LENGTH OF BOOLEAN ALGEBRAS AND ULTRAPRODUCTS

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Abstract. We prove the consistency with ZFC of "the length of an ultraproduct of Boolean algebras is smaller than the ultraproduct of the lengths". Similarly for some other cardinal invariants of Boolean algebras.

0. Introduction

On the length of Boolean algebras (the cardinality of linearly ordered subsets) see Monk [M1], [M2] (and Definition 1.1 below). In Shelah [Sh 345, §1] it is said that Koppelberg and Shelah noted that by the Łoś theorem for an ultrafilter $\mathcal{D}$ on κ and Boolean algebras B_i $(i<\kappa)$ we have

$$(*)\qquad |\prod_{i<\kappa}\mathrm{Length}(B_i)/\mathcal{D}| \leq \mathrm{Length}(\prod_{i<\kappa} B_i/\mathcal{D}),\ \text{and}$$
$$\mu_i < \mathrm{Length}(B_i) \quad\Rightarrow\quad |\prod_{i<\kappa}\mu_i/\mathcal{D}| < \mathrm{Length}(\prod_{i<\kappa} B_i/\mathcal{D}).$$

D. Peterson noted that the indicated proof fails, but holds for regular ultrafilters (see [Pe97]). Now the intention in [Sh 345] was for Length^+, i.e.

$$(*)^+\qquad |\prod_{i<\kappa}\mathrm{Length}^+(B_i)/\mathcal{D}| \leq \mathrm{Length}^+(\prod_{i<\kappa} B_i/\mathcal{D}),$$

where $\mathrm{Length}^+(B)$ is the first cardinal not represented as the cardinality of a linearly ordered subset of the Boolean Algebra (the only difference being the case the supremum is not attained).

Here we prove that the statement $(*)$ may fail (see Theorem 1.3 and Proposition 1.6). The situation is similar for many cardinal invariants.

Of course, if $(*)$ fails then (using ultraproducts of $\langle(\mathcal{H}(\chi),B_i):i<\kappa\rangle$ or see e.g. Rosłanowski, Shelah [RoSh 534, §1]) we have $\{i<\kappa:\mathrm{Length}^+(B)$ is a limit cardinal $\}\in\mathcal{D}$, and $\prod_{i<\kappa}\mathrm{Length}^+(B_i)/\mathcal{D}$ is λ–like for some successor cardinal λ. Hence

$$\{i<\kappa:\mathrm{Length}^+(B_i)\text{ is a regular cardinal}\}\in\mathcal{D}$$

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hence $\{i : \text{Length}^+(B_i) \text{ is an inaccessible cardinal}\} \in D$, so the example we produce is in some respect the only one possible. (Note that our convention is that "inaccessible" means regular limit ($> \aleph_0$), not necessarily strong limit.)

More results on cardinal invariants of ultraproducts of Boolean algebras can be found in [Sh 462], [Sh 479], [RoSh 534] and [Sh 620], [RoSh 651]. This paper is continued for other cardinal invariants (in particular spread) in [ShSi 677].

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1. The main result

Definition 1.1. (1) For a Boolean algebra B, let its length, Length(B), be $\sup\{|X| : X \subseteq B$, and X is linearly ordered (in B) $\}$.
(2) For a Boolean algebra B, let its strict length, Length$^+(B)$, be

$$\sup\{|X|^+ : X \subseteq B, \text{ and } X \text{ is linearly ordered (in } B)\}.$$

Remark 1.2. (1) In Definition 1.1, Length$^+(B)$ is (equivalently) the first λ such that for every linearly ordered $X \subseteq B$ we have $|X| < \lambda$.
(2) If Length$^+(B)$ is a limit cardinal then Length$^+(B) =$ Length(B); and if Length$^+(B)$ is a successor cardinal then Length$^+(B) =$ (Length$(B))^+$.

Theorem 1.3. *Suppose* $\mathbf{V}$ *satisfies* GCH *above* μ *(for simplicity),* κ *is measurable,* $\kappa < \mu$, μ *is* λ^+*-hypermeasurable (somewhat less will suffice),* F *is the function such that* $F(\theta) =$ *the first inaccessible* $> \theta$, *and* $\lambda = F(\mu)$ *is well defined, and* $\chi < \mu$, $\chi > 2^{2^\kappa}$.

Then for some forcing notion $\mathbb{P}$ *not collapsing cardinals, except those in the interval* (μ^+, λ) *[so in* $\mathbf{V}^{\mathbb{P}}$ *we have* $\mu^{++} = \lambda = F^{\mathbf{V}}(\mu)$*], and not adding subsets to* χ, *in* $\mathbf{V}^{\mathbb{P}}$, *we have:*

(α) *in* $\mathbf{V}^{\mathbb{P}}$ *the cardinal* μ *is a strong limit of cofinality* κ,
(β) *for some strictly increasing continuous sequence* $\langle \mu_i : i < \kappa\rangle$ *of (strong limit) singular cardinals* $> \chi$ *with limit* μ, *each* $\lambda_i =: F^{\mathbf{V}}(\mu_i)$ *is still inaccessible and for any normal ultrafilter* $\mathcal{D} \in \mathbf{V}$ *on* κ *we have:*
$$\prod_{i<\kappa} \lambda_i/\mathcal{D} \textit{ has order type } \mu^{++} = F^{\mathbf{V}}(\mu).$$

Definition 1.4. A forcing notion $\mathbb{Q}$ is directed μ-complete if: for a directed quasi-order I (so $(\forall s_0, s_1 \in I)(\exists t \in I)(s_0 \leq_I t \;\&\; s_1 \leq t)$) of cardinality $< \mu$, and $\bar{p} = \langle p_t : t \in I\rangle$ such that $p_t \in \mathbb{Q}$ and $s \leq_I t \Rightarrow p_s \leq_{\mathbb{Q}} p_t$, there is $p \in \mathbb{Q}$ such that $t \in I \Rightarrow p_t \leq_{\mathbb{Q}} p$.

PROOF Without loss of generality for every directed μ–complete forcing notion $\mathbb{Q}$ of cardinality at most λ satisfying the λ-c.c., in $\mathbf{V}^{\mathbb{Q}}$ the cardinal μ is still λ-hypermeasurable. [Why? If μ supercompact, use Laver [L], if μ is just λ-hypermeasurable see more in Gitik Shelah [GiSh 344].]

Let $\mathbb{Q}$ be the forcing notion adding λ Cohen subsets to μ, i.e., $\{f : f$ a partial function from λ to $\{0,1\}$, $|\mathrm{Dom}(f)| < \mu\}$.

In $\mathbf{V}$, let $\mathbb{R} = \mathrm{Levy}(\mu^+, <\lambda) = \{f : f$ a partial two place function such that $[f(\alpha,i)$ defined $\Rightarrow$ $0 < \alpha < \lambda$ & $i < \mu^+$ & $f(\alpha,i) < \alpha]$ and $|\mathrm{Dom}(f)| < \mu^+\}$ (so $\mathbb{R}$ collapses all cardinals in (μ^+, λ) and no others, so in $\mathbf{V}^{\mathbb{R}}$ the ordinal λ becomes μ^{++}). Clearly $\mathbb{R}$ is μ^+–complete and hence adds no sequence of length $\le \mu$ of members of $\mathbf{V}$.

In $\mathbf{V}^{\mathbb{Q}}$, there is a sequence $\bar{\mathcal{D}} = \langle \mathcal{D}_i : i < \kappa\rangle$ of normal ultrafilters on μ as in [Mg4] and, $\bar{g} = \langle g_{i,j} : i < j \le \kappa\rangle$, $g_{i,j} \in {}^{\mu}\mathcal{H}(\mu)$ witness this (that is $\mathcal{D}_i \in (\mathbf{V}^{\mathbb{Q}})^{\kappa}/\mathcal{D}_j$, in fact $\mathcal{D}_i$ is equal to $g_{i,j}/\mathcal{D}_j$ in the Mostowski collapse of $(\mathbf{V}^{\mathbb{Q}})^{\kappa}/\mathcal{D}_j$). Let $\underset{\sim}{\bar{\mathcal{D}}} = \langle \underset{\sim}{\mathcal{D}}_i : i \le \kappa\rangle$, $\underset{\sim}{\bar{g}} = \langle \underset{\sim}{g}_{i,j} : i < j \le \kappa\rangle$ be $\mathbb{Q}$–names of such sequences. Note that a $\mathbb{Q}$–name $\underset{\sim}{A}$ of a subset of μ is an object of size $\le \mu$, i.e., it consists of a μ–sequence of μ–sequences of members of $\mathbb{Q}$, say $\langle\langle p_{i,j} : j < \mu\rangle : i < \mu\rangle$, and function $f : \mu \times \mu \longrightarrow \{0,1\}$ such that each $\{p_{i,j} : j < \mu\}$ is a maximal antichain of $\mathbb{Q}$ and $p_{i,j} \Vdash_{\mathbb{Q}}$ "$i \in A \Leftrightarrow f(i,j) = 1$". So the set of members of $\mathbb{Q}$ and the set of $\mathbb{Q}$–names of subsets of μ are the same in $\mathbf{V}$ and in $\mathbf{V}^{\mathbb{R}}$. So in $\mathbf{V}^{\mathbb{R}\times\mathbb{Q}}$ the sequence $\bar{\mathcal{D}}$ still gives a sequence of normal ultrafilters as required in [Mg4] as witnessed by $\bar{g} = \langle g_{i,j} : i < j \le \kappa\rangle$. Also the Magidor forcing $\mathbb{P}(\bar{\mathcal{D}}, \bar{g})$ (from there) for changing the cofinality of μ to κ (not collapsing cardinals not adding subsets to χ, the last is just by fixing the first element in the sequence) is the same in $\mathbf{V}^{\mathbb{Q}}$ and $\mathbf{V}^{\mathbb{R}\times\mathbb{Q}}$ and has the same set of names of subsets of μ. We now use the fact that $\mathbb{P}(\bar{\mathcal{D}}, \bar{g})$ satisfies the μ^+-c.c. (see [Mg4]). Let $\mathbb{P} = (\mathbb{Q} \times \mathbb{R}) * \mathbb{P}(\underset{\sim}{\bar{\mathcal{D}}}, \underset{\sim}{\bar{g}})$, so again every $\mathbb{Q} * \mathbb{P}(\underset{\sim}{\bar{\mathcal{D}}}, \underset{\sim}{\bar{g}})$-name involves only μ decisions so also $\mathbf{V}^{\mathbb{Q}*\mathbb{P}(\underset{\sim}{\bar{\mathcal{D}}},\underset{\sim}{\bar{g}})}$, $\mathbf{V}^{(\mathbb{Q}\times\mathbb{R})*\mathbb{P}(\underset{\sim}{\bar{\mathcal{D}}},\underset{\sim}{\bar{g}})}$ have the same subsets of μ. So our only problem is to check conclusion (β) of Theorem 1.3.

Let $\mathcal{D} \in \mathbf{V}$ be any normal ultrafilter on κ (so this holds also in $\mathbf{V}^{\mathbb{R}}$, $\mathbf{V}^{\mathbb{R}\times\mathbb{Q}}$, $(\mathbf{V}^{\mathbb{R}\times\mathbb{Q}})^{\mathbb{P}(\underset{\sim}{\bar{\mathcal{D}}})}$).

Claim 1.4.1. *In $\mathbf{V}^{\mathbb{Q}*\mathbb{P}(\underset{\sim}{\bar{\mathcal{D}}},\underset{\sim}{\bar{g}})}$, the linear order $\prod\limits_{i<\kappa} F(\mu_i)/\mathcal{D}$ has true cofinality $F(\mu) = \lambda$.*

Proof of the Claim: Clearly $\Vdash_{\mathbb{Q}}$ "for $i < \kappa$ we have $\mu^{\mu}/\underset{\sim}{\mathcal{D}}_i$ is well ordered" (as $\underset{\sim}{\mathcal{D}}_i$ is $\aleph_1$–complete) and

$$\Vdash_{\mathbb{Q}} \quad \text{"for } i < \kappa \text{ we have } \mu^{\mu}/\underset{\sim}{\mathcal{D}}_i \text{ has cardinality } 2^{\mu} \text{ and even } \prod_{i<\mu} 2^{|i|}/\underset{\sim}{\mathcal{D}}_i \text{ has cardinality } 2^{\mu}\text{"}$$

[Why? As $\mu = \mu^{<\mu}$ and $\mathcal{D}_i$ is a uniform ultrafilter on μ. In details, let $h : {}^{\mu>}2 \to \mu$ be one-to-one, and for each $\eta \in {}^{\mu}2$ define $g_\eta \in {}^{\mu}\mu$ by $g_\eta(i) = h(\eta \restriction i)$. Then

$$\eta \neq \nu \in {}^{\mu}2 \quad \Rightarrow \quad \{i < \mu : g_\eta(i) = g_\nu(i)\} \text{ is a bounded subset of } \mu$$

and hence its complement belongs to $\mathcal{D}_i$ but $|\{g_\eta(i) : \eta \in {}^{\mu}2\}| = 2^{|i|}$].

Consequently, for some $F^* \in {}^{\mu}\mu$ we have

$$\Vdash_{\mathbb{Q}} \text{“} \prod_{\zeta<\mu} F^*(\zeta)/\underset{\sim}{\mathcal{D}}_i \text{ is isomorphic to } \lambda\text{”}.$$

If we look at the proof in [L] (or [GiSh 344]) which we use above, we see that w.l.o.g. F^* is the F above (and so does not depend on i). So let $\underset{\sim}{f}_{i,\alpha}$ be $\mathbb{Q}$-names such that

$$\Vdash_{\mathbb{Q}} \quad \begin{array}{l}\text{“for } i < \kappa \text{ and } \alpha < \lambda, \quad \underset{\sim}{f}_{i,\alpha} \in \prod_{\zeta<\mu} F(\zeta) \text{ and} \\ \quad f_{i,\alpha}/\underset{\sim}{\mathcal{D}}_i \text{ is the } \alpha\text{-th function in } \prod\limits_{\zeta<\mu} F(\zeta)/\underset{\sim}{\mathcal{D}}_i.\text{”}\end{array}$$

In $\mathbf{V}^{\mathbb{Q}}$ let $\mathcal{D}_\kappa = \bigcap\limits_{i<\kappa} \mathcal{D}_i$ and let $B_i \in \mathcal{D}_i \setminus \bigcup\limits_{j<i} \mathcal{D}_j$ be as in [Mg4] (you can also produce them straightforwardly), so $\langle B_i : i < \kappa \rangle$ is a sequence of pairwise disjoint subsets of μ. Define $f_\alpha \in {}^{\mu}\mu$ for $\alpha < \lambda$ as follows:

$$f_\alpha \restriction B_i = f_{i,\alpha} \restriction B_i \ \text{ and } \ f_\alpha \restriction (\mu \setminus \bigcup_{i<\kappa} B_i) \text{ is constantly zero.}$$

So $\langle f_\alpha : \alpha < \mu \rangle$ is $<_{\mathcal{D}_\kappa}$–increasing and cofinal in $\prod\limits_{\zeta<\mu} F(\zeta)$. Let $\underset{\sim}{\mathcal{D}}_\kappa$, $\underset{\sim}{B}_i$, $\underset{\sim}{f}_\alpha$ be $\mathbb{Q}$–names forced to be as above. Then $\Vdash_{\mathbb{Q}}$ “for $\alpha < \beta < \lambda$ the set $\underset{\sim}{A}_{\alpha,\beta} = \{\zeta < \mu : f_\alpha(\zeta) < f_\beta(\zeta)\}$ belongs to $\underset{\sim}{\mathcal{D}}_\kappa$”.

Now in $\mathbf{V}^{\mathbb{Q}}$, one of the properties of Magidor forcing $\mathbb{P}(\bar{\mathcal{D}}, \bar{g})$ is that

$$\Vdash_{\mathbb{P}(\bar{\mathcal{D}},\bar{g})} \quad \begin{array}{l}\text{“for every } A \in \mathcal{D}_\kappa = \bigcap\limits_{i<\kappa} \mathcal{D}_i \\ \qquad \text{for every } i < \kappa \text{ large enough we have } \underset{\sim}{\mu}_i \in A\text{”}\end{array}$$

(where $\langle \underset{\sim}{\mu}_i : i < \kappa \rangle$ is the increasing continuous κ–sequence cofinal in μ which $\mathbb{P}(\bar{\mathcal{D}}, \bar{g})$ adds).

Since for every $p \in \mathbb{P}(\bar{\mathcal{D}}, \bar{g})$, for some $q \geq p$ we have (recall from [Mg4] that $F^q(i)$ is the set which q “says” $\underset{\sim}{\mu}_i$ belongs to (when q does not forces a value to $\underset{\sim}{\mu}_i$))

$$[F^q(i) \subseteq A_{\alpha,\beta} \text{ for every } i < \kappa \text{ large enough}],$$

hence in $\mathbf{V}^{\mathbb{Q} * \mathbb{P}(\underset{\sim}{\bar{\mathcal{D}}},\bar{g})}$, for $\alpha < \beta < \lambda$ the set $\{i < \kappa : \neg(f_\alpha(\mu_i) < f_\beta(\mu_i))\}$ is bounded, i.e., $\langle \langle f_\alpha(\mu_i) : i < \kappa \rangle : \alpha < \lambda \rangle$ is $<_{J^{\mathrm{bd}}_\kappa}$–increasing in $\prod\limits_{i<\kappa} F(\mu_i)$.

On the other hand, in $\mathbf{V}^{\mathbb{Q}}$, assume $p \Vdash_{\mathbb{P}(\bar{\mathcal{D}},\bar{g})}$ "$\underset{\sim}{f} \in \prod\limits_{i<\kappa} F(\underset{\sim}{\mu}_i)$". W.l.o.g. p forces that $F^*(\underset{\sim}{\mu}_i) < \underset{\sim}{\mu}_{i+1}$, and so by [Mg4] (possibly increasing p), we have, for some function h, $\mathrm{Dom}(h) = \kappa$, $h(i) \in [i]^{<\aleph_0}$, that above p, we know: $\underset{\sim}{f}(i)$ depends on the value of $\underset{\sim}{\mu}_j$ for $j \in \{i\} \cup h(i)$. So we can define a function $f^* \in {}^{\mu}\mu$:

$$f^*(\zeta) = \sup\{\gamma : \begin{array}{l}\text{for some } i<\kappa, \text{ possibly } \underset{\sim}{\mu}_i \text{ is } \zeta \text{ and}\\ \gamma \text{ is a possible value (above } p) \text{ of } \underset{\sim}{f}(i)\}.\end{array}$$

So $f^*(\zeta) < F(\zeta)$, hence (in $\mathbf{V}^{\mathbb{Q}}$) for some α, $f^* <_{\mathcal{D}_\kappa} f_\alpha$ and consequently $p \Vdash$ "$f^* <_{\underset{\sim}{J}^{\rm bd}_\kappa} f_\alpha$". So $\langle f_\alpha : \alpha < \lambda\rangle$ is, in $\mathbf{V}^{\mathbb{Q}*\mathbb{P}(\bar{\mathcal{D}},\bar{g})}$, $<_{\underset{\sim}{J}^{\rm bd}_\kappa}$–increasing and cofinal in $\prod\limits_{i<\kappa} F(\mu_i)$ which is more than enough for 1.4.1.

Note that 1.4.1 holds in $\mathbf{V}^{(\mathbb{R}\times\mathbb{Q})*\mathbb{P}(\bar{\mathcal{D}},\bar{g})}$ too (remember that any sequence $\langle \alpha_i < F(\mu_i) : i < \kappa\rangle$ in $\mathbf{V}^{(\mathbb{Q}\times\mathbb{R})*\mathbb{P}(\bar{\mathcal{D}},\bar{g})}$ is bounded by a function $f \in \big(\prod\limits_{\theta\in\mathrm{Reg}\cap\mu} F(\theta)\big)^{\mathbf{V}}$ (see [Mg4]) and also is itself in $\mathbf{V}^{\mathbb{Q}*\mathbb{P}(\bar{\mathcal{D}},\bar{g})}$).

But why, if $\alpha_i < F(\mu_i)$ and $\langle \alpha_i : i < \kappa\rangle \in \mathbf{V}^{(\mathbb{R}\times\mathbb{Q})*\mathbb{P}(\bar{\mathcal{D}},\bar{g})}$, do we have that $\prod\limits_{i<\kappa} \alpha_i/\mathcal{D}$ has cardinality $\le \mu^+$ (this means $< \lambda$)?

It suffices to prove this inequality in the universe $\mathbf{V}_1 = \mathbf{V}^{\mathbb{Q}*\mathbb{P}(\bar{\mathcal{D}},\bar{g})}$.

Now $\mathbf{V}_1^\kappa/\mathcal{D}$ is well founded, hence there is an isomorphism from $\mathbf{V}_1^\kappa/\mathcal{D}$ onto a transitive class which we now call M and let j be the isomorphism (= the Mostowski collapse). As μ in $\mathbf{V}^{\mathbb{Q}*\mathbb{P}(\bar{\mathcal{D}},\bar{g})}$ is strong limit $> \kappa$, clearly $\alpha < \mu \quad\Rightarrow\quad j(\langle \alpha : i < \kappa\rangle/\mathcal{D}) < \mu$; and as $\mathcal{D}$ is normal, and $\langle \mu_i : i < \kappa\rangle$ is increasing continuous, we have $j(\langle \mu_i : i<\kappa\rangle/\mathcal{D}) = \mu$. As $\alpha_i < F(\mu_i)$ we have (by the Łoś theorem):

$$M \models \text{ "}j(\langle \alpha_i : i<\kappa\rangle/\mathcal{D}) \text{ is an ordinal smaller than the first inaccessible } >\mu\text{".}$$

But the property "not weakly inaccessible" is preserved by extending the universe (from M to $\mathbf{V}_1$). So we finish. $\blacksquare_{1.3}$

Remark 1.5. (1) The proof has little to do with our particular F. Assume $F : \mu \longrightarrow \mu$ and we add

$(*)$ $\quad F(\mu) = \lambda \quad$ for $\lambda = \big(j(F)\big)(\mu)$, $\quad j$ an appropriate elementary embedding.

Then all the proof of 1.3 works except possibly the last sentence, which use some absoluteness of the definition of F.

(2) We can also vary $\mathbb{R}$.

(3) Let $\lambda_i = F(\mu_i)$. In $\mathbf{V}' = \mathbf{V}^{(\mathbb{Q}\times\mathbb{R})*\mathbb{P}(\bar{\mathcal{D}},\bar{g})}$ we have $\mu = \mu^{<\kappa}$, moreover $(\forall \alpha < \lambda)[|\ \alpha\ |^{<\kappa} < \lambda]$ and μ is strong limit. Hence if in $\mathbf{V}'$, $\mathbb{P}$ is a forcing notion satisfying the κ-c.c. of cardinality $< \mu$, and $\mathcal{D}^*$

is an ultrafilter on κ extending $\mathcal{D}$ then (in $(\mathbf{V}')^{\mathbb{P}}$) we have: the ultraproduct $\prod\limits_{i<\kappa} \lambda_i/\mathcal{D}^*$ is λ–like.

Proposition 1.6. *Suppose $\langle \lambda_i : i < \kappa\rangle$ is a sequence of (weakly) inaccessible cardinals $> \kappa$, $\mathcal{D}$ an ultrafilter on κ, and the linear order $\prod\limits_{i<\kappa}(\lambda_i, <)/\mathcal{D}$ is λ–like, λ regular.*

(1) *There are Boolean algebras B_i (for $i < \kappa$) such that:*
  (a) $\mathrm{Length}(B_i) = \mathrm{Length}^+(B_i) = \lambda_i$,
  (b) $\mathrm{Length}^+(\prod\limits_{i<\kappa} B_i/\mathcal{D}) = \lambda$,
  (c) *if* $\lambda = \mu^+$ *then* $\mathrm{Length}(\prod\limits_{i<\kappa} B_i/\mathcal{D}) = \mu$.

(2) *Assume λ_i is regular $> \kappa$, $|B_i| = \lambda_i$, $B_i = \bigcup\limits_{\alpha<\lambda} B_{i,\alpha}$, $B_{i,\alpha}$ increasing continuous in α (the B_i, $B_{i,\alpha}$ are Boolean algebras of course) and we have*

$$(\forall \alpha < \lambda_i)(|B_{i,\alpha}| < \lambda_i)$$

and

$(*)_0$ *if $i < \kappa$, $\delta < \lambda_i$, $\mathrm{cf}(\delta) = \kappa^+$ then: $B_{i,\delta} \lessdot B_i$ ($\lessdot$ is complete subalgebra sign) i.e.*

$(*)^1_{B_i,B_{i,\delta}}$ *if $x \in B_i \setminus B_{i,\delta}$, $x \neq 0$ then for some y^* we have:*
  (a) *$y^* \in B_{i,\delta}$ and $y^* \neq 0$*
  (b) *for any $y \in B_{i,\alpha}$ such that $B_i \models$ "$y \le y^*$ & $y \neq 0$":*

$$B_i \models \text{“ } y \cap x \neq 0 \ \&\ x - y \neq 0 \text{ ”}$$

Then $\mathrm{Length}^+(\prod\limits_{i<\kappa} B_i/\mathcal{D}) \le \lambda$.

Proof (1) Let for $i < \kappa$ and $\alpha < \lambda_i$, B^i_α be the Boolean algebra generated freely by $\{x^i_\zeta : \zeta < \alpha\}$ except

$\otimes$ $x^{i,\alpha}_\zeta \le x^{i,\alpha}_\xi$ for $\zeta < \xi < \alpha$.

Let B_i be the free product of $\{B^i_\alpha : \alpha < \lambda_i\}$ so B_i is freely generated by $\{x^{i,\alpha}_\zeta : \alpha < \lambda_i,\ \zeta < \alpha\}$ except for $\otimes$.

Let $B_{i,\beta}$ be the subalgebra of B_i generated by $\{x^{i,\alpha}_\zeta : \alpha < \beta, \zeta < \alpha\}$.

Now clause (a) holds immediately, and the inequality $\ge$ in clause (b) holds by the Łoś theorem, and the other inequality follows by part (2) of the proposition. Lastly clause (c) follows.

(2) W.l.o.g. the set of members of B_i is λ_i, and the set of elements of $B_{i,\alpha}$ is an initial segment.

Let $S_i = \{\delta : \delta < \lambda_i$, the set of members of $B_{i,\delta}$ is δ and $\mathrm{cf}(\delta) = \kappa^+\}$

Let $(B, <^*, S) = \prod_{i<\kappa} (B_i, <_i, S_i)/\mathcal{D}$, with $<_i$ the order on the ordinals $< \lambda_i$ ($\leq$ is reserved for the order in the Boolean algebra). So $(|B|, <^*)$ is λ–like (where $|B|$ is the set of elements of B).

Let $\langle y_i : i < \lambda \rangle$ be an $<^*$ increasing sequence of members of B. Let

$$S' \overset{\text{def}}{=} \{\delta < \lambda : \ \mathrm{cf}(\delta) = \kappa^+ \text{ and } \{y_i : i < \delta\} \text{ has in } (|B|, <^*) \text{ a least upper bound which we call } y^\delta \text{ and it belongs to } S\}.$$

Now clearly (read [Sh 420, §1] if you fail to see; or assume $2^\kappa < \mu$ and [Sh 111, 2.3 p. 269]):

$\oplus$ $S' \subseteq \lambda$ is stationary.

Note: $\oplus$ is enough, as if $X \subseteq B$ is linearly ordered by $<^B$, let $y_i \in X$ for $i < \lambda$ be pairwise distinct; as $<^*$ is λ–like w.l.o.g. $\langle y_i : i < \lambda \rangle$ is $<^*$–increasing, and let S' be as above. For each $\delta \in S'$ apply $(*)^1_{B, B \restriction \{y : y <^* y_\delta\}}$ from the assumption to y_δ, y^δ (holds by Łoś theorem) and get y^*_i. Then apply Fodor lemma, and get a stationary subset S^2 of S' and an element y^* such that for every $i \in S^2$ we have $y^*_i = y^*$. Now the set of y_i for $i \in S^2$ is independent (check or see [Sh:92, 4.1]). $\blacksquare_{1.6}$

So putting Theorem 1.3 and Proposition 1.6 together

Conclusion 1.7. Assume CON(ZFC+ some μ is λ_1^+–hypermeasurable, for some strongly inaccessible $\lambda_1 > \mu$). Then it is consistent that for some κ, and sequence $\langle B_i : i < \kappa \rangle$ of Boolean algebras and ultrafilter $\mathcal{D}$ on κ we have, for some λ:

(a) $\prod_{i<\kappa} \mathrm{Length}(B_i)/\mathcal{D} = \lambda^+$
(b) $\mathrm{Length}(\prod_{i<\kappa} B_i/\mathcal{D}) = \lambda$.

Remark 1.8.
(1) We can say more on when ultraproducts of free products of Boolean algebras has not too large length.
(2) We can use the disjoint sum of $\langle B_{i,\alpha} : \alpha < \lambda_i \rangle$ instead.
(3) In the proof of 1.6 we actually have $\mathrm{Depth}^+(B_i) = \mathrm{Length}^+(B_i)$ and $\mathrm{Depth}^+(B) = \mathrm{Length}^+(B)$, and so similarly without $+$; where
$\mathrm{Depth}^+(B) = \sup\{|X|^+ : X \subseteq B \text{ is well ordered}\}$
$\mathrm{Depth}(B) = \sup\{|X| : X \subseteq B \text{ is well ordered}\}$.
So the parallel of 1.6 holds for Depth instead Length.
(4) Recal $c(B)$ is the cellularity of a Boolean algebra b, i.e., $\sup\{|X| : X$ is a set of pairwise disjoint non zero elements$\}$. If in the proof of 1.6 defining B^i_α we replace
$\otimes$ $x^{i,\alpha}_\zeta \leq x^{i,\alpha}_\xi$
by
$\otimes^4$ $x^{i,\alpha}_\zeta \cap x^{i,\alpha}_\xi = 0$

then $c(B_i) = \lambda_i = c^+(B_i)$, $c(B) = \mu^+$, $\lambda = c^+(B)$.
(Same proof.)

(5) We can also get the parallel to 1.6 for the independence number. Let B^i be the Boolean algebra generated by $\{x^{i,\alpha}_\zeta : \zeta < \alpha \text{ and } \alpha < \lambda_i\}$ freely except

($\otimes_5$) $x^{i,\alpha}_\zeta \cap x^{i,\beta}_\xi = 0$ if $\alpha < \beta < \kappa$, $\zeta < \alpha$, $\xi < \beta$.

Let I_i be the ideal of B^i which $\{x^{i,\alpha}_\zeta : \zeta < \alpha,\ \alpha < \lambda_i\}$ generates. Clearly it is a maximal ideal. Let $B_{i,\alpha}$ be the ideal of B^i generated by $\{x^{i,\beta}_\zeta : \zeta < \alpha \text{ and } \beta < \alpha\}$. Again w.l.o.g. the universe of B_i is λ_i and let

$$C_i = \{\delta < \lambda_i : \text{ for } x \in B_i \text{ we have: } x < \delta \text{ iff } x \in B_{i,\delta} \vee -x \in B_{i,\alpha}\}.$$

It is a club of λ_i.

The $B_{i,\beta}$ are not Boolean subalgebras of B_i, just *Boolean subrings*; now $(*)_0$ in proposition 1.6 is changed somewhat. We will have $P^{B_i} = I_i$ and $(B, <^*, P^*) = \prod_{i<\kappa} (B_i, <^*_i, P^{B_i})/\mathcal{D}$.

We know:

(α) P^* is a maximal ideal of B (by Łoś Theorem)

(β) if $i < \kappa$, $\delta < \lambda_i$ is a limit ordinal and $\delta \in C_i$ *then* for any $x \in P^{B_i}$ there are $x_0 < \delta$, $x_0 \in P^{B_i}$ and $x_1 \in P^{B_i}$ disjoint to all members of P^{B_i} which are $<_i \delta$ and $x = x_0 \cup x_1$. Similarly for B add if you like $Q_i = C_i \subseteq \lambda_i$.

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