FILTERS, COHEN SETS AND CONSISTENT EXTENSIONS OF THE ERDÖS-DUSHNIK-MILLER THEOREM

Saharon Shelah and Lee J. Stanley

Abstract. We present two different types of models where, for certain singular cardinals λ of uncountable cofinality, $\lambda \to (\lambda, \omega+1)^2$, although λ is not a strong limit cardinal. We announce, here, and will present in a subsequent paper, [7], that, for example, consistently, $\aleph_{\omega_1} \not\to (\aleph_{\omega_1},\ \omega+1)^2$ and consistently, $2^{\aleph_0} \not\to (2^{\aleph_0}, \omega+1)^2$.

§0. INTRODUCTION.

For regular uncountable κ, the Erdös-Dushnik-Miller theorem, Theorem 11.3 of [2], states that $\kappa \to (\kappa,\ \omega+1)^2$. For singular cardinals, κ, they were only able to obtain the weaker result, Theorem 11.1 of [1], that $\kappa \to (\kappa,\ \omega)^2$. It is not hard to see that if $cf\ \kappa = \omega$ then $\kappa \not\to (\kappa,\ \omega+1)^2$. If $cf\ \kappa > \omega$ and κ is a strong limit cardinal, then it follows from the General Canonization Lemma, Lemma 28.1 of [1], that $\kappa \to (\kappa,\ \omega+1)^2$. Question 11.4 of [1] is whether this holds without the assumption that κ is a strong limit cardinal, e.g., whether, in ZFC,

$$(1)\quad \aleph_{\omega_1} \to (\aleph_{\omega_1},\ \omega+1)^2.$$

Another natural question, which the second author first heard from Todorcevic, is whether, in ZFC,

$$(2)\quad 2^{\aleph_0} \to (2^{\aleph_0},\ \omega+1)^2.$$

In connection with (2), we note that the first author proved, [2], §2, the consistency of $2^{\aleph_0} \to [\aleph_1]^2_{n,2}$.

The research of the first author was partially supported by the NSF and the Basic Research Fund, Israel Academy of Science. This is paper number 419 in the first author's list of publications. Theorem 1 was proved in Fall, 1989 when both authors benefitted from the hospitality of MSRI, for which they record their gratitude. Theorem 3 was proved in Fall, 1993; Theorem 4 was was proved in Fall 1994.

In this paper we address these questions, by presenting two types of models where there is a singular cardinal λ of uncountable cofinality, such that $\lambda \to (\lambda,\ \omega+1)^2$ even though λ *is not a strong limit cardinal.* In either model, λ can be taken to be $\aleph_{\omega_1}$ and in the second, we can also have, simultaneously, $\lambda = 2^{\aleph_0}$. We also announce here, and will present in a subsequent paper, some very recent results that show that, consistently, (1) and (2) above may fail. For (1), this answers Question 11.4 of [1] negatively.

The first type of model seems specific to having the order type of the homogeneous set for the second color (green, for us, whereas the first color is the "traditional" red) be $\omega+1$, whereas the second model allows generalizations to green homogeneous sets of order type $\theta+1$ for cardinals, θ, with $\omega \leq \theta < cf\ \lambda$, under appropriate hypotheses. On the other hand, the proof for the first model is an outright implication from a hypothesis which follows from the existence of certain partition cardinals, either outright, or in inner models, and therefore, certainly, from the failure of the SCH, for example.

Theorem 1. *If $\omega < \kappa = cf\ \lambda,\ 2^\kappa < \lambda$ and there is a normal nice filter on κ, then $\lambda \to (\lambda,\ \omega+1)^2$.*

There is no assumption on powersets between κ and λ. We prove Theorem 1 in §1. The notion of *nice filter* is due to the first author. In (1.1), below, we will give a condensed definition, sufficient for our purposes, which is consistent with the more general treatment of §§0,1 of Chapter V of [3]. This is essentially clause (2) of Definition V.1.9 of [3]. The crucial property of nice filters, for the purposes of this paper, is that we can define a certain kind of rank function, $rk^2_D(f,\ \mathcal{E})$, with ordinal values, where D is any normal nice filter, $f : \kappa \to OR$ and $\mathcal{E}$ is the family of normal nice filters on κ. This rank function has the following important property:

(#) If $D \in \mathcal{E},\ f,\ g : \kappa \to OR,\ X$ is D-positive and for all $\gamma \in X,\ g(\gamma) < f(\gamma)$, then then there is $D' \in \mathcal{E}$ with $D \cup \{X\} \subseteq D'$ such that $rk^2_{D'}(g,\ \mathcal{E}) < rk^2_D(f,\ \mathcal{E})$.

This can be extracted from the following items of Chapter V of [3]: Claim V.2.13, and clause (1) of Fact V.3.16. The existence of a nice filter on ω_1, for example, is an outright consequence of the existence of a μ such that $\mu \to (\alpha)^{<\omega}_{\aleph_o}$ for all $\alpha < (2^{2^{\aleph_1}})^+$. It can also be obtained in forcing extensions starting from models with such large cardinals. For these results, see Conclusion V.1.13 and Remark V.1.13A of [3]. In view of the first fact, we easily have the following corollary to Theorem 1; a later result in a similar vein is Woodin's striking result that from CH and the existence of a measurable cardinal it follows that the club filter on $\aleph_1$ is not $\aleph_2$-saturated.

Corollary 2. *Assume that there is a measurable cardinal and that* $2^{\aleph_1} < \aleph_{\omega_1}$. *Then* $\aleph_{\omega_1} \to (\aleph_{\omega_1}, \omega+1)^2$.

In the second type of model, we have several parameters. We let $\omega < \kappa = cf\ \lambda < \lambda$. As mentioned above, we have a cardinal θ with $\omega \leq \theta < \kappa$. We have an additional cardinal parameter, σ, with $\sigma \neq \kappa$ and $\theta \leq \sigma$. The cases $\sigma < \kappa$ and $\sigma > \kappa$ require somewhat different different treatment, and lead to Theorems 3 and 4, below, respectively, proved in §2 and §3. However, much of the preliminary material developed for Theorem 3 carries over to the proof of Theorem 4. The main case of Theorem 3 is when $\theta = \sigma$, and the connection to Theorem 1 is when $\theta = \sigma = \omega$. Theorem 3 was proved in Fall 1993 and Theorem 4 was proved in Fall 1994.

For both Theorems, we assume that in V, λ is a strong limit cardinal, and that $\sigma^{<\sigma} = \sigma$. Our model is obtained by forcing with $\mathbf{P}$, which is the partial ordering for adding at least λ Cohen subsets of σ. When $\theta > \omega$, we need additional assumptions to guarantee, for example, that in $V^{\mathbf{P}}$, $\kappa \to (\kappa,\ \theta+1)^2$. When $\theta = \omega$, this is just the Erdös-Dushnik-Miller theorem for κ. The additional assumptions will involve cardinal exponentiation, and will be discussed below. We then have:

Theorem 3. *Suppose that in* V, $\omega \leq \theta \leq \sigma = \sigma^{<\sigma} < \kappa = cf\ \lambda < \lambda \leq \nu$, λ *is a strong limit cardinal and for all* $\mu < \kappa$, $\mu^{<\theta} < \kappa$. *Let* $\mathbf{P}$ *be the partial ordering for adding* ν *Cohen subsets of* σ. *Then, in* $V^{\mathbf{P}}$, $\lambda \to (\lambda,\ \theta+1)^2$.

Theorem 4. *Suppose that in* V, $\omega \leq \theta < \kappa < \sigma = \sigma^{<\sigma} < \lambda$, $\kappa = cf\ \lambda < \lambda \leq \nu$, λ *is a strong limit cardinal and for all* $\mu < \kappa$, $\mu^{<\theta} < \kappa$. *Let* $\mathbf{P}$ *be the partial ordering for adding* ν *Cohen subsets of* σ. *Then, in* $V^{\mathbf{P}}$, $\lambda \to (\lambda,\ \theta+1)^2$.

We shall deduce Theorem 4 from the following result about lifting certain positive partition relations on κ in V to λ in models, $V^{\mathbf{P}}$, where $\kappa,\ \lambda,\ \sigma,\ \mathbf{P}$ are as in Theorem 4.

Theorem 4*. *Suppose that in V, $\omega < \kappa < \sigma = \sigma^{<\sigma} < \lambda$, $\kappa = cf\ \lambda \leq \nu$, λ is a strong limit cardinal and $\mathbf{P}$ is the partial ordering for adding ν Cohen subsets of σ. Suppose, further, that $\zeta < \kappa$ and that $\kappa \to (\kappa,\ \zeta)^2$. Then, in $V^{\mathbf{P}}$, $\lambda \to (\lambda,\ \zeta)^2$.*

Of course, when we invoke Theorem 4* to obtain Theorem 4, we shall take $\zeta = \theta + 1$, and we will use the additional hypotheses on cardinal exponentiation in V to obtain the hypothesis of Theorem 4*, that $\kappa \to (\kappa,\ \zeta)^2$. Then, this relation will also hold in $V^{\mathbf{P}}$, since there are no new subsets of κ. In fact, it is even possible to factor Theorem 3 through a similar kind of result about lifting positive relations on κ to λ, but now lifting a $V^{\mathbf{P}}$ relation on κ to a $V^{\mathbf{P}}$ relation on λ, since this time, forcing with $\mathbf{P}$ will not necessarily preserve a positive V relation on κ. In what follows, we shall not proceed in this fashion; however, we do state the lifting theorem:

Theorem 3*. *Suppose that in V, $\zeta \leq \sigma + 1$, $\sigma = \sigma^{<\sigma} < \kappa = cf\ \lambda \leq \nu$, λ is a strong limit cardinal and $\mathbf{P}$ is the partial ordering for adding ν Cohen subsets of σ. Suppose, further, that in $V^{\mathbf{P}}$, $\kappa \to (\kappa,\ \zeta)^2$. Then, in $V^{\mathbf{P}}$, $\lambda \to (\lambda,\ \zeta)^2$.*

Once again, in order to obtain Theorem 3 from Theorem 3*, the additional hypotheses in Theorem 3 on cardinal exponentiation in V are designed to guarantee that the needed positive relation does hold in $V^{\mathbf{P}}$. It would, of course, be possible to combine Theorems 3* and 4* into a single statement, but the proof would certainly reflect the division into cases, which, here, is transparent in the statements.

Finally, though these more recent results will be presented in a subsequent paper, [7], we state here, as numbered theorems, the negative consistency results for questions (1) and (2), mentioned above and in the Abstract.

Theorem 5. *Suppose that, in L, $\mu \geq \lambda > \kappa$ are cardinals, $cf(\lambda) = \kappa > \omega$ (for example, $\lambda = (\aleph_\kappa)^L$, $\kappa = (\aleph_1)^L$). Let G be $\mathbf{P}$-generic over L, where $\mathbf{P}$ is (L's version of) the partial order for adding μ Cohen subsets of κ. Then, in $L[G]$, $\lambda \to (\lambda,\ \omega + 1)^2$ iff, in L, κ is weakly compact.*

Taking $\kappa = (\aleph_1)^L,\ \lambda = (\aleph_\kappa)^L$, we get the negative consistency result for (1). Combining the methods used to obtain Theorem 5 for this choice of κ and λ, an additional forcing to add λ Cohen reals, and a double Δ-system argument for the second forcing, we get:

Theorem 6. *Con(ZFC) implies Con(ZFC & $2^{\aleph_0} \not\to (2^{\aleph_0}, \omega+1)^2$).*

Remarks.

(1) The proof of the Erdös-Dushnik-Miller theorem proceeds by assuming that there is no homogeneous set of power κ for the first color (red, for us), and showing that a certain tree of homogeneous green sets must have a branch of length $\omega+1$, which naturally yields a homogeneous green set of order type $\omega+1$. If $\theta > \omega,\ \theta$ is a cardinal, $\tau > \theta$ is regular, and if:

$$(*)\ \text{for all } \nu < \tau,\ \nu^{<\theta} < \tau,$$

then we can carry out essentially the same proof to show that $\tau \to (\tau,\ \theta+1)^2$. Thus, taking $\tau = \kappa$, our hypotheses on cardinal exponentiation in Theorem 3, which remain true in $V^{\mathbf{P}}$, do guarantee that in $V^{\mathbf{P}},\ \kappa \to (\kappa,\ \theta+1)^2$.

Similary, if $\omega < \zeta,\ \theta = card\ \zeta,\ \theta < \tau,\ \tau$ is regular and if:

$$(**)\ \text{for all } \nu < \tau,\ \nu^{\theta} < \tau,$$

then a similar tree argument shows that $\tau \to (\tau,\ \zeta)^2$. Thus, the additional hypotheses on cardinal exponentiation in Theorem 4 do guarantee that we have the hypotheses of Theorem 4*.

For both theorems, we will also need to know that for many successor cardinals, τ, between κ and λ, we will have $\tau \to (\tau,\ \theta+1)^2$, or $\tau \to (\tau, \zeta)^2$ (for Theorem 4*). In view of the preceding paragraphs, it will suffice to have $(*)$ or $(**)$ for τ, in V.

One way of achieving this is to appeal to the fact that, in V, λ is a strong limit cardinal, and, for example, to take $\tau = \mu^+$, where $\mu = \mu^\theta$, and where μ is chosen to have various other properties, as desired.

(2) In all of what follows we shall have $\omega < \kappa = cf\ \lambda < \lambda$. We shall express λ as $sup\{\lambda_\eta | \eta < \kappa\}$, where $(\lambda_\eta | \eta < \kappa)$ is increasing and continuous, and for $\eta = 0$ or η a successor ordinal, λ_η is a successor cardinal. Various other properties of the λ_η for such η will

be introduced as needed. One such property will be that $\lambda_\eta = \mu^+$, where $\mu = \mu^\theta$ (and has various other properties, as desired). We also let $\Delta_0 = \lambda_0$ and for $\eta < \kappa,\ \Delta_{1+\eta} = [\lambda_\eta,\ \lambda_{\eta+1})$. For $\alpha < \lambda$ we will let $\eta(\alpha) =$ the unique $\eta < \kappa$ such that $\alpha \in \Delta_\eta$.

(3) Investigation of the case $\sigma = \kappa$, which is not treated in this paper, led to Theorems 5, 6, above, among other results. When $\sigma < \kappa$, we use the σ^+-chain condition of $\mathbf{P}$, whereas when $\sigma > \kappa$, we use the $< \sigma$-completeness of $\mathbf{P}$.

(4) In Theorems 3* and 4*, it is clearly necessary that in $V^{\mathbf{P}}$, $\kappa \to (\kappa,\ \theta + 1)^2$, respectively, that $\kappa \to (\kappa,\ \zeta)^2$.

(5) Our notation and terminology is intended to either be standard or have a clear meaning, e.g., *card* X for the cardinality of X, *o.t.* X for the order type of X, etc.

(6) Theorems 3 and 5 of [6] are close in spirit to some of the above material. There are also similarities to certain themes from [4] and [5].

§1. USING NICE FILTERS.

In this section we prove Theorem 1 of the Introduction, which, for convenience, we now restate.

Theorem 1. *If* $\omega < \kappa = cf\ \lambda,\ 2^\kappa < \lambda$ *and there is a normal nice filter on* κ*, then* $\lambda \to (\lambda,\ \omega + 1)^2$.

Proof. We begin by providing the promised definition of nice filter on κ. If D is a normal filter on κ and g is an ordinal valued function with domain κ, we first define the game $Gw^*(D,\ g)$, as follows. On move 0, player I chooses $D_0 := D$, and player II chooses $A_0 \in (D_0)^+$, and chooses $g_0 := g$. On move $n + 1$, player I chooses D_{n+1}, a normal filter on κ extending $D_n \cup \{A_n\}$, and player II chooses $A_{n+1} \in (D_{n+1})^+$, AND $g_{n+1} <_{D^*_{n+1}} g_n$, where $D^*_{n+1} :=$ the normal filter on κ generated by $D_{n+1} \cup \{A_{n+1}\}$. Player I wins if at some stage $n + 1$, Player II has no legal play. We then state:

(1.1) Definition. *D is nice if for all ordinal valued functions, g, with domain κ, Player I has a winning strategy in $Gw^*(D,\ g)$.*

Proceeding with the proof of the Theorem, we assume that $\omega < \kappa = cf\ \lambda < \lambda,\ 2^\kappa < \lambda$ and that there is a nice normal filter on κ. We will show that $\lambda \to (\lambda,\ \omega+1)^2$. There are no assumptions about powers of cardinals larger than κ, and, as noted in the Introduction, the interest of the result is when λ is not a strong limit cardinal. The simplest case, of course, is when $\kappa = \aleph_1$ and $\lambda = \aleph_\kappa$.

So, towards a contradiction, suppose that $c : [\lambda]^2 \to \{\text{red, green}\}$ but has no red set of power λ and no green set of order type $\omega+1$. Let $\lambda_\eta,\ \Delta_\eta,\ \eta < \kappa$ be as in Remark 2 of the Introduction. We can clearly assume, in addition, that $\lambda_0 > 2^\kappa$, for $\eta < \kappa,\ \lambda_{\eta+1} \geq \lambda_\eta^{++}$, and that each Δ_η is homogeneous red for c. The last is by the Erdös-Dushnik-Miller theorem for $\lambda_{\eta+1}$.

For $0 < \eta < \kappa$, we define Seq_η to be $\{(i_0, ..., i_{n-1}) | \eta(i_0) < ... < \eta(i_{n-1}) < \eta\}$. For $\zeta \in \Delta_\eta$ and $(i_0, ..., i_{n-1}) = \vec{i} \in Seq_\eta$, we say $\vec{i} \in T^\zeta$ iff $\{i_0, ..., i_{n-1},\ \zeta\}$ is homogeneous green for c. Note that an infinite decreasing (for reverse inclusion) branch in T^ζ violates the nonexistence of a green set of order type $\omega+1$, so, under reverse inclusion, T^ζ is well-founded. Therefore the following definition of a rank function, rk^ζ, on Seq_η can be carried out.

We define $rk^\zeta : Seq_\eta \to OR \cup \{-1\}$ by setting $rk^\zeta(\vec{i}\,)$ to be -1 if $\vec{i}\frown\zeta$ is not homogeneous green; otherwise, define $rk^\zeta(\vec{i}\,) \geq \eta$ iff for all $\tau < \eta$ there is j such that $rk^\zeta(\vec{i}\frown j) \geq \tau$. Of course, for limit ordinals, δ, if for all $\eta < \delta$, $rk^\zeta(\vec{i}\,) \geq \eta$, then $rk^\zeta(\vec{i}\,) \geq \delta$, and so for all $\vec{i} \in T^\zeta$, there is a largest η such that $rk^\zeta(\vec{i}\,) \geq \eta$. We take $rk^\zeta(\vec{i}\,)$ to be this largest η. In fact, it is clear that the range of rk^ζ is a proper initial segment of μ_η^+, where $\mu_\eta = card \bigcup\{\Delta_\tau | \tau < \eta\}$, and so, in particular, the range of rk^ζ has power at most λ_η. Note that $\lambda_{\eta+1} > \mu_\eta^+$.

But then, we can find B_η an end-segment of Δ_η such that for all $\vec{i} \in Seq_\eta$ and all $0 \leq \gamma < \mu_\eta^+$, if there is $\zeta \in B_\eta$ such that $rk^\zeta(\vec{i}\,) = \gamma$, then there are $\lambda_{\eta+1}$ such ζ. Recall that Δ_η and therefore also B_η are of order type $\lambda_{\eta+1}$, which is a successor cardinal. Everything is now in place for the main definition.

(1.2) Definition. $(\vec{i},\ Z,\ D,\ f) \in K$ *iff*

(1) *D is a nice, normal filter on κ,*
(2) *$f : \kappa \to OR$,*
(3) *$Z \in D$,*
(4) *for some $0 < \eta < \kappa$, $\vec{i} \in Seq_\eta$, and for all $\tau \in Z \setminus (\eta+1)$, there is $\zeta \in B_\tau$ such that $rk^\zeta(\vec{i}\,) = f(\tau)$ (so, in particular, $\vec{i} \in T^\zeta$).*

Note that $K \neq \emptyset$, since if we choose $\zeta_\tau \in B_\tau$, for $\tau < \kappa$, take $Z = \kappa$, $\vec{i} =$ the empty sequence, choose D to be any nice normal filter on κ and define f by $f(\tau) = rk^{\zeta_\tau}(\vec{i})$, then $(\vec{i},\ Z,\ D,\ f) \in K$.

Now, let $\mathcal{E}$ be the family of nice normal filters on κ. Since $rk^2_D(f,\ \mathcal{E}) \in OR$, clearly among the $(\vec{i},\ Z,\ D,\ f) \in K$, there is one with $rk^2_D(f,\ \mathcal{E})$ minimal.

So, fix one such, and denote it by $(\vec{i}^{\,*},\ Z^*,\ D^*,\ f^*)$. For $\tau \in Z^*$, set $C_\tau = \{\zeta \in B_\tau | rk^\zeta(\vec{i}) \leq f^*(\tau)\}$. Thus *card* $C_\tau = \lambda_{\tau+1}$, and for all $\zeta \in C_\tau$, *range*$(\vec{i}^{\,*} \cup \{\zeta\})$ is homogeneous green. Now suppose $\tau \in Z^*$. For all $\gamma \in Z^* \setminus (\tau+1)$ and $\zeta \in C_\tau$, let $C^+_\gamma(\zeta) = \{\xi \in C_\gamma | c(\{\zeta, \xi\}) = \text{green}\}$. Also, let $Z^+(\zeta) = \{\gamma \in Z^* \setminus (\beta+1) | \mathit{card}\ C^+_\gamma(\zeta) = \lambda_{\gamma+1}\}$. It is, perhaps, worth pointing out that we could just as well have required only that $C^+_\gamma \neq \emptyset$.

(1.3) Lemma. *For a D-positive set of $\tau \in Z^*$ and for $\lambda_{\tau+1}$ many $\zeta \in C_\tau$, $Z^+(\zeta)$ is D-positive.*

Proof. For $\tau \in Z^*$ and $\zeta \in C_\tau$, let $Y(\zeta) = \kappa \setminus Z^+(\zeta)$. Since $\lambda_0 > 2^\kappa$, for all $\tau \in Z^*$ there is $Y = Y_\tau \subseteq \kappa$ and $C'_\tau \subseteq C_\tau$ with *card* $C'_\beta = \lambda_{\tau+1}$ such that for all $\zeta \in C'_\tau$, $Y(\zeta) = Y_\beta$.

Let $\hat{Z} = \{\tau \in Z^* | Y_\tau \in D\}$. We now conclude by showing that $\hat{Z} \notin D$. If $\hat{Z} \in D$, then, since D is normal, we would have $Y^* \in D$, where $Y^* = \{\tau \in \hat{Z} | \text{for all } \eta \in \hat{Z} \cap \tau,\ \tau \in Y_\eta\}$. But then, by shrinking the C'_τ for $\tau \in Y^*$, as in the next paragraph, we would get a homogeneous red set of power λ, which is impossible.

We define $\hat{C}_\tau$ for $\tau \in Y^*$ by recursion on τ in such a way that $\hat{C}_\tau$ is a subset of C'_τ of power $\lambda_{\tau+1}$. So, let $\tau \in Y^*$, and set $\xi \in \hat{C}_\tau$ iff $\xi \in C'_\tau$ and for all $\eta \in Y^* \cap \tau$ and all $\zeta \in \hat{C}_\eta$, $\xi \notin C^+_\tau(\zeta)$. So, in fact, $\hat{C}_\tau$ is the result of removing at most λ_τ elements from C'_τ. But then, clearly the union of the $\hat{C}_\tau$ for $\tau \in Y^*$ is homogeneous red. This concludes the proof of Lemma 1.2.

We maintain the notation of the proof of Lemma 1.2. Fix τ as guaranteed by Lemma 1.2, i.e., such that Y_τ is defined, but $Y_\tau \notin D$. Let $X = Z^* \setminus Y_\tau$. Note that, for any $\zeta \in C'_\tau$, $X \setminus (\tau+1) = Z^+(\zeta)$ and X is D-positive. Now fix $\zeta \in C'_\tau$. For $\gamma \in X \setminus (\tau+1)$, note that by the definition of $C^+_\gamma(\zeta)$, there is $j \in C^+_\gamma(\zeta)$ such that $rk^j(\vec{i}^{\,*}) \leq f^*(\gamma)$. Choose one such and call it j_γ. Thus, again by the definition of $C^+_\gamma(\zeta)$, $\vec{i}^{\,*\frown}\zeta^\frown j_\gamma$ is homogeneous green, and so, by the definition of rk^{j_γ}, $rk^{j_\gamma}(\vec{i}^{\,*\frown}\zeta) < f^*(\gamma)$.

Now, define $g : \kappa \to OR$ by $g(\gamma) = rk^{j_\gamma}(\vec{i}^{\,*\frown}\zeta)$, if $\gamma \in X \setminus (\tau+1)$, and $g(\gamma) = 0$, otherwise. Now, by the definition of $rk^2_D(f^*,\ \mathcal{E})$, (again, see Chapter 5 of [3]) there is $D' \in \mathcal{E}$ with $D \cup \{X\} \subseteq D'$ and such that $rk^2_{D'}(g,\ \mathcal{E}) < rk^2_D(f^*,\ \mathcal{E})$. However, it is easily verified that $(\vec{i}\,^{*\frown}\zeta,\ X,\ D',\ g) \in K$, and, finally, this contradicts the choice of $(\vec{i}\,^*,\ Z^*,\ D,\ f^*)$, and thus completes the proof of Theorem 1.

§2. ADDING COHEN SETS BELOW THE COFINALITY.

In this section, we prove Theorem 3 of the Introduction, whose statement we now recall for convenience.

Theorem 3. *Suppose that in V, $\omega \le \theta \le \sigma = \sigma^{<\sigma} < \kappa = cf\ \lambda < \lambda \le \nu$, λ is a strong limit cardinal and for all $\mu < \kappa$, $\mu^{<\theta} < \kappa$. Let $\mathbf{P}$ be the partial ordering for adding ν Cohen subsets of σ. Then, in $V^{\mathbf{P}}$, $\lambda \to (\lambda,\ \theta+1)^2$.*

Proof. So, let $\lambda,\ \kappa,\ \theta,\ \sigma,\ \nu,\ \mathbf{P}$ be as in the statement of Theorem 3, and let $(\lambda_\eta | \eta < \kappa)$ be as in Remark (2) of the Introduction, and suppose, in addition that $\lambda_0 = \mu_0^+$, where $\mu_0 = (\beth_\omega(\kappa))^\theta$, and for $\eta < \kappa$, $\lambda_{\eta+1} = \mu_\eta^+$, where $\mu_\eta = (\mu_\eta)^{((\beth_\omega(\lambda_\eta))^\theta)}$. Thus, by Remark 1 of the Introduction, we will have that in $V^{\mathbf{P}}$, $\kappa \to (\kappa,\ \theta+1)^2$ and similarly:

(!) in $V^{\mathbf{P}}$, for each $\eta < \kappa$ which is either 0 or a successor ordinal, $\lambda_\eta \to (\lambda_\eta,\ \theta+1)^2$.

This follows from our choice of the λ_η since forcing with $\mathbf{P}$ adds no new sequences of ordinals of length $< \theta$. Also, let Δ_η, and $\eta(\alpha)$ be as in Remark 2 of the Introduction.

For $A \subseteq \nu$, we let $\mathbf{P}|A$ be the subordering of $\mathbf{P}$ with underlying set the set of $p \in P$ with domain included in A. If *card* $A =$ *card* B and T is a bijection from A to B, we abuse notation by also taking T to be the isomorphism from $\mathbf{P}|A$ to $\mathbf{P}|B$ induced by T.

Suppose, now, that $\mathbf{c}$ is a $\mathbf{P}$-name and that $p \in P$ forces that $\mathbf{c} : [\lambda]^2 \to \{\text{red, green}\}$. We now embark on an analysis of $\mathbf{c}$ as a $\mathbf{P}$-name culminating in (*), following (2.9). This analysis carries over to §3, and even in the case $\sigma = \kappa$. We use the latter case in our forthcoming paper, [7], when κ is weakly compact. Therefore, we temporarily drop the assumption the assumption $\sigma < \kappa$, or even that $\sigma \ne \kappa$, retaining only that $\sigma = \sigma^{<\sigma} < \lambda$.

By (!), we can assume, without loss of generality, that for each $\eta < \kappa$, p forces that Δ_η is homogeneous red for $\mathbf{c}$. In order to develop material that will carry over to the proof of Theorem 4*, in §3, for now, we make no additional hypotheses about $\mathbf{c}$.

For $\alpha < \beta < \lambda$, let $A(\alpha,\ \beta)$ be a subset of ν of power at most σ such that $\mathbf{c}(\alpha,\ \beta)$ is a $\mathbf{P}|A(\alpha,\ \beta)$-name. Such $A(\alpha,\ \beta)$ exists, since $\mathbf{P}$ has the σ^+-cc. Let $A^* = \bigcup\{A(\alpha,\ \beta)|\alpha < \beta < \lambda\}$ and let $\mathbf{P}^* = \mathbf{P}|A^*$. Without loss of generality, *dom* $p \subseteq A^*$. Thus, $\mathbf{c} \in V^{\mathbf{P}^*}$, so by arguing in $V^{\mathbf{P}^*}$, and remarking that *card* $A^* = \lambda$ and therefore that $\mathbf{P}^* \cong \mathbf{P}|\lambda$, we can assume, without loss of generality, that $\nu = \lambda$, which we do from here on.

For $\alpha < \beta < \lambda$, let $\pi(\alpha,\ \beta) = o.t.\ A(\alpha,\ \beta)$ and let $(\rho_\zeta^{\alpha,\ \beta}|\zeta < \pi(\alpha,\ \beta))$ be the increasing enumeration of $A(\alpha,\ \beta)$. Also, let $T(\alpha,\ \beta)$ be the order isomorphism from $A(\alpha,\ \beta)$ to $\pi(\alpha,\ \beta)$ (so $T(\alpha,\ \beta)(\rho_\zeta^{\alpha,\ \beta}) = \zeta$). Let $\mathbf{c}'(\alpha,\ \beta)$ be the $\mathbf{P}|\pi(\alpha,\ \beta)$-name which results from applying $T(\alpha,\ \beta)$ to $\mathbf{c}(\alpha,\ \beta)$ where $T(\alpha,\ \beta)$ is viewed as the isomorphism from $\mathbf{P}|A(\alpha,\ \beta)$ to $\mathbf{P}|\pi(\alpha,\ \beta)$, as in the previous paragraph. Fix functions $F_i : [\lambda]^2 \to \lambda$, for $i < \sigma$, such that for $\alpha < \beta < \lambda,\ A(\alpha,\ \beta) = \{F_i(\alpha,\ \beta)|i < \sigma\}$.

(2.1) Definition. *Let* $Y(\alpha,\ \beta) = \{(i,\ \zeta) \in \sigma \times \pi(\alpha,\ \beta)|F_i(\alpha,\ \beta) = \rho_\zeta^{\alpha,\ \beta}\}$. *We also let X be the set of ordered 4-tuples, $(\alpha,\ \beta,\ \gamma,\ \delta)$ from λ such that $\alpha < \beta$ and $\gamma < \delta$, and we define a function c^* with domain X by:*

$$c^*(\alpha,\ \beta,\ \gamma,\ \delta) = (\pi(\alpha,\ \beta),\ \pi(\gamma,\ \delta),\ Y(\alpha,\ \beta),\ Y(\gamma,\ \delta),\ \mathbf{c}'(\alpha,\ \beta),\ \mathbf{c}'(\gamma,\ \delta)).$$

Note that the following set is easily recoverable from $c^*(\alpha,\ \beta,\ \gamma,\ \delta)$:

$$\hat{c}(\alpha,\ \beta,\ \gamma,\ \delta) = \{(i,\ j)|F_i(\alpha,\ \beta) = F_j(\gamma,\ \delta)\}.$$

We abuse notation below by acting as if this were actually part of $c^*(\alpha,\ \beta,\ \gamma,\ \delta)$. Also note that range c^* has power at most 2^σ.

Applying the general canonization lemma, Lemma 28.1 of [1] to c^*, we get $B_\eta \subseteq \Delta_\eta$ with *card* $B_0 > \kappa + \sigma$ and for $0 < \eta < \kappa$, *card* $B_\eta > \lambda_\eta$, and such that $(B_\eta : \eta < \kappa)$ is canonical for c^*, i.e, letting $B = \bigcup\{B_\eta|\eta < \kappa\}$, if $(\alpha_n|n < 4),\ (\beta_n|n < 4) \in X \cap B^4$ and for all $n < 4$, $\eta(\alpha_n) = \eta(\beta_n)$, then $c^*((\alpha_n|n < 4)) = c^*((\beta_n|n < 4))$.

Further note that if $\eta(\alpha_1) = \eta(\alpha_2) < \eta(\beta_1) = \eta(\beta_2)$ and $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2 \in B$, then since $c^*(\alpha_1,\ \beta_1,\ \alpha_1,\ \beta_1) = c^*(\alpha_2,\ \beta_2,\ \alpha_2,\ \beta_2)$, we also have that $\mathbf{c}'(\alpha_1,\ \beta_1) = \mathbf{c}'(\alpha_2,\ \beta_2)$. This, in turn, means that if $p_1 \in P|A(\alpha_1,\ \beta_1)$, $p_2 = (T(\alpha_2,\ \beta_2))^{-1} \circ T(\alpha_1,\ \beta_1)(p_1)$, and $x \in \{\text{red, green}\}$, then p_1 forces that $\mathbf{c}(\alpha_1,\ \beta_1) = x$ iff p_2 forces that $\mathbf{c}(\alpha_2,\ \beta_2) = x$. We will use this observation in several places in what follows.

(2.2) Lemma. *Suppose that* $(\alpha,\ \beta,\ \gamma,\ \delta) \in X \cap B^4$, $\alpha \notin \{\gamma,\ \delta\}$, $\alpha' \in B_{\eta(\alpha)}$ *and* $F_i(\alpha,\ \beta) = F_j(\gamma,\ \delta)$. *Then also* $F_i(\alpha',\ \beta) = F_j(\gamma,\ \delta)$, *and analogous statements hold where the values of the other coordinates of* $(\alpha,\ \beta,\ \gamma,\ \delta)$ *are varied instead of varying the first coordinate.*

Proof. This is clear since $(B_\eta | \eta < \kappa)$ is canonical for c^*, $\eta(\alpha) = \eta(\alpha')$ and so, as noted at the end of Definition 2.1, $(i,\ j) \in \hat{c}(\alpha,\ \beta,\ \gamma,\ \delta)$ iff $(i,\ j) \in \hat{c}(\alpha',\ \beta,\ \gamma,\ \delta)$.

(2.3) Definition. *Suppose* $\eta < \tau < \kappa$, $i < \sigma$, $\alpha \in B_\eta$, $\beta \in B_\tau$. *We define* $F^\tau_{i,\ \alpha} : B_\tau \to \lambda$ *and* $F^\beta_{i,\ \eta} : B_\eta \to \lambda$ *by* $F^\tau_{i,\ \alpha}(\beta') = F_i(\alpha,\ \beta')$ *and* $F^\beta_{i,\ \eta}(\alpha') = F_i(\alpha',\ \beta)$.

(2.4) Lemma. *If* $\eta < \tau < \kappa,\ i < \sigma$ *then:*

(1) *either (for all* $\alpha \in B_\eta$, $F^\tau_{i,\ \alpha}$ *is constant) or (for all* $\alpha \in B_\eta$, $F^\tau_{i,\ \alpha}$ *is one-to-one).*

(2) *either (for all* $\beta \in B_\tau$, $F^\beta_{i,\ \eta}$ *is constant) or (for all* $\beta \in B_\tau$, $F^\beta_{i,\ \eta}$ *is one-to-one).*

Proof. We first argue that each is either constant or one-to-one. We consider the $F^\tau_{i,\ \alpha}$. Let $\beta_1 \neq \beta_2$ both in B_τ. We claim that if $F_i(\alpha,\ \beta_1) = F_i(\alpha,\ \beta_2)$ then $F^\tau_{i,\ \alpha}$ is constant, while if $F_i(\alpha,\ \beta_1) \neq F_i(\alpha,\ \beta_1)$, then $F^\tau_{i,\ \alpha}$ is one-to-one. In the first case, $(i,\ i) \in \hat{c}(\alpha,\ \beta_1,\ \alpha,\ \beta_2)$, while in the second case, $(i,\ i) \notin \hat{c}(\alpha,\ \beta_1,\ \alpha,\ \beta_2)$. But then, by canonicity, if $\beta \in B_\tau \setminus \{\beta_1,\ \beta_2\}$, $(i,\ i) \in \hat{c}(\alpha,\ \beta,\ \alpha,\ \beta_1)$ iff $(i,\ i) \in \hat{c}(\alpha,\ \beta,\ \alpha,\ \beta_2)$ iff $(i,\ i) \in \hat{c}(\alpha,\ \beta_1,\ \alpha,\ \beta_2)$. If $(i,\ i)$ is a member of none, then $F_{i,\ \alpha}$ is one-to-one. If $(i,\ i)$ is a member of all, then $F_{i,\ \alpha}$ is constant. The argument for the $F^\beta_{i,\ \eta}$ is completely analogous.

We now argue that if $\alpha_1 \neq \alpha_2$ both in B_η, and $F^\tau_{i,\ \alpha_1}$ is constant then so is $F^\tau_{i,\ \alpha_2}$. Once again, the argument for $F^{\beta_1}_{i,\ \eta}$ and $F^{\beta_2}_{i,\ \eta}$ is completely analogous. So, suppose that $F^\tau_{i,\ \alpha_1}$ is constant.

Choose $\gamma_1 \neq \gamma_2$ both in B_τ. Since $F^\tau_{i,\ \alpha_1}$ is constant, $(i,\ i) \in \hat{c}(\alpha_1,\ \gamma_1,\ \alpha_1,\ \gamma_2)$, so, by canonicity, $(i,\ i) \in \hat{c}(\alpha_2,\ \gamma_1,\ \alpha_2,\ \gamma_2)$ which means that $F^\tau_{i,\ \alpha_2}$ is constant.

(2.5) Remark. *In Lemma 2.4, we cannot conclude that if* $F^\tau_{i,\ \alpha}$ *is constant (resp. one-to-one) then* $F^\beta_{i,\ \eta}$ *is constant (resp. one-to-one), as this would involve an "illegal" application of canonization, comparing a "1,2" case to a "2,1" case. It is, however, worth noting that if all the* $F^\tau_{i,\ \alpha}$ *are constant, then all the* $F^\beta_{i,\ \eta}$ *are constant iff all the* $F^\tau_{i,\ \alpha}$ *have the same constant value;*

similarly, if all the $F^{\beta}_{i,\ \eta}$ *are constant, then all the* $F^{\tau}_{i,\ \alpha}$ *are constant iff all the* $F^{\beta}_{i,\ \eta}$ *have the same constant value. We argue for the first statement.*

Suppose that all the $F^{\tau}_{i,\ \alpha}$ *are constant. Let* $\alpha_1 \neq \alpha_2$ *both in* B_η *and* $\beta \in B_\tau$*. Then* $F^{\beta}_{i,\ \eta}$ *is constant iff* $F_i(\alpha_1,\ \beta) = F_i(\alpha_2,\ \beta)$ *and therefore, since the* $F^{\tau}_{i,\ \alpha_j}$ *are constant, this holds iff they have the same constant value.*

(2.6) Definition. *For* $\eta < \tau < \kappa,\ i < \sigma,\ \alpha \in B_\eta,\ \beta \in B_\tau$*, we define* $F_i(\alpha,\ \tau),\ F_i(\eta,\ \beta)$ *by* $F_i(\alpha,\ \tau) =$ *the constant value of* $F^{\tau}_{i,\ \alpha}$*, if* $F^{\tau}_{i,\ \alpha}$ *is a constant function, and undefined if it is a one-to-one function. Similarly,* $F_i(\eta,\ \beta) =$ *the constant value of* $F^{\beta}_{i,\ \eta}$*, if* $F^{\beta}_{i,\ \eta}$ *is a constant function and undefined if it is a one-to-one function.*

(2.7) Remark. *It is immediate from Lemma 2.4 that for fixed* $i < \sigma$*, and fixed* $\eta < \tau < \kappa$*, either all the* $F_i(\alpha,\ \tau)$ *are defined or all the* $F_i(\alpha,\ \tau)$ *are undefined, and similarly for the* $F_i(\eta,\ \beta)$*. Further, it is immediate from Remark 2.5 that if all the* $F_i(\alpha,\ \tau)$ *are defined then all the* $F_i(\eta,\ \beta)$ *are defined iff the function* $F_i(\cdot,\ \tau)$ *is constant (and, when both of these statements hold,* $F_i(\eta,\ \cdot)$ *is also constant, with the same constant value), and the analogous equivalence holds, starting from the hypothesis that all the* $F_i(\eta,\ \beta)$ *are defined.*

(2.8) Definition. *For* $\eta < \kappa$ *and* $\alpha \in B_\eta$*, we define* W_α *to be* $\{F_i(\eta',\ \alpha) | i < \sigma,\ \eta' < \eta\} \cup \{F_i(\alpha,\ \tau) | i < \sigma,\ \eta < \tau > \kappa\}$*.*

Note that for each $\eta < \kappa,\ \{W_\alpha | \alpha \in B_\eta\}$ is a system of sets of ordinals of power at most $\sigma + \kappa$. We have stated in terms of $\sigma + \kappa$ to emphasize that we are temporarily working without any assumptions as to the order relationship between σ and κ. Thus, for all $\eta < \kappa$, we can find $B^*_\eta \subseteq B_\eta$, with *card* $B^*_\eta =$ *card* B_η such that the $(W_\alpha | \alpha \in B^*_\eta)$ form a Δ-system whose heart we denote by H_η. We also set $H = \bigcup\{H_\eta | \eta < \kappa\}$. We further assume all of the following, for each $\eta < \kappa$:

(1) $(o.t.\ W_\alpha | \alpha \in B^*_\eta)$ has constant value, o_η; for $\alpha \in B^*_\eta$, we let $(\gamma^\alpha_\xi | \xi < o_\eta)$ be the increasing enumeration of W_α,
(2) there is fixed $a_\eta \subseteq o_\eta$ such that for all $\alpha \in B^*_\eta,\ a_\eta = \{\xi < o_\eta | \gamma^\alpha_\xi \in H_\eta\}$,
(3) there is fixed $b_\eta \subseteq (\sigma \times \eta \times o_\eta) \cup (\sigma \times (\kappa \setminus (\eta + 1)) \times o_\eta)$ such that for all $\alpha \in B^*_\eta$ and all $(i,\ \nu,\ \xi),\ (i,\ \nu,\ \xi) \in b_\eta$ iff either ($\nu < \eta$ and $\gamma^\alpha_\xi = F_i(\nu,\ \alpha)$) or ($\eta < \nu$ and $\gamma^\alpha_\xi = F_i(\alpha,\ \nu)$).

(2.9) Lemma. *If* $\alpha_k \in B^*_{\eta_k}$, $\beta_k \in B^*_{\tau_k}$, $\eta_k < \tau_k < \kappa$, $k = 0,\ 1$, *and* $\{\alpha_0,\ \beta_0\} \neq \{\alpha_1,\ \beta_1\}$, *then* $A(\alpha_0,\ \beta_0) \cap A(\alpha_1,\ \beta_1) \subseteq H$.

Proof. Suppose that $F_i(\alpha_0,\ \beta_0) = F_j(\alpha_1,\ \beta_1)$. Let $\alpha^* \in B^*_{\eta_1}$, $\alpha^* \notin \{\alpha_0,\ \alpha_1\}$ and let $\beta^* \in B^*_{\tau_1}$, $\beta^* \notin \{\beta_0,\ \beta_1\}$. Then, by canonicity, $F_i(\alpha_0,\ \beta_0) = F_j(\alpha^*,\ \beta_1)$, so $F_j(\alpha_1,\ \beta_1) = F_j(\alpha^*,\ \beta_1)$, which means that $F_i(\alpha_0,\ \beta_0) \in W_{\beta_1}$. By a similar argument, $F_i(\alpha_0,\ \beta_0) = F_j(\alpha_1,\ \beta^*) = F_j(\alpha^*,\ \beta^*) \in W_{\beta^*}$, and then, since $F_i(\alpha_0,\ \beta_0) \in W_{\beta_1} \cap W_{\beta^*}$, $F_i(\alpha_0,\ \beta_0) \in H_{\tau_1} \subseteq H$, as required.

Let $\mathbf{P}_0 = \mathbf{P}|(H \cup dom\ p)$ and let $V' = V^{\mathbf{P}_0}$. Note that all our hypotheses on V still hold in V' and $V^{\mathbf{P}} = (V')^{\mathbf{Q}}$, where, in V', $\mathbf{Q} \cong \mathbf{P}$. Thus, we can first force with $\mathbf{P}_0$ without changing anything relevant; therefore, we can assume that $H,\ p = \emptyset$, which we do, from here on. By Lemma 2.9, this, of course, guarantees that

$(*)$ For $\alpha_i,\ \beta_i$ as in Lemma 2.9, $A(\alpha_0,\ \beta_0) \cap A(\alpha_1,\ \beta_1) = \emptyset$.

Now choose $\alpha_\eta \in B^*_\eta$ for $\eta < \kappa$.

It is at this point that the proof of Theorem 4*, in §3, will begin to diverge. Here, we will assume that p also forces that $\mathbf{c}$ has no homogeneous green set of order type $\theta + 1$ and we will show that p forces that $\mathbf{c}$ has a homogeneous red set of power λ, while in §3, in the proof of Theorem 4*, our treatment of the colors will be more "symmetrical". However, the remainder of the argument, here, will be similar quite similar in spirit to the argument in Case 2 in §3, below.

Recall that here, we have already argued that, in $V^{\mathbf{P}}$, $\kappa \to (\kappa,\ \theta+1)^2$. Thus, in $V^{\mathbf{P}}$, there must be $S \subseteq \kappa$ of power κ such that $\{\alpha_\eta | \eta \in S\}$ is homogeneous red for $\mathbf{c}|\{\alpha_\eta | \eta < \kappa\}$ (and therefore also for $\mathbf{c}$).

So, let $\mathbf{S}$ be a $\mathbf{P}$-name and $q \in P$, $p \leq q$ be such that q forces that $\{\alpha_\eta | \eta \in \mathbf{S}\}$ is homogeneous red for $\mathbf{c}$ and that $\mathbf{S}$ has power κ. Then, in V, there are $S \subseteq \kappa$, and for $\eta \in S$, $q_\eta \in P$, $q \leq q_\eta$ such that q_η forces that $\alpha_\eta \in \mathbf{S}$.

We may assume, without loss of generality, that the $(q_\eta | \eta \in S)$ form a Δ-system with heart q (by which we mean that the q_η are pairwise isomorphic as well). Thus, the q_η, for $\eta \in S$ are pairwise compatible and whenever $\eta < \tau$ are both in S and $q_\eta,\ q_\tau \leq r \in P$, r forces that $\mathbf{c}(\alpha_\eta,\ \alpha_\tau) = $ red.

Let $A^* = \bigcup\{A(\alpha_\eta,\ \alpha_\tau) | \eta < \tau$, both in $S\}$. We may also assume that for all $\eta \in S$, $dom\ q_\eta \subseteq A^*$. This is because if this fails, then, letting $\overline{q}_\eta = q_\eta | A^*$, whenever $\eta < \tau$ are both in S and $\overline{q}_\eta,\ \overline{q}_\tau \leq \overline{r} \in P$, $\overline{r}$ forces that $\mathbf{c}(\alpha_\eta,\ \alpha_\tau)$ = red, because $\mathbf{c}(\alpha_\eta,\ \alpha_\tau)$ is a $\mathbf{P}|A(\alpha_\eta,\ \alpha_\tau)$-name and r forces that $\mathbf{c}(\alpha_\eta,\ \alpha_\tau)$ = red, where $r = \overline{r} \cup (q_\eta \setminus \overline{q}_\eta) \cup (q_\tau \setminus \overline{q}_\tau)$, and this is all that is required for the rest of the argument.

Further, we can clearly thin out S to obtain a subset, S', also of power κ, such that for $\tau \in S'$, letting $\tau' = min\ S' \setminus (\tau + 1),\ dom\ q_\tau \setminus dom\ q \subseteq \bigcup\{A(\alpha_{\eta_1},\ \alpha_{\eta_2})|\eta_1 < \eta_2 < \tau'$ both in $S\}$.

Finally, for $\tau \in S'$ and $\alpha \in B^*_\tau$, we make a copy q^τ_α of q_i, above q. We do this by moving only coordinates in the $A(\alpha_\eta,\ \alpha_\tau)$ and the $A(\alpha_\tau,\ \alpha_\eta)$ which are in $dom\ q_\tau \setminus dom\ q$. We move these coordinates according to the order-isomorphisms between the $A(\alpha_\eta,\ \alpha_\tau)$ and the $A(\alpha_\eta,\ \alpha)$, and the order-isomorphisms between the $A(\alpha_\tau,\ \alpha_\eta)$ and the $A(\alpha,\ \alpha_\eta)$. Clearly by $(*)$, above and by the previous paragraph, this is well-defined. Also, by Lemma 2.9 and $(*)$, the q^τ_α are pairwise compatible.

Further, arguing as in the paragraph immediately preceding Lemma 2.2, it is easy to see that whenever $\eta < \tau$ are both in S', $\alpha \in B^*_\eta$, $\beta \in B^*_\tau$ and $q^\eta_\alpha \cup q^\tau_\beta \le r$, r forces that $\mathbf{c}(\alpha,\ \beta) =$ red. Finally, clearly, whenever $q \le r \in P$, r is incompatible with fewer than λ many of the q^τ_α for $\tau \in S'$ and $\alpha \in B^*_\tau$. But then, letting $\mathbf{G}$ be the canonical $\mathbf{P}$-name for the generic, let $\mathbf{Y}$ be the following $\mathbf{P}$-name:

$$\{\alpha|\text{ there is } \tau \in S' \text{ such that } \alpha \in B^*_\tau \text{ and } q^\tau_\alpha \in \mathbf{G}\}$$

But then q forces that $\mathbf{Y}$ has power λ and is homogeneous red for $\mathbf{c}$. This concludes the proof of Theorem 3.

§3. ADDING COHEN SETS ABOVE THE COFINALITY.

Recall that in the Introduction we have already argued that Theorem 4 follows from Theorem 4*. Here, we will prove Theorem 4*, whose statement we recall.

Theorem 4*. *Suppose that in* V, $\omega < \kappa < \sigma = \sigma^{<\sigma} < \lambda,\ \kappa = cf\ \lambda \le \nu, \lambda$ *is a strong limit cardinal and* $\mathbf{P}$ *is the partial ordering for adding* ν *Cohen subsets of* σ. *Suppose, further, that* $\zeta < \kappa$ *and that* $\kappa \to (\kappa,\ \zeta)^2$. *Then, in* $V^{\mathbf{P}},\ \lambda \to (\lambda,\ \zeta)^2$.

Proof. We carry over from §2 all the material up to and including the choice of the $\alpha_\eta \in B^*_\eta$, for $\eta < \kappa$, and in particular, (2.1) - (2.9), except that here, the analogue of (!) of §2 is:

(!!) in $V^{\mathbf{P}}$, for each $\eta < \kappa$ which is either 0 or a successor ordinal, $\lambda_\eta \to (\lambda_\eta,\ \zeta)^2$.

The argument for this exactly follows that for (!) in §2. Also, as noted in the Introduction, it follows from the hypotheses of the Theorem, that in $V^{\mathbf{P}}$, $\kappa \to (\kappa,\ \zeta)^2$. Once again, (!!) enables us to assume, without loss of generality, that p forces that that each Δ_η is homogeneous red for $\mathbf{c}$.

Note that it is an easy consequence of Lemma 2.9 and our assumption that $H = \emptyset$ that if $\eta < \tau < \kappa$, $q \in P|H_{\alpha_\eta}$, $r \in P|H_{\alpha_\tau}$ then q and r are compatible. Recall that by the paragraph immediately preceding $(*)$, of §2, we are assuming that $p = \emptyset$. Let $s \in P$. We now argue, using the σ-completeness of $\mathbf{P}$ and the fact that $\sigma > \kappa$, that:

Lemma 3.1. *In V, there is $(p_\eta | \eta < \kappa)$ such that for all $\eta < \kappa$, $s|H_{\alpha_\eta} \leq p_\eta$, dom $p_\eta \subseteq H_{\alpha_\eta}$, and such that*

$(**)$ *if $\eta < \tau < \kappa$, $x \in \{$red, green$\}$ either $s \cup p_\eta \cup p_\tau$ forces $\mathbf{c}(\alpha_\eta,\ \alpha_\tau) = x$ or there is $q \in P|A(\alpha_\eta,\ \alpha_\tau)$ such that*

1. *$s \cup p_\eta \cup p_\tau \leq q$,*
2. *q forces $\mathbf{c}(\alpha_\eta,\ \alpha_\tau) \neq x$,*
3. *dom $q \setminus ($dom $p_\eta \cup$ dom $p_\tau) \subseteq A(\alpha_\eta,\ \alpha_\tau) \setminus (H_{\alpha_\eta} \cup H_{\alpha_\tau})$.*

Proof. Let $((\eta_\gamma,\ \tau_\gamma) | \gamma < \kappa)$ enumerate all the pairs $(\eta,\ \tau)$ with $\eta < \tau < \kappa$. For $\gamma \leq \kappa$, we define $(p^\gamma_\eta | \eta < \kappa)$ by recursion on γ so that $p^\gamma_\eta \in P|H_{\alpha_\eta}$, and for all $\eta < \kappa$ and all $\gamma_1 < \gamma_2 \leq \kappa$, $p^{\gamma_1}_\eta \leq p^{\gamma_2}_\eta$.

For $\eta < \kappa$, let $p^0_\eta = s|H_{\alpha_\eta}$ and for nonzero limit ordinals, $\delta < \kappa$, and $\eta < \kappa$, let $p^\delta_\eta = \bigcup \{p^\gamma_\eta | \gamma < \delta\}$. So, suppose that $\gamma = \xi + 1$. If $\eta \notin \{\eta_\xi,\ \tau_\xi\}$ we take $p^\gamma_\eta = p^\xi_\eta$. We construct $p^\gamma_{\eta_\xi}$, $p^\gamma_{\tau_\xi}$. Let $\eta = \eta_\xi$, $\tau = \tau_\xi$, $\alpha = \alpha_\eta$, $\alpha' = \alpha_\tau$, and let $p(0) = p^\xi_\eta$, $p'(0) = p^\xi_\tau$. Identify red with 0 and green with 1. We will have $p^\gamma_\eta = p(2)$, $p^\gamma_\tau = p'(2)$, where we define $p(i)$, $p'(i)$, $i = 1,\ 2$ by the following two-stage recursion. If $k = 0,\ 1$ and $p(k)$, $p'(k)$ are defined, and if $s \cup p(k) \cup p'(k)$ forces $\mathbf{c}(\alpha,\ \alpha') = k$, then we set $p(k+1) = p(k)$, $p'(k+1) = p'(k)$. Otherwise, choose $q \in P|A(\alpha,\ \alpha')$ such that $s \cup p(k) \cup p'(k) \leq q$ and such that q forces $\mathbf{c}(\alpha,\ \alpha') = 1 - k$. Finally, let $p(k+1) = q|H_\alpha$, $p'(k+1) = q|H_{\alpha'}$.

Clearly then, by construction, for $\eta < \kappa$, taking $p_\eta = p^\kappa_\eta$, p_η is as required. This completes the proof of the Lemma.

Remarks. *Although we have developed it for both colors, we only use the machinery of* $(**)$ *of Lemma 3.1 with* $x =$ *red. Also, in* $(**)$*, if* $s \cup p_\eta \cup p_\tau$ *does not force that* $\mathbf{c}(\alpha_\eta,\ \alpha_\tau) =$ *red, we choose* $q_{\eta,\ \tau}$ *to be some* q *whose existence is guaranteed by* $(**)$*.*

Now, still working in V, we define $d : [\kappa]^2 \to \{\text{red, green}\}$ by $d(\eta,\ \tau) =$ red iff $s \cup p_\eta \cup p_\tau$ forces $\mathbf{c}(\alpha_\eta,\ \alpha_\tau) =$ red. Now, in V, $\kappa \to (\kappa,\ \theta + 1)^2$, so either (Case 1) there is $Y \in [\kappa]^\zeta$ which is homogeneous green for d, or (Case 2) there is $Y \in [\kappa]^\kappa$ which is homogeneous red for d. We show that in Case 1, s has an extension which forces that there is a set of order type ζ which is homogeneous green for $\mathbf{c}$, while in Case 2, s, itself, forces that there is a set of power λ which is homogeneous red for $\mathbf{c}$. Clearly this suffices, since then the empty condition forces that $\lambda \to (\lambda,\ \zeta)^2$. We consider the cases separately.

Case 1: The Green Case.

Let $Y \in [\kappa]^\zeta$ be homogeneous green for d. For $\eta < \tau$ both in Y, note that $q_{\eta,\ \tau}$ is defined, since $d(\alpha_\eta,\ \alpha_\tau) = green$. Set

$$r = \bigcup\{q_{\eta,\ \tau} | \eta < \tau \text{ both in } Y\}$$

Once we have argued that r is a function, it will be clear that $r \in P,\ s \leq r$ (since for any $\eta < \tau$ which are both in $Y,\ s \leq q_{\eta,\ \tau}$) and further that r forces that $\{\alpha_\eta | \eta \in Y\}$ is homogeneous green for $\mathbf{c}$, since, again, whenever $\eta < \tau$ are both in Y, $q_{\eta,\ \tau}$ forces that $\mathbf{c}(\alpha_\eta,\ \alpha_\tau) =$ green. But, once again, it follows from the conjunction of Lemma 2.9 and $(*)$ that r is a function. This completes the proof in Case 1.

Case 2: The Red Case.

As we already noted there, the last part of the argument in §2 is quite similar in spirit to the argument we shall give for this case. Let $Y \in [\kappa]^\kappa$ be homogeneous red for d. As in §2, for $\eta \in Y$ and $\alpha \in B^*_\eta$, let $p^\eta_\alpha = T(p_\eta)$, where T is the order isomorphism between H_{α_η} and H_α. Once again, the p^η_α $(\eta \in Y,\ \alpha \in B^*_\eta)$ are pairwise compatible, by Lemma 2.9 and $(*)$, and whenever $\eta < \tau$ are both in Y, $\alpha \in B^*_\eta,\ \beta \in B^*_\tau$ and $s \cup p^\eta_\alpha \cup p^\tau_\beta \leq q,\ q$ forces that $\mathbf{c}(\alpha,\ \beta) =$ red, by the fact that $d(\eta,\ \tau) =$ red and by the argument of the paragraph immediately preceding Lemma 2.2. Also, once again, for all $s \leq q \in P,\ q$ is incompatible with at most σ of the p^η_α.

Now, let $\mathbf{G}$ again be the canonical $\mathbf{P}$-name of the generic, and for $\eta \in Y$, let $\mathbf{X}_\eta$ be the $\mathbf{P}$-name $\{\alpha \in B^*_\eta | p^\eta_\alpha \in \mathbf{G}\}$. Then, since *card* $B^*_\eta > \sigma$, s forces that *card* $\mathbf{X}_\eta =$ *card* B^*_η. We conclude by noting that by the previous paragraph, s also forces that "if $\eta < \tau$ are both in Y, $\alpha \in \mathbf{X}_\eta$, $\beta \in \mathbf{X}_\tau$ then $\mathbf{c}(\alpha, \beta) =$ red.". In other words, "as promised", s forces that $\bigcup\{\mathbf{X}_\eta | \eta \in Y\}$ is homogeneous red for $\mathbf{c}$ and has power λ. This concludes the proof of Case 2, and therefore of Theorem 4*.

References

1. P. Erdös, A. Hajnal, A. Mate and R. Rado, *Combinatorial Set Theory: Partition Relations for Cardinals*, North-Holland, New York, 1984.
2. S. Shelah, *276 Was Sierpinski Right, I?*, Israel J. Math **62** (1988), 335 - 380.
3. ______, *Cardinal Arithmetic*, Oxford Logic Guides: 29, Oxford University Press, Oxford, 1994.
4. ______, *On CH* $+ 2^{\aleph_1} \to (\alpha)^2_2$ *for* $\alpha < \omega_2$, Logic Colloquium '90. ASL Summer Meeting in Helsinki, Lecture Notes in Logic, vol 2, J. Oikkonen and J. Väänänen, eds., Springer Verlag, Berlin, 1993, pp. 281-289.
5. ______, *Borel sets with large squares*, Fundamenta Mathematica, submitted.
6. S. Shelah and L. Stanley, *A theorem and some consistency results in partition calculus*, Annals of Pure and Applied Logic **36** (1987 pages 119 - 152).
7. ______, *Consistent negative and positive partition relations for singular cardinals of uncountable cofinality*, in preparation.

Hebrew University, Rutgers University

Lehigh University