The Ehrenfeucht-Fraïssé-game of length ω_1

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Abstract

Let $\mathcal{A}$ and $\mathcal{B}$ be two first order structures of the same vocabulary. We shall consider the *Ehrenfeucht-Fraïssé-game of length* ω_1 *of* $\mathcal{A}$ *and* $\mathcal{B}$ which we denote by $\mathcal{G}_{\omega_1}(\mathcal{A},\mathcal{B})$. This game is like the ordinary Ehrenfeucht-Fraïssé-game of $L_{\omega\omega}$ except that there are ω_1 moves. It is clear that $\mathcal{G}_{\omega_1}(\mathcal{A},\mathcal{B})$ is determined if $\mathcal{A}$ and $\mathcal{B}$ are of cardinality $\leq \aleph_1$. We prove the following results:

Theorem 1 *If V=L, then there are models $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\aleph_2$ such that the game $\mathcal{G}_{\omega_1}(\mathcal{A},\mathcal{B})$ is non-determined.*

Theorem 2 *If it is consistent that there is a measurable cardinal, then it is consistent that $\mathcal{G}_{\omega_1}(\mathcal{A},\mathcal{B})$ is determined for all $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\leq \aleph_2$.*

Theorem 3 *For any $\kappa \geq \aleph_3$ there are $\mathcal{A}$ and $\mathcal{B}$ of cardinality κ such that the game $\mathcal{G}_{\omega_1}(\mathcal{A},\mathcal{B})$ is non-determined.*

1 Introduction.

Let $\mathcal{A}$ and $\mathcal{B}$ be two first order structures of the same vocabulary L. We denote the domains of $\mathcal{A}$ and $\mathcal{B}$ by A and B respectively. All vocabularies are assumed to be relational.

The *Ehrenfeucht-Fraïssé-game of length* γ *of* $\mathcal{A}$ *and* $\mathcal{B}$ denoted by $\mathcal{G}_\gamma(\mathcal{A},\mathcal{B})$ is defined as follows: There are two players called $\forall$ and $\exists$. First $\forall$ plays x_0 and then $\exists$ plays y_0. After this $\forall$ plays x_1, and $\exists$ plays y_1, and so on. If $\langle (x_\beta, y_\beta) : \beta < \alpha \rangle$ has been played and $\alpha < \gamma$, then $\forall$ plays x_α after which $\exists$ plays y_α. Eventually a sequence $\langle (x_\beta, y_\beta) : \beta < \gamma \rangle$ has been played. The rules of the game say that both players have to play elements of $A \cup B$. Moreover, if $\forall$ plays his x_β in A (B), then $\exists$ has to play his y_β in B (A). Thus the sequence $\langle (x_\beta, y_\beta) : \beta < \gamma \rangle$ determines a relation $\pi \subseteq A \times B$. Player $\exists$ wins this round of the game if π is a partial isomorphism. Otherwise $\forall$ wins. The notion of winning strategy is defined in the usual manner. We say that a player *wins* $\mathcal{G}_\gamma(\mathcal{A},\mathcal{B})$ if he has a winning strategy in $\mathcal{G}_\gamma(\mathcal{A},\mathcal{B})$.

Recall that

$$\begin{aligned} \mathcal{A} \equiv_{\omega\omega} \mathcal{B} &\iff \forall n < \omega (\exists \text{ wins } \mathcal{G}_n(\mathcal{A},\mathcal{B})) \\ \mathcal{A} \equiv_{\infty\omega} \mathcal{B} &\iff \exists \text{ wins } \mathcal{G}_\omega(\mathcal{A},\mathcal{B}). \end{aligned}$$

In particular, $\mathcal{G}_\gamma(\mathcal{A},\mathcal{B})$ is determined for $\gamma \leq \omega$. The question, whether $\mathcal{G}_\gamma(\mathcal{A},\mathcal{B})$ is determined for $\gamma > \omega$, is the subject of this paper. We shall concentrate on the case $\gamma = \omega_1$.

The notion

$$\exists \text{ wins } \mathcal{G}_\gamma(\mathcal{A},\mathcal{B}) \tag{1}$$

can be viewed as a natural generalization of $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$. The latter implies isomorphism for countable models. Likewise (1) implies isomorphism for models of cardinality $|\gamma|$:

Proposition 1 *Suppose $\mathcal{A}$ and $\mathcal{B}$ have cardinality $\leq \kappa$. Then $\mathcal{G}_\kappa(\mathcal{A},\mathcal{B})$ is determined: $\exists$ wins if $\mathcal{A} \cong \mathcal{B}$, and $\forall$ wins if $\mathcal{A} \ncong \mathcal{B}$.*

Proof. If $f : \mathcal{A} \cong \mathcal{B}$, then the winning strategy of $\exists$ in $\mathcal{G}_\kappa(\mathcal{A},\mathcal{B})$ is to play in such a way that the resulting π satisfies $\pi \subseteq f$. On the other hand, if $\mathcal{A} \ncong \mathcal{B}$, then the winning strategy of $\forall$ is to systematically enumerate $A \cup B$ so that the final π will satisfy $A = \text{dom}(\pi)$ and $B = rng(\pi)$. $\square$

For models of arbitrary cardinality we have the following simple but useful criterion of (1), namely in the terminology of [15] that they are "potentially isomorphic". We use $Col(\lambda,\kappa)$ to denote the notion of forcing which collapses $|\lambda|$ to κ (with conditions of cardinality less than κ).

Proposition 2 *Suppose $\mathcal{A}$ and $\mathcal{B}$ have cardinality $\leq \lambda$ and κ is regular. Player $\exists$ wins $\mathcal{G}_\kappa(\mathcal{A},\mathcal{B})$ if and only if $\Vdash_{Col(\lambda,\kappa)} \mathcal{A} \cong \mathcal{B}$.*

Proof. Suppose τ is a winning strategy of $\exists$ in $\mathcal{G}_\kappa(\mathcal{A},\mathcal{B})$. Since $Col(\lambda,\kappa)$ is $<\kappa$–closed,

$$\Vdash_{Col(\lambda,\kappa)} \text{“}\tau \text{ is a winning strategy of } \exists \text{ in } G_\kappa(\mathcal{A},\mathcal{B})\text{”}.$$

Hence $\Vdash_{Col(\lambda,\kappa)} \mathcal{A} \cong \mathcal{B}$ by Proposition 1. Suppose then $p \Vdash \tilde{f} : \mathcal{A} \cong \mathcal{B}$ for some $p \in Col(\lambda,\kappa)$. While the game $\mathcal{G}_\kappa(\mathcal{A},\mathcal{B})$ is played, $\exists$ keeps extending the condition p further and further. Suppose he has extended p to q and $\forall$ has played $x \in A$. Then $\exists$ finds $r \leq q$ and $y \in B$ with $r \Vdash \tilde{f}(x) = y$. Using this simple strategy $\exists$ wins. $\square$

Proposition 3 *Suppose T is an ω-stable first order theory with NDOP. Then $\mathcal{G}_{\omega_1}(\mathcal{A},\mathcal{B})$ is determined for all models $\mathcal{A}$ of T and all models $\mathcal{B}$.*

Proof. Suppose $\mathcal{A}$ is a model of T. If $\mathcal{B}$ is not $L_{\infty\omega_1}$-equivalent to $\mathcal{A}$, then $\forall$ wins $\mathcal{G}_{\omega_1}(\mathcal{A},\mathcal{B})$ easily. So let us suppose $\mathcal{A} \equiv_{\infty\omega_1} \mathcal{B}$. We may assume A and B are of cardinality $\geq \aleph_1$. If we collapse $|A|$ and $|B|$ to $\aleph_1$, T will remain ω-stable with NDOP, and $\mathcal{A}$ and $\mathcal{B}$ will remain $L_{\infty\omega_1}$-equivalent. So $\mathcal{A}$ and

$\mathcal{B}$ become isomorphic by [18, Chapter XIII, Section 1]. Now Proposition 2 implies that $\exists$ wins $\mathcal{G}_{\omega_1}(\mathcal{A},\mathcal{B})$. $\square$

Hyttinen [10] showed that $\mathcal{G}_{\gamma}(\mathcal{A},\mathcal{B})$ may be non-determined for all γ with $\omega < \gamma < \omega_1$ and asked whether $\mathcal{G}_{\omega_1}(\mathcal{A},\mathcal{B})$ may be non-determined. Our results show that $\mathcal{G}_{\omega_1}(\mathcal{A},\mathcal{B})$ may be non-determined for $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\aleph_3$ (Theorem 17), but for models of cardinality $\aleph_2$ the answer is more complicated.

Let $F(\omega_1)$ be the free group of cardinality $\aleph_1$. Using the combinatorial principle $\square_{\omega_1}$ we construct an abelian group G of cardinality $\aleph_2$ such that $\mathcal{G}_{\omega_1}(F(\omega_1),G)$ is non-determined (Theorem 4). On the other hand, we show that starting with a model with a measurable cardinal one can build a forcing extension in which $\mathcal{G}_{\omega_1}(\mathcal{A},\mathcal{B})$ is determined for all models $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\leq \aleph_2$ (Theorem 14).

Thus the free abelian group $F(\omega_1)$ has the remarkable property that the question

> Is $\mathcal{G}_{\omega_1}(F(\omega_1),G)$ determined for all G?

cannot be answered in ZFC alone. Proposition 3 shows that no model of an $\aleph_1$-categorical first order theory can have this property.

We follow Jech [11] in set theoretic notation. We use S^m_n to denote the set $\{\alpha < \omega_m : \mathrm{cf}(\alpha) = \omega_n\}$. Closed and unbounded sets are called cub sets. A set of ordinals is λ-*closed* if it is closed under supremums of ascending λ-sequences $\langle \alpha_i : i < \lambda\rangle$ of its elements. A subset of a cardinal is λ-*stationary* if it meets every λ-closed unbounded subset of the cardinal. The closure of a set A of ordinals in the order topology of ordinals is denoted by $\overline{A}$. The free abelian group of cardinality κ is denoted by $F(\kappa)$.

2 A non-determined $\mathcal{G}_{\omega_1}(F(\omega_1),G)$ with G a group of cardinality $\aleph_2$.

In this section we use $\square_{\omega_1}$ to construct a group G of cardinality $\aleph_2$ such that the game $\mathcal{G}_{\omega_1}(F(\omega_1),G)$ is non-determined (Theorem 4). For background on almost free groups the reader is referred to [4]. However, our presentation does not depend on special knowledge of almost free groups. All groups below are assumed to be abelian.

By $\Box_{\omega_1}$ we mean the principle, which says that there is a sequence $\langle C_\alpha : \alpha < \omega_2, \alpha = \cup\alpha \rangle$ such that

1. C_α is a cub subset of α.
2. If $\mathrm{cf}(\alpha) = \omega$, then $|C_\alpha| = \omega$.
3. If γ is a limit point of C_α, then $C_\gamma = C_\alpha \cap \gamma$.

Recall that $\Box_{\omega_1}$ follows from $V = L$ by a result of R. Jensen. For a sequence of sets C_α as above we can let $E_\beta = \{\alpha \in S^2_0 :$ the order type of C_α is $\beta\}$. For some $\beta < \omega_1$ the set E_β has to be stationary. Let us use E to denote this E_β. Then E is a so called *non-reflecting* stationary set, i.e., if $\gamma = \cup\gamma$ then $E \cap \gamma$ is non-stationary on γ. Indeed, then some final segment D_γ of the set of limit points of C_γ is a cub subset of γ disjoint from E. Moreover, $\mathrm{cf}(\alpha) = \omega$ for all $\alpha \in E$.

Theorem 4 *Assuming $\Box_{\omega_1}$, there is a group G of cardinality $\aleph_2$ such that the game $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ is non-determined.*

Proof. Let $\mathbb{Z}^{\omega_2}$ denote the direct product of ω_2 copies of the additive group $\mathbb{Z}$ of the integers. Let x_α be the element of $\mathbb{Z}^{\omega_2}$ which is 0 on coordinates $\neq \alpha$ and 1 on the coordinate α. Let us fix for each $\delta \in S^2_0$ an ascending cofinal sequence $\eta_\delta : \omega \to \delta$. For such δ, let

$$z_\delta = \sum_{n=0}^{\infty} 2^n x_{\eta_\delta(n)}.$$

Let $\langle C_\alpha : \alpha = \cup\alpha < \omega_2 \rangle$, $\langle D_\alpha : \alpha = \cup\alpha < \omega_2 \rangle$ and $E = E_\beta$ be obtained from $\Box_{\omega_1}$ as above. We are ready to define the groups we need for the proof: Let G be the smallest pure subgroup of $\mathbb{Z}^{\omega_2}$ which contains x_α for $\alpha < \omega_2$ and z_δ for $\delta \in E$, let G_α be the smallest pure subgroup of $\mathbb{Z}^{\omega_2}$ which contains x_γ for $\gamma < \alpha$ and z_δ for $\delta \in E \cap \alpha$, let F $(= F(\omega_2))$ be the subgroup of $\mathbb{Z}^{\omega_2}$ generated freely by x_α for $\alpha < \omega_2$, and finally, let F_α be the subgroup of $\mathbb{Z}^{\omega_2}$ generated freely by x_γ for $\gamma < \alpha$.

The properties we shall want of G_α are standard but for the sake of completeness we shall sketch proofs. We need that each G_α is free and for any $\beta \notin E$ any free basis of G_β can be extended to a free basis of G_α for all $\alpha > \beta$.

The proof is by induction on α. For limit ordinals we use the fact that E is non-reflecting. The case of successors of ordinals not in E is also easy. Assume now that $\delta \in E$ and the induction hypothesis has been verified up to δ. By the induction hypothesis for any $\beta < \delta$ such that $\beta \notin E$, there is n_0 so that

$$G_\delta = G_\beta \oplus H \oplus K$$

where K is the group freely generated by $\{x_{\eta_\delta(n)} \colon n_0 \leq n\}$ and $x_{\eta_\delta(m)} \in G_\beta$ for all $m < n_0$. Then

$$G_{\delta+1} = G_\beta \oplus H \oplus K'$$

where K' is freely generated by

$$\{\sum_{m=n}^{\infty} 2^{m-n} x_{\eta_\delta(m)} \colon n_0 \leq n\}.$$

On the other hand, if $\delta \in E$ and $\{x_{\eta_\delta(n)} : n < \omega\} \subseteq B$, where B is a subgroup of G such that $z_\delta \notin B$, then G/B is non-free, as $z_\delta + B$ is infinitely divisible by 2 in G/B.

Claim 1 $\exists$ *does not win* $\mathcal{G}_{\omega_1}(F, G)$.

Suppose τ is a winning strategy of $\exists$. Let $\alpha \in E$ such that the pair (G_α, F_α) is closed under the first ω moves of τ, that is, if $\forall$ plays his first ω moves inside $G_\alpha \cup F_\alpha$, then τ orders $\exists$ to do the same. We shall play $G_{\omega_1}(F, A)$ pointing out the moves of $\forall$ and letting τ determine the moves of $\exists$. On his move number $2n$ $\forall$ plays the element $x_{\eta_\alpha(n)}$ of G_α. On his move number $2n+1$ $\forall$ plays some element of F_α. Player $\forall$ plays his moves in F_α in such a way that during the first ω moves eventually some countable direct summand K of F_α as well as some countable $B \subseteq G_\alpha$ are enumerated. Let J be the smallest pure subgroup of G containing $B \cup \{z_\alpha\}$. During the next ω moves of $G_{\omega_1}(F, A)$ player $\forall$ enumerates J and $\exists$ responds by enumerating some $H \subseteq F$. Since τ is a winning strategy, H has to be a subgroup of F. But now H/K is free, whereas J/B is non-free, so $\forall$ will win the game, a contradiction.

Claim 2 $\forall$ *does not win* $\mathcal{G}_{\omega_1}(F, G)$.

Suppose τ is a winning strategy of $\forall$. If we were willing to use CH, we could just take α of cofinality ω_1 such that (F_α, G_α) is closed under τ, and derive a contradiction from the fact that $F_\alpha \cong G_\alpha$. However, since we do not want to assume CH, we have to appeal to a longer argument.

Let $\kappa = (2^\omega)^{++}$. Let $\mathcal{M}$ be the expansion of $\langle H(\kappa), \in\rangle$ obtained by adding the following structure to it:

(H1) The function $\delta \mapsto \eta_\delta$.

(H2) The function $\delta \mapsto z_\delta$.

(H3) The function $\alpha \mapsto C_\alpha$.

(H4) A well-ordering $<$ of the universe.

(H5) The winning strategy τ.

Let $\mathcal{N} = \langle N, \in, \ldots\rangle$ be an elementary submodel of $\mathcal{M}$ such that $\omega_1 \subseteq N$ and $N \cap \omega_2$ is an ordinal α of cofinality ω_1.

Let $D_\alpha = \{\beta_i : i < \omega_1\}$ in ascending order. Since $C_{\beta_i} = C_\alpha \cap \beta_i$, every initial segment of C_α is in N. By elementaricity, $G_{\beta_i} \in N$ for all $i < \omega_1$. Let ϕ be an isomorphism $G_\alpha \to F_\alpha$ obtained as follows: ϕ restricted to G_{β_0} is the $<$-least isomorphisms between the free groups G_{β_0} and F_0. If ϕ is defined on all G_{β_j}, $j < i$, then ϕ is defined on G_{β_i} as the $<$-least extension of $\bigcup_{j<i} \phi_{\beta_j}$ to an isomorphism between G_{β_i} and F_i. Recall that by our choice of D_α $G_{\beta_{i+1}}/G_{\beta_i}$ is free, so such extensions really exist.

We derive a contradiction by showing that $\exists$ can play ϕ against τ for the whole duration of the game $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$. To achieve this we have to show that, when $\exists$ plays his canonical strategy based on ϕ the strategy τ of $\forall$ directs $\forall$ to go on playing elements which are in N, that is, elements of $G_\alpha \cup F_\alpha$.

Suppose a sequence $s = \langle (x_\gamma, y_\gamma) : \gamma < \mu\rangle, \mu < \omega_1$, has been played. It suffices to show that $s \in N$. Choose β_i so that the elements of s are in $G_{\beta_i} \cup F_{\beta_i}$. Now s is uniquely determined by $\phi \restriction G_{\beta_i}$ and τ. Note that because $C_{\beta_i} = C_\alpha \cap \beta_i$, $\phi \restriction G_{\beta_i}$ can be defined inside N similarly as ϕ was defined above, using C_{β_i} instead of C_α. Thus $s \in N$ and we are done.

We have proved that $\mathcal{G}_{\omega_1}(F, G)$ is nondetermined. This clearly implies $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ is nondetermined. $\Box$

Remark. R. Jensen [14, p. 286] showed that if $\Box_{\omega_1}$ fails, then ω_2 is Mahlo in L. Therefore, if $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$ is determined for all almost free groups

$\mathcal{A}$ and $\mathcal{B}$ of cardinality $\aleph_2$, then ω_2 is Mahlo in L. If we start with $\Box_\kappa$, we get an almost free group A of cardinality κ^+ such that $\mathcal{G}_{\omega_1}(F(\omega_1), A)$ is nondetermined.

3 $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ can be determined for all G.

In this section all groups are assumed to be abelian. It is easy to see that $\exists$ wins $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ for any uncountable free group G, so in this exposition $F(\omega_1)$ is a suitable representative of all free groups. In the study of determinacy of $\mathcal{G}_{\omega_1}(F(\omega_1), \mathcal{A})$ for various $\mathcal{A}$ it suffices to study groups $\mathcal{A}$, since for other $\mathcal{A}$ player $\forall$ easily wins the game.

Starting from a model with a Mahlo cardinal we construct a forcing extension in which $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ is determined, when G is any group of cardinality $\aleph_2$. This can be extended to groups G of any cardinality, if we start with a supercompact cardinal.

In the proof of the next results we shall make use of *stationary logic* $L(aa)$. For the definition and basic facts about $L(aa)$ the reader is referred to [1]. This logic has a new quantifier $aa\, s$ quantifying over variables s ranging over countable subsets of the universe. A cub set of such s is any set which contains a superset of any countable subset of the universe and which is closed under unions of countable chains. The semantics of aa s is defined as follows:

$$aa\, s\phi(s, \ldots) \iff \phi(s, \ldots) \text{ holds for a cub set of } s.$$

Note that a group of cardinality $\aleph_1$ is free if and only if it satisfies

$$aa\, s\ aa\, s'(s \subseteq s' \rightarrow s'/s \text{ is free}). \tag{2}$$

Proposition 5 *Let G be a group. Then the following conditions are equivalent:*

(1) *$\exists$ wins $\mathcal{G}_{\omega_1}(F(\omega_1), G)$.*

(2) *G satisfies (2).*

(3) *G is the union of a continuous chain $\langle G_\alpha : \alpha < \omega_2\rangle$ of free subgroups with $G_{\alpha+1}/G_\alpha$ $\aleph_1$-free for all $\alpha < \omega_2$.*

Proof. (1) implies (2): Suppose $\exists$ wins $\mathcal{G}_{\omega_1}(F(\omega_1), G)$. By Proposition 2 we have $\Vdash_{Col(|G|,\omega_1)}$ "G is free." Using the countable completeness of $Col(|G|, \omega_1)$ it is now easy to construct a cub set S of countable subgroups of G such that if $A \in S$ then for all $B \in S$ with $A \subseteq B$ we have B/A free. Thus G satisfies (2). (2) implies (3) quite trivially. (3) implies (1): Suppose a continuous chain as in (3) exists. If we collapse $|G|$ to $\aleph_1$, then in the extension the chain has length $< \omega_2$. Now we use Theorem 1 of [8]:

> If a group A is the union of a continuous chain of $< \omega_2$ free subgroups $\{A_\alpha : \alpha < \gamma\}$ of cardinality $\leq \aleph_1$ such that each $A_{\alpha+1}/A_\alpha$ is $\aleph_1$-free, then A is free.

Thus G is free in the extension and (1) follows from Proposition 2. $\Box$

Let us consider the following principle:

(∗) For all stationary $E \subseteq S^2_0$ and countable subsets a_α of $\alpha \in E$ such that a_α is cofinal in α and of order type ω there is a closed $C \subseteq \omega_2$ of order type ω_1 such that $\{\alpha \in E : a_\alpha \setminus C \text{ is finite}\}$ is stationary in C.

Lemma 6 *The principle (∗) implies that $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ is determined for all groups G of cardinality $\aleph_2$.*

Proof. Suppose G is a group of cardinality $\aleph_2$. We may assume the domain of G is ω_2. Let us assume G is $\aleph_2$-free, as otherwise $\forall$ easily wins. If we prove that G satisfies (2), then Proposition 5 implies that $\exists$ wins $\mathcal{G}_{\omega_1}(F(\omega_1), G)$.

To prove (2), assume the contrary. By Proposition 5 we may assume that G can be expressed as the union of a continuous chain $\langle G_\alpha : \alpha < \omega_2\rangle$ of free groups with $G_{\alpha+1}/G_\alpha$ non-$\aleph_1$-free for $\alpha \in E$, $E \subseteq \omega_2$ stationary. By Fodor's Lemma, we may assume $E \subseteq S^2_0$. Also we may assume that for all α, every ordinal in $G_{\alpha+1} \setminus G_\alpha$ is greater than every ordinal in G_α. Finally by intersecting with a closed unbounded set we may assume that for all $\alpha \in E$ the set underlying G_α is α. Choose for each $\alpha \in E$ some countable subgroup b_α of $G_{\alpha+1}$ with $b_\alpha + G_\alpha/G_\alpha$ non-free. Let $c_\alpha = b_\alpha \cap G_\alpha$. We will choose a_α so that any final segment generates a subgroup containing c_α. Enumerate c_α as $\{g_n : n < \omega\}$ such that each element is enumerated infintely often. Choose an increasing sequence $(\alpha_n : n < \omega)$ cofinal in α so that for all n, $g_n \in G_{\alpha_n}$. Finally, for each n, choose $h_n \in G_{\alpha_n+1} \setminus G_{\alpha_n}$. Let $a_\alpha = \{h_n : n < \omega\} \cup \{h_n + g_n : n < \omega\}$. It is now easy to check that a_α is a sequence of order type ω which is cofinal in α and any subgroup of G which contains all but finitely many of the elements of a_α contains c_α.

By $(*)$ there is a continuous C of order type ω_1 such that $\{\alpha \in C : a_\alpha \setminus C \text{ is finite}\}$ is stationary in C. Let $D = \langle C \bigcup \sum_{\alpha \in C} b_\alpha \rangle$. Since $|D| \leq \aleph_1$, D is free.

For any $\alpha \in C$, let

$$D_\alpha = \langle (C \cap \alpha) \bigcup (\sum_{\beta \in (C \cap \alpha)} b_\beta) \rangle .$$

Note that $D = \bigcup_{\alpha \in C} D_\alpha$, each D_α is countable and for limit point δ of C, $D_\delta = \bigcup_{\alpha \in (C \cap \delta)} D_\alpha$. Hence there is an $\alpha \in C \cap E$ such that $a_\alpha \setminus C$ is finite and D/D_α is free. Hence $b_\alpha + D_\alpha / D_\alpha$ is free. But

$$b_\alpha + D_\alpha / D_\alpha \cong b_\alpha / b_\alpha \cap D_\alpha = b_\alpha / b_\alpha \cap G_\alpha ,$$

which is not free, a contradiction. □

For the next theorem we need a lemma from [6]. A proof is included for the convenience of the reader.

Lemma 7 *[6] Suppose λ is a regular cardinal and $\mathbb{Q}$ is a notion of forcing which satisfies the λ-c.c. Suppose $\mathcal{I}$ is a normal λ-complete ideal on λ and $\mathcal{I}^+ = \{S \subseteq \lambda : S \notin \mathcal{I}\}$. For all sets $S \in \mathcal{I}^+$ and sequences of conditions $\langle p_\alpha : \alpha \in S \rangle$, there is a set C with $\lambda \setminus C \in \mathcal{I}$ so that for all $\alpha \in C \cap S$,*

$$p_\alpha \Vdash_{\mathbb{Q}} \text{“}\{\beta : p_\beta \in \tilde{G}\} \in \mathcal{J}^+, \text{ where } \mathcal{J} \text{ is the ideal generated by } \mathcal{I}\text{”}.$$

Proof. Suppose the lemma is false. So there is an $\mathcal{I}$-positive set $S' \subseteq S$ such that for all $\alpha \in S'$ there is an extension r_α of p_α and a set $I_\alpha \in \mathcal{I}$ (note: I_α is in the ground model) so that

$$r_\alpha \Vdash \{\beta : p_\beta \in \tilde{G}\} \subseteq I_\alpha .$$

Let I be the diagonal union of $\{I_\alpha : \alpha \in S'\}$.

Suppose now that $\alpha < \beta$ and $\alpha, \beta \in (S' \setminus I)$. Since $\beta \notin I$, $r_\alpha \Vdash p_\beta \notin \tilde{G}$. Hence $r_\alpha \Vdash r_\beta \notin \tilde{G}$. So r_α, r_β are incompatible. Hence $\{r_\alpha : \alpha \in S' \setminus I\}$ is an antichain which, since S' is $\mathcal{I}$-positive, is of cardinality λ. This is a contradiction. □

Theorem 8 *Assuming the consistency of a Mahlo cardinal, it is consistent that* $(*)$ *holds and hence that* $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ *is determined for all groups* G *of cardinality* $\aleph_2$.

Proof. By a result of Harrington and Shelah [7] we may start with a Mahlo cardinal κ in which every stationary set of cofinality ω reflects, that is, if $S \subseteq \kappa$ is stationary, and $\mathrm{cf}(\alpha) = \omega$ for $\alpha \in S$, then $S \cap \lambda$ is stationary in λ for some inaccessible $\lambda < \kappa$.

For any inaccessible λ let $\mathbb{P}_\lambda$ be the Levy-forcing for collapsing λ to ω_2. The conditions of $\mathbb{P}_\lambda$ are countable functions $f : \lambda \times \omega_1 \to \lambda$ such that $f(\alpha, \beta) < \alpha$ for all α and β and each f is increasing and continuous in the second coordinate. It is well-known that $\mathbb{P}_\lambda$ is countably closed and satisfies the λ-chain condition [11, p. 191].

Let $\mathbb{P} = \mathbb{P}_\kappa$. Suppose $p \in \mathbb{P}$ and

$$p \Vdash \text{“}\tilde{E} \subseteq S^2_0 \text{ is stationary and } \forall\alpha \in \tilde{E}(\tilde{a}_\alpha \subseteq \alpha \text{ is cofinal in } \alpha \text{ and of order type } \omega)\text{”}.$$

Let

$$S = \{\alpha < \kappa : \exists q \leq p(q \Vdash \alpha \in \tilde{E})\}.$$

For any $\alpha \in S$ let $p_\alpha \leq p$ such that $p_\alpha \Vdash \alpha \in \tilde{E}$. Since $\mathbb{P}$ is countably closed, we can additionally require that for some countable $a_\alpha \subseteq \alpha$ we have $p_\alpha \Vdash \tilde{a}_\alpha = a_\alpha$.

The set S is stationary in κ, for if $C \subseteq \kappa$ is cub, then $p \Vdash C \cap \tilde{E} \neq \emptyset$, whence $C \cap S \neq \emptyset$. Also $\mathrm{cf}(\alpha) = \omega$ for $\alpha \in S$. Let λ be inaccessible such that $S \cap \lambda$ is stationary in λ. We may choose λ in such a way that $\alpha \in S \cap \lambda$ implies $p_\alpha \in \mathbb{P}_\lambda$. By Lemma 7 there is a $\delta \in S \cap \lambda$ such that

$$p_\delta \Vdash_{\mathbb{P}_\lambda} \text{“}\tilde{E}_1 = \{\alpha < \lambda : p_\alpha \in \tilde{G}\} \text{ is stationary.”}$$

Let $\mathbb{Q}$ be the set of conditions $f \in \mathbb{P}$ with $\mathrm{dom}(f) \subseteq (\kappa \setminus \lambda) \times \omega_1$. Note that $\mathbb{P} \cong \mathbb{P}_\lambda \otimes \mathbb{Q}$. Let G be $\mathbb{P}$-generic containing p_δ and $G_\lambda = G \cap \mathbb{P}_\lambda$ for any inaccessible $\lambda \leq \kappa$. Then G_λ is $\mathbb{P}_\lambda$-generic and ω_2 of $V[G_\lambda]$ is λ. Let us work now in $V[G_\lambda]$. Thus λ is the current ω_2, $E_1 = \{\alpha < \lambda : p_\alpha \in G_\lambda\}$ is stationary, and we have the countable sets $a_\alpha \subseteq \alpha$ for $\alpha \in E_1$. Since $\mathbb{Q}$ collapses λ there is a name $\tilde{f}$ such that

$$\Vdash_{\mathbb{Q}} \text{“}\tilde{f} : \omega_1 \to \lambda \text{ is continuous and cofinal.”}$$

More precisely $\tilde{f}$ is the name for the function f defined by $f(\alpha) = \beta$ if and only if there is some $g \in G$ so that $g(\lambda, \alpha) = \beta$. Let $\tilde{C}$ denote the range of $\tilde{f}$. We shall prove the following statement:

Claim: $\Vdash_{\mathbb{Q}} \{\alpha < \lambda : a_\alpha \setminus \tilde{C}$ is finite $\}$ is stationary in $\tilde{C}$.

Suppose $q \in \mathbb{Q}$ so that $q \Vdash$ "$\tilde{D} \subseteq \omega_1$ is a cub." Let $\mathcal{M}$ be an appropriate expansion of $\langle H(\kappa), \in\rangle$ and $\langle \mathcal{N}_i : i < \lambda\rangle$, $\mathcal{N}_i = \langle N_i, \in, \ldots\rangle$, a sequence of elementary submodels of $\mathcal{M}$ such that:

(i) Everything relevant is in N_0.

(ii) If $\alpha_i = N_i \cap \lambda$, then $\alpha_i < \alpha_j$ for $i < j < \lambda$.

(iii) N_{i+1} is closed under countable sequences.

(iv) $|N_i| = \omega_1$.

(v) $N_i = \bigcup_{j<i} N_j$ for i a limit ordinal.

Choose $\gamma = \alpha_i \in E_1$ and let $\langle i_n : n < \omega\rangle$ be a sequence of successor ordinals such that $\gamma = \sup\{\alpha_{i_n} : n < \omega\}$. Let $q_0 \leq q$ and $\beta_0 \in \omega_1$ such that $q_0, \beta_0 \in N_{i_0}$,

$$q_0 \Vdash \text{“}\beta_0 \in \tilde{D}\text{”}$$

and q_0 decides the value of $\tilde{f}''\beta_0$ (which will by elementaricity necessarilly be a subset of α_{i_0}).

If q_n and β_n are defined we choose $q_{n+1} \leq q_n$ and $\beta_{n+1} \in \omega_1$ such that $q_{n+1}, \beta_{n+1} \in N_{i_{n+1}}$,

$$q_{n+1} \Vdash \text{“}\beta_{n+1} \in \tilde{D} \text{ and } a_\gamma \cap (\alpha_{i_{n+1}} \setminus \alpha_{i_n}) \subseteq \tilde{f}\text{''}\beta_{n+1} \subseteq \alpha_{i_{n+1}}\text{”}$$

and q_{n+1} decides $\tilde{f}$''β_{n+1}. Finally, let $q_\omega = \bigcup\{q_n : n < \omega\}$ and $\beta = \bigcup\{\beta_n : n < \omega\}$. Then

$$q_\omega \Vdash \text{“}\beta \in \tilde{D} \text{ and } a_\gamma \setminus \tilde{f}\text{''}\beta \text{ is finite.”}$$

The claim, and thereby the theorem, is proved. □

Corollary 9 *The statement that $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$ is determined for every structure $\mathcal{A}$ of cardinality $\aleph_2$ and every uncountable free group $\mathcal{B}$, is equiconsistent with the existence of a Mahlo cardinal.*

Remark. If $\mathcal{G}_{\omega_1}(A, F(\omega_1))$ is determined for all groups A of cardinality κ^+, κ singular, then $\Box_\kappa$ fails. This implies that the Covering Lemma fails for the Core Model, whence there is an inner model for a measurable cardinal. This shows that the conclusion of Theorem 8 cannot be strengthened to arbitrary G. However, by starting with a larger cardinal we can make this extension:

Theorem 10 *Assuming the consistency of a supercompact cardinal, it is consistent that $\mathcal{G}_{\omega_1}(F(\omega_1), G)$ is determined for all groups G.*

Proof. Let us assume that the stationary logic $L_{\omega_1\omega}(aa)$ has the Löwenheim-Skolem property down to $\aleph_1$. This assumption is consistent relative to the consistency of a supercompact cardinal [2]. Let G be an arbitrary $\aleph_2$-free group. Let H be an $L(aa)$-elementary submodel of G of cardinality $\aleph_1$. Thus H is a free group. The group H satisfies the sentence (2), whence so does G. Now the claim follows from Proposition 5. $\Box$

Corollary 11 *Assuming the consistency of a supercompact cardinal, it is consistent that $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$ is determined for every structure $\mathcal{A}$ and every uncountable free group $\mathcal{B}$.*

4 $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$ can be determined for all $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\aleph_2$.

We prove the consistency of the statement that $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$ is determined for all $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\leq \aleph_2$ assuming the consistency of a measurable cardinal. Actually we make use of an assumption that we call $I^*(\omega)$ concerning stationary subsets of ω_2. This assumption is known to imply that ω_2 is measurable in an inner model. It follows from the previous section that some large cardinal axioms are needed to prove the stated determinacy.

Let $I^*(\omega)$ be the following assumption about ω_1-stationary subsets of ω_2:

$I^*(\omega)$ Let $\mathcal{I}$ be the ω_1-nonstationary ideal NS_{ω_1} on ω_2. Then $\mathcal{I}^+$ has a σ-closed dense subset K.

Hodges and Shelah [9] define a principle $I(\omega)$, which is like $I^*(\omega)$ except that $\mathcal{I}$ is not assumed to be the ω_1-nonstationary ideal. They use $I(\omega)$ to prove the determinacy of an Ehrenfeucht-Fraïssé-game played on several boards simulataneously.

Note that $I^*(\omega)$ implies $\mathcal{I}$ is precipitous, so the consistency of $I^*(\omega)$ implies the consistency of a measurable cardinal [12].

Theorem 12 *([12])The assumption $I^*(\omega)$ is consistent relative to the consistency of a measurable cardinal.*

We shall consider models $\mathcal{A}, \mathcal{B}$ of cardinality $\aleph_2$, so we may as well assume they have ω_2 as universe. For such $\mathcal{A}$ and $\alpha < \omega_2$ we let $\mathcal{A}_\alpha$ denote the structure $\mathcal{A} \cap \alpha$. Similarly $\mathcal{B}_\alpha$.

Lemma 13 *Suppose $\mathcal{A}$ and $\mathcal{B}$ are structures of cardinality $\aleph_2$. If $\forall$ does not have a winning strategy in $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$, then*

$$S = \{\alpha : \mathcal{A}_\alpha \cong \mathcal{B}_\alpha\}$$

is ω_1-stationary.

Proof. Let $C \subseteq \omega_2$ be ω_1-closed and unbounded. Suppose $S \cap C = \emptyset$. We derive a contradiction by describing a winning strategy of $\forall$: Let $\pi : \omega_1 \to \omega_1 \times \omega_1 \times 2$ be onto with $\alpha, \beta, d \leq \pi(\alpha, \beta, d)$ for all $\alpha, \beta < \omega_1$ and $d < 2$. If $\alpha < \omega_2$, let $\theta_\alpha : \omega_1 \to \alpha$ be onto. Suppose the sequence $\langle (x_i, y_i) : i < \alpha \rangle$ has been played. Here x_i denotes a move of $\forall$ and y_i a move of $\exists$. During the game $\forall$ has built an ascending sequence $\{c_i : i < \alpha\}$ of elements of C. Now he lets c_α be the smallest element of C greater than all the elements $x_i, y_i, i < \alpha$. Suppose $\pi(\alpha) = (i, \gamma, d)$. Now $\forall$ will play $\theta_{c_i}(\gamma)$ as an element of $\mathcal{A}$, if $d = 0$, and as an element of $\mathcal{B}$ if $d = 1$.

After all ω_1 moves of $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$ have been played, some $\mathcal{A}_\alpha$ and $\mathcal{B}_\alpha$, where $\alpha \in C$, have been enumerated. Since $\alpha \notin S$, $\forall$ has won the game. $\Box$

Theorem 14 *Assume $I^*(\omega)$. The game $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$ is determined for all $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\leq \aleph_2$.*

Proof. Suppose $\forall$ does not have a winning strategy. By Lemma 13 the set $S = \{\alpha : \mathcal{A}_\alpha \cong \mathcal{B}_\alpha\}$ is ω_1-stationary. Let I and K be as in $I^*(\omega)$. If $\alpha \in S$, let $h_\alpha : \mathcal{A}_\alpha \cong \mathcal{B}_\alpha$. We describe a winning strategy of $\exists$. The idea of this strategy is that $\exists$ lets the isomorphisms h_α determine his moves. Of course, different h_α may give different information to $\exists$, so he has to decide which h_α to follow. The key point is that $\exists$ lets some h_α determine his move only if there are stationarily many other h_β that agree with h_α on this move.

Suppose the sequence $\langle (x_i, y_i) : i < \alpha \rangle$ has been played. Again x_i denotes a move of $\forall$ and y_i a move of $\exists$. Suppose $\forall$ plays next x_α and this is (say) in A. During the game $\exists$ has built a descending sequence $\{S_i : i < \alpha\}$ of elements of K with $S_0 \subseteq S$. The point of the sets S_i is that $\exists$ has taken care that for all $i < \alpha$ and $\beta \in S_i$ we have $y_i = h_\beta(x_i)$ or $x_i = h_\beta(y_i)$ depending on whether $\forall$ played x_i in A or B. Now $\exists$ lets $S'_\alpha \subseteq \bigcap_{i<\alpha} S_i$ so that $S'_\alpha \in K$ and $\forall i \in S'_\alpha (x_\alpha < i)$. For each $i \in S'_\alpha$ we have $h_i(x_\alpha) < i$. By normality, there are an $S_\alpha \subseteq S'_\alpha$ in K and a y_α such that $h_i(x_\alpha) = y_\alpha$ for all $i \in S_\alpha$. This element y_α is the next move of $\exists$. Using this strategy $\exists$ wins. $\Box$

5 A non-determined $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$ with $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\aleph_3$.

We construct directly in ZFC two models $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\aleph_3$ with $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$ non-determined. It readily follows that such models exist in all cardinalities $\geq \aleph_3$. The construction uses a square-like principle (Lemma 16), which is provable in ZFC.

Lemma 15 *[17, 19] There is a stationary $X \subseteq S^3_1$ and a sequence $\langle D_\alpha : \alpha \in X \rangle$ such that*

1. *D_α is a cub subset of α for all $\alpha \in X$.*

2. *The order type of D_α is ω_1.*

3. *If $\alpha, \beta \in X$ and $\gamma < min\{\alpha, \beta\}$ is a limit of both D_α and D_β, then $D_\alpha \cap \gamma = D_\beta \cap \gamma$.*

4. *If $\gamma \in D_\alpha$, then γ is a limit point of D_α if and only if γ is a limit ordinal.*

Proof. We shall sketch, for completeness, a proof of this given by Burke and Magidor [3, Lemma 7.7].

Let $<^*$ be a well-ordering of $H(\omega_3)$. For each $\alpha \in S^3_1$, let $\langle N^\alpha_\delta : \delta < \omega_2 \rangle$ be a continuously increasing chain of elementary submodels of $\langle H(\omega_3), \in, <^* \rangle$ such that

(N1) $(\omega_1 + 1) \cup \{\omega_2, \alpha\} \subseteq N^\alpha_0$.

(N2) $|N^\alpha_\delta| \leq \omega_1$.

(N3) $N^\alpha_\delta \cap \omega_2 \in \omega_2$.

(N4) $\overline{N^\alpha_\delta \cap \omega_3} \in N^\alpha_{\delta+1}$.

Let $A^\alpha_\delta = N^\alpha_\delta \cap \alpha$ for each $\alpha \in S^3_1$. Since, $\alpha \in N^\alpha_\delta$, A^α_δ is cofinal in α. Let $X \subseteq S^3_1$ be stationary such that for some $\delta, \rho < \omega_2$ and for all $\alpha \in X$ we have

1. δ= least ordinal of cofinality ω_1 with $N^\alpha_\delta \cap \omega_2 = \delta$.
2. The order type of $\overline{A^\alpha_\delta}$ is $\rho + 1$.

Let $f : \omega_1 \to \rho$ be cofinal and continuous. Let $g : \rho + 1 \cong \overline{A^\alpha_\delta}$ such that gf maps successors to successors. Let D_α be the image of ω_1 under gf. $\square$

Lemma 16 *There are sets S, T and C_α for $\alpha \in S$ such that the following hold:*

1. *$S \subseteq S^3_0 \cup S^3_1$ and $S \cap S^3_1$ is stationary.*
2. *$T \subseteq S^3_0$ is stationary and $S \cap T = \emptyset$.*
3. *If $\alpha \in S$, then $C_\alpha \subseteq \alpha \cap S$ is closed and of order-type $\leq \omega_1$.*
4. *If $\alpha \in S$ and $\beta \in C_\alpha$, then $C_\beta = C_\alpha \cap \beta$.*
5. *If $\alpha \in S \cap S^3_1$, then C_α is cub on α.*

Proof. Let S and $\langle D_\alpha : \alpha \in S \rangle$ be as in Lemma 15. Let $S' = X \cup Y$, where Y consists of ordinals which are limit points $< \alpha$ of some $D_\alpha, \alpha \in X$. If $\alpha \in X$, we let C_α be the set of limit points $< \alpha$ of D_α. If $\alpha \in Y$, we let C_α be the set of limit points $< \alpha$ of $D_\beta \cap \alpha$, where $\beta > \alpha$ is chosen arbitrarily from X.

Now claims 1,3,4 and 6 are clearly satisfied.

Let $S^3_0 = \bigcup_{i<\omega_2} T_i$ where the T_i are disjoint stationary sets. Since $|\overline{C_\alpha}| \leq \omega_1$, there is $i_\alpha < \omega_2$ such that $i \geq i_\alpha$ implies $\overline{C_i} \cap T_i = \emptyset$. Let $S'' \subseteq S'$ be stationary such that $\alpha \in S''$ implies i_α is constant i. Let $T = T_i$. Finally, let $S = S'' \cup \bigcup\{C_\alpha : \alpha \in S''\}$. Claim 2 is satisfied, and the Lemma is proved. □

Theorem 17 *There are structures $\mathcal{A}$ and $\mathcal{B}$ of cardinality $\aleph_3$ with one binary predicate such that the game $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$ is non-determined.*

Proof. Let S, T and $\langle C_\alpha : \alpha \in S\rangle$ be as in Lemma 16. We shall construct a sequence $\{M_\alpha : \alpha < \omega_3\}$ of sets and a sequence $\{G_\alpha : \alpha \in S\}$ of functions such that the conditions (M1)–(M6) below hold. Let W_α be the set of all mappings

$$G^{d_0}_{\gamma_0} \ldots G^{d_n}_{\gamma_n},$$

where $\gamma_0, \ldots, \gamma_n \in S \cap \alpha$, $d_i \in \{-1, 1\}$, G^1_γ means G_γ and G^{-1}_γ means the inverse of G_γ. Let $W = W_{\omega_3}$. (Note that W consists of a set of partial functions.)

The conditions on the M_α's and the G_α's are:

(M1) $M_\alpha \subseteq M_\beta$ if $\alpha < \beta$, and $M_\alpha \subset M_{\alpha+1}$ if $\alpha \in S$.

(M2) $M_\nu = \bigcup_{\alpha<\nu} M_\alpha$ for limit ν.

(M3) G_α is a bijection of $M_{\alpha+1}$ for $\alpha \in S$.

(M4) If $\beta \in S$ and $\alpha \in C_\beta$, then $G_\alpha \subseteq G_\beta$.

(M5) If for some β, $G_\beta(a) = b$ and for some $w \in W$, $w(a) = b$, then there is some γ so that $w \subseteq G_\gamma$. Furthermore if β is the minimum ordinal so that $G_\beta(a) = b$ then $\gamma = \beta$ or $\beta \in C_\gamma$.

In order to construct the set $M = \bigcup_{\alpha<\omega_3} M_\alpha$ and the mappings G_α we define an oriented graph with M as the set of vertices. We use the terminology of Serre [16] for graph-theoretic notions. If x is an edge, the origin of x is denoted by $o(x)$ and the terminus by $t(x)$. Our graph has an inverse edge $\overline{x}$ for each edge x. Thus $o(\overline{x}) = t(x)$ and $t(\overline{x}) = o(x)$. Some edges are called *positive*, the rest are called *negative*. An edge is positive if and only if its inverse is negative. For each edge x of M there is a set $L(x)$ of labels. The set of possible labels for positive edges is $\{g_\alpha : \alpha < \omega_3\}$. The negative edges can have elements of $\{g_\alpha^{-1} : \alpha < \omega_3\}$ as labels. The labels are assumed to be given in such a way that a positive edge gets g_α as a label if and only if its inverse gets the label g_α^{-1}. During the construction the sets of labels will be extended step by step.

The construction is analogous to building an acyclic graph on which a group acts freely. The graph then turns out to be the Cayley graph of the group. The labelled graph we will build will be the "Cayley graph" of W which will be as free as possible given (M1)–(M4). Condition (M5) is a consequence of the freeness of the construction.

Let us suppose the sets $M_\beta, \beta < \alpha$, of vertices have been defined. Let $M_{<\alpha} = \bigcup_{\beta<\alpha} M_\alpha$. Some vertices in $M_{<\alpha}$ have edges between them and a set $L(x)$ of labels has been assigned to each such edge x.

If α is a limit ordinal, we let $M_\alpha = M_{<\alpha}$. So let us assume $\alpha = \beta + 1$. If $\beta \notin S$, $M_\alpha = M_\beta$. So let us assume $\beta \in S$. Let $\gamma = \sup(C_\beta)$. Notice that since S consists entirely of limit ordinals and $C_\beta \subseteq S$, either $\gamma = \beta$ or $\gamma + 1 < \beta$.

Case 1. $\gamma = \beta$: We extend M_β to M_α by adding new vertices $\{P_z : z \in \mathbb{Z}\}$ and for each $z \in \mathbb{Z}$ a positive edge $x_\alpha^{P_z}$ with $o(x_\alpha^{P_z}) = P_z$ and $t(x_\alpha^{P_z}) = P_{z+1}$. We also let $L(x_\alpha^{P_z}) = \{g_\beta\} \cup \{g_\delta : \beta \in C_\delta\}$.

Case 2. $\gamma + 1 < \beta$: We extend M_β to M_α by adding new vertices $\{P'_z : z \in \mathbb{Z} \setminus \{0\}\}$ for each $P \in M_\beta \setminus M_{\gamma+1}$. For notational convenience let $P'_0 = P$. Now we add for each $P \in M_\beta \setminus M_{\gamma+1}$ new edges as follows. For each $z \in \mathbb{Z}$ we add a positive edge $x_\alpha^{P'_z}$ with

$$o(x_\alpha^{P'_z}) = P'_z, t(x_\alpha^{P'_z}) = P'_{z+1}, L(x_\alpha^{P'_z}) = \{g_\beta\} \cup \{g_\delta : \beta \in C_\delta\}$$

This determines completely the inverse of $x_\alpha^{P'_z}$.

This ends the construction of the graph. In the construction each vertex P in $M_{\alpha+1}$,$\alpha \in S$, is made the origin of a unique edge x_α^P with $g_\alpha \in L(x_\alpha^P)$. We define

$$G_\alpha(P) = t(x_\alpha^P).$$

The construction of the sets M_α and the mappings G_α is now completed. It follows immediately from the construction that each G_α, $\alpha \in S$, is a bijection of $M_{\alpha+1}$. So (M1)–(M3) hold. (M4) holds, because g_α is added to the labels of any edge with g_β, where $\beta \in C_\alpha$, as a label. Finally, (M5) is a consequence of the fact that the graph is circuit-free.

Let us fix $a_0 \in M_1$ and $b_0 = G_{\beta_0}(a_0)$, where $\beta_0 \in C_\alpha$ for all $\alpha \in S$. Note that we may assume, without loss of generality, the existence of such a β_0.

If $a_0, a_1 \in M$, let

$$R_{(a_0,a_1)} = \{(a'_0, a'_1) \in M^2 : \exists w \in W(w(a_0) = a'_0 \wedge w(a_1) = a'_1)\}.$$

We let

$$\mathcal{M} = \langle M, (R_{(a_0,a_1)})_{(a_0,a_1)\in M^2} \rangle$$
$$\mathcal{A} = \langle \mathcal{M}, a_0 \rangle$$
$$\mathcal{B} = \langle \mathcal{M}, b_0 \rangle$$

and show that $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$ is non-determined.

The reduction of the language of $\mathcal{A}$ and $\mathcal{B}$ to one binary predicate is easy. One just adds a copy of ω_3, together with its ordering, and a copy of $M \times M$ to the structures with the projection maps. Then fix a bijection ϕ from ω_3 to M^2. Add a new binary predicate R to the language and interpret R to be contained in $\omega_3 \times M^2$ such that $R(\beta, (a, b))$ holds if and only if $R_{\phi(\alpha)}(a, b)$ holds. We can now dispense with the old binary predicates. We have replaced our structure by one in a finite language without making any difference to who wins the game $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$. The extra step of reducing to a single binary predicate is standard.

An important property of these models is that if $\alpha \in S \cap S^3_1$, then $G_\alpha \restriction M_\alpha$ is an automorphism of the restriction of $\mathcal{M}$ to M_α and takes a_0 to b_0.

Claim 3 $\forall$ *does not win* $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$.

Suppose $\forall$ has a winning strategy τ. Again, there is a quick argument which uses CH: Find $\alpha \in S$ such that M_α is closed under τ and $\mathrm{cf}(\alpha) = \omega_1$. Now C_α is cub on α, whence G_α maps M_α onto itself. Using G_α player $\exists$ can easily beat τ, a contradiction.

In the following longer argument we need not assume CH. Let κ be a large regular cardinal. Let $\mathcal{H}$ be the expansion of $\langle H(\kappa), \in \rangle$ obtained by adding the following structure to it:

(H1) The function $\alpha \mapsto M_\alpha$.

(H2) The function $\alpha \mapsto G_\alpha$.

(H3) The function $\alpha \mapsto C_\alpha$.

(H4) A well-ordering $<^*$ of the universe.

(H5) The winning strategy τ.

(H6) The sets S and T.

Let $\mathcal{N} = \langle N, \in, \ldots \rangle$ be an elementary submodel of $\mathcal{H}$ such that $\alpha = N \cap \omega_3 \in S \cap S^3_1$.

Now C_α is a cub of order-type ω_1 on α and G_α maps M_α onto M_α. Moreover, G_α is a partial isomorphism from $\mathcal{A}$ into $\mathcal{B}$. Provided that τ does not lead $\forall$ to play his moves outside M_α, $\exists$ has on obvious strategy: he lets G_α determine his moves. So let us assume a sequence $\langle (x_\xi, y_\xi) : \xi < \gamma \rangle$ has been played inside M_α and $\gamma < \omega_1$. Let $\beta \in C_\alpha$ such that M_β contains the elements x_ξ, y_ξ for $\xi < \gamma$. The sequence $\langle y_\xi : \xi < \gamma \rangle$ is totally determined by G_β and τ. Since $G_\beta \in N$, $\langle y_\xi : \xi < \gamma \rangle \in N$, and we are done.

Claim 4 *$\exists$ does not win $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$.*

Suppose $\exists$ has a winning strategy τ. Let $\mathcal{H}$ be as above and $\mathcal{N} = \langle N, \in, \ldots \rangle$ be an elementary submodel of $\mathcal{H}$ such that $\alpha = N \cap \omega_3 \in T$. We let $\forall$ play during the first ω moves of $\mathcal{G}_{\omega_1}(\mathcal{A}, \mathcal{B})$ a sequence $\langle a_n : n < \omega \rangle$ in $\mathcal{A}$ such that if α_n is the least α_n with $a_n \in M_{\alpha_n}$, then the sequence $\langle \alpha_n : n < \omega \rangle$ is ascending and $\sup\{\alpha_n : n < \omega\} = \alpha$. Let $\exists$ respond following τ with $\langle b_n : n < \omega \rangle$. As his move number ω player $\forall$ plays some element $a_\omega \in M \setminus M_\alpha$ in $\mathcal{A}$ and $\exists$ answers according to τ with b_ω.

For all $i \leq \omega$, $R_{(a_0,a_i)}(a_0, a_i)$ holds. Hence $R_{(a_0,a_i)}(b_0, b_i)$ holds. So there is w_i such that $w_i(a_0) = b_0$ and $w_i(a_i) = b_i$. Since $G_{\beta_0}(a_0) = b_0$, by (M5), for each i there is β_i so that $G_{\beta_i}(a_i) = b_i$. We can assume that β_i is chosen to be minimal. Notice that for all i, $\beta_i > \alpha_i$ and for $i < \omega$, $\beta_i \in \mathcal{N}$. So $\sup\{\beta_i : i < \omega\} = \alpha$.

Also, by the same reasoning as above, for each $i < \omega$, $R_{(a_i,a_\omega)}(b_i, b_\omega)$ holds. Applying (M5), we get that $G_{\beta_\omega}(a_i) = b_i$. Using (M5) again and the minmality of β_i, for all $i < \omega$, $\beta_i \in C_{\beta_\omega}$. Thus α is a limit of elements of C_{β_ω}, contradicting $\alpha \in T$. $\square$

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