so

$$2^{c^{m-1}}a_1^{c^{m-2}}=2^{2c^{m-1}}a^{c^{m-1}}\le2^{c^m}a^{c^{m-1}}\le\exp(2^{-k}/\gamma_j)\ \ \forall\ \ i\ge k.$$

The inductive hypothesis therefore tells us that there are $\hat{p}_1 \le_{K\cap\xi,k} p{\restriction}\xi$, $H^* : T_k^{K\cap\xi} \to A^{T_k}$, $F^* : T_k^{K\cap\xi} \to \mathcal{P}T_k$ and a countable $D \subseteq \kappa$ such that

$$\hat{p}_1 \text{ is } (K\cap\xi,k)\text{-fixed},$$
$$\hat{p}_1^{(\mathbf{u})} \Vdash_{\mathbb{P}_\xi} \tilde{H} = H^*(\mathbf{u}) \;\;\&\;\; \tilde{q}\cap T_k = F^*(\mathbf{u})$$

for every $\mathbf{u} \in \mathbf{U}(\hat{p}_1; K\cap\xi, k)$, and

$$\hat{p}_1 \Vdash_{\mathbb{P}_\xi} \tilde{\Delta} \subseteq D.$$

At this point we observe that

$$\hat{p}_1 \Vdash_{\mathbb{P}_\xi} \big(\tilde{q} \Vdash_{\mathbb{Q}_\xi} \operatorname{supp}(\hat{r}_0) \subseteq D\big).$$

Now the only difference between $\mathbb{P}_{\xi+1} * \mathbb{P}'$ and $\mathbb{P}_\zeta$ is that for members of the former their supports have to be regarded as $\mathbb{P}_{\xi+1}$-names for countable subsets of ζ, and these are not always reducible to countable ground-model sets. But in the present case this difficulty does not arise and we have a $p_1 \in \mathbb{P}_\zeta$ defined by saying that $p_1{\restriction}\xi = \hat{p}_1$, $p_1{\restriction}\xi \Vdash_{\mathbb{P}_\xi} p_1(\xi) = \tilde{q}$, and $p_1{\restriction}\eta \Vdash_{\mathbb{P}_\eta} p_1(\eta) = \hat{r}_0(\eta)$ for $\xi < \eta < \zeta$; then $\operatorname{supp}(p_1) \subseteq \operatorname{supp}(\hat{p}_1) \cup \{\xi\} \cup (D\cap\zeta)$ is countable.

Consequently $p_1 \in \mathbb{P}_\zeta$ is well-defined and now, setting $H(\mathbf{u}^\frown t) = H^*(\mathbf{u})(t)$ for $\mathbf{u} \in T_k^{K\cap\xi}$, $t \in T_k$,

$$p_1 \le_{K,k} p,$$
$$p_1 \Vdash_{\mathbb{P}} \Delta \subseteq D,$$
$$\mathbf{U}(p_1; K, k) = \{\mathbf{u}^\frown t : \mathbf{u} \in \mathbf{U}(\hat{p}_1; K\cap\xi, k),\, t \in F^*(\mathbf{u})\},$$
$$p_1^{(\mathbf{v})} \Vdash_{\mathbb{P}_\zeta} \tau = H(\mathbf{v}) \;\;\forall\;\; \mathbf{v} \in \mathbf{U}(p_1; K, k)$$

and finally

$$p_1^{(\mathbf{u})}{\restriction}\xi \Vdash_{\mathbb{P}_\xi} p_1(\xi)\cap T_k = F^*(\mathbf{u}) \;\;\forall\;\; \mathbf{u} \in \mathbf{U}(p_1; K\cap\xi, k),$$

so that p_1 is (K,k)-fixed, and the induction proceeds.

1N Lemma Suppose $\zeta \le \kappa$, $\langle K_k\rangle_{k\in\mathbb{N}}$ is an increasing sequence of finite subsets of ζ, $\langle p_k\rangle_{k\in\mathbb{N}}$ is a sequence in $\mathbb{P}_\zeta$; suppose that

$$p_{2k+2} \le_{K_k,k+1} p_{2k+1} \le_{K_k,k} p_{2k}$$

for every $k \in \mathbb{N}$ and that $\bigcup_{k\in\mathbb{N}} \operatorname{supp}(p_k) \subseteq \bigcup_{k\in\mathbb{N}} K_k$. Then there is a $\hat{p} \in \mathbb{P}_\zeta$ such that $\hat{p} \le p_k$ for every $k \in \mathbb{N}$, $\operatorname{supp}(\hat{p}) \subseteq \bigcup_{k\in\mathbb{N}} K_k$ and

$$\hat{p}{\restriction}\xi \Vdash_{\mathbb{P}_\xi} \hat{p}(\xi)\cap T_k = p_{2k+1}\cap T_k \;\;\forall\;\; k\in\mathbb{N},\, \xi\in K_k,$$
$$\hat{p}{\restriction}\xi \Vdash_{\mathbb{P}_\xi} \hat{p}(\xi)\cap T_{k+1} = p_{2k+2}\cap T_{k+1} \;\;\forall\;\; k\in\mathbb{N},\, \xi\in K_{k+1},$$

so that

$$\mathbf{U}(\hat{p}; K_k, k) \supseteq \mathbf{U}(p_{2k+1}; K_k, k) \text{ and } \mathbf{U}(\hat{p}; K_k, k+1) \supseteq \mathbf{U}(\hat{p}_{k+2}; K_k, k+1)$$

for every $k \in \mathbb{N}$.

proof For each $\xi < \zeta$ choose $\hat{p}(\xi)$ such that

$$\mathbb{1}_{\mathbb{P}_\xi} \Vdash_{\mathbb{P}_\xi} \hat{p}(\xi) = \textstyle\bigcap_{k\in\mathbb{N}} p_k(\xi).$$

An easy induction on ξ shows that $\hat{p}{\restriction}\xi \in \mathbb{P}_\xi$ for every $\xi \le \zeta$; for if $\xi \in \zeta\setminus\bigcup_{k\in\mathbb{N}} K_k$ then

$$\mathbb{1}_{\mathbb{P}_\xi} \Vdash_{\mathbb{P}_\xi} \hat{p}(\xi) = T = \mathbb{1}_{\mathbb{Q}_\xi},$$

while if $k \in \mathbb{N}$ and $\xi \in K_k$ then

$$\hat{p}{\restriction}\xi \Vdash_{\mathbb{P}_\xi} p_{2l+2}(\xi) \le_{l+1} p_{2l+1}(\xi) \le_l p_{2l}(\xi) \;\;\forall\;\; l \ge k,$$

so that by Lemma 1H,

$$\begin{aligned}\hat{p}{\restriction}\xi \Vdash_{\mathbb{P}_\xi} \hat{p}(\xi) \in \mathbb{Q}_\xi \;\&\; \hat{p}(\xi)\cap T_{k+1} &= p_{2k+1}\cap T_{k+1} = p_{2k+2}\cap T_{k+1}\\ \&\; \hat{p}(\xi)\cap T_k &= p_{2k+1}\cap T_k.\end{aligned}$$

It follows at once that $\mathbf{U}(\hat{p}; K_k, k) \supseteq \mathbf{U}(p_{2k+1}; K_k, k)$, $\mathbf{U}(\hat{p}; K_k, k+1) \supseteq \mathbf{U}(p_{2k+2}; K_k, k+1)$ for every $k \in \mathbb{N}$.

1O Lemma Suppose that $0 < \zeta \le \kappa$, σ is a $\mathbb{P}_\zeta$-name for a member of $\prod_{k\in\mathbb{N}} n_k$, and $p \in \mathbb{P}_\zeta$. Then we can find a $\hat{p}$ and sequences $\langle K_k\rangle_{k\in\mathbb{N}}$, $\langle H_k\rangle_{k\in\mathbb{N}}$ such that

$\hat{p} \in \mathbb{P}_\zeta$, $\hat{p} \le p$;
$\langle K_k\rangle_{k\in\mathbb{N}}$ is an increasing sequence of subsets of ζ, $\#(K_k) \le k+1$ for every k, $K_0 = \{0\}$;
$\operatorname{supp}(\hat{p}) \subseteq \bigcup_{k\in\mathbb{N}} K_k$;
$\hat{p}$ is (K_k, k)-fixed and $(K_k, k+1)$-fixed for every k;
H_k is a function from $T_{k+1}^{K_k}$ to n_k for every k;
$\hat{p}^{(\mathbf{u})} \Vdash_{\mathbb{P}_\zeta} \sigma(k) = H_k(\mathbf{u})$ whenever $k \in \mathbb{N}$ and $\mathbf{u} \in \mathbf{U}(\hat{p}; K_k, k+1)$.

proof Using Lemma 1M, we can find sequences $\langle p_k\rangle_{k\in\mathbb{N}}$, $\langle K_k\rangle_{k\in\mathbb{N}}$ and $\langle H_k\rangle_{k\in\mathbb{N}}$ such that

$p = p_0$, $K_0 = \{0\}$;
$\#(K_{k+1}) \le k+2$, $K_{k+1} \supseteq K_k$;
$p_{2k+1} \le_{K_k,k} p_{2k}$, p_{2k+1} is (K_k,k)-fixed;
$H_k : T_{k+1}^{K_k} \to n_k$ is a function;
$p_{2k+2} \le_{K_k,k+1} p_{2k+1}$, p_{2k+2} is $(K_k,k+1)$-fixed,
$p_{2k+2}^{(\mathbf{u})} \Vdash_{\mathbb{P}_\zeta} \sigma(k) = H_k(\mathbf{u}) \ \forall \ \mathbf{u} \in \mathbf{U}(p_{2k+2}; K_k, k+1)$
for every $k \in \mathbb{N}$. Furthermore, we may do this in such a way that $\bigcup_{k\in\mathbb{N}} K_k$ includes $\bigcup_{k\in\mathbb{N}} \mathrm{supp}(p_k)$. We need of course to know that the n_k are rapidly increasing; specifically, that

$$2^{c_k^{k+1}} \le \exp(2^{-k}/\gamma_i) \ \forall \ i \ge k$$

(when choosing p_{2k+1}) and that

$$2^{c_{k+1}^{k+1}} n_k^{c_{k+1}^k} \le \exp(2^{-k-1}/\gamma_i) \ \forall \ i \ge k+1$$

(when choosing p_{2k+2}), where we write $c_k = \#(T_k)$. But as $c_k \le \prod_{i<k} 2^{n_i}$, this is a consequence of 1E(a)(i) and (iii).

Armed with the sequences $\langle p_k\rangle_{k\in\mathbb{N}}$, $\langle K_k\rangle_{k\in\mathbb{N}}$ we may now use Lemma 1N to find a $\hat p$ as described there. Because p_{2k+1} is (K_k,k)-fixed and $\mathbf{U}(\hat p; K_k,k) \supseteq \mathbf{U}(p_{2k+1}; K_k, k)$ we must have equality here and $\hat p$ is (K_k,k)-fixed for every $k \in \mathbb{N}$. Similarly, $\hat p$ is $(K_k,k+1)$-fixed for every k. Moreover, if $\mathbf{u} \in \mathbf{U}(\hat p; K_k, k+1) = \mathbf{U}(p_{2k+2}; K_k, k+1)$ we have $\hat p^{(\mathbf{u})} \le p_{2k+2}^{(\mathbf{u})}$, so

$$\hat p^{(\mathbf{u})} \Vdash_{\mathbb{P}_\zeta} \sigma(k) = H_k(\mathbf{u})$$

as required.

1P Lemma Suppose that $\zeta \le \kappa$, $p \in \mathbb{P}_\zeta$, $k \in \mathbb{N}$, $K \in [\zeta]^{<\omega}$ and $\mathbf{V}$ is a non-empty subset of $\mathbf{U}(p; K, k)$. Then we have a $p_1 = \bigvee_{\mathbf{v}\in\mathbf{V}} p^{(\mathbf{v})}$ defined (up to $\le$-equivalence in $\mathbb{P}_\zeta$) by saying
if $\xi \in \zeta \setminus K$ then $p_1(\xi) = p(\xi)$;
if $\xi \in K$ then

$$(p_1{\restriction}\xi)^{(\mathbf{u})} \Vdash_{\mathbb{P}_\xi} p_1(\xi) = \bigcup\{p(\xi) \cap T^{(t)} : \exists \ \mathbf{v} \in \mathbf{V} \text{ such that } \mathbf{v}{\restriction}\xi+1 = \mathbf{u}^\frown t\}$$

for $\mathbf{u} \in \{\mathbf{v}{\restriction}\xi : \mathbf{v} \in \mathbf{V}\}$.
Now $p_1 \le p$ and if $\xi < \zeta$, $t \in p_1(\xi)$, $\mathrm{rank}(t) \ge k$ we shall have

$$p_1{\restriction}\xi \Vdash_{\mathbb{P}_\xi} \mathrm{suc}(t) \cap p_1(\xi) = \mathrm{suc}(t) \cap p(\xi);$$

so if $\xi < \zeta$, $i \ge k$ we have

$$p_1{\restriction}\xi \Vdash_{\mathbb{P}_\xi} \delta_i(p_1(\xi)) \ge \delta_i(p(\xi)).$$

If $p_2 \le p_1$ there is some $\mathbf{v} \in \mathbf{V}$ such that p_2 is compatible with $p_1^{(\mathbf{v})} = p^{(\mathbf{v})}$. If $k \le l \in \mathbb{N}$, $K \subseteq L \in [\zeta]^{<\omega}$ then

$$\mathbf{U}(p_1; L, l) = \{\mathbf{w} : \mathbf{w} \in \mathbf{U}(p; L, l), \ \exists \ \mathbf{v} \in \mathbf{V} \text{ such that } \mathbf{v}(\xi) \le \mathbf{w}(\xi) \ \forall \ \xi \in K\},$$

and $p_1^{(\mathbf{w})} = p^{(\mathbf{w})}$ for every $\mathbf{w} \in \mathbf{U}(p_1; L, l)$; consequently, p_1 is (L,l)-fixed if p is.

proof Requires only a careful reading of the definitions.

Remark Note that 1G-1P are based just on the fact that $\mathbb{Q}$ is a p.o.set of rapidly branching trees; the exact definition of 'rapidly branching' in 1E(c) is relevant only to some of the detailed calculations. Similar ideas may be found in [BJSp89]and [Sh326].

1Q Construction: part 3 (a) Set $X = \prod_{k\in\mathbb{N}} n_k$. Then X, with its product topology, is a compact metric space. Let μ be the natural Radon probability on X, the product of the uniform probabilities on the factors.

(b) For each $k \in \mathbb{N}$ set $l_k = \lceil (\ln n_k)^2 \rceil$. Take $W_k' \subseteq n_k \times n_k$ such that $\#(W_k') \le 2^{-k-1} n_k^2$ and whenever $I \in [n_k]^{l_k}$ and $J_0, \dots, J_{l_k-1}$ are disjoint members of $[n_k]^{\le k}$, there are $i \in I$ and $j < l_k$ such that $\{i\} \times J_j \subseteq W_k'$. (This is possible by Lemma 1B and 1E(a)(iv).) Set $W_k = W_k' \cup \{(i,i) : i < n_k\}$.

Write R for

$$\{(x,y) : x, y \in X, \ (x(k), y(k)) \in W_k \ \forall \ k \in \mathbb{N}, \ \{k : x(k) = y(k)\} \text{ is finite}\};$$

then R is negligible for the product measure of $X \times X$. For $r \in \mathbb{N}$ write R_r for the set

$$\{(x, \langle y_i\rangle_{i<r}) : x \in X, \ (x, y_i) \in R \ \forall \ i < r\} \subseteq X \times X^r.$$

We shall frequently wish to interpret the formulae for the sets X, R_r in $V^{\mathbb{P}}$; when doing so we will write $\ulcorner X \urcorner$, $\ulcorner R_r \urcorner$.

(c) Write

$$\mathcal{L} = \{\langle L_k\rangle_{k\in\mathbb{N}} : L_k \subseteq n_k \ \forall\ k \in \mathbb{N}, \prod_{k\in\mathbb{N}} \#(L_k)/n_k > 0\}.$$

Again, we shall wish to distinguish between the ground-model set $\mathcal{L}$ and a corresponding $\mathbb{P}$-name $\ulcorner\mathcal{L}\urcorner$.

(d) For each $k \in \mathbb{N}$ let Φ_k be the $\mathbb{P}$-name for a subset of n_k defined (up to equivalence) by saying that

$$p \Vdash_{\mathbb{P}} \Phi_k = C_k(t)$$

whenever $\mathrm{rank}(t) > k$ and $p(0) \subseteq T^{(t)}$. (Here $C_k(t)$ is the kth factor of t, as described in 1E(b).) Let Ψ_k, Ψ be $\mathbb{P}$-names for the subsets of $\ulcorner X\urcorner$ given by

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \Psi_k = \{\sigma : \sigma \in \ulcorner X\urcorner,\ \sigma(i) \in \Phi_i \ \forall\ i \geq k\},\ \Psi = \bigcup_{k\in\mathbb{N}} \Psi_k.$$

Then we have

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \#(\Phi_k) \geq (1 - 2^{-k-1})n_k \ \forall\ k \in \mathbb{N},$$

so that

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \ulcorner\mu\urcorner(\Psi) = 1.$$

1R Main Lemma If $r \in \mathbb{N}$ and $D \subseteq X^r$ is a (ground-model) set such that $D \cap (\prod_{k\in\mathbb{N}} L_k)^r \neq \emptyset$ for every (ground-model) sequence $\langle L_k\rangle_{k\in\mathbb{N}} \in \mathcal{L}$, then for every (ground-model) sequence $\langle L_k\rangle_{k\in\mathbb{N}} \in \mathcal{L}$

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \Psi \cap \prod_{k\in\mathbb{N}} L_k \subseteq \ulcorner R_r\urcorner^{-1}[D \cap (\prod_{k\in\mathbb{N}} L_k)^r].$$

proof (a) Let $\langle L_k\rangle_{k\in\mathbb{N}} \in \mathcal{L}$, let σ be a $\mathbb{P}$-name such that

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma \in \Psi \cap \prod_{k\in\mathbb{N}} L_k,$$

and let $p \in \mathbb{P}$. Write D' for $D \cap (\prod_{k\in\mathbb{N}} L_k)^r$. Let $k_0 \geq r$, $p_1 \leq p$ be such that $p_1 \Vdash_{\mathbb{P}} \sigma \in \Psi_{k_0}$. By Lemma 1O, we have a $p_2 \leq p_1$, an increasing sequence $\langle K_k\rangle_{k\in\mathbb{N}}$ of finite subsets of κ, and a sequence $\langle H_k\rangle_{k\in\mathbb{N}}$ of functions such that

p_2 is (K_k, k)-fixed and $(K_k, k+1)$-fixed for every $k \in \mathbb{N}$,
$p_2^{(\mathbf{u})} \Vdash_{\mathbb{P}} \sigma(k) = H_k(\mathbf{u})$ whenever $\mathbf{u} \in \mathbf{U}(p_2; K_k, k+1)$, $k \in \mathbb{N}$;
$\bigcup_{k\in\mathbb{N}} K_k \supseteq \mathrm{supp}(p_2)$;
$0 \in K_0$, $\#(K_k) \leq k+1$ for every $k \in \mathbb{N}$.

(b) For $k \geq k_0$, let Z_k be the cartesian product set n_k^r and take $\tilde{W}_k$ to be

$$\{(i, z) : i < n_k,\ z \in Z_k,\ (i, z(j)) \in W_k \ \forall\ j < r\}.$$

Set $\mathcal{A}_k = \mathcal{P}n_k \setminus \{\emptyset\}$ and $\mathbb{T}_k = \mathcal{P}(\mathcal{A}_k^{K_k}) \setminus \{\emptyset\}$; define $\preccurlyeq_k$ on $\mathbb{T}_k$ as in Lemma 1C, taking l_k and K_k (with the order induced by that of κ) in place of l and m there.

For each $\mathbf{u} \in \mathbf{U}(p_2; K_k, k)$ set

$$\mathcal{T}_{\mathbf{u}} = \{\mathbf{c} : \mathbf{u}^\wedge\mathbf{c} \in \mathbf{U}(p_2; K_k, k+1)\} \in \mathbb{T}_k,$$

where for $\mathbf{u} \in T_k^K$, $\mathbf{c} \in (\mathcal{P}n_k)^K$ we write

$$\mathbf{u}^\wedge\mathbf{c} = \langle \mathbf{u}(\xi) \times \mathbf{c}(\xi)\rangle_{\xi\in K}.$$

By Corollary 1D, we may find for each such $\mathbf{u}$ a $w_{\mathbf{u}} < n_k$ and a set $J_{\mathbf{u}} \subseteq n_k$ such that $\#(J_{\mathbf{u}}) \leq rl_k$ and

either there is a $\mathcal{T} \preccurlyeq_k \mathcal{T}_{\mathbf{u}}$ such that $H_k(\mathbf{u}^\wedge\mathbf{c}) = w_{\mathbf{u}}$ for every $\mathbf{c} \in \mathcal{T}$
or for every $z \in (n_k \setminus J_{\mathbf{u}})^r$ there is a $\mathcal{T} \preccurlyeq_k \mathcal{T}_{\mathbf{u}}$ such that $(H_k(\mathbf{u}^\wedge\mathbf{c}), z) \in \tilde{W}_k$, that is,

$$(H_k(\mathbf{u}^\wedge\mathbf{c}), z(j)) \in W_k \ \forall\ j < r,$$

for every $\mathbf{c} \in \mathcal{T}$.

Set

$$\tilde{I}_k = \{w_{\mathbf{u}} : \mathbf{u} \in \mathbf{U}(p_2; K_k, k)\},$$
$$\tilde{J}_k = \bigcup\{J_{\mathbf{u}} : \mathbf{u} \in \mathbf{U}(p_2; K_k, k)\},$$

so that

$$\#(\tilde{I}_k) \leq \#(\mathbf{U}(p_2; K_k, k)) \leq \#(T_k^{K_k}) \leq (\prod_{i<k} 2^{n_i})^{k+1},$$
$$\#(\tilde{J}_k) \leq kl_k(\prod_{i<k} 2^{n_i})^{k+1} \leq 2^{-k}n_k$$

by 1E(a)(vi).

(c) Let $k_1 \geq \max(k_0, 1)$ be such that

$$\delta_k(p_2(0)) \geq 2,\ \tilde{J}_k \not\supseteq L_k$$

for every $k \geq k_1$. Take any $\mathbf{v}^* \in \mathbf{U}(p_2; K_{k_1-1}, k_1)$ and set $p_3 = p_2^{(\mathbf{v}^*)}$. Then p_3 is (K_k, k)-fixed and $(K_k, k+1)$-fixed for every $k \geq k_1$, and

$$\mathcal{T}_{\mathbf{u}} = \{\mathbf{c} : \mathbf{u}^\wedge\mathbf{c} \in \mathbf{U}(p_3; K_k, k+1)\}$$

whenever $k \geq k_1$, $\mathbf{u} \in \mathbf{U}(p_3; K_k, k)$.

(d) We have

$$p_3 \Vdash_{\mathbb{P}} \sigma(i) = H_i(\mathbf{v}_i^*) \ \forall \ i < k_1,$$

where $\mathbf{v}_i^*$ is that member of $T_{i+1}^{K_i}$ such that $\mathbf{v}_i^*(\eta) \le \mathbf{v}^*(\eta)$ for every $\eta \in K_i$. Set $L'_k = \{H_k(\mathbf{v}_k^*)\}$ for $k < k_1$, $L'_k = L_k \setminus \tilde{J}_k$ for $k \ge k_1$; then $\prod_{k\in\mathbb{N}} \#(L'_k)/n_k > 0$, because $\prod_{k\in\mathbb{N}} \#(L_k)/n_k > 0$ and $\sum_{k\in\mathbb{N}} \#(\tilde{J}_k)/n_k < \infty$. So there is a $\tilde{z} \in D \cap (\prod_{k\in\mathbb{N}} L'_k)^r \subseteq D'$. Writing $z_k = \langle \tilde{z}(j)(k)\rangle_{j<r}$ for $k \in \mathbb{N}$, we have $z_k \in (n_k \setminus \tilde{J}_k)^r$ for $k \ge k_1$, and

$$p_3 \Vdash_{\mathbb{P}} (\sigma(k), z_k) \in \tilde{W}_k$$

for $k < k_1$, because $(i, i) \in W_k$ for $i < n_k$.

(e) For each $k \ge k_1$, $\mathbf{u} \in \mathbf{U}(p_3; K_k, k)$ choose $\mathcal{T}'_{\mathbf{u}} \preccurlyeq_k \mathcal{T}_{\mathbf{u}} \in \mathbb{T}_k$ such that
either $H_k(\mathbf{u}^\frown\mathbf{c}) = w_{\mathbf{u}} \in \tilde{I}_k$ for every $\mathbf{c} \in \mathcal{T}'_{\mathbf{u}}$
or $(H_k(\mathbf{u}^\frown\mathbf{c}), z_k) \in \tilde{W}_k$ for every $\mathbf{c} \in \mathcal{T}'_{\mathbf{u}}$.
Define $\langle \mathcal{S}_k\rangle_{k\ge k_1}$, $\langle \tilde{p}_k\rangle_{k\ge k_1}$ by
$\tilde{p}_{k_1} = p_3$,
$\mathcal{S}_k = \{\mathbf{u}^\frown\mathbf{c} : \mathbf{u} \in \mathbf{U}(\tilde{p}_k; K_k, k),\ \mathbf{c} \in \mathcal{T}'_{\mathbf{u}}\}$,
$\tilde{p}_{k+1} = \bigvee\{\tilde{p}_k^{(\mathbf{v})} : \mathbf{v} \in \mathcal{S}_k\}$
for every $k \ge k_1$, as in Lemma 1P. An easy induction on k shows that

$$\mathcal{T}'_{\mathbf{u}} = \{\mathbf{c} : \mathbf{u}^\frown\mathbf{c} \in \mathbf{U}(\tilde{p}_{k+1}; K_k, k+1)\}$$

whenever $\mathbf{u} \in \mathbf{U}(\tilde{p}_k; K_k, k)$, $k \ge k_1$, that $\tilde{p}_k$ is (K_l, l)-fixed and $(K_l, l+1)$-fixed whenever $k_1 \le k \le l$, and that $\tilde{p}_k^{(\mathbf{v})} = p_3^{(\mathbf{v})}$ whenever $k_1 \le k \le l$ and $\mathbf{v} \in \mathbf{U}(\tilde{p}_k; K_l, l) \cup \mathbf{U}(\tilde{p}_k; K_l, l+1)$. Also $\mathrm{supp}(\tilde{p}_k) \subseteq \bigcup_{l\in\mathbb{N}} K_l$ for every $k \ge k_1$.

(f) It is likewise easy to see that, for $k \ge k_1$,
$\tilde{p}_{k+1} \le \tilde{p}_k$,
$\tilde{p}_{k+1}{\restriction}\xi \Vdash_{\mathbb{P}_\xi} \tilde{p}_{k+1}(\xi) \cap T_k = \tilde{p}_k(\xi) \cap T_k \ \forall \ \xi < \kappa$,
$\tilde{p}_{k+1}{\restriction}\xi \Vdash_{\mathbb{P}_\xi} \tilde{p}_{k+1}(\xi) \cap \mathrm{suc}(t) = \tilde{p}_k(\xi) \cap \mathrm{suc}(t) \ \forall \ t \in \tilde{p}_{k+1}(\xi) \cap T_i$
unless $i = k$ and $\xi \in K_k$,
$\tilde{p}_{k+1} \Vdash_{\mathbb{P}} \sigma(k) \in \tilde{I}_k$ *or* $(\sigma(k), z_k) \in \tilde{W}_k$.

(g) On the other hand, if $k \ge k_1$ and $\xi \in K_k$,

$$\tilde{p}_{k+1}{\restriction}\xi \Vdash_{\mathbb{P}_\xi} \mathrm{dp}(\{C : t \times C \in \tilde{p}_{k+1}(\xi)\}) \ge \mathrm{dp}(\{C : t \times C \in \tilde{p}_k(\xi)\})/2l_k$$
$$\forall \ t \in \tilde{p}_{k+1}(\xi) \cap T_k.$$

To see this, take any $q \le \tilde{p}_{k+1}{\restriction}\xi$ and t such that

$$q \Vdash_{\mathbb{P}_\xi} t \in \tilde{p}_{k+1}(\xi) \cap T_k = \tilde{p}_k(\xi) \cap T_k.$$

We may suppose that $\mathbf{v}_0 \in \mathcal{S}_k$ is such that $q \le \tilde{p}_k^{(\mathbf{v}_0)}{\restriction}\xi = \tilde{p}_{k+1}^{(\mathbf{v}_0)}{\restriction}\xi = q_1$. Now $\tilde{p}_{k+1}$ is $(K_k, k+1)$-fixed so there must be a $t' \ge t$ such that

$$q_1 \Vdash_{\mathbb{P}_\xi} t' \in T_{k+1} \cap \tilde{p}_{k+1}(\xi).$$

There is accordingly a $\mathbf{v}_1 \in \mathcal{S}_k$ such that $\mathbf{v}_0{\restriction}\xi = \mathbf{v}_1{\restriction}\xi$ and $\mathbf{v}_1(\xi) = t'$. Express $\mathbf{v}_1$ as $\mathbf{u}^\frown\mathbf{c}_1$ where $\mathbf{u} \in \mathbf{U}(\tilde{p}_k; K_k, k)$ and $\mathbf{c}_1 \in \mathcal{T}'_{\mathbf{u}}$. Of course $\mathbf{u}(\xi) = t$.

Now
$q_1 \Vdash_{\mathbb{P}_\xi} \{C : t \times C \in \tilde{p}_{k+1}(\xi)\} \supseteq \{\mathbf{c}(\xi) : \mathbf{c} \in \mathcal{T}'_{\mathbf{u}},\ \mathbf{c}{\restriction}\xi = \mathbf{c}_1{\restriction}\xi\}$,
$q_1 \Vdash_{\mathbb{P}_\xi} \{C : t \times C \in \tilde{p}_k(\xi)\} = \{\mathbf{c}(\xi) : \mathbf{c} \in \mathcal{T}_{\mathbf{u}},\ \mathbf{c}{\restriction}\xi = \mathbf{c}_1{\restriction}\xi\}$
because $\tilde{p}_k$ and $\tilde{p}_{k+1}$ are both $(K_k, k+1)$-fixed, while $\mathbf{v}_0{\restriction}\xi = (\mathbf{u}{\restriction}\xi)^\frown(\mathbf{c}_1{\restriction}\xi)$. But because $\mathcal{T}'_{\mathbf{u}} \preccurlyeq_k \mathcal{T}_{\mathbf{u}}$,

$$\mathrm{dp}(\{\mathbf{c}(\xi) : \mathbf{c} \in \mathcal{T}'_{\mathbf{u}},\ \mathbf{c}{\restriction}\xi = \mathbf{c}_1{\restriction}\xi\}) \ge \mathrm{dp}(\{\mathbf{c}(\xi) : \mathbf{c} \in \mathcal{T}_{\mathbf{u}},\ \mathbf{c}{\restriction}\xi = \mathbf{c}_1{\restriction}\xi\})/2l_k.$$

So we get

$$q \le q_1 \Vdash_{\mathbb{P}_\xi} \mathrm{dp}(\{C : t \times C \in \tilde{p}_{k+1}(\xi)\}) \ge \mathrm{dp}(\{C : t \times C \in \tilde{p}_k(\xi)\})/2l_k.$$

As q and t are arbitrary, we have the result.

(h) Because $\gamma_k \ln(2l_k) \le 2^{-k}$ (by 1E(a)(v)),

$$\tilde{p}_{k+1} \le_{K_k,k} \tilde{p}_k$$

for every $k \ge k_1$. Also, $\mathrm{supp}(\tilde{p}_k) \subseteq \bigcup_{l\in\mathbb{N}} K_l$ for every $k \ge k_1$. By Lemma 1N, there is a $p_4 \in \mathbb{P}$ such that $p_4 \le \tilde{p}_k$ for every $k \ge k_1$. Moreover, we may take it that

$$p_4(0) = \bigcap_{k\ge k_1} \tilde{p}_k(0)$$

(as in Lemma 1H), so that

$$\delta_i(p_4(0)) \ge \delta_i(p_3(0)) - \gamma_i \ln(2l_i) \ge 1$$

whenever $i \ge k_1$. Note that for $k \ge k_1$,

$$p_4 \Vdash_{\mathbb{P}} \sigma(k) \in \tilde{I}_k \textit{ or } (\sigma(k), z_k) \in \tilde{W}_k,$$

while for $k < k_1$,

$$p_4 \Vdash_{\mathbb{P}} (\sigma(k), z_k) \in \tilde{W}_k.$$

(i) Now define $p_5 \in \mathbb{P}$ by setting
$\tilde{I}'_i = \tilde{I}_i \cup \{\tilde{z}(j)(i) : j < r\}\ \forall\ i \in \mathbb{N}$,
$p_5(0) = \{t : t \in p_4(0),\ C_i(t) \cap \tilde{I}'_i = \emptyset$ whenever $k_1 \le i < \mathrm{rank}(t)\}$,
$p_5(\xi) = p_4(\xi)$ if $0 < \xi < \kappa$.

Of course we must check that $p_5(0)$, as so defined, belongs to $\mathbb{Q}_0 \cong \mathbb{Q}$; but because $\delta_k(p_4(0)) \ge 1$ for $k \ge k_1$, we have

$$\mathrm{dp}(\{C : t \times C \in p_4(0)\}) \ge \exp(1/\gamma_k) \ge 2(\textstyle\prod_{i<k} 2^{n_i})^{k+1} + 2k \ge 2\#(\tilde{I}'_k)$$

for every $k \ge k_1$, $t \in p_4(0) \cap T_k$, using 1E(a)(vii). Of course $p_5(0) \cap T_{k_1-1} = p_4(0) \cap T_{k_1-1} = \{\mathbf{v}^*(0)\}$, so every element of $p_5(0)$ has a successor in $p_5(0)$, and also

$$\delta_i(p_5(0)) \ge \delta_i(p_4(0)) - \gamma_i \ln 2$$

for $i \ge k_1$. (Here at last is the key step which depends on using dp in our measure of 'rapidly branching' given in 1E(d).) Thus $p_5(0) \in \mathbb{Q}$ and $p_5 \in \mathbb{P}$. But also

$$p_5 \Vdash_{\mathbb{P}} \Phi_k \cap \tilde{I}'_k = \emptyset\ \forall\ k \ge k_1,$$

so that

$$p_5 \Vdash_{\mathbb{P}} \sigma(k) \notin \tilde{I}_k,\ \sigma(k) \ne \tilde{z}(j)(k)\ \forall\ j < r,\ (\sigma(k), z_k) \in \tilde{W}_k$$

for $k \ge k_1$; finally

$$p_5 \Vdash_{\mathbb{P}} (\sigma, \tilde{z}) \in \ulcorner R_r \urcorner,\ \sigma \in \ulcorner R_r \urcorner^{-1}[D'];$$

as $p_5 \le p$ and p, σ are arbitrary,

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \Psi \cap \textstyle\prod_{k \in \mathbb{N}} L_k \subseteq \ulcorner R_r \urcorner^{-1}[D'],$$

as claimed.

1S Theorem For each $r \in \mathbb{N}$,

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \text{if } D_i \subseteq \ulcorner X \urcorner \text{ and } D_i \cap \prod_{k \in \mathbb{N}} L_k \ne \emptyset\ \forall\ \langle L_k \rangle_{k \in \mathbb{N}} \in \ulcorner \mathcal{L} \urcorner,\ i < r,$$
$$\text{then } \forall\ \langle L_k \rangle_{k \in \mathbb{N}} \in \ulcorner \mathcal{L} \urcorner\ \exists\ \langle x_i \rangle_{i<r} \in \prod_{i<r} (D_i \cap \prod_{k \in \mathbb{N}} L_k)$$
$$\text{such that } (x_j, x_i) \in \ulcorner R \urcorner\ \forall\ i < j < r.$$

proof Induce on r. If $r = 0$ the result is trivial. For the inductive step to $r+1$, take $\mathbb{P}$-names Δ_i for subsets of $\ulcorner X \urcorner$ such that

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \Delta_i \cap \textstyle\prod_{k \in \mathbb{N}} L_k \ne \emptyset\ \forall\ \langle L_k \rangle_{k \in \mathbb{N}} \in \ulcorner \mathcal{L} \urcorner,\ i \le r.$$

Take a $\mathbb{P}$-name $\mathfrak{L}$ for a member of $\ulcorner \mathcal{L} \urcorner$. Because members of $\mathcal{L}$ can be coded by simple sequences, we may suppose that $\mathfrak{L}$ is a $\mathbb{P}_\alpha$-name for some $\alpha < \kappa$ (1Je). The inductive hypothesis tells us that

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \forall\ \langle L_k \rangle_{k \in \mathbb{N}} \in \ulcorner \mathcal{L} \urcorner\ \exists\ \langle x_i \rangle_{i<r} \in \prod_{i<r} (\Delta_i \cap \prod_{k \in \mathbb{N}} L_k)$$
$$\text{such that } (x_j, x_i) \in \ulcorner R \urcorner \text{ if } i < j < r.$$

Next, using 1Jc-e, we can find a $\beta \in \kappa \setminus \alpha$ and a $\mathbb{P}_\beta$-name Δ for a subset of $\ulcorner X^r \urcorner$ such that

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \Delta \subseteq \textstyle\prod_{i<r} \Delta_i,$$

$$\mathbb{1}_{\mathbb{P}_\beta} \Vdash_{\mathbb{P}_\beta} (x_j, x_i) \in \ulcorner R \urcorner^{(\beta)} \text{ whenever } \langle x_l \rangle_{l<r} \in \Delta,\ i < j < r,$$
$$\Delta \cap (\prod_{k \in \mathbb{N}} L_k)^r \ne \emptyset\ \forall\ \langle L_k \rangle_{k \in \mathbb{N}} \in \ulcorner \mathcal{L} \urcorner^{(\beta)},$$

where we write $\ulcorner \ldots \urcorner^{(\beta)}$ to indicate that we are interpreting some formula in $V^{\mathbb{P}_\beta}$.

Now we remark that by 1Jb $\mathbb{P}$ can be regarded, for forcing purposes, as an iteration $\mathbb{P}_\beta * \mathbb{P}'$, where $\mathbb{P}'$ is a $\mathbb{P}_\beta$-name for a p.o.set with the same definition, interpreted in $V^{\mathbb{P}_\beta}$, as $\mathbb{P}$ has in the ground model. So we may use Lemma 1R in $V^{\mathbb{P}_\beta}$ to say that

$$\mathbb{1}_{\mathbb{P}_\beta} \Vdash_{\mathbb{P}_\beta} \big(\mathbb{1}_{\mathbb{P}'} \Vdash_{\mathbb{P}'} \Psi^{(\beta)} \cap \prod \mathfrak{L} \subseteq \ulcorner R_r \urcorner^{-1}[\Delta \cap (\prod \mathfrak{L})^r]\big),$$

using the notation $\Psi^{(\beta)}$ to indicate which version of the $\mathbb{P}$-name Ψ we are trying to use. Moving to $V^{\mathbb{P}}$ for a moment, we have $\ulcorner \mu \urcorner \Psi^{(\beta)} = 1$ and $\ulcorner \mu \urcorner (\prod \mathfrak{L}) > 0$, so

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \exists\ l \in \mathbb{N},\ \ulcorner \mu \urcorner (\Psi_l^{(\beta)} \cap \prod \mathfrak{L}) > 0.$$

Also, of course, every $\Psi_l^{(\beta)} \cap \prod \mathfrak{L}$ can be regarded (in $V^{\mathbb{P}}$) as the product of a sequence belonging to $\ulcorner \mathcal{L} \urcorner$. By the original hypothesis on Δ_r,

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \Delta_r \cap \Psi^{(\beta)} \cap \prod \mathfrak{L} \neq \emptyset.$$

We can therefore find a $\mathbb{P}$-name σ_r for a member of $\Delta_r \cap \Psi^{(\beta)} \cap \prod \mathfrak{L}$, and now further $\mathbb{P}_\beta$-names σ_i, for $i < r$, such that

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \langle \sigma_i \rangle_{i<r} \in \Delta \cap (\prod \mathfrak{L})^r,\ (\sigma_r, \langle \sigma_i \rangle_{i<r}) \in \ulcorner R_r \urcorner.$$

But of course we now have

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \langle \sigma_i \rangle_{i \le r} \in \prod_{i \le r} (\Delta_i \cap \prod \mathfrak{L}),\ (\sigma_j, \sigma_i) \in \ulcorner R \urcorner\ \forall\ i < j \le r.$$

As $\langle \Delta_i \rangle_{i \le r}$, $\mathfrak{L}$ are arbitrary, this shows that the induction proceeds.

Version of 6.8.91

2. Pointwise compact sets of measurable functions We turn now to the questions in analysis which the construction in §1 is designed to solve. We begin with some definitions and results taken from [Ta84].

2A Definitions (a) Let (X, Σ, μ) be a probability space. Write $\mathcal{L}^0 = \mathcal{L}^0(\Sigma) \subseteq \mathbb{R}^X$ for the set of Σ-measurable real-valued functions on X. Let $\mathfrak{T}_p$ be the topology of pointwise convergence, the usual product topology, on $\mathbb{R}^X$. Let $\mathfrak{T}_m$ be the (non-Hausdorff, non-locally-convex) topology of convergence in measure on $\mathcal{L}^0$, defined by the pseudometric

$$\rho(f, g) = \int \min(|f(x) - g(x)|, 1) \mu(dx)$$

for f, $g \in \mathcal{L}^0$.

(b) A set $A \subseteq \mathbb{R}^X$ is **stable** if whenever $\alpha < \beta$ in $\mathbb{R}$, $E \in \Sigma$ and $\mu E > 0$ there are k, $l \ge 1$ such that

$$\mu^*_{k+l}\{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in E^k,\ \mathbf{y} \in E^l,\ \exists\ f \in A,\ f(\mathbf{x}(i)) \le \alpha \,\&\, f(\mathbf{y}(j)) \ge \beta\ \forall\ i < k,\ j < l\} < (\mu E)^{k+l},$$

writing μ^*_{k+l} for the usual product outer measure on $X^k \times X^l$. (See [Ta84], 9-1-1.)

2B Stable sets Suppose that (X, Σ, μ) is a probability space and that $A \subseteq \mathbb{R}^X$ is a stable set.

(a) If (X, Σ, μ) is complete, then $A \subseteq \mathcal{L}^0(\Sigma)$. ([Ta84], §9.1.)

(b) The $\mathfrak{T}_p$-closure of A in $\mathbb{R}^X$ is stable.

(c) If A is bounded above and below by members of $\mathcal{L}^0$, its convex hull is stable ([Ta84], 11-2-1).

(d) If $A \subseteq \mathcal{L}^0$ (as in (a)), then $\mathfrak{T}_m \restriction A$, the subspace topology on A induced by $\mathfrak{T}_m$, is coarser than $\mathfrak{T}_p \restriction A$. ([Ta84], 9-5-2.)

For more about stable sets, see [Ta84]and [Ta87].

2C Pettis integration Let (X, Σ, μ) be a probability space and B a (real) Banach space.

(a) A function $\phi : X \to B$ is **scalarly measurable** if $g\phi : X \to \mathbb{R}$ is Σ-measurable for every $g \in B^*$, the continuous dual of B.

(b) In this case, ϕ is **Pettis integrable** if there is a function $\theta : \Sigma \to B$ such that

$$\int_E g\phi \, d\mu \text{ exists } = g(\theta E)\ \forall\ E \in \Sigma,\ g \in B^*.$$

(c) If $\phi : X \to B$ is bounded and scalarly measurable, then

$$A = \{g\phi : g \in B^*,\ \|g\| \le 1\} \subseteq \mathcal{L}^0$$

is $\mathfrak{T}_p$-compact. In this case ϕ is Pettis integrable iff

$$f \mapsto \int_E f : A \to \mathbb{R}$$

is $\mathfrak{T}_p{\restriction}A$-continuous for every $E \in \Sigma$ ([Ta84], 4-2-3). In particular (by 2B(d)) ϕ is Pettis integrable if A is stable.

2D The rivals Write μ_L for Lebesgue measure on $[0,1]$, and Σ_L for its domain. Consider the following two propositions:

(*) $[0,1]$ is not the union of fewer than $\mathfrak{c}$ closed negligible sets;

(†) there are sequences $\langle n_k\rangle_{k\in\mathbb{N}}$, $\langle W_k\rangle_{k\in\mathbb{N}}$ such that
$n_k \ge 2^k$, $W_k \subseteq n_k \times n_k$, $\#(W_k) \le 2^{-k}n_k^2 \;\; \forall \; k \in \mathbb{N}$;
taking $X = \prod_{k\in\mathbb{N}} n_k$, μ the usual Radon probability on X,

$$R = \{(x,y) : x,\, y \in X,\, (x(k), y(k)) \in W_k \;\; \forall \;\; k \in \mathbb{N},\, \{k : x(k) = y(k)\} \text{ is finite}\},$$

then whenever $D \subseteq X$, $\mu^*D = 1$ and $r \in \mathbb{N}$ there are $x_0, \ldots, x_r \in D$ such that $(x_j, x_i) \in R$ whenever $i < j \le r$.

Evidently (*) is a consequence of CH, while in the language of §1, $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ (†), this being a slightly weaker version of Theorem 1S.

Thus both (*) and (†) are relatively consistent with ZFC. Consequences of (*) are explored in [Ta84], where it is called Axiom L; we list a few of them in 2E below. Our purpose in this paper is to show that (†) leads to a somewhat different world.

2E Theorem Assume (*). Write $\mathcal{L}^0$ for $\mathcal{L}^0(\Sigma_L)$.

(a) If $A \subseteq \mathcal{L}^0$ is separable and compact for $\mathfrak{T}_p$, it is stable.

(b) If $A \subseteq \mathcal{L}^0$ is separable and compact for $\mathfrak{T}_p$, its closed convex hull in $\mathbb{R}^{[0,1]}$ lies within $\mathcal{L}^0$.

(c) If $A \subseteq \mathcal{L}^0$ is separable and compact for $\mathfrak{T}_p$, then $\mathfrak{T}_m{\restriction}A$ is coarser than $\mathfrak{T}_p{\restriction}A$.

(d) If $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ is a separable compact Radon measure space and $f : [0,1] \times Y \to \mathbb{R}$ is measurable in the first variable and continuous in the second, then it is measurable for the (completed) product measure $\mu_L \times \nu$.

(e) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a stochastically independent sequence of measurable subsets of $[0,1]$, with $\lim_{n\to\infty} \mu E_n = 0$ but $\sum_{n\in\mathbb{N}} (\mu E_n)^k = \infty$ for every $k \in \mathbb{N}$, then there is an ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that

$$\lim_{n\to\mathcal{F}} E_n = \{x : \{n : x \in E_n\} \in \mathcal{F}\}$$

is non-measurable.

proof (a) See [Ta84], 9-3-1(b). **(b)** Use (a) and 2B(c). **(c)** Use (a) and 2B(d). **(d)** Use (a) and [Ta84], 10-2-1. **(e)** Observe that, writing χE_n for the characteristic function of E_n, the set $\{\chi E_n : n \in \mathbb{N}\}$ is not stable, and use (a).

2F Theorem Assume (†).

(a) There is a bounded Pettis integrable function $\phi : [0,1] \to \ell^\infty$ such that $\{g\phi : g \in (\ell^\infty)^*,\, \|g\| \le 1\}$ is not stable in $\mathcal{L}^0(\Sigma_L)$.

(b) There is a separable convex $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0(\Sigma_L)$ which is not stable.

proof (We write ℓ^∞ for the Banach space of bounded real sequences.) Take $\langle n_k\rangle_{k\in\mathbb{N}}$, $\langle W_k\rangle_{k\in\mathbb{N}}$, X, μ, R from the statement of (†). Because $([0,1], \mu_L)$ is isomorphic, as measure space, to (X, μ), we may work with X rather than with $[0,1]$. Write Σ for the domain of μ, $\mathcal{L}^0 = \mathcal{L}^0(\Sigma)$.

(a) For $k \in \mathbb{N}$ write

$$\mathcal{I}_k = \{I : I \subseteq n_k,\, \#(I) \le k,\, (i,j) \notin W_k \text{ for all distinct } i,\, j \in I\},$$

For $k \in \mathbb{N}$, $I \subseteq n_k$ set

$$H_{kI} = \{x : x \in X,\, x(k) \in I\}.$$

Let A be

$$\{\chi H_{kI} : k \in \mathbb{N},\, I \in \mathcal{I}_k\},$$

writing $\chi H : X \to \{0,1\}$ for the characteristic function of $H \subseteq X$; let Z be the $\mathfrak{T}_p$-closure of A in $\mathbb{R}^X$. Because A is uniformly bounded, Z is $\mathfrak{T}_p$-compact. For $E \in \Sigma$ define $f_E : Z \to \mathbb{R}$ by setting $f_E(h) = \int_E h(x)\mu(dx)$ for $h \in A$, $f_E(u) = 0$ for $u \in Z \setminus A$. Enumerate A as $\langle h_m\rangle_{m\in\mathbb{N}}$, and define $\phi : X \to \ell^\infty$, $\theta : \Sigma \to \ell^\infty$ by setting

$$\phi(x)(m) = h_m(x) \;\; \forall \;\; m \in \mathbb{N},\, x \in X,$$
$$\theta(E)(m) = \int_E h_m(x)\mu(dx) \;\; \forall \;\; m \in \mathbb{N},\, E \in \Sigma.$$

We aim to show

(i) that A is not stable;
(ii) that if ν is a Radon probability on $A' = Z \setminus A$ then $\int u(x)\,\nu(du) = 0$ for μ-almost every x;
(iii) $f_E : Z \to \mathbb{R}$ is continuous for every $E \in \Sigma$;
(iv) θ is the indefinite Pettis integral of ϕ, so that ϕ is Pettis integrable;
(v) $K = \{g\phi : g \in C(Z)^*,\ \|g\| \le 1\}$ includes A so is not stable.

ad (i) Suppose that k, $l \ge 1$. Take any $m \ge l$. Set

$$\begin{aligned} G &= \{\mathbf{y} : \mathbf{y} \in X^l,\ \exists\ I \in \mathcal{I}_m,\ \mathbf{y}(j) \in H_{mI}\ \ \forall\ j < l\} \\ &\supseteq \{\mathbf{y} : \mathbf{y} \in X^l,\ (\mathbf{y}(i)(m), \mathbf{y}(j)(m)) \notin W_m \text{ for distinct } i,\ j < l\}. \end{aligned}$$

Because $\#(W_m) \le 2^{-m} n_m^2$, $\mu_l G \ge (1 - 2^{-m})^{l^2}$. If $\mathbf{y} \in G$, set $I = \{\mathbf{y}(j)(m) : j < l\} \in \mathcal{I}_m$; then
$$\mu_k\{\mathbf{x} : \mathbf{x} \in X^k,\ \mathbf{x}(i)(m) \notin I\ \ \forall\ i < k\} \ge (1 - n_m^{-1} l)^k.$$
So we conclude that
$$\begin{aligned} \mu_{k+l}\{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in X^k,\ \mathbf{y} \in X^l,\ \exists\ f \in A,\ f(\mathbf{x}(i)) = 0\ \ \forall\ i < k,\ f(\mathbf{y}(j)) = 1\ \ \forall\ j < l\} \\ \ge (1 - 2^{-m})^{l^2} (1 - n_m^{-1} l)^k \end{aligned}$$
(by Fubini's theorem). Because k, l and m are arbitrary, A cannot be stable.

ad (ii) Because each $\mathcal{I}_m$ is finite, any member of A' must be of the form χE where $E \subseteq X$ and
$$x \in E,\ x' \in X,\ \{k : x(k) \ne x'(k)\} \text{ is finite } \ \Rightarrow x' \in E.$$
Note also that if x, $y \in E$ then $(x, y) \notin R$; because either $x(k) = y(k)$ for infinitely many k, or there are k, I such that $x(k) \ne y(k)$, $I \in \mathcal{I}_k$ and x, y both belong to H_{kI}, in which case $(x(k), y(k)) \notin W_k$.

Now let ν be a Radon probability on A', and set $w(x) = \int u(x)\,\nu(du)$ for each $x \in X$, so that w belongs to the closed convex hull of A' in $\mathbb{R}^X$. If x, x' are two members of X differing on only finitely many coordinates, then $u(x) = u(x')$ for every $u \in A'$; consequently $w(x) = w(x')$. Also $0 \le w(x) \le 1$ for every $x \in X$.

Take $\delta > 0$ and set $D = \{x : w(x) \ge \delta\}$. By the zero-one law, $\mu^* D$ must be either 0 or 1. Suppose, if possible, that $\mu^* D = 1$. Let $r \in \mathbb{N}$ be such that $r\delta \ge 1$. By (†), there are $x_0, \dots, x_r \in D$ such that $(x_j, x_i) \in R$ for $i < j \le r$. But in this case $\sum_{i \le r} u(x_i) \le 1$ for every $u \in A'$, while $\sum_{i \le r} w(x_i) \ge (r+1)\delta > 1$, and w cannot belong to the closed convex hull of A'.

Accordingly $\mu^* D$ must be 0. As δ is arbitrary, $w = 0$ a.e.

ad (iii) Because $f_E(\chi H_{kI}) \le k n_k^{-1}$ for every $I \in \mathcal{I}_k$, $\lim_{m\to\infty} f_E(h_m) = 0$ and f_E is continuous.

ad (iv) We need to show that
$$\int_E g(\phi(x))\,\mu(dx) \text{ exists } = g(\theta(E))\ \ \forall\ g \in (\ell^\infty)^*,\ E \in \Sigma.$$
It is enough to consider positive linear functionals g of norm 1. For any such g we have a Radon probability ν on Z such that
$$g(\langle f(h_m)\rangle_{m\in\mathbb{N}}) = \int_Z f(u)\,\nu(du) \text{ for every } f \in C(Z),$$
using the Riesz representation of positive linear functionals on $C(Z)$. Set $\epsilon_m = \nu\{h_m\}$, $\epsilon = 1 - \sum_{m\in\mathbb{N}} \epsilon_m = \nu A'$. Then we can find a Radon probability ν' on A' such that
$$g(\langle f(h_m)\rangle_{m\in\mathbb{N}}) = \sum_{m\in\mathbb{N}} \epsilon_m f(h_m) + \epsilon \int_{A'} f(u)\,\nu'(du)$$
for every $f \in C(Z)$. Now an easy calculation (using (ii)) shows that
$$g(\theta(E)) = g(\langle f_E(h_m)\rangle_{m\in\mathbb{N}}) = \sum_{m\in\mathbb{N}} \epsilon_m \int_E h_m(x)\,\mu(dx) = \int_E g(\phi(x))\,\mu(dx)$$
for every $E \in \Sigma$.

ad (v) If $m \in \mathbb{N}$ then $h_m = e_m\phi \in K$ where $e_m \in (\ell^\infty)^*$ is defined by setting $e_m(z) = z(m)$ for every $z \in \ell^\infty$. This completes the proof.

(b) The unit ball of $(\ell^\infty)^*$ is w*-separable and its continuous image $K \subseteq \mathcal{L}^0$ is separable; so K witnesses the truth of (b).

2G Further properties of the model Returning to 1R/1S, we see that the model of §1 has some further striking characteristics closely allied to, but not obviously derivable from, (†). Consider for instance

(‡) there is a closed negligible set $Q \subseteq [0,1]^2$ such that whenever $D \subseteq [0,1]$ and $\mu_L^* D = 1$ then $\mu_L Q^{-1}[D] = 1$;

(‡)′ there is a negligible set $Q' \subseteq [0,1]^2$ such that whenever C, $D \subseteq [0,1]$ and $(C \times D) \cap Q' = \emptyset$ then one of C, D is negligible.

Then $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} (\ddagger)$. For start by taking Q_1 to be

$$\{(x,y) : x,\, y \in X,\ (x(k), y(k)) \in W_k\ \ \forall\ \ k \in \mathbb{N}\},$$

the closure of R in $X \times X$. Then the argument for 1S shows that

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \text{ if } D \subseteq \ulcorner X \urcorner \text{ and } D \cap \prod_{k\in\mathbb{N}} L_k \neq \emptyset\ \forall\ \langle L_k \rangle_{k\in\mathbb{N}} \in \ulcorner \mathfrak{L} \urcorner$$

$$\text{then } \exists\ \beta < \kappa \text{ such that } \Psi^{(\beta)} \subseteq \ulcorner Q_1 \urcorner^{-1}[D].$$

Consequently

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \text{if } D \subseteq \ulcorner X \urcorner \text{ and } \ulcorner \mu \urcorner^* D = 1 \text{ then } \ulcorner \mu \urcorner (\ulcorner Q_1 \urcorner^{-1}[D]) = 1.$$

Accordingly we have in $V^{\mathbb{P}}$ the version of $(\ddagger)$ in which $([0,1], \mu_L)$ is replaced by (X, μ). However there is now a continuous inverse-measure-preserving function $f : X \to [0,1]$, and taking

$$Q = \{(f(x), f(y)) : (x,y) \in Q_1\}$$

we obtain $(\ddagger)$ itself. Evidently $(\ddagger)$ implies $(\ddagger)'$, taking Q' to be

$$\{(x+q, y+q') : (x,y) \in Q,\ q,\ q' \text{ are rational}\} \cap [0,1]^2.$$

Of course $(^*)$ and $(\ddagger)$ are mutually incompatible (the argument for 2E(a) from $(^*)$, greatly simplified, demolishes $(\ddagger)$ also). The weaker form $(\ddagger)'$ is incompatible with CH or MA, but not with $(^*)$, both $(\ddagger)'$ and $(^*)$ being true in Cohen's original model of not-CH (see [Frp89]).

2H Problems The remarkable results quoted in 2E depend on the identification of separable relatively pointwise compact sets with stable sets ('Axiom F' of [Ta84]). In models satisfying $(\dagger)$, this identification breaks down. But our analysis does not seem to touch any of 2E(b)-(e). We therefore spell out the obvious problems still outstanding. Write $\mathcal{L}^0$ for $\mathcal{L}^0(\Sigma_L)$.

(a) Is it relatively consistent with ZFC to suppose that there is a separable $\mathfrak{T}_p$-compact set $A \subseteq \mathcal{L}^0$ such that the closed convex hull of A in $\mathbb{R}^{[0,1]}$ does not lie within $\mathcal{L}^0$?

(b) Is it relatively consistent with ZFC to suppose that there is a separable $\mathfrak{T}_p$-compact set $A \subseteq \mathcal{L}^0$ such that $\mathfrak{T}_m \restriction A$ is not coarser than $\mathfrak{T}_p \restriction A$? Does it make a difference if A is assumed to be convex? (This question seems first to have been raised by J.Bourgain and F.Delbaen.)

(c) Is it relatively consistent with ZFC to suppose that there are a separable compact Radon measure space $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ and a function $f : [0,1] \times Y \to \mathbb{R}$ which is measurable in the first variable, continuous in the second variable, but not jointly measurable for $\mu_L \times \nu$?

(d) Is it relatively consistent with ZFC to suppose that there is a stochastically independent sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ in Σ_L such that $\sum_{n\in\mathbb{N}} (\mu_L E_n)^k = \infty$ for every $k \in \mathbb{N}$, but $\mu_L(\lim_{n\to\mathcal{F}} E_n) = 0$ for every non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$? (This question is essentially due to W.Moran; see also [Ta84], 9-1-4 for another version.)

Here we note only that a positive answer to (a) would imply the same answer to (c), and that the word 'separable' in (a)-(c) is necessary, as is shown by examples 3-2-3 and 10-1-1 in [Ta84].

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