MORE ON CARDINAL ARITHMETIC
SH410

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An Annotated Table of Contents

§6 The Existence of Strongly Almost Disjoint Families

We characterize such existence questions by pp's. An example is the question of the existence of a family of λ^+ subsets of $\lambda > \kappa^{\aleph_0}$, each of cardinality $\kappa\,(> \aleph_0)$ such that the intersection of any two is finite.

§1 On Normal Filters

The following Lemma 1.2 is similar to [Sh 355, 5.4], [Sh 400, 3.5], but deal with normal ideals (see [Sh 371, §4], in particular [Sh 371, Definition 4.1,Claim 4.6]). Remember prc is defined in [Sh 371, §4] as:

1.1 Definition. 1) For a regular uncountable cardinal κ, normal ideal J on κ, $\bar{\mu}$ a κ-sequence of cardinals $> \kappa$, and $f \in {}^{\kappa}\mathrm{Ord}$, we define:

$$\begin{aligned}\mathrm{prc}_J(f,\bar{\mu}) = \mathrm{Min}\Big\{|\mathscr{P}| : &\mathscr{P} \text{ is a family of } \kappa\text{-sequences of sets of ordinals,}\\ &\bar{B} = \langle B_i : i<\kappa\rangle, |B_i| < \mu_i \text{ or at least}\\ &\{i<\kappa : |B_i| \geqq \mu_i\} \in J, \text{ such that: for every } g \in {}^{\kappa}\mathrm{Ord},\\ &g \leqq_J f \text{ there is a sequence } \langle \bar{A}^{\zeta} : \zeta<\kappa\rangle \text{ of members}\\ &\mathscr{P} \text{ satisfying } \{i<\kappa : g(i) \notin \bigcup_{\zeta<i} A_i^{\zeta}\} \in J\Big\}.\end{aligned}$$

2) We may write f as a sequence of ordinals say $\langle \lambda_i : i<\kappa\rangle$, and if $\lambda_i = \lambda$ for each i, we may write λ.
3) $\mathrm{prc}'_J(f,\mu)$ is defined similarly but from $\bar{B} = \langle B_i : i<\kappa\rangle$ we demand this time $|B_i| < \mu$.

Remark. See there ([Sh 371, 4.2,4.3]) for some basic properties. But 1.2 below substantially improves [Sh 371, Claim 4.6] there.

1.2 Lemma. *1) Let κ be a regular uncountable cardinal, $f : \kappa \to$ ordinals, J a normal ideal on κ, and $\bar{\mu} = \langle \mu_i : i<\kappa\rangle$ a sequence of regular cardinals. Then*

$$\begin{aligned}\mathrm{prc}_J(f,\bar{\mu}) = \sup\Big\{\mathrm{tcf}[\textstyle\prod_{i<\kappa} \mu'_i/J_1] : &J_1 \text{ a normal ideal on } \kappa \text{ extending } J,\\ &\text{such that the tcf is well defined and}\\ &\{i<\kappa : \text{ not } ``\mu_i \leqq \mu'_i = \mathrm{cf}(\mu'_i) \leqq f(i)''\} \in J\Big\}\end{aligned}$$

provided that:

- (α) $\mu_i = \mu > \kappa$.

2) We can replace assumption (α) by (β) below, and $\bigwedge\limits_i \mu_i > \kappa$.

- (β) *μ_i strictly increasing and for limit i:*
  if $i = \sum\limits_{j<i} \mu_j$ is regular then $i = \mu_i$, otherwise $\mu_i = (\sum\limits_{j<i} \mu_j)^+$.

1.3 Remark. 1) On getting $=^+$ see [Sh 420, 6.1](C) and [Sh 430, §4]. The problem is when $\mathrm{pcf}(\mathfrak{a})$ has an accumulation point which is inaccessible.
2) In the case (β) holds, if $\bar{\mu}^* = \langle \mu_i^* : i < \kappa \rangle$ is (strictly) increasing continuous, $\sup_{i<\kappa} \mu_i^* = \sup_{i<\kappa} \mu_i$ then $\mathrm{prc}(f, \bar{\mu}) + (\sum_{i<\kappa} \mu_i)^+ = \mathrm{prc}(f, \bar{\mu}^*) + (\sum_{i<\kappa} \mu_i)^+$, by [Sh 371, 4.10](2) + [Sh 355, 2.1].
3) If in (β) we place "$\mu_i > \kappa$" by $\sup_{i<\kappa} \mu_i = \kappa$, $\bigwedge_{i<\kappa^*} \mu_i < \kappa$ (so κ is inaccessible we can get:

$$\mathrm{prc}_J(f, \bar{\mu}) = \sup\big\{\text{nor-cf}_i \prod_{i<\kappa} \mu_i'/J : \{i : \mu_i \le \mu_i' = \ \mathrm{cf}(\mu_i') \le f(i)\} \in J\big\}.$$

Proof. The inequality $\geq$:
Same proof as that of "$\lambda(1) \le \lambda(2)$" in the proof of [Sh 371, 4.6].

The inequality $\leq$:
Let λ^* be the successor of the sup.

1.4 Fact. There is a family $\mathscr{P}^*$ such that:

(i) members of $\mathscr{P}^*$ are of the form

$$\langle B_{i,\zeta} : i < \kappa, \zeta < \zeta_i \rangle \text{ or } \langle < B_{i,\zeta} : \zeta < \zeta_i > : i < \kappa \rangle$$

where $\zeta_i < \mu_i$ and each $B_{i,\zeta}$ is a non-empty subset of $f(i)+1$

(ii) $|\mathscr{P}^*| < \lambda^*$

(iii) if $\langle B_{i,\zeta} : i < \kappa, \zeta < \zeta_i \rangle \in \mathscr{P}^*, g \in \prod_{i<\kappa} (f(i)+1), A \subseteq \kappa$ and for $i \in A$ we have $\xi_i < \zeta_i$ satisfying $g(i) \in B_{i,\xi_i}$ then there are $E, \langle A_j : j < \kappa \rangle$ and for $j < \kappa$ sequences $\bar{B}^j = \langle B^j_{i,\zeta} : i < \kappa, \zeta < \zeta_i^j \rangle$ and $\langle \xi_i^j : i \in A_j \rangle$ such that:

- (a) $E \subseteq \kappa$ and $\kappa \backslash E \in J$;
- (b) $A \cap E = \{i < \kappa : i \in \bigcup_{j<i} A_j\}$;
- (c) $\bar{B}^j \in \mathscr{P}^*$ for $j < \kappa$;
- (d) for $j < \kappa$ and $i \in A_j$ we have: $\xi_i^j < \zeta_i^j$ and $B^j_{i,(\xi_i^j)} \subseteq B_{i,\xi_i}$ and

$$[|B^i_{i,(\xi_i^j)}| \ge \mu_i \Rightarrow |B^j_{i,(\xi_i^j)}| < |B_{i,(\xi_i^j)}|]$$

- (e) for $i \in A_j$ we have $g(i) \in B^j_{i,(\xi_i^j)}$

(iv) $\langle < f(i)+1 > : i < \kappa \rangle \in \mathscr{P}^*$ (so $\zeta_i = 1$ here).

Proof of the Inequality from the Fact. Let us define a family $\mathscr{P}'$:

$$\mathscr{P}' = \big\{\langle \cup\{B_{i,\zeta} : \zeta < \zeta_i \text{ and } |B_{i,\zeta}| < \mu_i\} : i < \kappa\rangle : \langle B_{i,\zeta} : i < \kappa, \zeta < \zeta_i\rangle \in \mathscr{P}^*\big\}.$$

Now each member of $\mathscr{P}'$ has the right form as each μ_i is regular and $\langle B_{i,\zeta} : i < \kappa, \zeta < \zeta_i\rangle \in \mathscr{P}^*$ implies $\zeta_i < \mu_i$. Also the cardinality of $\mathscr{P}'$ is $< \lambda^*$ (by (ii) of 1.3).

Let $g \le f$ and it is enough to find $\langle B^\epsilon_i : i < \kappa\rangle \in \mathscr{P}'$ for $\epsilon < \kappa$ such that $\{i : g(i) \notin \bigcup\limits_{\epsilon<i} B^\varepsilon_i\} \in J$.

We choose by induction on n, for every $\eta \in {}^n\kappa$ the following: $\bar{B}^\eta = \langle B^\eta_{i,\zeta} : i < \kappa, \zeta < \zeta^\eta_i\rangle \in \mathscr{P}^*, \langle \xi^\eta_i : i \in A_\eta\rangle$ with $\xi^\eta_i < \zeta^\eta_i$ and $A_\eta \subseteq \kappa$ such that:

- (α) $\bar{B}^{<>} = \langle < f(i)+1 > : i < \kappa\rangle, \zeta^{<>}_i = 1, \xi^{<>}_i = 0, A_{<>} = \kappa$,
- (β) $[i \in A_{\eta^\frown<j>} \Rightarrow g(i) \in B^{\eta^\frown<j>}_{i,(\xi^{\eta^\frown<j>}_i)} \subseteq B^\eta_{i,\xi^\eta_i}]$,
- (γ) $[i \in A_{\eta^\frown<j>} \& |B^\eta_{i,\xi^\eta_i}| \ge \mu_i \Rightarrow |B^{\eta^\frown<j>}_{i,(\xi^{\eta^\frown<j>}_i)}| < |B^\eta_{i,\xi^\eta_i}|]$ and
- (δ) $\{i \in A_\eta : i \notin \bigcup\limits_{j<i} A_{\eta^\frown<j>}\} \in J$.

The induction step is by (iii) of the fact; in the end let for $\eta \in {}^{\omega>}\kappa$ and $i < \kappa$: $B^*_{\eta,i} =: \cup\{B^\eta_{i,\zeta} : \zeta < \zeta^\eta_i \text{ and } |B^\eta_{i,\zeta}| < \mu_i\}$.

Clearly for each $\eta \in {}^{\omega>}\kappa$ we have $\langle B^*_{\eta,i} : i < \kappa\rangle \in \mathscr{P}'$; let us enumerate ${}^{\omega>}\kappa$ as $\{\rho_\epsilon : \epsilon < \kappa\}$ such that $[\rho \lhd \rho_\epsilon \Rightarrow \rho \in \{\rho_\zeta : \zeta < \epsilon\}]$, and let us define $\bar{B}^\epsilon = \langle B^*_{\rho_\epsilon,i} : i < \kappa\rangle \in \mathscr{P}'$, hence by Definition 1.1 it is enough to show that $E =: \{i < \kappa : g(i) \notin \bigcup\limits_{\epsilon<i} B^*_{\rho_\epsilon,i}\}$ belongs to J. We know that for every $\eta \in {}^{\omega>}\kappa$ the set $X_\eta =: \{i \in A_\eta : i \notin \bigcup\limits_{j<i} A_{\eta^\frown<j>}\}$ belongs to J. Also the sets

$$Y =: \{\delta < \kappa : \delta \text{ limit and } \neg(\forall\epsilon)[\epsilon < \delta \Rightarrow \rho_\epsilon \in {}^{\omega>}\delta]\}$$

$$X =: \{i < \kappa : \text{if } i \notin Y \text{ then for some } \eta \in {}^{\omega>}i \text{ we have } i \in X_\eta\}$$

belong to J. It suffices to show

- $(*)$ for every $\delta \in \kappa\backslash X$, for some $\eta \in {}^{\omega>}\delta$ we have $\delta \in B^*_{\eta,i}$.

Why $(*)$ holds? Choose by induction on $n < \omega, \rho_n \in {}^n\delta$ such that: $\delta \in A_{\rho_n}$. For $n = 0$ remember $A_{<>} = \kappa$. For $n+1$, as $\delta \in A_{\rho_n}$ and $\delta \notin X$ clearly $\delta \notin X_{\rho_n}$ so necessarily $\delta \in \bigcup\limits_{j<i} A_{\rho_n{}^\frown<j>}$.

Now $\langle |B^{\rho_n}_{\delta,\xi^{\rho_n}_\delta}| : n < \omega\rangle$ is non-increasing hence (by (β) above) is eventually constant hence (by (γ) above) for some $n, |B^{\rho_n}_{\delta,\xi^{\rho_n}_\delta}| < \mu_\delta$, hence $g(\delta) \in B^*_{\rho_n,\delta}$. So $(*)$ holds and we have finished proving that $\mathscr{P}'$ exemplify the inequality from the fact.

Proof of the Fact 1.1A. It suffices to prove that for any $\bar{B}^* = \langle B^*_{i,\zeta} : i < \kappa, \zeta < \zeta_i\rangle$ satisfying the requirements in (i), we can find $\mathscr{P}_{\bar{B}^*}$ satisfying $(i) + (ii)$ and (iii)

for the given $\bar{B}^*$ and any $g, \langle \xi_i : i < \kappa \rangle$ as there. Let $Y^i_0 = \{\zeta < \zeta_i : |B^*_{i,\zeta}| < \mu_i\}, Y^i_1 = \{\zeta < \zeta_i : |B^*_{\zeta,i}| \geq \mu_i > \text{ cf}|B^*_{\zeta,i}|\}$ and $Y^i_2 = \{\zeta < \zeta_i : \text{ cf}|B^*_{i,\zeta}| \geq \mu_i\}$ (for each $i < \kappa, \zeta < \zeta_i$).

Clearly $\langle Y^i_0, Y^i_1, Y^i_2 \rangle$ is a partition of $\{\zeta : \zeta < \zeta_i\}$, now for $\zeta \in Y^i_1 \cup Y^i_2$, let $\mu^i_\zeta =: \text{ cf}|B^*_{\zeta,i}|$, and $\langle B_{i,\zeta,\epsilon} : \epsilon < \mu^i_\zeta \rangle$ be an increasing continuous sequence of subsets of $B^*_{i,\zeta}$ of cardinality $< |B^*_{\zeta,i}|$ and $\bigcup\limits_{\epsilon < \mu^i_\zeta} B_{i,\zeta,\epsilon} = B^*_{i,\zeta}$.

Now let $\mathfrak{a} =: \{\mu^i_\zeta : i < \kappa, \zeta \in Y^i_2\}$, so $\mathfrak{a}$ is a set of regular cardinals.

Case 1: Assume assumption (α) of 1.2;
so $\mathfrak{a}$ is a set of $< \mu + \kappa^+ = \mu$ regular cardinals, each $\geq \mu$, so the pcf analysis of [Sh 371, §2] apply. Let us get $\big\langle \langle f^\theta_\alpha : \alpha < \theta \rangle : \theta \in \text{ pcf}(\mathfrak{a}) \big\rangle$.

Now for each $\theta \in \text{ pcf}(\mathfrak{a}) \cup \{1\}$ which is $< \lambda^*$ and $\alpha < \theta$, we choose $\bar{B}^{\theta,\alpha} = \langle B^{\theta,\alpha}_{i,\zeta} : i < \kappa, \zeta < \zeta^{\theta,\alpha}_i \rangle$ such that:

$$\begin{aligned}\{B^{\theta,\alpha}_{i,\zeta} : \zeta < \zeta^{\theta,\alpha}_i\} =& \{B^*_{i,\zeta} : \zeta \in Y^i_0\} \cup \{B^*_{i,\zeta,\epsilon} : \epsilon < \mu^i_\zeta, \zeta \in Y^i_1\} \\ & \cup \{B_{i,\zeta,f^\theta_\alpha(\mu^i_\zeta)} : \zeta \in Y^i_2 \text{ and } \theta \in \text{ pcf}(\mathfrak{a}) (\text{i.e. } \theta \neq 1)\}.\end{aligned}$$

Let $\mathscr{P}_{\bar{B}^*} =: \{\bar{B}^{\theta,\alpha} : \theta \in \lambda^* \cap \text{ pcf}(\mathfrak{a}) \text{ and } \alpha < \theta \text{ or } \theta = 1, \alpha = 0\}$. Now it is as required, in particular $|\mathscr{P}_{\bar{B}^*}| < \lambda^*$ because $\lambda^* > \sup(\lambda^* \cap \text{ pcf}(\mathfrak{a})$ holds as λ^* is a successor cardinal.

Case 2: Assume assumption. Here we partition $\mathfrak{a}$ to κ sets diagonally; i.e. without loss of generality each ζ_i is a cardinal hence by clause (β) we have $\zeta_i \leq \bigcup\limits_{j<i} \mu_{j+1} + \mu_0$. So for every limit $i < \kappa$ we have $\zeta_i \leq \bigcup\limits_{j<i} \mu_j$. Remember $\kappa < \bigcup\limits_{i<\kappa} \mu_i$ and even $\kappa < \mu_0$ and let for $j < \kappa$:
$\mathfrak{a}_j =: \{\mu^i_\zeta : i \in (j, \kappa) \text{ and } \zeta \in Y^i_2 \text{ and } \zeta < \mu_j\}$. So $|\mathfrak{a}_j| \leq \kappa + \mu_j < \text{ Min}(\mathfrak{a}_j)$ because and relevant μ^i_ζ is $\geq \mu_i > \mu_j \geq \kappa$ and if $\langle \xi_i : i < \kappa \rangle \in \prod\limits_{i<\kappa} (1 + \zeta_i)$ then we can define $h : \kappa \to \kappa, h(i) < 1 + i$ such that: $[i < \kappa \ \& \ i \text{ limit } \Rightarrow \mu^i_{\xi_i} \in \mathfrak{a}_{h(i)}]$. $\square_{1.2}$

The following lemma generalizes [Sh 371, 1.5].

1.5 Lemma. *Suppose $\sigma_1 \leq \sigma_2 \leq \kappa < \theta < \lambda$ are cardinals, $\sigma_1, \sigma_2, \kappa$ are regular, $\lambda > \ cf(\lambda) = \kappa > \aleph_0, \lambda < \mu = \ cf(\mu) < \ pp^+_{\Gamma(\theta,\sigma_2)}(\lambda)$, and for every large enough $\lambda' < \lambda, [\sigma_1 \leq \ cf(\lambda') < \theta \Rightarrow \ pp_{\Gamma(\theta,\sigma_1)}(\lambda') < \lambda]$.*

Then there is an increasing sequence $\langle \mu_i : i < \kappa \rangle$ of regular cardinals $< \lambda, \lambda = \sup\limits_{i<\kappa} \mu_i$ and an ideal J on κ satisfying $\lambda = \ tlim_J \mu_i$ and $\mu = \ tcf(\prod\limits_{i<\kappa} \mu_i / J)$ such that:

- *(a) J is σ_2-complete and extend $J^{\rm bd}_\kappa$*
- *(b) if $\kappa \geq \sigma_1^+$ and $(\forall \alpha < \kappa)[cov(|\alpha|, \sigma_2, \sigma_1, 2) < \kappa]$ then J is normal;*
- *(c) if $\sigma_1 = \aleph_0$ then $J = J^{\rm bd}_\kappa$.*

Proof. For (c) see [Sh 371, 1.6], so we can assume $\sigma_1 > \aleph_0$ as otherwise we have there gotten a conclusion stronger than (a) + (b) + (c).

Let $\mathfrak{a} \subseteq \text{ Reg } \cap \lambda$ have cardinality $< \theta$, be unbounded in λ, I a σ_2-complete ideal on a, and $(\forall \lambda' < \lambda)[\mathfrak{a} \cap \lambda' \in I]$ and $\mu = \text{ tcf}(\Pi\mathfrak{a}/I)$. As $\text{cf}(\lambda) = \kappa < \theta < \lambda$ without loss of generality $\theta < \text{ Min}(\mathfrak{a})$, and let $\langle \lambda_i : i < \kappa \rangle$ be increasing continuous with limit $\lambda, \aleph_0 \leq \text{ cf}(\lambda_i) < \kappa$ (remember $\kappa > \aleph_0$); without loss of generality $\theta < \lambda_0 < \text{ Min})\mathfrak{a})$ and $\text{pp}_{\Gamma(\theta,\sigma_1)}(\lambda_i) < \lambda_{i+1}$ and, if $i > 0$ is a limit ordinal then $\mathfrak{a} \cap \lambda_i$ is unbounded in λ_i. Also without loss of generality for every $i < \kappa$:

$$(*)_1 \ \lambda_0 \leq \lambda' < \lambda_i \ \& \ \sigma_1 \leq \text{ cf}(\lambda') < \theta \Rightarrow \text{ pp}_{\Gamma(\theta,\sigma_1)}(\lambda') < \lambda_{i+1}$$

hence

$$(*)_2 \ \mathfrak{b} \subseteq (\lambda_0, \lambda_i) \ \& \ |\mathfrak{b}| < \theta \Rightarrow \text{ sup pcf}_{\sigma_1\text{-complete}}(\mathfrak{b}) < \lambda_{i+1}.$$

Let $\langle \mathfrak{b}_\sigma[\mathfrak{a}] : \sigma \in \text{ pcf}(\mathfrak{a}) \rangle$ be a generating sequence (exists by [Sh 371, 2.6]) and without loss of generality $\mu = \text{ max pcf}(\mathfrak{a})$. By [Sh 345a, 3.6] (and 3.1(7)) without loss of generality $\sigma \in \mathfrak{b}_\chi[\mathfrak{a}] \Rightarrow \mathfrak{b}_\sigma[\mathfrak{a}] \subseteq \mathfrak{b}_\chi[\mathfrak{a}]$. Let $i < \kappa$ satisfy $\text{cf}(i) \geq \sigma_1$, as $|\mathfrak{a}| < \theta$ and $(*)_1$ we have $\text{sup pcf}_{\sigma_1\text{-complete}}(\lambda_i \cap \mathfrak{a}) < \lambda_{i+1}$ hence for some $\mathfrak{c}_i$ we have: $\mathfrak{c}_i \subseteq \lambda_{i+1} \cap \text{ pcf}(\mathfrak{a} \cap \lambda_i), |\mathfrak{c}_i| < \sigma_1$ and $\mathfrak{a} \cap \lambda_i \subseteq \bigcup_{\theta \in \mathfrak{c}_i} \mathfrak{b}_\theta[\mathfrak{a}]$ (otherwise we can find a σ_1-complete proper ideal J on $\lambda_i \cap \mathfrak{a}$ such that

$$[\sigma < \text{pp}^+_{\Gamma(\theta,\sigma_1)}(\lambda_i) \ \& \ \sigma \in \text{pcf}(\lambda_i \cap \mathfrak{a}) \Rightarrow \mathfrak{b}_\sigma[\mathfrak{a}] \cap (\lambda_i \cap \mathfrak{a} \backslash \mu_i) \in J],$$

a contradiction to $(*)_1$. Note that $\mathfrak{c}_i \subseteq \lambda_{i+1}$.

Let $S_0 = \{\delta < \kappa : \text{cf}(\delta) \geqq \sigma_1\}$, so for some $i(*) < \kappa$ we have:

$$S_1 = \{\delta \in S_0 : \mathfrak{c}_\delta \cap \lambda_\delta \subseteq \lambda_{i(*)}\}.$$

is a stationary subset of κ.

By renaming, without loss of generality $i(*) = 0$ and so $\mathfrak{c}_i \subseteq (\lambda_i, \lambda_{i+1})$, and for $i \in S_1$ let $\langle (\theta_{i,\zeta}, \mathfrak{e}_{i,\zeta}) : \zeta < \zeta_i \rangle$ list $\{(\theta, \mathfrak{b}_\theta[\mathfrak{a}]) : \theta \in \mathfrak{c}_i\}$; so:

$$(*)_1 \ \mathfrak{a} \cap \lambda_i \subseteq \bigcup_{\zeta < \zeta_i} \mathfrak{e}_{i,\zeta}, \text{ max pcf}(\mathfrak{e}_{i,\zeta}) = \theta_{i,\zeta}, \lambda_i < \theta_{i,\zeta} < \lambda_{i+1}$$
$$\text{and } \theta_{i,\zeta} \in \text{ pcf}_{\sigma\text{-complete}}(\mathfrak{a} \cap \lambda_i).$$

1.6 Fact. There are finite $\mathfrak{d}_{i,\zeta} \subseteq \text{ pcf}(\mathfrak{e}_{i,\zeta}) \backslash \lambda_i$ for $i \in S_1, \zeta < \zeta_i$ and stationary $S_2 \subseteq S_1$ such that letting $\mathfrak{d}_i = \bigcup_{\zeta < \zeta_i} d_{i,\zeta}$ we have: if $S \subseteq S_2, \kappa = \text{ sup}(S)$ then $\mu \in \text{ pcf}_{\sigma_2\text{-complete}} \left(\bigcup_{i \in S} \mathfrak{d}_i \right)$.

Proof of 1.2 from the Fact. Now the preliminary part of 1.5 is easy; as $\mathfrak{d}_i \subseteq (\lambda_i, \lambda_{i+1})$, and $|\mathfrak{d}_i| < \sigma_1 < \kappa = \text{ cf}(\kappa)$, clearly $\mathfrak{d} =: \bigcup_{i \in S_1} \mathfrak{d}_i$ has order type κ,

and by 1.6 $\mu \in \operatorname{pcf}_{\sigma_2\text{-complete}}\left(\bigcup\limits_{i\in S_1}\mathfrak{d}_i\right)$ and by $(*)_2$ above, for $j<\kappa$ implies $\mu \notin \operatorname{pcf}_{\sigma_1\text{-complete}}\left(\bigcup\limits_{i\in S_1}\mathfrak{d}_i\cap\lambda_j\right)$; also $\mu = \max\operatorname{pcf}(\mathfrak{d})$. So we are left with clauses $(a)+(b)$. Let $\mathfrak{d}_\delta = \{\lambda_{\delta,\zeta}:\zeta<\zeta_\delta<\sigma_1\}$, as $\sigma_1<\kappa$, clearly without loss of generality for some $\zeta(*)$

$$S_3 = \{\delta\in S_2 : \zeta_\delta = \zeta(*)\} \text{ is stationary.}$$

For each $\zeta<\zeta(*)$ let

$$\mathscr{P}_\zeta =: \{S\subseteq S_2 : \max\operatorname{pcf}\{\lambda_{\delta,\zeta}:\delta\in S\}<\mu\}.$$

If for some $\zeta<\zeta(*)$, the normal ideal on κ which $\mathscr{P}_\zeta$ generates is proper, we have finished. If not, for each $\zeta<\zeta(*)$ there are members $S_{\zeta,i}(i<\kappa)$ of $\mathscr{P}_\zeta$ and club C_ζ of κ such that:

$$\delta\in S_3\cap C_\zeta \Rightarrow \bigvee_{i<\delta}\delta\in S_{\zeta,i}.$$

Clearly $C=\bigcap\limits_{\zeta<\zeta(*)}C_\zeta$ is a club of κ, now remembering $S_3\subseteq S_0$ we know that on $S_3\cap C$ the function $\delta\mapsto \sup\{\operatorname{Min}\{i:\delta\in S_{\zeta,i}\}:\zeta<\zeta(*)\}$ is a pressing down function, so for some stationary $S_4\subseteq S_3$ and ordinal $j(*)<\operatorname{Min}(S_4)<\kappa$ we have:

$$\delta\in S_4 \Leftrightarrow \bigwedge_{\zeta<\zeta(*)}\bigvee_{j<j(*)}\delta\in C\cap S_{\zeta,j}.$$

But as $\operatorname{cov}(|j(*)|,\sigma_2,\sigma_1,2)<\kappa$, there is $w\subseteq j(*)$ such that $|w|<\sigma_2$ and S_5 is a stationary subset of κ where

$$S_5 = \big\{\delta\in S_3 : \bigwedge_{\zeta<\zeta(*)}\bigvee_{i\in w}\delta\in S_{\zeta,i}\big\}.$$

Let $\mathfrak{b}=\{\lambda_{i,\zeta}$: for some j and ζ we have $i\in S_{\zeta,j}, j\in w$ and $\zeta<\zeta(*)\}$, it is the union of $<\sigma_2$ sets ($\mathfrak{b}_{j,\zeta}=\{\lambda_{i,\zeta}:i\in S_{\zeta,j}\}$ for $j\in w,\zeta<\zeta(*)$), each with max pcf $<\mu$.

This contradicts the fact (1.6).

Proof of Fact 1.2A. Similar to the proof of [Sh 371, 1.5]. $\square_{1.5}$

§2 On measures of the size of $\mathscr{D}_{<\kappa}(\lambda)$

SAHARON: on 2.1 see 430?
Improving a little [Sh 400, 5.9].

2.1 Claim. *Assume* $\lambda > \mathrm{cf}(\lambda) = \aleph_0, \lambda > 2^\theta$ *and* $[\lambda' < \lambda \ \ \& \ \ \mathrm{cf}(\lambda') \leqq \theta \Rightarrow \mathrm{pp}_\theta(\lambda') < \lambda]$ *and* $\mathrm{pp}(\lambda) < \mathrm{cov}(\lambda, \lambda, \aleph_1, 2)$. *Then* $\{\mu : \lambda < \mu = \aleph_\mu \leq \mathrm{pp}_\theta(\lambda)\}$ *has order type* $\geq \theta$.

We shall return to this in [Sh 430, §1] so we do not elaborate.

2.2 Claim. *Suppose* θ, κ *are regular,* $\aleph_0 < \theta \leq \kappa < \lambda \leq \lambda_1 \leq \lambda^*$, *and* $(\forall \mu)[\lambda \leq \mu \leq \lambda_1 \ \ \& \ \ \mathrm{cf}(\mu) < \theta \Rightarrow \mathrm{pp}_{<\theta}(\mu) \leqq \lambda^*]$ *and* $\mathrm{cov}(\kappa, \theta, \theta, 2) = \kappa$. *Then there is a family* $\mathscr{P}$ *of* $\leq \lambda^*$ *subsets of* λ_1, *each of cardinality* $< \lambda$ *such that:*

- $(*)_1$ *for every* $Y \subseteq \lambda_1, |Y| \leq \kappa$ *there are* $Z_n (n < \omega)$ *such that:* $Y \subseteq \bigcup\limits_{n<\omega} Z_n, |Z_n| = \kappa$ *and for each* n

$$(\forall Z)[Z \subseteq Z_n \ \ \& \ \ |Z| < \theta \Rightarrow (\exists X \in \mathscr{P})[Z \subseteq X]],$$

- $(*)_2$ *for every* $Y \subseteq \lambda_1, |Y| \geq \theta_1$ *and* $\aleph_1 < \mathrm{cf}(\theta_1) \ \ \& \ \ \theta_1 \leq \theta$ *there is* $X \in \mathscr{P}$ *such that:* $X \cap Y$ *has cardinality* $\geq \theta_1$.

Remark. 1) Here and later we can replace $\leq \lambda^*$ by $< \lambda^* = \mathrm{cf}(\lambda^*)$. [Saharon: check.]
2) See [Sh 430, x.x.].

Proof. It suffices to prove $(*)_1$ as $(*)_2$ follows. Let $\Theta = \{\mu : \lambda \leq \mu \leq \lambda_1$ and $\mathrm{cf}(\mu) < \theta\}$. Clearly if $\Theta = \emptyset$, the conclusion is straightforward (by induction on λ_1).

Without loss of generality $\lambda^* = \sup\{\mathrm{pp}_{<\theta}(\mu) : \mu \in \Theta\}$. Now each $\mathrm{pp}_{<\theta}(\mu)$ (for $\mu \in \Theta$) has cofinality $\geq \theta$, and if $\delta < \theta, \langle \mu_i : i < \delta \rangle$ increasing, $\langle \mathrm{pp}_{<\theta}(\mu_i) : i < \delta \rangle$ strictly increasing then $\mathrm{pp}_{<\theta}(\bigcup\limits_{i<\delta} \mu_i) > \sum\limits_{i<\delta} \mathrm{pp}_{<\theta}(\mu_i)$, hence $\mathrm{cf}(\lambda^*) \geq \theta$ and, by [Sh 355, 2.3], without loss of generality $\lambda^* = \lambda_1$. Let χ be regular large enough and B be an elementary submodel of $(\mathscr{H}(\chi), \in, <^*_\chi)$ of cardinality λ^* such that $\lambda^* + 1 \subseteq B$. Let

$$\mathscr{P} = \{X \in B : X \subseteq \lambda^* \text{ and } |X| < \lambda\}.$$

Now repeat the proof of [Sh 400, 3.5], noting

2.3 Observation. Suppose:

- (i) $\mathfrak{a}$ is a set of regular cardinals $|\mathfrak{a}| \leq \kappa < \mathrm{Min}(\mathfrak{a})$;

(*ii*) the function $\mathfrak{b} \mapsto \langle f^{\mathfrak{b}}_\alpha : \alpha < \max \operatorname{pcf}(\mathfrak{b})\rangle$ (for $\mathfrak{b} \subseteq \mathfrak{a}$) is as in [Sh 371, §1]; i.e. satisfies:

$(*)_1$ $f^{\mathfrak{b}}_\alpha \in \Pi\mathfrak{b}$, we stipulate $f^b_\alpha \restriction (\mathfrak{a}\backslash\mathfrak{b}) = 0_{\mathfrak{a}\backslash\mathfrak{b}}$ and $\langle f^{\mathfrak{b}}_\alpha : \alpha < \theta\rangle$ is $<_{J_{<\max \operatorname{pcf}[\mathfrak{b}]}}$-increasing and cofinal in $(\Pi\mathfrak{b}, <_{J_{<\max \operatorname{pcf}[\mathfrak{b}]}})$, and $(\forall g \in \Pi\mathfrak{b})(\exists^\theta \alpha < \theta)[g < f^\theta_\alpha]$

$(*)_2$ if $\delta < \theta \in \operatorname{pcf}(\mathfrak{a})$, $\operatorname{cf}(\delta) \in (|\mathfrak{b}|, \operatorname{Min}\mathfrak{b}]$ and $\sigma \in \mathfrak{a}\backslash\{\operatorname{cf}(\delta)\}$ then $f^{\mathfrak{b}}_\delta(\sigma) = \operatorname{Min}\{\bigcup_{\alpha\in C} f^{\mathfrak{b}}_\alpha(\sigma) : C$ a club of $\delta\}$

(*iii*) $\mathfrak{a} \subseteq \lambda^*, \lambda \le \lambda^*$, and
$(\forall\mathfrak{b})[\mathfrak{b} \subseteq \mathfrak{a}\ \&\ |\mathfrak{b}| < \theta\ \&\ \sup(\mathfrak{b}) \ge \lambda\ \&\ \sup(\mathfrak{b}) \notin \mathfrak{b} \Rightarrow \max \operatorname{pcf}_{J^{bd}_{\mathfrak{b}}}\mathfrak{b} \le \lambda^*]$

(*iv*) We let for $\theta \in \operatorname{pcf}(\mathfrak{a})$, and $\alpha < \theta$: $f^\theta_\alpha = f^{\mathfrak{b}_\theta[\mathfrak{a}]}_\alpha$.

Then for every $\{\mathfrak{a}_i : i < \kappa\} \subseteq \{\mathfrak{b} : \mathfrak{b} \subseteq \mathfrak{a}, |\mathfrak{b}| < \theta\}, g_i \in \Pi\mathfrak{a}_i (i < \kappa)$ we can find g and $\lambda'_i < \lambda$ such that: $g \in \Pi\mathfrak{a}$, for each $i < \kappa, g_i \le g$ and for every $i < \kappa$ there are $\lambda'_i < \lambda$ such that letting $\mathfrak{c}^i = \mathfrak{a}^i\backslash\lambda'_i$ we have $g \restriction \mathfrak{c}^i$ is Max of finitely many functions from $\{f^\mu_\alpha \restriction c^i : i < \kappa, \alpha < \mu, \mu \in \operatorname{pcf}(\mathfrak{a})$ and $\mu \le \lambda^*\}$. Moreover for some $\mathfrak{d} \subseteq [\lambda, \lambda^*] \cap \operatorname{pcf}(\mathfrak{a})$, for every $\theta \in \operatorname{Reg} \cap [\lambda, \lambda^*] \cap \mathfrak{d}$, for some $\lambda_\theta < \lambda, g \restriction (\mathfrak{b}_\theta[\mathfrak{a}]\backslash\lambda_\theta)$ is (the suitable restriction of a) Max of finitely many functions from $\{f^\sigma_{g(\sigma)} : \sigma \in (\lambda, \lambda^*] \cap \operatorname{pcf}(\mathfrak{a}\backslash\lambda')$ for every $\lambda' < \lambda\}$.

Proof. Without loss of generality $\kappa^+ < \operatorname{Min}(\mathfrak{a})$. Use [Sh 371, 1.4] with $\kappa^+, \mathfrak{a}, \langle\mathfrak{a}_i : i < \kappa\rangle$, here standing for $\delta, \mathfrak{a}, \langle\mathfrak{b}_i : i < \zeta^*\rangle$ there and we get $\langle < \lambda_{i,\ell}, \mathfrak{c}_{i,\ell} : \ell \le n(i) >: i < \kappa\rangle$. Let $\lambda'_i = \sup[\cup\{\mathfrak{c}_{i,\ell} : \mathfrak{c}_{i,\ell}$ a bounded subset of $\lambda\}]$. $\square_{2.3}$ $\square_{2.2}$

Similarly

2.4 Claim. *1) In 2.2, if $\theta = \kappa$ and $(\forall\mu)[\kappa \le \mu \le \lambda_1\ \&\ \operatorname{cf}(\mu) < \theta \Rightarrow \operatorname{pp}_{<\theta}(\mu) \le \lambda^*]$ (i.e. $\kappa = \lambda$) then we can add*

$(*)_3$ *for every $Y \subseteq \lambda, |Y| \le \kappa$ there are $Z_n, \langle Z_{n,i} : i < \theta\rangle$ for $n < \omega$ such that $Y \subseteq \bigcup_{n<\omega} Z_n, Z_n = \bigcup_{i<\theta} Z_{n,i}, |Z_{n,i}| < \theta, \langle Z_{n,i} : i < \theta\rangle$ increasing continuous and each $Z_{n,i}$ belongs to $\mathscr{P}$.*

Hence

2.5 Conclusion. If $\lambda > \operatorname{cf}(\lambda) = \aleph_0$ then there is a family $\mathscr{P}$ of cardinality $\le \sup\{pp(\mu) : \mu \le \lambda, \operatorname{cf}(\mu) = \lambda_0\}$ consisting of countable subsets of λ such that:

$(*)$ if $Y \subseteq \lambda, |Y| = \aleph_1$ then for some $Z \in \mathscr{P}, Y \cap Z$ is infinite. Moreover, we can find α^n_i $(n < \omega, i < \omega_1)$ such that $Y \subseteq \{\alpha^n_i : n < \omega, i < \omega_1\}$ and for each n for arbitrarily large $i < \omega_1, \{\alpha^n_j : j < i\} \in \mathscr{P}$.

2.6 Conclusion. If $\lambda > \kappa \ge \operatorname{cf}(\lambda)$, and $\operatorname{pp}_\kappa(\lambda) < \operatorname{cov}(\lambda, \lambda, \kappa^+, 2)$ then $\operatorname{pp}_\kappa(\lambda) < \operatorname{cf}_\kappa(\Pi(\lambda \cap \operatorname{Reg}\backslash\kappa^+), <_{J^{bd}_\lambda})$.

Proof. Let $\chi = \beth_3(\lambda)^+$, and for $\zeta \leq \kappa^+$ let $\mathfrak{B}_\zeta$ be an elementary submodel of $(\mathscr{H}(\chi), \in, <^*_\chi)$ of cardinality $\mathrm{pp}_\kappa(\lambda)$ such that $\mathrm{pp}_\kappa(\lambda) \subseteq \mathfrak{B}_\zeta, \mathfrak{B}_\zeta$ increasing continuous, and $\langle \mathfrak{B}_\epsilon : \epsilon \leqq \zeta \rangle \in \mathfrak{B}_{\zeta+1}$. Let $\mathfrak{B} = \mathfrak{B}_{\kappa^+}$. Assume that the conclusion fail and we shall prove that $\mathrm{cov}(\lambda, \lambda, \kappa^+, 2) \leqq \mathrm{pp}_\kappa(\lambda)$, in fact that $\mathscr{P} =: \mathfrak{B}_{\kappa^+} \cap \mathscr{S}_{<\lambda}(\lambda)$ exemplify it. Let $a \subseteq \lambda, |a| \leq \kappa$ and we shall find $A \in \mathscr{P}$ such that $a \subseteq A$; this suffice. Choose by induction on $\zeta < \omega_1, f_\zeta \in \mathfrak{B} \cap \Pi(\mathrm{Reg} \cap \lambda)$ such that letting N_ζ be the Skolem Hull of $a \cup \{f_\xi : \xi < \zeta\} \cup \{i : i \leq \kappa\}$, we have: for every large enough $\sigma \in \mathrm{Reg} \cap \lambda, [\sigma \in N_\zeta \Rightarrow \sup(\sigma \cap N_\zeta) < f_\zeta(\sigma)]$. Now use 2.5 + proof of [Sh 400, 3.5]. $\square_{2.6}$

* * *

We now return to the issue of strong covering (from [Sh:b, Ch.XIII,§1-§4]) (better version [Sh:g, Ch.VII]). It influenced the first proof of a bound on $\aleph_\omega^{\aleph_0}$, and is clearly related to computing $\mathrm{Min}\{|S| : S \subseteq \mathscr{S}_{\leqq\kappa}(\lambda)$ is stationary$\}$.

2.7 Lemma. *Suppose $\mathbf{W} \subseteq \mathbf{V}$ is a transitive class of $\mathbf{V}$ including all the ordinals and is a model of ZFC.*
1) For every set $Y \in \mathbf{V}$ of ordinals of cardinality $< \kappa$ (in $\mathbf{V}$) there are $Y_n \in \mathbf{W}$ $n < \omega$ (so we know only that $\langle Y_n : n < \omega \rangle \in \mathbf{V}$!), $\mathbf{W} \models$ "Y_n a set of $< \kappa$ ordinals" such that $Y \subseteq \bigcup\limits_{n<\omega} Y_n$ provided that:

$(*)_\kappa$ (*i*) *κ is a regular uncountable cardinal in $\mathbf{V}$,*

(*ii*) *if $a \in \mathbf{V}$ is a subset of $Reg^{\mathbf{W}} \backslash \kappa, |a| < \kappa$ and $g \in (\Pi a)^{\mathbf{V}}$ then*

$\otimes$ *there is a function $h \in \mathbf{W}$ such that $\theta \in Dom(g) \Rightarrow g(\theta) < h(\theta) < \theta$ (so $Dom(g) \subseteq Dom(h)$)*

or even just

$(*)^-_\kappa$ *like $(*)_\kappa$ but in (ii) we demand only:*

$\otimes^-$ *there are functions $h_n \in \mathbf{W}$ (for $n < \omega$) such that $(\forall \theta \in Dom(g))[\bigvee\limits_{n<\omega} g(\theta) < h_n(\theta) < \theta]$.*

2) For every set $Y \in \mathbf{V}$ of ordinals of cardinality $< \kappa$ (in $\mathbf{V}$) there is $Z \in \mathbf{W}$ satisfying $\mathbf{W} \models$ "Z a set of $< \kappa$ ordinals" such that $Y \subseteq Z$ provided that:

$$(*)_\kappa + \aleph_2^{\mathbf{V}} \leq \kappa.$$

3) Assume that $\kappa = \aleph_1^{\mathbf{V}}$ and $(*)_\kappa$ holds and

$\oplus_0$ $\mathbf{V} \models$ "A *a set of ordinals of power κ*" $\Rightarrow (\exists B \in \mathbf{W})[A \cap B$ *infinite* & $\mathbf{W} \models$ "$|B| < \kappa$"$]$.

Then the conclusion of part (2) holds.
4) Assume $\kappa = \aleph_1^{\mathbf{V}}, (*)_\kappa, (\kappa^+)^{\mathbf{V}} = (\kappa^+)^{\mathbf{W}}$ and $(*)_{\kappa^+}$. Then the conclusion of part

(2) holds.
5) Assume

- (a) $(\kappa^+)^{\mathbf{V}} = (\kappa^+)^{\mathbf{W}}, (*)_\kappa, (*)_{\kappa^+}$;
- (b) *there is* $\langle C_\delta : \delta \in (\lambda+1)\backslash(\kappa+1), \aleph_0 \leq cf^{\mathbf{W}}\delta \leq \kappa\rangle \in \mathbf{W}$ *satisfying* C_δ *a club of* δ *for each* δ *such that* $[\alpha \in acc(C_\delta) \Rightarrow C_\alpha = C_\delta \cap \alpha]$ *and* $otp(C_\delta) \leq \kappa$. *[nec?]*

Then $(\mathbf{W}, \mathbf{V})$ *satisfies the* κ*-strong covering (see [Sh:g, Ch.VII]) which means:*

- $\otimes$ *for every ordinal* α *and model* $M \in \mathbf{V}$ *with universe* α*, with countable vocabulary, there is* $N \prec M$ *of power* $< \kappa, N \cap \kappa$ *an ordinal and the universe of* N *belongs to* $\mathbf{W}$.

6) Moreover, in (5)

- $\oplus^+$ *in the game where a play last* κ *moves, in the* i*th move (for* $i < \kappa$*) the first and second players choose* $a_i, b_i \in [\lambda]^{<\kappa}$*, respectively preserving* $\bigcup_{j<i} b_j \subseteq a_i \subseteq b_i$*, the first player has a winning strategy where the first player winning a play means* $\{\delta < \kappa : \bigcup_{i<\delta} a_i \in \mathbf{W}\} \in \mathscr{D}_\kappa$ *(= the clubs filter on* κ *(in* $\mathbf{V}$*)).*

Remark. 1) Note that parts (3), (4) hold for $\kappa \geq \aleph_1^{\mathbf{V}}$, but this is covered already by part (2).
2) Note that in part (9), $\aleph_2^{\mathbf{V}} = \aleph_2^{\mathbf{W}}$.

Proof. Should be straightforward (if you read till here). [Originally we say only; for (1) imitate the proof of [Sh 400, 3.5], for (2) — repeat the proof of [Sh 400, 3.5] by doing the induction for $i < \aleph_1$, then use part (1). For (3) — instead using part (1) in the end, use the assumption, for (4), (5) imitate the proof of [Sh 400, 3.6].]

For the proof of 1)-5), let Y be a subset of the ordinal λ, a cardinal of $\mathbf{V}$ (for part (5) λ is given), and let $\chi =: [(2^\lambda)^+]^{\mathbf{W}}$. Let $(\mathscr{H}(\chi)^{\mathbf{W}}, \in, <^*_\chi) \in \mathbf{W}$.
1) In $\mathbf{V}$ we choose by induction on $n < \omega, N_n, \alpha_n, h_{n,\ell}, g_n$ such that:

- (a) $N_n \prec (\mathscr{H}(\chi)^{\mathbf{W}}, \in, <^*_\chi)$
- (b) $\mathbf{V} \models \|N_n\| < \kappa$ and $N_n \cap \kappa = \alpha_n$;
- (c) $Y \subseteq N_0$ and $\{\kappa, \lambda\} \in N_0$
- (d) $N_n \prec N_{n+1}$
- (e) $h_{n,\ell} \in \mathbf{W}$ is a (partial) function (for $\ell < \omega$) and from λ to λ
- (f) g_n is a function, $\mathrm{Dom}(g_n) = (\lambda \cap \mathrm{Reg}^{\mathbf{W}}\backslash\kappa) \cap N_n, g_n(\theta) =: \sup(N_n \cap \theta)$
- (g) for every $\theta \in \mathrm{Dom}(g_n)$ for some $\ell, g_n(\theta) < h_{n,\ell}(\theta)$
- (h) $h_{n,\ell} \in N_{n+1}$ for $\ell < \omega$.

There is no problem to carry the definition. Let $N = N_\omega =: \bigcup_{n<\omega} N_n$ and $\alpha^*_\omega = \bigcup_{n<\omega} \alpha_n = N \cap \kappa$. For $\alpha \le \omega$ let M_α be defined as the Skolem Hull in $(\mathscr{H}(\chi)^{\mathbf{N}}, \in, <^*_\chi)$ of $\{i : i < \alpha_n\} \cup \{h_{n,\ell} : n, \ell < \alpha\} \cup \{\kappa, \lambda\}$. Clearly $[\alpha < \omega \Rightarrow M_\alpha \in \mathbf{W}]$, and $[\alpha \le \omega \Rightarrow M_\alpha \prec N_\alpha]$ and $\mathbf{V} \models$ "$\|M_\alpha\| < \kappa$". Now $M = M_\omega = \bigcup_{n<\omega} M_n \prec N, M \cap \kappa = \alpha_\omega = N \cap \kappa$ and for every $\theta \in (\lambda^+ \cap \operatorname{Reg}^{\mathbf{W}} \backslash \kappa) \cap M$ and n for some $m < \omega, \theta \in N_m$ & $m > n$ so for some $\ell < \omega$ we have $\sup(N_n \cap \theta) < h_{m,\ell}(\theta) \in M$ hence $\sup(N \cap \theta) = \sup_{n<\omega}(N_n \cap \theta) \le \sup(M \cap \theta) \le \sup(N \cap \theta)$. So by [Sh 400, 3.3A], $M \cap \lambda = N \cap \lambda$, so $Y_n =: \lambda \cap M_n$ for $n < \omega$ are as required.

2)-5) The following will be used in proving during the proof of 2) - 5). Let $\delta(*) \le \kappa$ be given (i.e. we shall choose it for each part) and we assume $(*)_\kappa$. We fix $(\mathscr{H}(\chi)^{\mathbf{W}}, \in, <^*_\chi) \in \mathbf{W}$ as in part (1).

In $\mathbf{V}$ we choose by induction on $i < \delta(*), N_i, \alpha_i, h_i, g_i$ such that:

- (a) $N_i \prec (\mathscr{H}(\chi)^{\mathbf{W}}, \in, <^*_\chi)$
- (b) $\mathbf{V} \models$ "$\|N_i\| < \kappa$" and $N_i \cap \kappa = \alpha_i$
- (c) $Y \subseteq N_0, \{\kappa, \lambda\} \subseteq N_0$
- (d) N_i is increasing continuous
- (e) $h_i \in \mathbf{W}$ is a partial function from $\lambda + 1$ to $\lambda + 1$
- (f) g_i is the function with $\operatorname{Dom}(g_i) = (\lambda^+ \cap \operatorname{Reg}^{\mathbf{W}} \backslash \kappa) \cap N_i$ such that

$$g_i(\theta) = \sup(N_i \cap \theta)$$

- (g) $\theta \in \operatorname{Dom}(g_i) \Rightarrow g_i(\theta) < h_i(\theta) < \theta$
- (h) $h_i \in N_{i+1}$.

There is no problem to carry the construction. We let $N = N_{\delta(*)} = \bigcup_{i<\delta(*)} N_i$.

Proof of Part (2). Choose $\delta(*) = \aleph_1^{\mathbf{V}}$, so under the assumptions of part (2) we have $\delta(*) < \kappa$, hence $\mathbf{V} \models$ "$\|N\| < \kappa$". Apply part (1) to $Y = \{h_i : i < \delta(*)\}$ (which is $\subseteq \mathbf{W}$), so we can find $Y_n \in \mathbf{W}$ (and if you like $Y_n \subseteq H =: \{h \in \mathbf{W} : h$ a partial function from $\lambda + 1$ to $\lambda + 1\}$), for $n < \omega$ such that $\mathbf{V} \models$ "$|Y_n| < \kappa$" and $Y \subseteq \bigcup_{n<\omega} Y_n$; (well, Y is not a set of ordinals, but we can code it as one). So for some $n = n(*)$, we hvae $\mathbf{V} \models$ "$|Y \cap Y_n| = \aleph_1$". Let M be the Skolem Hull in $(\mathscr{H}(\chi)^{\mathbf{W}}, \in, <^*)$ of $\{\alpha : \alpha < \bigcup_{i<\delta(*)} \alpha_i\} \cup Y_{n(*)} \cup \{\kappa, \lambda\}$; so $M \in \mathbf{W}, \mathbf{V} \models$ "$\|M\| < \kappa$" and so $\mathbf{W} \models$ "$\|M\| < \kappa$". Let $M' = M \cap N$; so clearly $M' \prec N$, $\bigcup_{i<\delta(*)} \alpha_i \subseteq M' \cap \kappa \subseteq N \cap \kappa = \bigcup_{i<\delta(*)} \alpha_i$. Lastly, if $i < \delta(*)$ and $\theta \in (\lambda^+ \cap \operatorname{Reg}^{\mathbf{W}} \backslash \kappa) \cap M'$ then for some $j \in (i, \delta(*)), \theta \in N_j$ and for some ϵ we have $j < \epsilon < \delta(*)$ & $h_\epsilon \in Y_{n(*)}$, so $\theta \in \operatorname{Dom}(h_\epsilon)$ and $\sup(N_i \cap \theta) \le \sup(N_j \cap \theta) \le \sup(N_\epsilon \cap \theta) < h_\epsilon(\theta) \in M'$. So

by [Sh 400, 3.3A] we know $M' \cap \lambda = N \cap \lambda$, so $Y \subseteq M' \cap \lambda \subseteq M \cap \lambda \in \mathbf{W}$ and $\mathbf{W} \models$ "$|M \cap \lambda| \leq \|M\| < \kappa$ as required. [Saharon: fill details]

Proof of Part (3). As in part (2), we choose $\delta(*) = \aleph_1^{\mathbf{V}}$ and get N_i, α_i, g_i, h_i (for $i < \delta(*)$). By $\oplus_0$ applied to $\{h_i : i < \delta(*)\}$ (again translating to a set of ordinals) there is a set $B \in \mathbf{W}$ such that $\mathbf{W} \models$ "$|B| < \kappa$" and $A \cap B$ infinite, without loss of generality $B \subseteq H$ (see proof of part (2)). So there is a limit ordinal $\zeta < \delta(*)$ such that $\zeta = \sup\{i < \zeta : h_i \in B\}$. Now let M be the Skolem Hull in $(\mathscr{H}(\chi)^{\mathbf{W}}, \in, <^*_\chi)$ of $\{\alpha : \alpha < \bigcup_{i<\zeta} \alpha_i\} \cup B \cup \{\kappa, \lambda\}$, and let $M' = M \cap N_\zeta$ so clearly $M' \cap \lambda \subseteq M \cap \lambda \in \mathbf{W}, |M' \cap \lambda| \leq |M \cap \lambda| \leq \|M\| < \kappa$ and as above $M' \cap \lambda = N_\zeta \cap \lambda$, so $N_\zeta \cap \lambda = M' \cap \lambda \in \mathbf{W}$ but $Y \subseteq N_\zeta \cap \lambda = M' \cap \lambda \subseteq M \cap \lambda$ and are finished.

Proof of Part (4). We let $\delta(*) = \kappa$ and get N_i, α_i, g_i, h_i (for $i < \delta(*)$) be as before. Now we apply part (1) with $\kappa^+ (= (\kappa^+)^{\mathbf{V}} = (\kappa^+)^{\mathbf{W}}), \{h_i : i < \kappa\}$ here standing for κ and Y there, and get $\langle Y_n : n < \omega\rangle$. So for some $n(*)$ the set $\{i : g_i \in Y_{n(*)}\}$ is unbounded in κ. Let M be the Skolem Hull in $(\mathscr{H}(\chi)^{\mathbf{W}}, \in, <^*_\chi)$ of $\{\alpha : \alpha < \bigcup_{i<\delta(*)} \alpha_i\} \cup Y_{n(*)} \cup \{\kappa, \lambda\}$. As before $N \subseteq M, M \in \mathbf{W}$ and $\mathbf{W} \models$ "$\|M\| < \kappa^+$". So $\mathbf{W} \models$ "$\|M\| = \kappa$" hence there is a one to one function $f \in W$ from $\{i : i < \kappa\}$ onto M, so for some club $E \in \mathbf{V}$ of κ (in $\mathbf{V}$)

$$i \in E \Rightarrow N_i \subseteq \text{ Rang}(f \restriction i).$$

So for each $i \in E, Y \subseteq$ Rang$(f \restriction i)$ hence we are done.
5) So we have already chosen N_i, α_i, g_i, h_i for $i < \kappa$. By parts (2) + (4) we can assume without loss of generality that Dom(h_i) has cardinality $< \kappa$. Let $N_\kappa = \bigcup_{i<\kappa} N_i$, by part (1) there is a sequence $\langle Y_n : n < \omega\rangle$ satisfying $Y_n \in \mathbf{W}$ such that:

$$\{h_i : i < \kappa\} \subseteq \bigcup_{n<\omega} Y_n \subseteq H = \{f \in \mathbf{W} : f \text{ a partial function from } (\lambda+1) \cap \text{ Reg}^{\mathbf{W}} \backslash \kappa \text{ to } \lambda+1\}$$

and for each $n, \mathbf{W} \models$ "$|Y_n| < \kappa^+$". So for some $n(*), \{i < \kappa : h_i \in Y_{n(*)}\}$ is unbounded in κ. So in $\mathbf{W}$ there is a list $\langle f_i : i < \kappa\rangle$ of $Y_{n(*)}$. In $\mathbf{V}$, for each $i < \kappa$ let $j_i < \kappa$ be minimal such that:

- (a) $h_i \in \{f_\zeta : \zeta < j_i\}$;
- (b) if for some $\zeta < \kappa, f_i \leq h_\zeta$ (i.e. Dom$(f_i) \subseteq$ Dom(h_ζ) and $(\forall \theta \in$ Dom$(f_i))[f_i(\theta) \leq h_\zeta(\theta)]$ then there is such $\zeta < j_i$
- (c) $j_i \geq i + 1$.

Let $E = \{\zeta < \kappa :$ for every $i < \zeta, j_i < \zeta$, and ζ is a limit ordinal$\}$. Now for each $\zeta \in E$ note that

$(*)_1(\alpha)$ $\{h_i : i < \zeta\} \subseteq \{f_i : i < \zeta\}$;

(β) $\bigwedge_{i<\zeta} [\bigvee_{j<\kappa} f_i \leq h_j \leftrightarrow \bigvee_{j<\zeta} f_i \leq h_j]$.

As $[i<j \Rightarrow h_i<h_j]$ clearly we get:

$(\beta)'$ $\bigwedge\limits_{i<\zeta}[\bigvee\limits_{j<\zeta} f_i \le h_j \Leftrightarrow f_i \le h_\zeta]$.

Define for $\zeta \le \kappa$ a function f^*_ζ as follows:

$(*)_2$ $\mathrm{Dom}(f^*_\zeta) = \bigcup\{\mathrm{Dom}(f_i) : i<\zeta \text{ and } f_i \leqq h_\zeta\}$
$f^*_\zeta(\theta) = \sup\{f_i(\theta) : i<\zeta \text{ and } f_i \leqq h_\zeta\}$.

So clearly

$(*)_3$ for $\zeta \in E$, $\mathrm{Dom}(f^*_\zeta) = \cup\{\mathrm{Dom}(h_i) : i<\zeta\}, f^*_\zeta(\theta) = \sup\{h_i(\theta) : i<\zeta\}$ hence $\mathrm{Dom}(f^*_\zeta) = N_{\zeta'} \cap (\lambda+1) \cap \mathrm{Reg}^{\mathbf{W}} \backslash (\kappa+1)$

$(*)_4$ $f^*_\zeta \in \mathbf{W}$, and even for $\langle f^*_\zeta : \zeta<\kappa \text{ limit}\rangle \in \mathbf{W}$. Let us define for $\zeta \le \kappa$ a function g^*_ζ:

$$\mathrm{Dom}(g^*_\zeta) = \bigcup_{i<\zeta} \mathrm{Dom}(h_i) \subseteq (\lambda+1) \cap \mathrm{Reg}^{\mathbf{W}} \backslash \kappa,$$

$$g^*_\zeta(\theta) = \bigcup_{i<\zeta} g_i(\theta).$$

Clearly for ζ limit:

$(*)_5$ $\mathrm{Dom}(g^*_\zeta) = (\lambda+1) \cap \mathrm{Reg}^{\mathbf{W}} \cap N_\zeta \backslash \kappa$

and if $\zeta \in E$

$$\begin{aligned}\theta \in \mathrm{Dom}(g^*_\zeta) \Rightarrow g^*_\zeta(\theta) &= \bigcup\{g_i(\theta) : i<\zeta, \theta \in N_i\} = \bigcup\{h_i(\theta) : i<\zeta, \theta \in N_i\} \\ &= \bigcup\{f_i(\theta) : i<\zeta \text{ and } f_i \le h_\zeta\} = f^*_\zeta(\theta).\end{aligned}$$

So for $\zeta \in E$ we have $g^*_\zeta \subseteq f^*_\zeta$ but by $(*)_3 + (*)_5$ they have the same domain hence $g^*_\zeta = f^*_\zeta$.

For every $\theta \in N_\kappa \cap ((\lambda+1) \cap \mathrm{Reg}^{\mathbf{W}} \backslash (\kappa+1)), g_\kappa(\theta)$ is an ordinal $\in (\kappa,\theta) \subseteq (\kappa,\lambda)$ of cofinality κ, so $C_{g_\kappa(\theta)}$ is a set of ordinals of order type κ; let $C_{g_\kappa(\theta)} = \{\alpha^\theta_\zeta : \zeta<\kappa\}$, (increasing); it is strictly increasing continuous and has limit $g_\kappa(\theta)$; also $\langle g^*_\zeta(\theta) : \zeta<\kappa\rangle$ is also strictly increasing continuous with limit $g^*_\kappa(\theta)$. Clearly $E_\theta = \{\zeta<\kappa : \zeta \text{ limit ordinal and } \alpha^\theta_\zeta = g^*_\zeta(\theta) \text{ (so } \theta \in N_\zeta)\}$ is a club of κ (in V; as $V \models$ "κ regular uncountable"). So for $\zeta \in E_\theta, C_{\alpha^\theta_\zeta} = C_{g_\kappa(\theta)} \cap \alpha^\theta_\zeta \subseteq N_{\zeta+1}$, hence $[\zeta \in \text{ acc } E_\theta \Rightarrow C_{g_\kappa(\theta)} \cap \alpha^\theta_\zeta \subseteq N_\zeta]$. Let $E^* = \{\zeta \in E : (\forall i<\zeta)(\forall \theta \in N_i)[\theta \in (\lambda+1) \cap \mathrm{Reg}^W \backslash (\kappa+1) \Rightarrow \zeta \in \text{ acc } E_\theta\}$. So for $\zeta \in E^*, \zeta$ is a limit ordinal and:

$$\begin{aligned}\theta \in N_\zeta \cap ((\lambda+1) \cap \mathrm{Reg}^W \backslash (\kappa+1)) \Rightarrow C_{f^*_\zeta(\theta)} &= C_{g^*_\zeta(\theta)} \\ &= C_{g^*_\kappa(\theta)} \cap g^*_\zeta(\theta) \subseteq N_\zeta.\end{aligned}$$

Now we shall show that for $\zeta \in E^*, N_\zeta \cap \lambda \in W$. For $\zeta \in E^*$ we define by induction on $n < \omega, M^n_\zeta$:

$$M^0_\zeta \text{ is the Skolem Hull in } (\mathscr{H}(\chi)^{\mathbf{W}}, \in, <^*) \text{ of } \{\alpha : \alpha < \alpha_\zeta\}.$$

Let M^{n+1}_ζ be the Skolem Hull in $(\mathscr{H}(\chi)^{\mathbf{W}}, \in, <^*_\chi)$ of

$$M^n_\zeta \cup \bigcup \{C_{f^*_\zeta(\theta)} : \theta \in (\lambda+1) \cap \text{ Reg}^{\mathbf{W}} \backslash (\kappa+1) \text{ and } \theta \in M^n_\zeta \cap \text{ Dom } f^*_\zeta\}.$$

Let $M_\zeta = \bigcup\limits_{n<\omega} M^n_\zeta$, clearly $M_\zeta \in \mathbf{W}$ (as $f^*_\zeta \in \mathbf{W}$) we can prove by induction on n that $M^n_\zeta \subseteq N_\zeta$, hence: $M_\zeta \subseteq N_\zeta$. Also $\alpha_\zeta \subseteq M_\zeta \cap \kappa \subseteq N_\zeta \cap \kappa = \alpha_\zeta$, and:

> $\theta \in M_\zeta \cap ((\lambda+1) \cap \text{ Reg}^W \backslash (\kappa+1)) \Rightarrow C_{g^*_\zeta(\theta)}$ is an unbounded of subsets of both $N_\zeta \cap \theta, M_\zeta \cap \theta$.

So by [Sh 400, 5.1A](1) we get $N_\zeta \cap \lambda = M_\zeta \cap \lambda$. Alternatively, let for $\zeta \in E, M^*_\zeta$ is the Skolem hull of $\alpha_\zeta \cup \{f^*_i : i < \zeta\}$ in $(\mathscr{H}(\chi)^{\mathbf{W}}, \in, <^*_\chi)$; so clearly $M^*_\zeta \subseteq N_\zeta$ and again by [Sh 400, 5.1A] we have $M^*_\zeta \cap \lambda = N_\zeta \cap \lambda$.
6) The winning strategy for the second player is to choose "on the side" also N_i, α_i, h_i, g_i as in the common part of the proof of parts (2)-(5) and guaranteeing: a_i include $\bigcup\limits_{j<i}(N_j \cap \lambda) \cup \bigcup\limits_{j<i} b_j, N_{i+1}$ include b_i and $N_i \cap \lambda$ is the universe of an elementary submodel of $(\mathscr{H}(\chi)^{\mathbf{W}}, \in, <^*_\chi) \in \mathbf{W}$. $\square_{2.7}$

2.8 Remark. 1) We can put λ as a parameter of the Lemma 2.7, then in $(*)_\kappa, a \subseteq (\lambda+1) \cap \text{ Reg}^{\mathbf{W}} \backslash \kappa$, etc., (so we may write $(*)_\kappa$) and $Y \subseteq \lambda$ (in parts (1)-(4)) and $\alpha \leq \lambda$ (in $\otimes$ of part (5)).
2) Note that $(*)^\lambda_\kappa$ follows easily from the relevant covering property in [Sh:g, Ch.VII]:

> $(*)$ if $a \in \mathbf{V}, a \subseteq \lambda, \mathbf{V} \models$ "$|a| < \kappa$" then for some $b \in \mathbf{W}, a \subseteq b, \mathbf{W} \models$ "$|b| < \kappa$".

§3 pcf Inaccessibility and Characterizing the Existence of Non <-decreasing Sequences (for a Topological Problem)

3.1 Definition. 1) We say λ is (μ,θ,σ)-inaccessible if $\lambda > \mu \geq \theta > \sigma$ and for any $\mathfrak{a} \subseteq \text{Reg}$ we have: if $|\mathfrak{a}| < \theta$, $\text{Min}(\mathfrak{a}) > \mu$ and $\mathfrak{a} \subseteq \lambda$, even $\sup(\mathfrak{a}) < \lambda$ and I is a σ-complete ideal on $\mathfrak{a}$, then $\lambda \not\leq \text{ tcf}(\Pi\mathfrak{a}/I)$ (when tcf is well defined).
2) If we write $*$ instead of μ we mean "for some $\mu, \theta \leq \mu < \lambda$".
3) If we omit σ we mean $\sigma = \aleph_0$.
4) "accessible" is just the negation of "inaccessible".

We now rephrase various old results.

3.2 Claim. *1) For λ regular, in the definition, "and σ-complete $I, \lambda \not\leq \text{tcf}(\Pi\mathfrak{a}/I)$" can be replaced by "$\lambda \notin \text{pcf}_{\sigma\text{-complete}}(\mathfrak{a})$" and also by "$\lambda \neq \text{tcf}(\Pi\mathfrak{a}/I)$ for any $\mathfrak{a} \subseteq$ Reg $\cap\,(\mu,\lambda), |\mathfrak{a}| < \theta, I$ being σ-complete"; also if $\text{cf}(\lambda) \notin [\sigma,\theta)$ then "$\sup(\mathfrak{a}) < \lambda$" is not necessary just "$\lambda \notin \mathfrak{a}$".*
2) Assume $\lambda > \mu \geq \theta > \sigma$ and $\text{cf}(\lambda) \geq \sigma$. Then λ is (μ,θ,σ)-inaccessible iff $[\lambda' \in (\mu,\lambda)$ & $\sigma \leq \text{cf}(\lambda') < \theta \Rightarrow \text{pp}_{\Gamma(\theta,\sigma)}(\lambda') < \lambda]$.
3) If $\lambda = \text{cf}(\lambda) > \mu \geq \theta = \text{cf}(\theta) > \sigma, \lambda$ is (μ,θ,σ)-accessible then there is a set $\mathfrak{a} \subseteq$ Reg $\cap\,(\mu,\lambda)$ of (μ,θ,σ)-inaccessible cardinals each $> \mu, \mathfrak{a}$ of cardinality $< \theta$ such that $\lambda \in \text{pcf}_{\sigma\text{-complete}}(\mathfrak{a})$.
4) If $\lambda = \text{cf}(\lambda) \geq \kappa > \mu \geq \theta > \sigma = \text{cf}(\sigma)$, and $(\exists\mathfrak{a})[\mathfrak{a} \in$ Reg $\cap\,(\mu,\kappa)$ & $|\mathfrak{a}| < \theta$ & $\lambda \in \text{pcf}_{\sigma\text{-complete}}(\mathfrak{a})]$ then there is a set $\mathfrak{a}$ of (μ,θ,σ)-inaccessible cardinals $\in (\mu,\kappa)$ with $|\mathfrak{a}| < \theta$ such that $\lambda = \max \text{pcf}(\mathfrak{a})$, and $\lambda \in \text{pcf}_{\sigma\text{-complete}}(\mathfrak{a})$. If κ is (μ,θ,σ)-inaccessible then necessarily $\sup(\mathfrak{a}) = \kappa, J^{bd}_{\mathfrak{a}} \subseteq J_{<\lambda}[\mathfrak{a}]$. If $\sigma = \aleph_0$, also there is a tree of cardinality $\leqq \kappa$ and $\geq \lambda$ ($(\text{cf}\kappa)$-branches if κ is (μ,θ,σ)-inaccessible).
5) If $\lambda =$ max pcf$(\mathfrak{a}), \kappa = |\mathfrak{a}| \leq \mu <$ Min$(\mathfrak{a})$, each $\theta \in \mathfrak{a}$ is $(\mu,\kappa^+,2)$-inaccessible then there is a tree of cardinality sup$(\mathfrak{a})$ and $\geq \lambda\text{cf}(\text{otp})$-branches. If we have κ pairwise disjoint subsets of $\mathfrak{a}$ not in $J_{<\lambda}[\mathfrak{a}], 2^\kappa \geq \sup(\mathfrak{a})$ or on each $\theta \in \mathfrak{a}$ there is an entangled linear order then there is an entangled linear order of cardinality λ.
6) If $\mu < \lambda <$ pp$^+(\mu)$, then there is a tree with $\leq \mu$ nodes and $\geq \lambda$ branches. If μ is $(*,(\text{cf}\mu)^+,2)$-inaccessible we can demand "$\geq \lambda\,(\text{cf}\mu)$-branches".

Proof. 1) Easy (using pcf analysis and [Sh 355, §1]).
2) Easy, too (use [Sh 355, 2.4]).
3) Prove by induction on λ using [Sh 345a, 1.10] (so in [Sh 345a, 1.12] we can replace pcf by $\text{pcf}_{\sigma\text{-complete}}$).
4) Similar to (3).
5) By citeSh:355,§4.
6) Easy, too.

We state some variants of [Sh 400, §2,§4]; specifically combining [Sh 400, 2.4,4.2]:

3.3 Claim. *Suppose:*

(*i*) $\langle\lambda_\zeta : \zeta < \zeta(*)\rangle$ *is a strictly increasing sequence of regular cardinals $> \sigma$*

(ii) *for* ζ *limit,* $\lambda_\zeta = (\bigcup\limits_{\xi<\zeta} \lambda_\xi)^+$

(iii) $\lambda_\zeta \in \mathrm{pcf}(\mathfrak{a}_\zeta)$

(iv) $\mathfrak{a}_\zeta \subseteq$ *Reg* $\cap (\sigma^+, \lambda_0)$ *and* $|\mathfrak{a}_\zeta| < \sigma, \sigma$ *regular.*

Then $\zeta(*) < \sigma^{+3}$.

Similarly combining [Sh 400, 2.4,4.2]

3.4 Claim. *Suppose:*

(i) $\langle \lambda_\zeta : \zeta < \zeta(*)\rangle$ *is a strictly increasing sequence of regular cardinals* $> \kappa$

(ii) *for* ζ *limit,* $\lambda_\zeta = (\bigcup\limits_{\xi<\zeta} \lambda_\xi)^+$ *sn*

(iii) $\lambda_\zeta \in$ *pcf*$_{\sigma\text{-complete}}(a_\zeta)$ *where* σ *is regular*

(iv) $\mathfrak{a}_\zeta \subseteq$ *Reg* $\cap (\kappa^+, \lambda_0), |\mathfrak{a}_\zeta| < \kappa, \kappa$ *regular*

(v) $\mathfrak{a}_\zeta \subseteq \mathfrak{b}$ *and if* $\langle \mu_\zeta : \zeta \leqq \kappa^{+2}\rangle$ *is strictly increasing sequence of regular* $\mathfrak{a} \subseteq \mathfrak{b}, |\mathfrak{a}| < \kappa, \{\mu_\zeta : \zeta < \kappa^{+2}\} \subseteq$ *pcf*$(\mathfrak{a}), \mu_{\kappa^{+2}} =$ *max pcf*$\{\mu_\zeta : \zeta < \kappa^{+2}\}$ *and* $(\lambda_0 + \sum\limits_{\zeta<\kappa^{+2}} \mu_\zeta)^+ < \mu_{\kappa^{+2}} < \bigcup\limits_{\zeta<\xi} \{\lambda_\zeta : \zeta < \kappa^{+3}\}$, *then there are* $\mu'_\zeta < \mu_\zeta$ *regular, as for* $\zeta \leqq \kappa^{+2}$ *and* $\mathfrak{a}' \subseteq \mathfrak{b}$ *with the same properties and* $\Sigma\{\mu_\zeta : \zeta < \kappa^{+2}\} = \Sigma\{\mu'_\zeta : \zeta < \kappa^{+2}\}$.

Then $\zeta(*) <$ *Max*$\{\kappa^{+3}, cov(\bigcup\limits_{\zeta<\xi} \mathfrak{a}_\zeta, \kappa, \kappa, \sigma)^+\}$.

3.5 Claim. *The following is impossible:*

(i) $\sigma < \kappa < \theta < \mu$ *are regular,* $\kappa^+ < \theta$

(ii) $\langle \lambda_\zeta : \zeta < \mu\rangle$ *is a strictly increasing sequence of regular cardinals* $> \mu$

(iii) $S = \{\epsilon < \mu : \mathrm{cf}(\epsilon) = \theta$ *and for some club* C *of* $\epsilon, \sup \mathrm{pcf}_{\sigma\text{-complete}}\{\lambda_\zeta : \zeta \in C\} < \sum\limits_{\zeta<\mu} \lambda_\zeta\}$ *is stationary;*

(iv) (a) *if* $\delta < \mu$, *cf*$(\delta) = \kappa$ *then for every club* C *of* δ, *there is* $\alpha \in C$ *such that*

$$\sup \mathrm{pcf}_{\sigma\text{-complete}}\{\lambda_\zeta : \zeta \in \alpha \cap C\} \geq \bigcup_{\zeta\in C} \lambda_\zeta$$

or

(b) $\lambda_\zeta \in$ *pcf*$_{\sigma\text{-complete}}(\mathfrak{a}_\zeta), |\mathfrak{a}_\zeta| < \kappa$ *and* $\mu > \mathrm{cov}(\bigcup\limits_{\zeta<\mu} \mathfrak{a}_\zeta, \kappa, \kappa, \sigma)$.

3.6 Claim. *Assume* $|\mathfrak{a}| < \mathrm{Min}(\mathfrak{a})$, *then*

$$\mathrm{cf}_{\leq |\mathfrak{a}|}(\Pi\,\mathrm{pcf}(\mathfrak{a})) \leq \max\,\mathrm{pcf}(\mathfrak{a}).$$

Proof. More is proved in [Sh 371, §3].

* * *

The following answers a question of Gerlits, Hajnal and Szentmiklossy in [GHS]. They dealt with "κ-good topological spaces X" (i.e. every subset is the union of $\leq \kappa$ compact sets) and "weakly κ-good spaces" (every $Y \subseteq X$ of cardinality $> \kappa$ contains a compact subset of cardinality $> \kappa$). [GHS] has the easy implication. We return to this in [Sh 513, §6].

3.7 Theorem. *The following conditions on $\kappa < \sigma < \theta$ are equivalent: (κ is an infinite cardinal, σ and θ are ordinals)*

$(A)_{\kappa,\sigma,\theta}$ *there are functions $f_\alpha : \kappa \to \sigma$ for $\alpha < \theta$ such that:*

$$\alpha < \beta \Rightarrow \bigvee_{i<\kappa} f_\alpha(i) < f_\beta(i)$$

$(B)_{\kappa,\sigma,\theta}$ $2^\kappa \geq |\theta|$ *or for every regular $\mu_1 \leq \theta$ for some singular cardinal $\lambda^* \leq \sigma$ we have:*

$$\mathrm{cf}(\lambda^*) \leq \kappa, \lambda^* > 2^\kappa, \mathrm{pp}^+(\lambda^*) > \mu_1.$$

Proof. First note

3.8 Observation. Let $\kappa < \sigma, \kappa$ an infinite cardinal, σ, θ are ordinals. If for every regular $\theta_1, \sigma \leq \theta_1 < \theta$ the statement $(A)_{\kappa,\sigma,\theta_1}$ holds and θ is singular (e.g. $\theta > |\theta|$), then $(A)_{\kappa,\sigma,\theta}$ holds.

Proof. We prove this by induction on θ; if $\theta \leq \sigma$ – trivial: use the constant functions. As θ is singular $\theta = \sum_{\alpha<\theta(*)} \theta_\alpha$ where $\theta(*) < \theta, \theta_\alpha < \theta, \theta_\alpha$ increasing continuous, $\theta_0 = 0$. By the assumption for each $\alpha < \theta(*)$, there is a sequence $\langle f^\alpha_i : i < \theta_\alpha\rangle$ as required in $(A)_{\kappa,\sigma,\theta_\alpha}$, [why? if θ_α is singular by the induction hypothesis, if θ_α is regular by an assumption of 3.8]. Similarly there is $\langle f_i : i < \theta(*)\rangle$ exemplifying $(A)_{\kappa,\sigma,\theta(*)}$.

For $i < \theta$ let $i = \sum_{\beta<\alpha(i)} \theta_\beta + j(i), j(i) < \theta_{\alpha(i)}, \alpha(i) < \theta(*)$ and let $g_i : \kappa \to \lambda$ be $g_i(2\zeta) = f_{\alpha(i)}(\zeta), g_i(2\zeta+1) = f^{\alpha(i)}_{j(i)}(\zeta)$. $\square_{3.8}$

Continuation of the Proof of 3.7. First we do the easy direction.

$(B) \Rightarrow (A)$

First Case: $2^\kappa \geq |\theta|$.

Let $\{A_\alpha : \alpha < \theta\}$ be a family of $|\theta|$ distinct subsets of κ, let $A'_\alpha = \{2i : i \in A_\alpha\} \cup \{2i+1 : i \notin A_\alpha, i < \kappa\}$ and let $f_\alpha : \theta \to \{0,1\} \subseteq \lambda$ be

$$f_\alpha(i) = \begin{cases} 0 & \text{if } i \in A'_\alpha \\ 1 & \text{if } i \notin A'_\alpha. \end{cases}$$

Second Case: $\lambda^* \leqq \sigma$,$\mathrm{cf}(\lambda^*) \leq \kappa, \lambda^* > 2^\kappa$, $\mathrm{pp}^+(\lambda^*) > \theta, \theta$ a regular cardinal.

So there are regular cardinals $\lambda_i < \lambda^*$ for $i < \kappa$ (such that $\lambda_i > \kappa$) and an ideal J on $\kappa, \kappa \notin J$ such that $\prod\limits_{i<\kappa} \lambda_i/J$ has true cofinality $\theta_1 \geq \theta$. So there is a sequence $\langle f_\alpha : \alpha < \theta_1 \rangle, f_\alpha \in \prod\limits_{i<\kappa} \lambda_i$ and $\alpha < \beta \Rightarrow f_\alpha < f_\beta \bmod J$. Now $\langle f_\alpha : \alpha < \theta \rangle$ is a sequence as required.

By 3.8 those two cases suffice.

$(A) \Rightarrow (B)$

Let $\langle f_\alpha : \alpha < \theta \rangle$ be as in (A).

We can assume that (B) fails, σ minimal for which this occurs (for a given κ for some θ) and θ minimal for the given κ and σ. So $\theta \geq (2^\kappa)^+$. By Observation 3.8, θ is a regular cardinal. So $2^\kappa < \theta$ (hence $2^\kappa < \sigma$) and $[a \subseteq \mathrm{Reg} \cap \sigma^+ \backslash \kappa^+ \ \& \ |a| \leq \kappa \Rightarrow \max \mathrm{pcf}\, a < \theta]$, and $\sigma < \theta$.

As θ is a regular cardinal necessarily $\mathrm{cf}(\sigma) \leq \kappa$ (otherwise for some $\sigma_1 < \sigma$ the set $\{\alpha < \theta : \mathrm{Rang}(f_\alpha) \subseteq \sigma_1\}$ is unbounded in θ, contradicting the minimality of σ). Also σ is a limit ordinal as $2^\kappa < \theta = \mathrm{cf}(\theta)$ (as if $\sigma = \beta + 1$, for some $A \subseteq \kappa, B = \{\alpha : \bigwedge\limits_{i<\kappa} [i \in A \Leftrightarrow f_\alpha(i) = \beta]\}$ has cardinality θ, so $\{f_\alpha \restriction (\kappa \backslash A) : \alpha \in B\}$ essentially contradicts the minimality of σ).

Let χ be regular large enough. We choose by induction on $i < (2^\kappa)^+$, a model N_i such that:

$N_i \prec (\mathscr{H}(\chi), \in, <^*_\chi)$

$\|N_i\| = 2^\kappa$

$2^\kappa \subseteq N_0$

$\kappa, \sigma, \theta \in N_0, \langle f_\alpha : \alpha < \theta \rangle \in N_0$

$i < j \Rightarrow N_i \prec N_j$

$\langle N_j : j \leq i \rangle \in N_{i+1}$

N_i increasing continuous.

Let $\delta_i =: \sup(\theta \cap N_i)$ so $\langle \delta_i : i < (2^\kappa)^+ \rangle$ is strictly increasing continuous (as θ is regular, $\theta > \sigma$ and $\sigma > 2^\kappa$, necessarily $\delta_i < \theta$). We define for $i < (2^\kappa)^+$, a function $g_i \in {}^\kappa\sigma$ by

$$g_i(\zeta) = \mathrm{Min}(N_i \cap \sigma \backslash f_{\delta_i}(\zeta))$$

(it is well defined as $\sigma \in N_0 \subseteq N_i$ and $N \cap \sigma$ is unbounded in σ as $\mathrm{cf}\, \sigma \leq \kappa$).

Now $i < (2^\kappa)^+$, $\mathrm{cf}(i) = \kappa^+$ implies $N_i = \bigcup\limits_{j<i} N_j$ and $\mathrm{Rang}\ g_i \subseteq \bigcup\limits_{j<i} N_j$ hence $\bigvee\limits_{j<i} [\mathrm{Rang}(g_i) \subseteq N_j]$; but every subset of N_j of cardinality $\leq \kappa$ belongs to N_{j+1},

hence $g_i \in \bigcup_{j<i} N_j$. So by Fodor Lemma for some stationary subset S of $\{i < (2^\kappa)^+ : \mathrm{cf}(i) = \kappa^+\}$ and some $g^* : \kappa \to \sigma$ and some $A \subseteq \kappa$ and some $i(*) < (2^\kappa)^+$ we have: $[i \in S \Rightarrow g_i = g^*], (\forall i \in S)(\forall \zeta < \kappa)[f_{\delta_i}(\zeta) = g^*(\zeta) \Leftrightarrow \zeta \notin A]$ and $g^* \in N_{i(*)}$; note $A \in N_0 \subseteq N_{i(*)}$ as $A \subseteq \kappa$.

Clearly $i \in S \ \ \& \ \ \zeta \in A \Rightarrow \ \mathrm{cf}[g^*(\zeta)] > 2^\kappa$ (otherwise $g^*(\zeta) = \sup(N_i \cap g^*(\zeta))$ (as $N_i \prec (\mathscr{H}(\chi), \in, <^*_\chi), 2^\kappa + 1 \subseteq N_0 \subseteq N_i$) and easy contradiction). Also, as the f_α's are pairwise distinct, clearly $A \neq \emptyset$.

Question: What is $\mathrm{cf}[\prod_{\zeta \in A} \mathrm{cf}[g^*(\zeta)]]$?

(I.e. cofinality of the partial ordered set).

By [Sh 355, 3.1] it is $\max \mathrm{pcf}\{\mathrm{cf}[g^*(\zeta)] : \zeta \in A\}$, which by an assumption is $< \theta$, so there is a family $G \subseteq \prod_{\zeta \in A} g^*(\zeta)$ of cardinality $< \theta$ such that $(\forall f \in \prod_{\zeta \in A} g^*(\zeta))(\exists g \in G)[f < g]$. As the parameters in the demands on G belongs to $N_{i(*)}$, without loss of generality $G \in N_{i(*)}$.

Now we can define a partial function H from the family G to θ:

if $g \in G$ and for some α the condition $(*)$ below holds then $H(g)$ is such an ordinal

if $g \in G$ and for no α the condition $(*)$ below holds then $H(g)$ is not defined where

$$(*) \ \ \alpha < \theta, f_\alpha \restriction (\kappa \backslash A) = g^* \restriction (\kappa \backslash A) \text{ and } g = g \restriction A \le f_\alpha \restriction A \le g^* \restriction A.$$

Now we can choose an ordinal $j(*)$ such that

$$i(*) < j(*) < (2^\kappa)^+, \ \ j(*) \in S$$

(possible as S is a stationary subset of $(2^\kappa)^+$).

We know that there is a function $h \in G$ such that $f_{\delta_{j(*)}} \restriction A < h$.

Question: Is $H(h)$ well defined?

Possibility A: The answer is yes.

Then $H(h) < \cup\{H(g) + 1 : g \in (\mathrm{Dom}(H)) \subseteq G\}$. This union is an ordinal $< \theta$ (as $|G| < \theta$ and $\mathrm{Rang}(H) \subseteq \theta$ and θ is regular); also this union belongs to $N_{i(*)}$ (as $G, H \in N_{i(*)}$), hence the union is an ordinal $< \delta_{i(*)} < \delta_{j(*)}$. So $H(h) < \delta_{j(*)}$.

But (by the choice of h for the first inequality, and definition of $H(h)$ for the second inequality)

$$\otimes_1 \ \ f_{\delta_{j(*)}} \restriction A \le g_{j(*)} \restriction A < h \restriction A \le f_{H(h)} \restriction A$$

and (by the definition of H for the first equality, choice of g^* and $j(*) \in S$ for the second):

$$\otimes_2 \ \ f_{H(h)} \restriction (\kappa \backslash A) = g^* \restriction (\kappa \backslash A) = f_{\delta_{j(*)}} \restriction (\kappa \backslash A).$$

Now $\otimes_1, \otimes_2$ together implies $f_{\delta_{j(*)}} \leq f_{H(h)}$, but as said earlier $H(h) < \delta_{j(*)} < \theta$, together they contradict the choice of $\langle f_\alpha : \alpha < \theta \rangle$.

Possibility B: The answer is no.

So $H(h)$ is not well defined and without loss of generality $h \in N_{j(*)+1}$ (as all parameters in the requirements on it are in $N_{j(*)+1}$). Choose $j \in S, j > j(*)$; as $H(h)$ is not well defined, no $\alpha < \theta$ satisfies the requirements in $(*)$. But of the three demands on α, δ_j trivially satisfy two and a half: "$\alpha < \theta, f_\alpha \restriction (\kappa \backslash A) = g^* \restriction (\kappa \backslash A)$ and $f_\alpha \restriction A \leq g^* \restriction A$"; so the remaining one should fail, i.e. $\neg[h \restriction A \leq f_{\delta_j} \restriction A]$. So for some $\zeta \in A$ we have $h(\zeta) > f_{\delta_j}(\zeta)$; now $h \in N_{j_{j(*)}+1} \subseteq N_j$ hence $h(\zeta) \in N_j$ hence $h(\zeta) \in N_j \cap \sigma \backslash f_{\delta_j}(\zeta)$, hence (by the definition of g_j), $g_j(\zeta) \leq h(\zeta)$ hence (as $j \in S$) we have $g^*(\zeta) \leq h(\zeta)$ but $h \in G \subseteq \prod_{\xi \in A} g^*(\xi)$, so $h(\zeta) < g^*(\zeta)$, contradiction.

$\square_{3.7}$

§4 Entangled Orders — Narrow Order Boolean Algebra Revisited

4.1 Theorem. *1) If $\kappa^{+4} \le \mathrm{cf}(\lambda) < \lambda \le 2^\kappa$ then there is an entangled linear order of cardinality λ.*
2) Moreover, if $\chi_0 < \lambda$ we can demand that the linear order has density character $\ge \chi_0$ (in fact, in every interval of the linear order).

Remark. See more [Sh 462] and [Sh 666].

Proof. Without loss of generality $\chi_0 > \mathrm{cf}(\lambda) \ge \kappa^{+4}$. By [Sh 355, 2.1] there is an increasing continuous sequence $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ of singular cardinals with limit λ such that $\mathrm{tcf}(\prod\limits_{i<\mathrm{cf}(\lambda)} \lambda_i^+, <_{J^{\rm bd}_{\mathrm{cf}(\lambda)}}) = \lambda^+$ and $\lambda_0 > \chi_0$. The proof will be split to cases (one of them relies on the solution to others for smaller cardinals, so you may want to say we are proving 4.1 by induction on λ). Without loss of generality $\chi_0 > \mathrm{cf}(\lambda)$.

Case I: For $i < \mathrm{cf}(\lambda)$ we have $\max \mathrm{pcf}\{\lambda_j^+ : j < i\} < \lambda$.

So for some unbounded $A \subseteq \mathrm{cf}(\lambda)$ we have $i \in A \Rightarrow \lambda_i^+ > \max \mathrm{pcf}\{\lambda_j^+ : j \in A \cap i\}$.

So $\mathfrak{a} = \{\lambda_i^+ : i \in A\}$ is as required in [Sh 355, 4.12] (with λ^+, $\mathrm{cf}(\lambda)$ here standing for λ, κ there, noting that $2^{\mathrm{cf}(\lambda)} \ge 2^\kappa \ge \lambda$).

So we can assume:

Assumption — not Case I

So there is $\mu, \chi_0 < \mu < \lambda$, $\mathrm{cf}(\mu) < \mathrm{cf}(\lambda)$, $\mathrm{pp}_{<\mathrm{cf}(\lambda)}(\mu) > \lambda$. Choose a minimal such μ, so by 3.2(2):

$(*)$ $\mathfrak{a} \subseteq \mathrm{Reg} \setminus \chi_0$ & $\sup(\mathfrak{a}) < \mu$ & $|\mathfrak{a}| < \mathrm{cf}(\lambda) \Rightarrow \max \mathrm{pcf}(\mathfrak{a}) < \lambda$.

Clearly (by [Sh 355, 2.3]) in $(*)$'s conclusion we can replace "$< \lambda$" by "$< \mu$" i.e.

$(*)'$ $\mathfrak{a} \subseteq \mathrm{Reg} \setminus \chi_0$ & $\sup(\mathfrak{a}) < \mu$ & $|\mathfrak{a}| < \mathrm{cf}(\lambda) \Rightarrow \max \mathrm{pcf}(\mathfrak{a}) < \mu$.

Let $\sigma =: \mathrm{cf}(\mu)$, so $\mathrm{pp}(\mu) = \mathrm{pp}_{<\mathrm{cf}(\lambda)}(\mu)$ (by [Sh 371, 1.6](3)) and remember $\mathrm{pp}_{<\mathrm{cf}(\lambda)}(\mu) > \lambda$.

Case II: $\sigma \ge \kappa$ (and not Case I; actually $2^\sigma \ge \mu$ suffices).

First assume $\sigma > \aleph_0$. As said above $\mathrm{pp}_{<\mathrm{cf}(\lambda)}(\mu) > \lambda$ and by [Sh 371, 1.7] there is a strictly increasing sequence $\langle \mu_i^* : i < \sigma\rangle$ of regular cardinals satisfying $\mu = \bigcup\limits_{i<\sigma} \mu_i^*$, and $\lambda^+ = \max \mathrm{pcf}\{\mu_i^* : i < \sigma\} = \mathrm{tcf} \prod\limits_{i<\sigma} \mu_i^*/J^{\rm bd}_\delta$. Now as we can replace $\langle \mu_i^* : i < \sigma\rangle$ by $\langle \mu_i^* : i \in A\rangle$ for any $A \subseteq \sigma$ unbounded, by $(*)'$ without loss of generality $\mu_i^* > \max \mathrm{pcf}\{\mu_j^* : j < i\}$, so we can apply again [Sh 355, 4.12] (or 3.2(5)).

When $\sigma = \aleph_0$, 4.1 follows from [Sh 355, 4.13](1).

Case III: $\mathrm{cf}(\lambda) = \kappa^{+4}$ and $\sigma < \kappa$ (instead $\sigma < \mu, 2^\sigma < \lambda$ suffice).

So $\sigma^{+4} \leq \operatorname{cf}(\lambda)$. Let $\mathscr{P} =: \{A : A \subseteq \operatorname{cf}(\lambda), \operatorname{otp}(A) = \kappa^+, A$ is a closed subset of $\sup(A)$ and $\max \operatorname{pcf}\{\lambda_i^+ : i \in c\} = \lambda^+_{\sup(A)}\}$. For any $C \in \mathscr{P}$, try to choose by induction on $i < \kappa^+ (\mathfrak{b}_i) = \mathfrak{b}_i[C]$ and $\gamma_i = \gamma_i[C]$ such that:

- (i) $\mathfrak{b}_i \subseteq \operatorname{Reg} \cap \mu \setminus \bigcup_{j<i} \mathfrak{b}_j \setminus \chi_0$
- (ii) $\gamma_i \in C \setminus \bigcup_{j<i} (\gamma_j + 1)$
- (iii) $\lambda^+_{\gamma_i} \in \operatorname{pcf}(b_i)$
- (iv) $|\mathfrak{b}_i| \leq \kappa$
- (v) all members of b_i are $(\chi_0, \kappa^+, \aleph_0)$-inaccessible
- (vi) γ_i is minimal under those requirements.

Subcase IIIa: For some $j < \kappa^+$.

For every $C \in \mathscr{P}$ such that $\operatorname{Min}(C) \geq j$. For some $\varepsilon(*) < \operatorname{cf}(\lambda)$, we cannot define $\mathfrak{b}_i, \gamma_i$ are defined iff $\varepsilon(*)$.

Let $C, i(*)$ be as above. Let $\gamma^* = \bigcup_{i<i(*)} \gamma_i$, so $\gamma^* \in C$. Now if $\gamma \in C \setminus \gamma^*$ then (by [Sh 355, 1.5B]) as $\operatorname{pp}_\sigma(\mu) \geq \lambda^+ > \lambda^+_\gamma$, there is $\mathfrak{a}_\gamma \subseteq \operatorname{Reg} \cap (\chi_0, \mu), |\mathfrak{a}_\gamma| \leq \sigma$ such that $\lambda^+_\gamma \in \operatorname{pcf}(\mathfrak{a}_\gamma)$. By 3.2(3) there is $\mathfrak{c}_\gamma \subseteq \operatorname{Reg} \cap (\chi_0, \mu)$ of cardinality $\leq \kappa$ consisting of $(\chi_0, \kappa^+, \aleph_0)$-inaccessible cardinals such that $\lambda^+_\gamma \in \operatorname{pcf}(\mathfrak{c}_\gamma)$. Now $\gamma, c_\gamma \setminus \bigcup_{i<i(*)} \mathfrak{b}_i$ cannot serve as $\gamma_{i(*)}, \mathfrak{b}_{i(*)}$ so necessarily $\lambda^+_\gamma \notin \operatorname{pcf}(\mathfrak{c}_\gamma \setminus \bigcup_{i<i(*)} \mathfrak{b}_i)$ hence without loss of generality $\mathfrak{c}_\gamma \subseteq \bigcup_{i<i(*)} \mathfrak{b}_i$.

Version 2: So $\{\lambda^+_\gamma : \gamma \in C \setminus i(*)\} \subseteq \operatorname{pcf}\{\bigcup_{i<i(*)} \mathfrak{b}_i\}$ and $|\bigcup_{i<i(*)} \mathfrak{b}_i| \leq \kappa$. By the proof of [Sh 400, 4.2] we get a contradiction.

Subcase IIIb: For every $j < \operatorname{cf}(\lambda)$ there are $C \in \mathscr{P}$ with $\operatorname{Min}(C) > j$ such that for C, the pair $(\mathfrak{b}_i, \gamma_i)$ defined for every $i < \kappa^+$.

We shall now show

- $\otimes$ for every $i(*) < \operatorname{cf}(\lambda)$ there is $\lambda' \in \lambda \cap \operatorname{pcf}\{\lambda^+_j : j < \operatorname{cf}(\lambda)\} \setminus \lambda_{i(*)}$ such that $\operatorname{Ens}(\lambda', \lambda')$ (exemplified by linear order which has density character $> \chi_0$ in every interval).

Why $\otimes$ is sufficient: We can for $i < \operatorname{cf}(\lambda)$, choose $\mu^*_i, \lambda_i < \mu^*_i = \operatorname{cf}(\mu^*_i) \in \lambda \cap \operatorname{pcf}\{\lambda^+_j : j < \operatorname{cf}(\lambda)\}$, as required in $\otimes$. As $\bigwedge_i \mu^*_i < \lambda$ without loss of generality $\langle \mu^*_i : i < \operatorname{cf}(\lambda)\rangle$ is (strictly) increasing. We try to choose by i induction on $\epsilon < \operatorname{cf}(\lambda), i(\epsilon) < \operatorname{cf}(\lambda)$ strictly increasing such that $\mu^*_{i(\epsilon)} > \max \operatorname{pcf}\{\mu^*_{i(\zeta)} : \zeta < \epsilon\}$.

Let $i(\epsilon)$ be defined iff $\epsilon < \epsilon(*)$. So $\epsilon(*)$ is $\leq \operatorname{cf}(\lambda)$ and is a limit ordinal λ and $\max \operatorname{pcf}\{\mu^*_{i(\epsilon)} : \epsilon < \epsilon(*)\} \geq \lambda$ hence $\geq \lambda^+$, but $\operatorname{pcf}\{\mu^*_{i(\varepsilon)} : \varepsilon < \varepsilon(*)\} \subseteq \operatorname{pcf}\{\lambda^+_j : j < \operatorname{cf}(\lambda^+)\} \subseteq \lambda^+ + 1$ hence $\lambda^+ = \max \operatorname{pcf}\{\mu^*_{i(\varepsilon)} : \varepsilon < \varepsilon(*)\}$. Note that

$\mu^*_{i(\epsilon)} > \max \operatorname{pcf}\{\mu^*_{i(\zeta)} : \zeta < \epsilon\}, \mu^*_{i(\epsilon)}$ is strictly increasing, and $\operatorname{Ens}(\mu^*_{i(\epsilon)}, \mu^*_{i(\epsilon)})$ for $\varepsilon < \varepsilon(*)$. So applying [\Sh:355] , 4.12] we finish.

Why $\otimes$ holds: Let $i(*) < \operatorname{cf}(\lambda)$ be given. Choose $C \subseteq (i(*), \operatorname{cf}(\lambda))$ from $\mathscr{P}$ such that $(\mathfrak{b}_\varepsilon[C], \gamma_\varepsilon[C])$ is defined iff $\varepsilon < \varepsilon[C]$ and $\varepsilon[C] < \kappa^+$. By the definition of $\mathscr{P}$ we have $\max \operatorname{pcf}\{\lambda^+_\gamma : \gamma \in C\} < \lambda$. Let $\mathfrak{d} =: \{\lambda^+_\gamma : \gamma \in C\}$, let $\langle \mathfrak{b}_\theta[\mathfrak{d}] : \theta \in \operatorname{pcf}(\mathfrak{d})\rangle$ be as in [Sh 371, 2.6]. Let θ be minimal such that $\operatorname{otp}(\mathfrak{b}_\theta[\mathfrak{d}])$ is $\geq \kappa$. We can find $B_\epsilon \subseteq C$ (for $\epsilon < \kappa$) such that $\{\lambda^+_\gamma : \gamma \in B_\epsilon\} \subseteq \mathfrak{b}_\theta[\mathfrak{d}]$, $\operatorname{otp}|B_\epsilon| = \kappa$ and the B_ϵ's are pairwise disjoint. Clearly $\max \operatorname{pcf}\{\lambda^+_\gamma : \gamma \in B_\epsilon\} = \theta$ as $\{\lambda^+_\gamma : \gamma \in B_\epsilon\}$ is $\subseteq \mathfrak{b}_\theta[\mathfrak{d}]$, but is not a subset of any finite union of $\mathfrak{b}_{\theta'}[\mathfrak{c}], \theta' < \theta$. Now letting $\mathfrak{a}^* =: \cup\{\mathfrak{b}_\varepsilon[C] : \varepsilon < \varepsilon[C]\}$, there is (by [Sh 371, 2.6]) a subset $\mathfrak{a}$ of $\mathfrak{a}^*$ such that $\theta = \max \operatorname{pcf}(\mathfrak{a})$ but $\theta \notin \operatorname{pcf}(\mathfrak{a}^* \backslash \mathfrak{a})$. Now as $\theta \in \operatorname{pcf}\{\lambda^+_\gamma : \gamma \in B_\epsilon\}, \lambda^+_\gamma \in \operatorname{pcf}(\mathfrak{b}_\gamma)$ we have (by [Sh 345a, 1.12]) $\theta \in \operatorname{pcf}(\bigcup\limits_{\gamma \in B_\epsilon} \mathfrak{b}_\gamma)$ hence by the previous sentence $\theta \in \operatorname{pcf}(\mathfrak{a} \cap \bigcup\limits_{\gamma \in B_\epsilon} \mathfrak{b}_\gamma)$. Let $\mathfrak{c}_\epsilon =: \mathfrak{a} \cap \bigcup\limits_{\gamma \in B_\epsilon} \mathfrak{b}_\gamma, \lambda' = \theta$, we can apply [Sh 355, 4.12] and get that there is an entangled linear order of cardinality λ' (which is more than required, see [Sh 345b]); and, of course, $\lambda_{i(*)} < \lambda' \in \lambda \cap \operatorname{pcf}\{\lambda_j : j < \operatorname{cf}(\lambda)\}$. The assumptions of [Sh 355, 4.12] holds as the $\mathfrak{c}_\epsilon$ are pairwise disjoint (by (i) above), $\theta \in \operatorname{pcf}\{\lambda^+_\gamma : \gamma \in B_\epsilon\{\subseteq \operatorname{pcf}(\bigcup\limits_{\gamma \in B_\epsilon} \mathfrak{b}_\gamma) = \operatorname{pcf}(\mathfrak{c}_\epsilon)$ and $[\theta_1 \in \mathfrak{a} \Rightarrow \max \operatorname{pcf}(\mathfrak{a} \cap \theta_1) < \theta_1]$ as θ_1 is $(\chi_0, \kappa^+, \aleph_0)$-inaccessible and $\theta = \lambda' \geq \sup\{\lambda^+_\gamma : \gamma \in C\} > \lambda_{i(*)} > \chi_0$. So $\otimes$ holds and we finish Subcase IIIb hence Case III.

Case IV: $\operatorname{cf}(\lambda) > \kappa^{+4}$ and $\sigma \leq \kappa$ (and not Case I).

For each $\delta < \operatorname{cf}(\lambda)$ of cofinality κ^{+4} we can apply the previous cases (or the induction hypothesis on λ) and get an entangled linear order of power λ^+_δ. So $\otimes$ holds and we finish as in Subcase IIIb. $\square_{4.1}$

4.2 Claim. *Assume $\kappa^{+4} \leq \theta = cf(\theta), \langle \lambda_i : i < \theta\rangle$ is a strictly increasing sequence of regular cardinals, $\theta < \lambda_i \leq 2^\kappa$ and $\lambda_\theta = tcf(\prod\limits_{i<\theta} \lambda_i/J^{\rm bd}_\theta)$.*

1) If $\sup\limits_{i<\theta} \lambda_i \leqq 2^\kappa$ then there is an entangled linear order of cardinality λ_θ.

2) Ens $(\lambda_\theta, 2^\kappa)$.

Remark. Remember that if there is an entangled linear order in λ then $\operatorname{Ens}(\lambda, \lambda)$ (so [Sh 345b, 7](5)).

Proof. Same proof as 4.1.

4.3 Claim. *Assume*

- *(i) λ is regular, uncountable*
- *(ii) $\kappa < \lambda \Rightarrow 2^\kappa < 2^\lambda$*
- *(iii) for some regular $\chi \leqq 2^\lambda$ there is no linear order of cardinality λ with $\geqq \chi$ Dedekind cuts or even no tree of cardinality $\lambda \geqq \chi$ λ-branches.*

Then ($2^{<\lambda} < 2^\lambda$ *and) for some* μ

(a) *for every regular* χ *in* $(2^{<\lambda}, 2^\lambda]$ *(or even* $(\mu, 2^\lambda]$*) there is an entangled linear order of cardinality* χ *and density* μ*,*

(b) $\mu \in (\lambda, 2^{<\lambda}]$*,* $cf(\mu) = \lambda$*,* $pp_{\Gamma(\lambda)}(\mu) = 2^\lambda, \mu$ *is* $(\lambda, \lambda^+, 2)$*-inaccessible*

(the linear order is $(T, <_{\ell x}), T \subseteq {}^{\mu>}2$ *has* $\leqq \mu$ *nodes and* $\geqq \chi$ λ*-branches).*

[Saharon: see also [Sh 430, §3].

Proof. Note: $2^{<\lambda} < 2^\lambda$ [if $(\exists \theta < \lambda)(2^\theta = 2^{<\lambda})$ by (ii), otherwise $\mathrm{cf}(2^{<\lambda}) = \lambda$ and by classical cardinal arithmetic, $\mathrm{cf}(2^\lambda) > \lambda$, hence $2^{<\lambda} < 2^\lambda$]. By [Sh 355, Lemma 5.11] if the conclusion fails then for every regular χ in $(2^{<\lambda}, 2^\lambda]$ there is $\mu, \lambda = \mathrm{cf}(\mu) < \mu \leqq 2^{<\lambda}$, $\mathrm{pp}_{\Gamma(\mu)}(\mu) \geqq \chi$. Choose a minimal μ such that $\lambda < \mu \leqq 2^{<\lambda}$, $\mathrm{cf}(\mu) \leqq \mathrm{cf}(\lambda)$ and $\mathrm{pp}^+(\mu) > \chi$ (note: μ does not depend on χ, by [Sh 355, 2.3]). So necessarily μ is $(\lambda, \lambda^+, 2)$-inaccessible. Let $\chi \in (\mu, 2^\lambda]$ be regular. As $(2^{<\lambda})^{<\lambda} = 2^{<\lambda}$ necessarily $\mathrm{cf}(\mu) = \lambda$, so by [Sh 371, 1.6](3) there is a strictly increasing sequence $\langle \mu_i : i < \mathrm{cf}(\mu) = \lambda \rangle$ of regular cardinals, $\lambda < \mu_i < \mu, \mu = \sum\limits_i \mu_i$ and $\chi = \mathrm{tcf}(\Pi \mu_i / J^{\mathrm{bd}}_{\mathrm{cf}\lambda})$. As μ is $(\lambda, \lambda^+, 2)$-inaccessible without loss of generality $\mu_i > \max \mathrm{pcf}\{\mu_j : j < i\}$. So by [Sh 355, 4.12] we finish. $\square_{4.3}$

4.4 Conclusion. 1) For a class of cardinals μ, there is an entangled linear order of cardinality μ^+.
2) Assume λ is strong limit singular. Then for some successor cardinal in $(\lambda, 2^\lambda]$ there is an entangled linear order.

Proof. 1) By part (2).
2) If $\aleph_{\lambda^{+4}} < 2^\lambda$ then apply theorem 4.1 (with $\lambda, \aleph_{\lambda^{+4}}$ here standing for κ, λ there) so there is an entangled linear order of cardinality $\aleph_{\lambda^{+4}+1} (\leqq 2^\lambda)$, which is as required. So assume $2^\lambda \leqq \aleph_{\lambda^{+4}}$. We know that there is a linear order of cardinality 2^λ and density character λ; hence (see [Sh:g, AP,§1]) there is an entangled linear order of cardinality $\mathrm{cf}(2^\lambda)$. But as $2^\lambda \leqq \aleph_{\lambda^{+4}}$ necessarily $\mathrm{cf}(2^\lambda)$ is a successor cardinal.

§5 prd: Measuring $\Pi f(i)$ by a family Γ of ideals and family sequences $\langle B_i : i < \kappa\rangle, |B_i| < \mu_i$

In [Sh 371, §4], and here in §1 we have dealt with generalizations of the measuring $\prod_{i<\kappa} f(i)/I$, i.e. whereas defining $\mathrm{cov}(\lambda,\lambda,\theta,\sigma)$ we cover a set $a \in [\lambda]^{<\theta}$ by $< \sigma$ subsets of cardinality $< \lambda$; there we ask that κ belongs to the closure to a normal ideal of J union the family of $A \subseteq \kappa$ for which we succeed to cover. Here we replace "normal" by an abstract property Γ (and phrase the required properties). We also generalize normality to ideals on $\mathscr{Y}$ with $\iota : \mathscr{Y} \to \kappa$, a generalization used in [Sh 420, §4], [Sh 430].

5.1 Context. 1) κ is a regular uncountable cardinal, $\mathscr{Y}$ a set, ι a function from $\mathscr{Y}$ onto $\kappa, \mathscr{Y}_i = \iota^{-1}(\{i\})$. Here I, J vary on ideals on $\mathscr{Y}, \Gamma$ a family of proper ideals on $\mathscr{Y}$.

5.2 Definition. 1) $\Gamma_{\mathscr{Y},\kappa,\sigma} = \{J : J \text{ a } \sigma\text{-complete ideal on } \mathscr{Y}\}$ (if $(\mathscr{Y},\iota) = (\kappa, \mathrm{id}_\kappa)$ this $\Gamma_{\kappa,\sigma}$ is essentially $\Gamma(\kappa^+,\sigma)$).
2) $\Gamma_{\mathscr{Y},\kappa} = \Gamma^{\mathrm{nor}}_{\mathscr{Y},\kappa} =: \{J : J \text{ a normal ideal on } \mathscr{Y}\}$ (normal — see 5.3(0) below). If $\mathscr{Y} = \kappa, \iota = \mathrm{id}$ we write $\Gamma^{\mathrm{nor}}_\kappa$.

5.3 Definition. 0) An ideal I on $\mathscr{Y}$ is normal if: for any club C of κ the set $\bigcup_{i\notin C} \mathscr{Y}_i$ belong to I and for any sequence $\langle A_i : i < \kappa\rangle$ of sets from I the set $\nabla_i A_i =: \{x \in \mathscr{Y} : x \in \bigcup_{j<\iota(x)} A_j\}$ belongs to I. (So normal implies κ-complete).
1) We say Γ is σ-complete if every $J \in \Gamma$ is σ-complete.
2) We say Γ is normal if every $J \in \Gamma$ is normal.
3) We say Γ is restriction closed when: $J \in \Gamma, A \subseteq \kappa, A \neq \emptyset \bmod J$ implies there is $I \in \Gamma, J \cup \{\kappa\backslash A\} \subseteq I$.
4) We say Γ is closed if for every $\mathscr{P} \subseteq \mathscr{P}(\mathscr{Y}), c\ell_\Gamma(\mathscr{P})$ is well defined where $c\ell_\Gamma(\mathscr{P})$ is the minimal member of $\Gamma \cup \{\mathscr{P}(\kappa)\}$ which include it, i.e. $(\forall I \in \Gamma)[\mathscr{P} \subseteq I \Leftrightarrow c\ell_\Gamma(\mathscr{P}) \subseteq I]$.

Note: $c\ell_\Gamma$ for a not necessarily closed Γ, is a partial function.
5) We say Γ has character $\leqq \mu$ when: $\mathscr{P}(\kappa) = c\ell_\Gamma(\mathscr{P})$ where $\mathscr{P} \subseteq \mathscr{P}(\kappa)$ implies that for some $\mathscr{P}' \subseteq \mathscr{P}$ of cardinality $\leqq \mu$, we have $\mathscr{P}(\kappa) = c\ell_\Gamma(\mathscr{P})$.
6) The character of Γ is the minimal cardinal μ such that Γ has character $\leqq \mu$.

5.4 Definition. 1) We say Γ is suitable if it is θ-suitable for every θ? We say that Γ is θ-suitable when: for every ideal $J \in \Gamma$ on $\mathscr{Y}$, if then $\mathscr{P}(\kappa) = c\ell_\Gamma(J \cup \{f^{-1}_\eta(\{0\}) : \eta \in T\})$ where

- (i) T is a (non-empty) set of finite sequences of ordinals $< \theta$ closed under initial segments
- (ii) $A_\eta \subseteq \mathscr{Y}$ for $\eta \in T$ and $A_{<>} = \mathscr{Y}$
- (iii) for each $\eta \in T$ of length n,

$$\mathscr{P}(\kappa) = c\ell_\Gamma(J \cup \{\kappa\backslash A_{\eta\restriction 0}, \ldots, \kappa\backslash A_{\eta\restriction n}\} \cup \{A_{\eta^\frown<i>} : \eta^\frown < i > \in T\})$$

(iv) $\eta \triangleleft \nu \Rightarrow A_\eta \supseteq A_\nu$
(v) $f_\eta : A_\eta \to \mathrm{Ord}$
(vi) if $\eta^\frown < \zeta > \in T$ and $y \in A_{\eta^\frown<\zeta>}, f_\eta(y) \neq 0$ then $f_\eta(y) > f_{\eta^\frown<\zeta>}(y)$.

5.5 Remark. 1) Clearly for $\theta_1 \leqq \theta_2$ if Γ is θ_2-suitable then Γ is θ_1-suitable.
2) If $\theta \geqq 2^{|\mathscr{Y}|}$ its value is immaterial, so we can omit it.

5.6 Claim. *1)* $\Gamma_{\mathscr{Y},\kappa,\sigma}$, *if* $\kappa \geqq \sigma = \mathrm{cf}(\sigma) > \aleph_0$ *is closed, restriction closed, of character* σ *and is suitable. For* $\mathscr{P} \subseteq \mathscr{P}(\mathscr{Y})$,

$$cl_{\Gamma_{\mathscr{Y},\kappa,\sigma}}(\mathscr{P}) = \big\{Z : \text{ for some } \alpha < \sigma, A_i \in \mathscr{P} \text{ for } i < \alpha \text{ we have } Z \subseteq \bigcup_{i<\alpha} A_i\big\}.$$

2) If $\mathscr{P}$ *is a family of subsets of* $\mathscr{Y}, \Gamma = \Gamma^{\mathrm{nor}}_{\mathscr{Y},\kappa}$ *then*

$$cl_\Gamma(\mathscr{P}) = \Big\{ Z : Z \subseteq \mathscr{Y} \text{ and for some club } C \text{ of } \kappa \text{ and sequence } \langle A_i : i < \kappa\rangle \text{ of member of } \mathscr{P} \text{ we have : } Z \subseteq \{x \in \mathscr{Y} : \iota(x) \in C \text{ and } x \in \bigcup_{j<\iota(x)} A_j\Big\}.$$

3) $\Gamma^{\mathrm{nor}}_{\mathscr{Y},\kappa}$ *(remember* $\kappa = cf(\kappa) > \aleph_0$*) is closed, restriction closed, suitable and of character* κ.
4) If Γ *is* θ*-suitable and has character* $\leqq \theta$ *then it is suitable (we shall use this freely).*

Proof. 1) Let us check suitability leaving the rest to the reader; so let $J, \theta, \langle A_\eta, f_\eta : \eta \in T\rangle$ be as in Definition 5.4. If the required conclusion in Definition 5.4 fails then there is a σ-complete filter I on κ containing $J \cup \{f_\eta^{-1}(\{0\}) : \eta \in T\}$. For each η, by condition (iii), for some set w_η of ordinals, $|w_\eta| < \sigma$ and $B_\eta \in J$ we have

$(*)$ $\mathscr{Y} = B_\eta \cup \{\kappa \backslash A_{\eta\restriction 0} \backslash \ldots \backslash A_{\eta\restriction \ell g\eta}\} \cup \{A_{\eta^\frown<\zeta>} : \zeta \in w_\eta\}$
(and $\zeta \in w_\eta \Rightarrow \eta^\frown < \zeta > \in T$).

Let $T^* =: \{\eta \in T : \ell < \ell g(\eta) \Rightarrow \eta(\ell) \in w_{\eta\restriction\ell}\}$, as σ is regular uncountable, clearly $|T^*| < \sigma$. Let

$$B =: \cup\{B_\eta : \eta \in T^*\} \cup \{A_\eta : \eta \in T^* \text{ and } A_\eta \in I\} \cup \{f_\eta^{-1}(\{0\}) : \eta \in T^*\}.$$

Now B is the union of $< \sigma$ members of I and I is σ-complete, so $B \in I$. Choose $y \in \kappa \backslash B$.

We now choose by induction on n, η_n such that:

$$\eta_0 =<>$$

$$\eta_n \in T^*,$$

$$\ell g(\eta_n) = n$$

$$\eta_n \trianglelefteq \eta_{n+1}$$

$$y \in A_{\eta_n}.$$

For $n = 0, \eta_0 =<>$ so $y \in \mathscr{Y} = A_{<>}$. For $n+1, y \in A_{\eta_n \restriction n}, \eta_n \in T^*$ and $y \notin B$ hence $y \notin B_{\eta_n}$, so by $(*)$ there is $\zeta_n \in w_{\eta_n}$ such that $y \in A_{\eta_n \char 94 <\zeta_n>}$ so $\eta_{n+1} = \eta_n \char 94 \langle \zeta \rangle$ is as required.

In the end for each n we have $f_{\eta_n}(y) > 0$ as $f^{-1}_{\eta_n}(\{0\}) \subseteq B$, (remember $\eta_n \in T^*$) hence by condition (vi) from 5.4, $\langle f_{\eta_n}(y) : n < \omega \rangle$ is a strictly decreasing sequence of ordinals, contradiction.

2) Left to the reader.

3) Again we leave the proof of restriction closed and closed and having character κ to the reader and prove suitability. The proof is similar but use diagonal union. So by part (2) and condition (iii) of 5.4 for each $\eta \in T$ for some $B_\eta \in J$ and function h_η from κ to ordinals such that $\eta \char 94 \langle h_\eta(i) \rangle \in T$ for $i < \kappa$, we have

$$(*)\ \ \mathscr{Y} = B_\eta \cup \{\kappa \backslash A_{\eta \restriction 0} \backslash A_{\eta \restriction 1} \backslash \ldots \backslash A_{\eta \restriction \ell g \eta}\} \cup \{x \in \mathscr{Y} : x \in \bigcup_{i < \iota(x)} A_{\eta \char 94 \langle h_\eta(i) \rangle}\}.$$

By renaming, without loss of generality $T \subseteq {}^{\omega >}\kappa$ and each h_η the identity function.

Let I be the normal ideal on $\mathscr{Y}$ generated by $J \cup \{f_n^{-1}(\{0\}) : \eta \in T\}$, so we assume I is a normal proper ideal and we shall get a contradiction.

Now define $\mathscr{Y}^*$

$$\begin{aligned} \mathscr{Y}^* = \{x \in \mathscr{Y} : &(a) \quad \iota(x) \text{ a limit ordinal } < \kappa \\ &(b) \quad \text{if } \eta \in T \cap {}^{\omega >}\iota(x) \text{ then } x \notin B_\eta \\ &(c) \quad \text{if } \eta \in T \cap {}^{\omega >}\iota(x) \text{ and } A_\eta \in I \text{ then } x \notin A_\eta\}. \end{aligned}$$

Clearly $\mathscr{Y}^* \equiv \mathscr{Y} \bmod I$, hence we can find $x(*) \in \mathscr{Y}^*$. Now we choose by induction on $n \eta_n \in {}^{\omega >}\iota(x(*))$ as in the proof of part (1) and get similar contradiction.

4) Left to the reader. $\square_{5.6}$

5.7 Definition. 1) For $\bar{\mu} = \langle \mu_i : i < \kappa \rangle, J$ an ideal on $\mathscr{Y}$ and $f : \mathscr{Y} \to \text{ord}$ we define

$$\begin{aligned}\text{prd}^{\Gamma}_J(f,\bar{\mu}) = \ \text{Min}\big\{|\mathscr{P}| : &\mathscr{P} \text{ is a family of sequences of the form } \langle B_x : x \in \mathscr{Y}\rangle,\\ &\text{each } B_x \text{ a set of ordinals, } B_x \text{ of cardinality}\\ &< \mu_{\iota(x)} \text{ such that for every } g \in {}^{x}\text{ord satisfying}\\ &g \leqq f, \text{ we have :}\\ &\mathscr{P}(\kappa) = c\ell_{\Gamma}[J \cup \{\{x \in \mathscr{Y} : g(x) \in B_x\} : \langle B_x : x \in \mathscr{Y}\rangle \in \mathscr{P}\}]\big\}.\end{aligned}$$

2) If above $J \subseteq I, I$ an ideal on κ

$$\begin{aligned}\text{prd}^{\Gamma}_{J,I}(f,\bar{\mu}) = \ \text{Min}\big\{|\mathscr{P}| : &\mathscr{P} \text{ is a family of sequences } \langle B_x : x \in \mathscr{Y}\rangle,\\ &\text{each } B_x \text{ a set of ordinals of cardinality } < \mu_{\iota(x)}\\ &\text{and for every } g \in {}^{x}\text{ord satisfying } g \leqq f,\\ &\text{we have :}\\ &I \subseteq c\ell_{\Gamma}(J \cup \{\{x \in \mathscr{Y} : g(x) \in B_x\} : \langle B_x : x \in X\rangle \in \mathscr{P}\})\big\}.\end{aligned}$$

3) If $\bar{\mu}$ is constantly μ, we may write μ instead. We can use also $\bar{\mu} = \langle \mu_x : x \in \mathscr{Y}\rangle$, (but usually do not) with the obvious meaning.

5.8 Claim. *1) If $\{x \in \mathscr{Y} : f(x) < \omega\} \in J$ we can in 5.7 demand $g <_J f$, and if in addition $\bigwedge\limits_x f(x) \neq 0$ we can demand $g < f$ (without changing the values).*

2) If $\{x : f(x) \geqq \mu_{\iota(x)}\} \in J$ then $\text{prd}^{\Gamma}_J(f,\bar{\mu}) = 1$.

3) If $\bigwedge \mu_i = \mu, \text{cf}(\mu_{>}|\mathscr{Y}|, \{x : f(x) > \mu\} \in J$ then $\text{prd}^{\Gamma}_J(f,\bar{\mu}) \leqq \text{cf}(\mu)$.

4) If $f_1 \leqq_J f_2$ or just $\{x : |f_1(x)| > |f_2(x)|\} \in J$ then $\text{prd}^{\Gamma}_J(f_1,\bar{\mu}) \leqq \text{prd}^{\Gamma}_J(f_2,\bar{\mu})$ (and the other obvious monotonicity properties).

5) If $\mu_i = \mu$, $\text{cf}(\mu) > |\mathscr{Y}|$ we can in Definition 5.7 demand $\bigwedge\limits_{x \in \mathscr{Y}} B_x = B_0$ for $\bar{B} \in \mathscr{P}$ (i.e. without changing the value).

6) If Γ is σ-complete and restriction closed, $\epsilon(*) < \sigma, \langle A_\epsilon : \epsilon < \epsilon(*)\rangle$ is a partition of $\mathscr{Y}$ ($J, \bar{\mu}$ as in 5.7) then

$$\sup_{\epsilon} \ prd^{\Gamma}_{J+(\kappa \setminus A_\epsilon)}(f,\bar{\mu}) \leqq \text{prd}^{\Gamma}_J(f,\bar{\mu}) \leqq \sum_{\epsilon} \text{prd}^{\Gamma}_{J+(\kappa \setminus A_\epsilon)}(f,\bar{\mu}).$$

7) If Γ is normal and restriction closed, $A_\epsilon \subseteq \{x \in \mathscr{Y} : \iota(x) > \epsilon\}$ for $\epsilon < \kappa, \langle A_\epsilon : \epsilon < \kappa\rangle$ a partition of $\mathscr{Y} \setminus \iota^{-1}(\{0\})$ then

$$\sup_{\epsilon<\kappa} \text{prd}^{\Gamma}_{J+(\mathscr{Y} \setminus A_\epsilon)} \leqq \text{prd}^{\Gamma}_J(f,\bar{\mu}) \leqq \sum_{\epsilon<\kappa} \text{prd}^{\Gamma}_{J+(\mathscr{Y} \setminus A_\epsilon)}(f,\bar{\mu}).$$

8) $\text{prd}^{\Gamma^{\text{nor}}_\kappa}_J(f,\bar{\mu}) = \text{prc}_J(f,\bar{\mu})$ if $J \in \Gamma^{\text{nor}}_\kappa$.

9) Assume Γ is normal, $\bar{\mu}^\ell = \langle \mu^\ell_i : i < \kappa\rangle$ for $\ell = 1, 2, \bar{\mu}^1$ increasing continuous and

for each i, $\mathrm{cf}(\mu_i^1) < \mu_i^1$ & $\mu_1^2 = (\mu_i^1)^+$. *Then for (any* $\Gamma, J \in \Gamma$ *and* $f \in {}^{\mathscr{Y}}\mathrm{Ord}$*) we have* $\mathrm{prd}^\Gamma_J(f, \bar\mu^1) = \mathrm{prd}^\Gamma_J(f, \bar\mu^2)$.

10) If $\bar\mu = \langle \mu_i : i < \kappa\rangle$ *is increasing continuous with limit* μ*, and* Γ *is normal,* $J \in \Gamma$ *then* $\mathrm{prd}^\Gamma_J(\bar\mu, \bar\mu) \leqq \mu$ *and even* $\mathrm{prd}^\Gamma_J(\bar\mu, \bar\mu) = \mathrm{cf}(\mu)$.

11) If $\bigwedge\limits_{i<\kappa} \mu_i = \mu$, $\mathrm{cf}(\mu) > |\mathscr{Y}|, \Gamma = \Gamma_{\mathscr{Y},\kappa,\sigma}$ *then for any* α, $\mathrm{prd}^\Gamma_{\{\emptyset\}}(\alpha, \bar\mu) = \mathrm{cov}(|\alpha|, \mu, |\mathscr{Y}|^+, \sigma)$.

Proof. E.g.

6) The first inequality should be clear. Also the second: assume it fails, let $\lambda_\epsilon = \mathrm{prd}^\Gamma_{J+(\kappa\setminus A_\epsilon)}(f, \bar\mu)$, $\sum\limits_{\epsilon<\epsilon(*)} \lambda_\epsilon < \mathrm{prd}^\Gamma_J(f, \bar\mu)$, let $\mathscr{P}_\epsilon$ exemplify the definition of λ_ϵ and $\mathscr{P}$ be $\bigcup\limits_{\epsilon<\epsilon(*)} \mathscr{P}_\epsilon$. As $\sum\limits_\epsilon \lambda_\epsilon < \mathrm{prd}^\Gamma_J(f, \bar\mu)$, $\mathscr{P}$ cannot exemplify $\sum\limits_\epsilon \lambda_\epsilon \geqq \mathrm{prd}^\Gamma_J(f, \bar\mu)$, so there is a function $g \in {}^\kappa\mathrm{ord}$ exemplifying this, so there is a proper ideal $I \in \Gamma$ extending

$$J \cup \big\{\{x \in \mathscr{Y} : g(x) \in B_x\} : \langle B_x : x \in \mathscr{Y}\rangle \in \mathscr{P}\big\}.$$

As Γ is σ-complete also I is σ-complete so for some $\epsilon < \epsilon(*)$ we have $A_\epsilon \notin I$; but Γ is restriction closed so there is $I_1 \in \Gamma, I \cup \{\mathscr{Y}\setminus A_\epsilon\} \in I_1$. So $I_1 \in \Gamma$ extend

$$[J \cup \{\mathscr{Y}\setminus A_\epsilon\}] \cup \{\{x \in \mathscr{Y} : g(x) \in B_x\} : \langle B_x : x \in \mathscr{Y}\rangle \in \mathscr{P}_\epsilon\}$$

contradicting the choice of $\mathscr{P}_\epsilon$. $\square_{5.8}$

5.9 Lemma. *1) Suppose*

(∗) $\mu_i = \mu = \mathrm{cf}(\mu) > |\mathscr{Y}|, \Gamma$ *is suitable, restriction closed,* $f \in {}^{\mathscr{Y}}\mathrm{Ord}$ *and* J *an ideal on* κ.

Then:

$$\otimes \qquad \mu + \mathrm{prd}^\Gamma_J(f, \bar\mu) = \mu + \sup\big\{\mathrm{tcf}\prod_{x\in\mathscr{Y}} \lambda_x/I : I \text{ an ideal on } \mathscr{Y} \text{ in } \Gamma \text{ extending } J \text{ such that } \mu < \lambda_x = \mathrm{cf}(\lambda_x) \leqq f(x)\big\}.$$

2) If Γ *is normal,* $\bar\mu = \langle \mu_i : i < \kappa\rangle, \mu_i = \theta_i^+, \langle \theta_i : i < \kappa\rangle$ *is increasing continuous,* $\theta_i > \kappa, \mu_i > |\mathscr{Y}_i|, \mu = \bigcup\limits_{i<\kappa} \mu_i, \Gamma$ *suitable and restriction closed and* $f \in {}^{\mathscr{Y}}\mathrm{Ord}$*,* J *an ideal on* $\mathscr{Y}$ *then* $(\otimes)$ *above holds.*

Proof. Like the proof of 1.2.

5.10 Conclusion. 1) Suppose $\mu, \bar{\mu}, \Gamma, J$ are as in 5.9, $f \in {}^{\mathscr{Y}}\text{ord}$ and $\{x : \text{cf}[f(x)] = f(x) \geqq \mu_{\iota(x)}\} \in J$ <u>then</u>

$$\mu + \ \text{prd}^{\Gamma}_J(f, \bar{\mu}) = \mu + \sup\{\text{prd}^{\Gamma}_J(g, \bar{\mu}) : g <_J f\} + \sup\{\text{tcf}_x \prod_{x \in \mathscr{Y}} f(x)/I : J \subseteq I \in \Gamma \text{ (and the tcf well defined)}\}.$$

2) If in addition $G \subseteq \{g : g \in {}^{\mathscr{Y}}\text{Ord}, \ g <_J f\}$ is cofinal or at least $\mathscr{P}(\kappa) = c\ell_{\Gamma}[I \cup \{\{i : h(i) < g(i)\} : g \in G\}]$ for every $h <_J f$, (and e.g. $\bigwedge\limits_i f(i) \geqq \theta$) <u>then</u>

$$\mu + \ \text{prd}^{\Gamma}_J(f, \bar{\mu}) = \mu + \sup\{\text{prd}^{\Gamma}_J(g, \bar{\mu}) : g \in G\} + \sup\{\text{tcf} \prod_{x \in \mathscr{Y}} f(x)/I : J \subseteq I \in \Gamma\}.$$

5.11 Claim. *Suppose $\mu, \bar{\mu}, \Gamma, J$ are as in 5.9((1) or (2)), $g : \mathscr{Y} \to$ card and $f(x) = g(x)^+ \geqq \mu^+_{\iota(x)}$.*

<u>Then</u> $\text{prd}^{\Gamma}_J(f, \bar{\mu}) \leqq [\text{prd}^{\Gamma}_J(g, \bar{\mu})]^+ + \mu$.

Proof. Let $\mathscr{P}$ exemplify the value of $\text{prd}^{\Gamma}_J(g, \bar{\mu})$, say $|\mathscr{P}| = \chi$. So for every $h <_J f$, clearly $\{x \in \mathscr{Y} : |h(x)| \leqq g(x)\} = \mathscr{Y} \mod J$, hence there is $\mathscr{P}_h \subseteq \prod\limits_{x \in \mathscr{Y}} h(x)$ exemplifying $\text{prd}^{\Gamma}_J(h, \bar{\mu}) \leqq \text{prd}^{\Gamma}_J(g, \bar{\mu}) = \chi$. Assume $\text{prd}_J(f, \bar{\mu}) > \chi^+ + \mu$; by 5.9 there is $f' : \mathscr{Y} \to \text{ord}$, each $f'(x)$ a regular cardinal $\geqq \mu_i, f' \leqq f \mod I$ where $I \in \Gamma$ an ideal extending J such that $\chi^+ < \text{tcf} \prod\limits_{x \in \mathscr{Y}} f'(x)/I$. Let $\langle h_\zeta : \zeta < \chi' \rangle$ be $<_I$-increasing cofinal in $\prod\limits_{x \in \mathscr{Y}} f'(x)/I$. As in [Sh 355, 1.5] without loss of generality for some $\zeta(*) < \text{tcf} \prod\limits_{x \in \mathscr{Y}} f'(x)/I$ of cofinality χ^+ we have: $\langle h_\zeta : \xi < \zeta(*) \rangle$ has a $<_I$-lub h' such that: for 5.9(1) $\{x \in \mathscr{Y} : \text{cf}[f(x)] \leqq \mu\} \in I$ and for 5.9(2) $\{x \in \mathscr{Y} : \text{cf}[f(x)] < \mu_{\iota(x)}\} \in I$; without loss of generality it is $h_{\zeta(*)}$ and $\bigwedge\limits_{x \in \mathscr{Y}} \text{cf}[h_{\zeta(*)}(x)] \geqq \mu_{\iota(x)}$, and without loss of generality: $\xi < \zeta(*) \Rightarrow h_\xi < h_{\zeta(*)}$. For each $\bar{B} = \langle B_x : x \in \mathscr{Y} \rangle \in \mathscr{P}_{h_{\zeta(*)}}$ define a function $f_{\bar{B}} : f_{\bar{B}}(x) = \sup(h_{\zeta(*)}(x) \cap B_x)$. So $f_{\bar{B}} < h_{\zeta(*)}$ hence for some $\xi(\bar{B}) < \zeta(*)$ we have $f_{\bar{B}} < f_{\xi(\bar{B})} \mod I$. Let $\bigcup\limits_{\bar{B}} \xi(\bar{B}) < \xi < \zeta(*)$ — possible as the number of $\bar{B}$'s is $\leqq |\mathscr{P}_{h_{\zeta(*)}}| \leqq \chi < \chi^+ = \text{cf}(\zeta(*))$. So for $\bar{B} \in \mathscr{P}_{h_{\zeta(*)}}$ we have $\{x \in \mathscr{Y} : f_\xi(x) \in B_x\} \in I$. But $f_\xi < f_{\zeta(*)}$ so we get contradiction to the choice of $\mathscr{P}_{h_{\zeta(*)}}$. $\square_{5.11}$

5.12 Definition. Let $1 - \mathrm{cf}^{\Gamma}_{J}(\langle a_x : x \in \mathscr{Y}\rangle)$ be

$$\sup\{\mathrm{tcf}\prod_{x\in\mathscr{Y}}\lambda_x/I : J \subseteq I \in \Gamma \text{ and } \lambda_x \in a_x \text{ for } x \in \mathscr{Y}\}.$$

5.13 Claim. *$\bar{\mu} = \langle \mu_i : i < \kappa\rangle$ is non-decreasing, Γ is a suitable restriction close family of ideals on $\mathscr{Y}$, $J \in \Gamma$, $f \in {}^{\mathscr{Y}}$ ord and $\bigwedge\limits_{x\in\mathscr{Y}} f(x) \geqq \mu_{\iota(x)} = \mathrm{cf}\,\mu_{\iota(x)}$.*

1) If $\lambda \leqq \mathrm{prd}^{\Gamma}_{J}(f,\bar{\mu})$ is regular, then for some $\langle \mathfrak{a}_x : x \in \mathscr{Y}\rangle$ we have:

- *(i)* $\mathfrak{a}_x \subseteq \mathrm{Reg} \cap f(x)^+ \backslash \mu_{\iota(x)}$
- *(ii)* $|\bigcup\limits_{x\in\mathscr{Y}} \mathfrak{a}_x| < \mu$ *if* $(*)$ *of 5.9(1) and* $|\mathfrak{a}_x|^+ < \mu_{\iota(x)}$ *when 5.9(2)'s assumptions hold*
- *(iii)* $\lambda = 1 - \mathrm{cf}^{\Gamma}_{J}(\langle a_x : x \in \mathscr{Y}\rangle)$.

2) If λ is inaccessible, Γ, σ-complete and $[\chi < \mu \Rightarrow \mathrm{cov}(\chi,\chi_0,\kappa,\sigma) < \lambda]$ then without loss of generality $|\bigcup\limits_{x\in\mathscr{Y}} a_x| < \chi_0$.

Proof. 1) Like the proof of 1.1.
2) Straight.

5.14 Claim. *Assume the hypothesis of 5.9.*

If $g \in {}^{\mathscr{Y}}\mathrm{Ord}$, and each $g(s)$ is an ordinal $\geqq \mu_{\iota(x)}$ and $f(i) = \aleph_{g(i)}$ and let $\lambda = \mathrm{prd}^{\Gamma}_{J}(g,\bar{\mu}) + \bigcup\limits_{\alpha<\mu} |\alpha|^{+3} + |\mathscr{Y}|$ then $\mathrm{prd}^{\Gamma}_{J}(f,\bar{\mu}) \leqq \aleph_{\lambda^+}$.

Proof. Assume not, so $\mathrm{prd}^{\Gamma}_{J}(f,\bar{\mu}) \geqq \aleph_{\lambda^+ +1}$ hence by 5.9 there is $I \in \Gamma, J \subseteq I$ and $f^* \leqq f$ such that: each $f^*(x)$ is a regular cardinal $\geqq \mu_{\iota(x)}$ and $\aleph_{\lambda^+ +1} \leqq \mathrm{tcf}(\prod\limits_{x\in\mathscr{Y}} f^*(x)/I)$. By [Sh 355, 1.5] for each $\alpha < \lambda^+$ such that $\aleph_\alpha \geqq |\mathscr{Y}|$ (e.g. $\alpha \geqq \lambda$) there is $f_\alpha <_I f^*, f_\alpha \leqq f^*$, each $f_\alpha(x)$ a regular cardinal $> \mu_{\iota(x)}$ and $\mathrm{tcf}(\prod\limits_{x\in\mathscr{Y}} f^*(x)/I) = \aleph_{\alpha+1}$. Clearly $\alpha \neq \beta \Rightarrow f_\alpha \neq_I f_\beta$. Now let $f_\alpha = \aleph_{g_\alpha}$, so $g_\alpha <_I g, g_\alpha \leqq g$. Let $\mathscr{P}$ exemplify $\mathrm{prd}^{\Gamma}_{J}(g,\bar{\mu}) \leqq \lambda$, so for each $\alpha < \lambda^+$ for some $B^\alpha \in \mathscr{P}, \{y : f_\alpha(y) \in \{\aleph_j : j \in B^\alpha_y\}\} \notin I$ (and, by 5.8(9), and normality without loss of generality $\sup\{|B^\alpha_y| : y \in \mathscr{Y}\} < \sum\limits_{i<\kappa}\mu_i$). By 3.3 we get contradiction. $\square_{5.14}$

§6 The Existence of Strongly Almost Disjoint Families

See [Sh 355, §0] on the history of the subject.

6.1 Theorem. *Assume J is an ideal on κ, κ not the union of $\aleph_0$ members of $J, \mu > \kappa^{<\sigma}$ where*

$\otimes$ $\sigma = \kappa^+$ *or at least* $\forall A \in J^+ \exists B \in J^+[B \subseteq A \ \& \ |B| < \sigma]$

and $\sigma = cf(\sigma) > \aleph_0$.

Then

$$T^3_J(\mu) = T^2_J(\mu) \leqq T^1_J(\mu) \leqq T^4_J(\mu)$$

where

$$T^1_J(\mu) = T_J(\mu) = \sup\{|\mathscr{F}| : \mathscr{F} \text{ is a family of functions from } \kappa \text{ to } \mu \text{ such that for } f \neq g \text{ from } \mathscr{F} \text{ we have } f \neq_J g\}.$$

$$T^2_J(\mu) = \sup\big\{\lambda : \text{there are } n_i < \omega \text{ for } i < \kappa \text{ and regular } \lambda_{i,\ell} > \kappa^{<\sigma} \text{ for } i < \kappa, \ell < n_i \text{ such that : } \lambda_{i,\ell} \leqq \mu \text{ and } \lambda \leqq \max pcf\{\lambda_{i,\ell} : i < \kappa, \ell < n_i\};\ \text{moreover if } A \in J^+ (= \mathscr{P}(\kappa)\backslash J) \ \underline{\text{then}}\ \lambda \leqq \max pcf\{\lambda_{i,\ell} : i \in A, \ell < n_i\}\}.$$

$T^3_J(\mu)$ is defined similarly but for $A \in J^+$ we demand:

$$\lambda = \max \operatorname{pcf}\{\lambda_{i,\ell} : i \in A, \ell < n_i\}$$

$$T^4_J(\mu) = \operatorname{Min}\{\sup_s T^2_{J+(\kappa\backslash A_n)}(\lambda) : A_n \subseteq A_{n+1} \subseteq \kappa = \bigcup_{n<\omega} A_n, A_n \notin J\}.$$

6.2 Remark. 1) Note that usually the four terms in the conclusion of the theorem are equal.

- (α) If J is $\aleph_1$-complete then $T^2_J(\mu) = T^4_J(\mu)$ hence all are equal
- (β) all terms are equal if for $\langle A_n : n < \omega\rangle$ such that $A_n \subseteq \kappa, A_n \notin J, A_n \subseteq A_{n+1}, \kappa = \bigcup_n A_n$ we have: for some n and $B \subseteq A_n$ we have $(\kappa, J), (B, J \cap \mathscr{P}(B))$ are isomorphic.

2) The supremum in the definition of $T^1_J(\mu)$ is always obtained.
[Why? If $\mathscr{F}_1, \mathscr{F}_2$ are as there, $\mathscr{F}_1$ maximal $|\mathscr{F}_1| < |\mathscr{F}_2|$ then for every $f \in F_2$ there is $g_f \in \mathscr{F}$, etc.].

Proof. $\underline{T^2_J(\mu) = T^3_J(\mu)}$.

Trivially $T^3_J(\mu) \leqq T^2_J(\mu)$; for the other direction let λ appear in the sup defining $T^2_J(\mu)$, as exemplified by $\langle < \lambda_{i,\ell} : \ell < n_i > : i < \kappa \rangle$; as $\max \operatorname{pcf}\{\lambda_{i,\ell} : i \in A, \ell < n_i\}$ is always regular, without loss of generality λ is regular.

By [Sh 355, §1]; more elaborately, for some $\mathfrak{a} \subseteq \mathfrak{a}^* =: \{\lambda_{i,\ell} : \ell < n_i, i < \kappa\}$ we have $[\mathfrak{a} \neq \emptyset \Rightarrow \lambda = \max \operatorname{pcf}(\mathfrak{a})]$ and $\lambda \notin \operatorname{pcf}[\mathfrak{a}^* \backslash \mathfrak{a}]$ and $\mathfrak{a} \neq \mathfrak{a}^* \Rightarrow \lambda < \max \operatorname{pcf}(\mathfrak{a}^* \backslash \mathfrak{a})$. Define $\lambda'_{i,\ell}$ to be $\lambda_{i,\ell}$ if $\lambda_{i,\ell} \in a^*$. If $\mathfrak{a} \neq \mathfrak{a}^*$ let $u^* = \{(i,\ell) : \lambda_{i,\ell} \in \mathfrak{a}^* \backslash \mathfrak{a})\}$, $J = \{u \subseteq u^* : \max \operatorname{pcf}\{\lambda_{(i,\ell)} : (i,\ell) \in u\} < \lambda\}$. By [Sh 355, 1.5] we can find regular $\lambda'_{i,\ell} < \lambda_{i,\ell}$ for $(i,\ell) \in u^*$ such that $\lambda = \operatorname{tcf} \prod\limits_{(i,\ell) \in u^*} \lambda'_{i,\ell}/J$. Now $\langle \lambda'_{i,\ell} : \ell < n_i, i < \kappa \rangle$ exemplifies $\lambda \leqq T^3_J(\mu)$.

$\underline{T^3_J(\mu) \leq T^1_J(\mu)}$. Very easy; of course, instead $F \subseteq {}^\kappa\mu$ we can have $F \subseteq {}^\kappa Y$ as long as $|Y| \leqq \mu$. For $\lambda, \langle < \lambda_{i,\ell} : \ell < n_i > : i < \kappa \rangle$ as in the definition of $T^3_J(\mu)$, let $a = \{\lambda_{i,\ell} : i < \kappa, \ell < n_i\}$ and $\langle f_\alpha : \alpha < \lambda \rangle$ be a sequence of members of Πa which is $<_{J_{<\lambda}[a]}$-increasing and cofinal. Now let $Y = {}^{\omega >}(\mu + 1)$ and for each $\alpha < \lambda$ we define $g_\alpha \in {}^\kappa Y$ as follows:

$$g_\alpha(i) =: \langle \lambda_{i,\ell} : \ell < n_i \rangle^\frown \langle f_\alpha(\lambda_{i,\ell}) : \ell < n_i \rangle.$$

We leave the checking to the reader. We now turn to the main case.

$\underline{T^1_J(\mu) \leq T^4_J(\mu)}$. Let λ be the right side expression $-T^4_J(\mu)$ (so clearly $\lambda \geqq \mu$), $\chi =: \beth_3(\lambda)^+$ and for $\zeta \leqq \omega + 1$ let: $M^*_\zeta \prec (H(\chi), \in, <^*_\chi)$, $\|M^*_\zeta\| = \lambda, \lambda + 1 \subseteq M^*_\zeta, \zeta < \xi \leqq \omega + 1 \Rightarrow M^*_\zeta \in M^*_\xi$. Suppose $F \subseteq {}^\kappa\mu$ exemplify $T^1_J(\mu) > \lambda$ and we shall get a contradiction, without loss of generality $F \in M^*_0$. Clearly for every $f \in {}^\kappa\mu$ we have: $\{g \in F : \neg g \neq_J f\}$ has cardinality $\leqq \kappa^{<\sigma}$ (remember $\otimes$), hence necessarily there is $f^* \in F$ such that for every $g \in {}^\kappa\mu \cap M^*_{\omega+1}$ (e.g. $g \in F \cap M^*_{\omega+1}$) we have $f^* \neq_J g$. Moreover, if $A \subseteq \kappa, A \notin J, B \subseteq \mu, |B| \leqq \kappa^{<\sigma}$ then $\{f \in F : \{\alpha \in A : f(\alpha) \in B\} \notin J\}$ has cardinality $\leqq \kappa^{<\sigma}$ (again, remember $\otimes$), so if in addition $A, B \in M^*_\zeta$ then $[f \in F \ \& \ \{\alpha \in A : f(\alpha) \in B\} \notin J \Rightarrow f \in M^*_\zeta]$.

We define by induction on $k < \omega, N^a_k, N^b_k, a_k, f^k$ such that:

- (a) $N^a_k \prec M^*_0, N^b_k \prec (H(\chi), \in, <^*_\chi)$
- (b) N^b_0 is the Skolem Hull of $\{f^*\} \cup \{i : i \leqq \kappa^{<\sigma}\}$ in $(\mathscr{H}(\chi), \in, <^*_\chi)$
- (c) N^a_0 is the Skolem Hull of $\{i : i \leqq \kappa^{<\sigma}\}$ (in $(\mathscr{H}(\chi), \in, <^*_\chi)$, equivalently in M^*_0)
- (d) N^b_{n+1} is the Skolem Hull of $N^b_n \cup \{f^n(\theta) : \theta \in a_n\}$
- (e) N^a_{n+1} is the Skolem Hull of $N^a_n \cup \{f^n(\theta) : \theta \in a_n\}$
- (f) $\mathfrak{a}_n = N^a_n \cap \lambda^+ \cap \operatorname{Reg} \backslash (\kappa^{<\sigma})^+$
- (g) $f^n \in \Pi \mathfrak{a}_n$ and for each $\theta \in a_n, f^n(\theta) > \sup(\theta \cap N^b_n)$

(h) if $\mathfrak{b} \subseteq \mathfrak{a}_n$, $\max \operatorname{pcf}(\mathfrak{b}) \leqq \lambda$ and $|\mathfrak{b}| < \sigma$ then
$f^n \restriction \mathfrak{b} \in \{\operatorname{Max}\{f^{c_\ell}_{\lambda_\ell,\alpha_\ell} \restriction \mathfrak{b} : \ell < n\} : n < \omega, \alpha_\ell < \max \operatorname{pcf}(\mathfrak{b})$ and
$\lambda_\ell \in \operatorname{pcf}(\mathfrak{a}_n)$ and $\mathfrak{c}_\ell \in \{\mathfrak{b}_\theta[\mathfrak{a}_n] : \theta \in \operatorname{pcf}(\mathfrak{a}_n)\}\}$

where $\mathfrak{b} \mapsto \langle f^{\mathfrak{b}}_\alpha : \alpha < \max \operatorname{pcf}(\mathfrak{b})\rangle$ for $\mathfrak{b} \subseteq \mathfrak{a}_n$ is a definable function (in $(\mathscr{H}(\chi), \in, <^*_\chi), \langle f^b_\alpha : \alpha < \max \operatorname{pcf}(\mathfrak{b})\rangle$ as in [Sh 371, §1].

By 2.3 (i.e. [Sh 371, §1]) there is no problem to do it, $N^a_n \prec N^b_n, N^a_n \prec M^*_0, N^a_n \prec N^a_{n+1}, N^b_n \prec N^b_{n+1}$ and (as in [Sh 400, 3.3A,5.1A]) we get $\bigcup\limits_n N^a_n = \bigcup\limits_n N^b_n$, hence $\operatorname{Rang}(f^*) \subseteq \bigcup\limits_{n<\omega} N^a_n$. Now for each $i < \kappa$ let $m(i) = \min\{m : f^*(i) \in N^a_m\}$, and we can find finite $\mathfrak{e}(i) \subseteq \bigcup\limits_{\ell<m(i)} \mathfrak{a}_\ell, y(i) \subseteq \kappa^{<\sigma} + 1$ such that $f^*(i) \in M^{e(i),y(i)}_{m(i)}$ where for any $\mathfrak{e} \subseteq \bigcup\limits_n \mathfrak{a}_k$ and $y \subseteq \kappa^{<\sigma} + 1$ (we define by induction of ℓ):

$M^{e,y}_0$ is the Skolem Hull of y in $(\mathscr{H}(\chi), \in, <^*_\chi)$,
$M^{e,y}_{\ell+1}$ = Skolem Hull of $M^e_\ell \cup \{f^m(\theta) : m \leqq \ell$ and $\theta \in e \cap M^e_\ell\}$.

Clearly: $[\mathfrak{e} \subseteq \bigcup\limits_m \mathfrak{a}_m$ & $\mathfrak{e} \in M^*_\ell$ & $\max \operatorname{pcf}(\mathfrak{e}) \leqq \lambda$ & $\ell < \omega \Rightarrow M^{e,y}_\ell \in M^*_{\ell+1}]$, and $[\mathfrak{e} \subseteq \mathfrak{d}$ & $y \subseteq z \Rightarrow M^{\mathfrak{e},y}_\ell \subseteq M^{d,z}_\ell]$ and $M^{e,y}_\ell \subseteq N^a_\ell$.

Let $A_n = \{i < \kappa : m(i) \leqq n\}$. Clearly $A_n \subseteq A_{n+1}, \kappa = \bigcup\limits_{n<\omega} A_n$, but κ is not the union of $\aleph_0$ members of J, so for some $n(*) < \omega, [n \geqq n(*) \Rightarrow A_n \notin J]$. It suffices to prove:

$(*)$ if $m(*) < \omega, A \subseteq \kappa, \bigwedge\limits_{i\in A} m(i) \leqq m(*), A \notin J$ then $\max \operatorname{pcf}(\bigcup\limits_{i\in A} \mathfrak{e}(i) > \lambda$

[as this means $n(*) \leqq n < \omega \Rightarrow T^2_{J+(\kappa\setminus A_n)}(\mu) > \lambda$, hence $\langle A_{n(*)+\ell} : \ell < \omega\rangle$ and $\langle e(i) : i < \kappa\rangle$ exemplified $T^4_J(\mu) > \lambda$ contradiction].

We can replace A by any subset which is not in J.

By the assumption $\otimes$ without loss of generality $|A| < \sigma$, and suppose A contradicts $(*)$. Let $\mathfrak{e}^* = \bigcup\limits_{i\in A} \mathfrak{e}(i), y^* = \bigcup\limits_{i\in A} y(i)$. As σ is infinite, clearly $|\mathfrak{e}^*| < \sigma, |y^*| < \sigma, \bigwedge\limits_n \|M^{\mathfrak{e}^*,y^*}_n\| < \sigma$ (remember $\sigma > \aleph_0$).

Prove by induction on ℓ (suffice for $\ell \leqq m(*)$) that $\mathfrak{e}^* \cap M^{\mathfrak{e}^*,y^*}_n \in M^*_{\ell+1}, y^* \cap M^{\mathfrak{e}^*,y^*} \in M^*_{\ell+1}$ and $M^{\mathfrak{e}^*,y^*}_\ell \subseteq M^{\mathfrak{e}^*,y^*}_{\ell+1}$. For $\ell = 0$ this holds as $M^{\mathfrak{e}^*,y^*}_0 \prec N^a_0, \|M^{\mathfrak{e}^*,y^*}_0\| < \sigma$ and $\mathfrak{e}^* \cap M^{\mathfrak{e}^*}_0 = \mathfrak{e}^* \cap N^a_0$ is a subset of N^a_0 of cardinality $< \sigma, N^a_0 \in M^*_1, \|N^a_0\|^{<\sigma} = \|N^a_0\| \leqq \kappa^{<\sigma}$. Similarly for y^*. For $\ell + 1$, as we know $M^{\mathfrak{e}^*,y^*}_\ell \in M^*_{\ell+1}$ and $f^\ell \restriction (\mathfrak{e}^* \cap M^{\mathfrak{e}^*,y^*}_\ell) \in M^*_{\ell+1}$ by (h) as $\max \operatorname{pcf}(\mathfrak{e}^*) \leqq \lambda$ by an assumption hence $M^{\mathfrak{e}^*,y^*}_{\ell+1} \in M^*_{\ell+2}$. As $\|M^{\mathfrak{e}^*,y^*}_{\ell+1}\| = \kappa^{<\sigma}, |\mathfrak{e}^*| < \sigma$ and $\kappa^{<\sigma} + 1 \subseteq M^*_0$ necessarily $e^* \cap M^{e^*,y^*}_{\ell+1} \in M^*_{\ell+2}$. So $M^{\mathfrak{e}^*,y^*}_{m(*)} \in M^*_{m(*)+1}$ so $\operatorname{Rang}(f^* \restriction A) \subseteq M^{\mathfrak{e}^*,y^*}_{m(*)} \in M^*_{m(*)+1}$, so by the choice of $f^*, A \in I$.

Proof of 6.1A(1). $T^4_J(\mu) \leqq T^1_J(\mu)$. Let $A_n \subseteq \kappa, A_n \subseteq A_{n+1}, \kappa = \bigcup\limits_{n<\omega} A_n, A_n \notin J$ and $T^2_{J+(\kappa \setminus A_n)}(\mu) \geqq \lambda$. For each n, by earlier parts of the proof, there is $\mathscr{F}_n \subseteq {}^\kappa\mu$ such that $|\mathscr{F}_n| \geqq \lambda$ and $[f \neq g \in F_n \Rightarrow f \neq_{J+(\kappa \setminus A_n)} g]$.

Let $F_n = \{f^n_\alpha : \alpha < \alpha_n\}, \alpha_n \geqq \lambda$ exemplify this. Now define $f_\alpha \in {}^\kappa\mu$ for $\alpha < \lambda$ as follows: for $\zeta < \kappa$ let $n(\zeta) = \operatorname{Min}\{n : \zeta \in A_n\}$ and $f_\alpha(\zeta) = \omega f^{n(\zeta)}_\alpha(\zeta) + n(\zeta)$. $\square_{6.1}$

6.3 Conclusion. Suppose $\operatorname{cf}(\kappa) > \aleph_0, \kappa > \sigma \geqq \aleph_0$ and $I = [\kappa]^{<\sigma}, \mu > \kappa^\sigma$. Then $T^4_I(\mu)$ is $T^2_I(\mu)$ hence is $T^1_J(\mu)$.

Proof. Apply 6.1 (σ^+ here corresponds to σ there), more exactly by 6.1(A)(2).

6.4 Remark. Asking on almost disjoint sets is an inessential change.

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