Intersection of $< 2^{\aleph_0}$ ultrafilters may have measure zero

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Abstract

We show that it is consistent with ZFC that the intersection of some family of less than $2^{\aleph_0}$ ultrafilters can have measure zero. This answers a question of D. Fremlin.

The goal of this paper is to prove the theorem in the title. The solution is due to the second author.

Throughout the paper we use standard notation. All the filters are assumed to be non-principal filters on ω, i.e., they contain the Frechet filter $\mathcal{F}_0 = \{X \subseteq \omega : |\omega - X| < \aleph_0\}$. We identify filters with subsets of 2^ω. In this way the question about measurability and the Baire property of filters on ω makes sense. Let μ denote the standard measure on 2^ω and $\mu^\star$ and $\mu_\star$ the outer and inner measure respectively.

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1 Introduction

In this section we present several results concerning intresections of filters on ω. The following classical result is a starting point for all subsequent theorems.

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Theorem 1.1 (Sierpinski) *Suppose that $\mathcal{F}$ is a filter on ω. Then $\mathcal{F}$ has measure zero or is nonmeasurable. Similarly, it either is meager or does not have the Baire property. If $\mathcal{F}$ is an ultrafilter then $\mathcal{F}$ is nonmeasurable and does not have the Baire property.* ■

In particular if $\mathcal{F}$ is a filter then $\mu_\star(\mathcal{F}) = 0$ and if $\mathcal{F}$ is an ultrafilter then $\mu^\star(\mathcal{F}) = 1$.

First consider the filters which do not have the Baire property.

Theorem 1.2 (Talagrand [T])

1. *The intersection of countably many filters without the Baire property is a filter without the Baire property.*
2. *Assume* **MA**. *Then the intersection of $< 2^{\aleph_0}$ filters without the Baire property is a filter without the Baire property.* ■

For ultrafilters we have a much stronger result.

Theorem 1.3 (Plewik [P]) *The intersection of $< 2^{\aleph_0}$ ultrafilters is a filter which does not have the Baire property.* ■

For Lebesgue measure the situation is more complicated.

Theorem 1.4 (Talagrand [T]) *Let $\{\mathcal{F}_n : n \in \omega\}$ be a family of nonmeasurable filters. Then $\mathcal{F} = \bigcap_{n\in\omega} \mathcal{F}_n$ is nonmeasurable.*

If we consider uncountable families of filters the analog of 1.2 is no longer true.

Theorem 1.5 (Fremlin [F]) *Assume Martin's Axiom.*

Then there exists a family $\left\{\mathcal{F}_\xi : \xi < 2^{\aleph_0}\right\}$ of nonmeasurable filters such that $\bigcap_{\xi\in I} \mathcal{F}_\xi$ is a measurable filter for every uncountable set $I \subset 2^{\aleph_0}$. In particular there exists a family of $\aleph_1$ nonmeasurable filters with measurable intersection. ■

The next theorem shows that the above pathology cannot happen if we assume stronger measurability properties.

Let $\vec{p} = \langle p_n : n \in \omega\rangle$ be a sequence of reals such that $p_n \in (0, \frac{1}{2}]$ for all $n \in \omega$.

Define $\mu_{\vec{p}}$ to be the product measure on 2^ω such that for all n,

$$\mu_{\vec{p}}(\{x \in 2^\omega : x(n) = 1\}) = p_n$$

and

$$\mu_{\vec{p}}(\{x \in 2^\omega : x(n) = 0\}) = 1 - p_n.$$

Notice that if $p_n = \frac{1}{2}$ for all n then $\mu_{\vec{p}}$ is the usual measure on 2^ω.

Theorem 1.6 (Bartoszynski [Ba]) *Assume* **MA**. *Let $\mu_{\vec{p}}$ be a measure such that $\lim_{n\to\infty} p_n = 0$ and let $\{\mathcal{F}_\xi : \xi < \lambda < 2^\omega\}$ be a family of $\mu_{\vec{p}}$-nonmeasurable filters. Then*

$$\bigcap_{\xi<\lambda} \mathcal{F}_\xi \text{ is a Lebesgue nonmeasurable filter. } \blacksquare$$

Finally notice that the additional assumptions in 1.2 and 1.5 are necessary. Namely, we have the following result.

Theorem 1.7 *It is consistent with ZFC that there exists a family of filters $\mathcal{A}$ of size $2^{\aleph_0}$ such that*

1. *$\mathcal{A}$ consists of filters which do not have the Baire property and the intersection of any uncountable subfamily of $\mathcal{A}$ is equal to $\mathcal{F}_0$,*
2. *$\mathcal{A}$ consists of non-measurable filters and the intersection of any uncountable subfamily of $\mathcal{A}$ is equal to $\mathcal{F}_0$.*

Proof 1) Let $\mathbf{V}$ be a model of ZFC satisfying CH. Let $\langle c^\xi_\eta : \xi < \omega_1, \eta < \kappa\rangle$ be a generic sequence of Cohen reals over $\mathbf{V}$. Let $\mathcal{F}_\eta$ be the filter generated by $\left\{c^\xi_\eta : \xi < \omega_1\right\}$ for $\eta < \kappa$.

One easily checks that this family of filters has the required properties.

2) Use a sequence of random instead of Cohen reals. $\blacksquare$.

2 Intersection of ultrafilters

In this section we show that the analog of 1.3 is not true.

Theorem 2.1 *It is consistent with ZFC that the intersection of some family of $< 2^{\aleph_0}$ ultrafilters has measure zero.*

Proof We start with the following observation.

Lemma 2.2 *Suppose that $\{\mathcal{F}_\xi : \xi < \kappa\}$ is a family of filters on ω such that $\mu_\star(\bigcup_{\xi<\kappa} \mathcal{F}_\xi) > 0$. Then*

$$\mu(\bigcap_{\xi<\kappa} \mathcal{F}_\xi) = 0 \ .$$

Moreover, if $\mathcal{F}_\xi$'s are ultrafilters, $\bigcap_{\xi<\kappa} \mathcal{F}_\xi$ has measure zero iff $\bigcup_{\xi<\kappa} \mathcal{F}_\xi$ has measure 1.

Proof Suppose that $\mu_\star(\bigcup_{\xi<\kappa} \mathcal{F}_\xi) > 0$. Since all filters are assumed to be non-principal we know that $\mu_\star(\bigcup_{\xi<\kappa} \mathcal{F}_\xi) = 1$. Let $A \subseteq 2^\omega$ be a measure 1 set contained in $\bigcup_{\xi<\kappa} \mathcal{F}_\xi$. Define $A^\star = \{\omega - X : X \in A\}$. Clearly $\mu(A^\star) = 1$.

We claim that

$$A^\star \cap \bigcap_{\xi<\kappa} \mathcal{F}_\xi = \emptyset \ .$$

Suppose that $X \in A^\star$. Then $\omega - X \in A$ and hence $\omega - X \in \mathcal{F}_\eta$ for some $\eta < \kappa$. It follows that $X \notin \bigcap_{\xi<\kappa} \mathcal{F}_\xi$.

Conversely, suppose that $\bigcap_{\xi<\kappa} \mathcal{F}_\xi$ has measure zero. Let $A \subseteq 2^\omega$ be a set of measure 1 which is disjoint with $\bigcap_{\xi<\kappa} \mathcal{F}_\xi$. If $\mathcal{F}_\xi$'s are ultrafilters then it follows that $A^\star \subseteq \bigcup_{\xi<\kappa} \mathcal{F}_\xi$. ■

Let $\{I_n : n \in \omega\}$ be a partition of ω into finite sets such that $|I_n| \geq 2^n$ for $n \in \omega$. Define

$$A = \left\{X \subseteq \omega : \exists a > 1 \ \forall^\infty n \ \frac{|X \cap I_n|}{|I_n|} > \frac{a}{3}\right\}.$$

It is not hard to see that $\mu(A) = 1$.

Let $\mathcal{F}$ be an ultrafilter on ω. Define a notion of forcing $\mathcal{Q}_\mathcal{F}$ as follows.

$$\mathcal{Q}_\mathcal{F} = \left\{q \in [A]^{<\omega} : \exists a > 1 \ \left\{n : \frac{|I_n \cap \bigcap q|}{|I_n|} > \frac{a}{3^{|q|}}\right\} \in \mathcal{F}\right\}.$$

Elements of $\mathcal{Q}_\mathcal{F}$ are ordered by inclusion.

Lemma 2.3 *$\mathcal{Q}_\mathcal{F}$ is powerfully ccc.*

Proof We have to show that $\mathcal{Q}^n_\mathcal{F}$ satisfies countable chain condition for every natural number n.

Let's start with the case when $n = 1$.

Suppose that $\mathcal{A} \subseteq \mathcal{Q}_\mathcal{F}$ is an uncountable subset. Find $k \in \omega$, $a > 1$ and an uncountable set $\mathcal{A}' \subseteq \mathcal{A}$ such that $|q| = k$ and

$$\left\{n : \frac{|I_n \cap \bigcap q|}{|I_n|} \geq \frac{a}{3^k}\right\} \in \mathcal{F}$$

for all $q \in \mathcal{A}'$.

We show that any sufficiently big subset of $\mathcal{A}'$ contains two compatible conditions. We will use the following general observation.

Lemma 2.4 *Let (X, ν) be a measure space with probability measure ν. Suppose that $1 < b < a$ and $\varepsilon > 0$. There exists a number l such that if $A_1, \ldots, A_l$ are subsets of X of measure $\geq a \cdot \varepsilon$ then there are $i \neq j$ such that*

$$\nu(A_i \cap A_j) \geq b^2 \cdot \varepsilon^2.$$

Proof Let $A_1, \ldots, A_l$ be sets of measure $a \cdot \varepsilon$. Consider random variables X_i given by characteristic function of A_i for $i \leq l$. Note that $X_i^2 = X_i$ for $i \leq l$.

Suppose that $\nu(A_i \cap A_j) < b^2 \cdot \varepsilon^2$ for $i \neq j$. In particular $E(X_i X_j) < b^2 \cdot \varepsilon^2$ for $i \neq j$. Recall that $\mathbf{E}(X^2) \geq \big(\mathbf{E}(X)\big)^2$ for every random variable X.

We compute

$$\begin{aligned}
0 &\leq \mathbf{E}\Big(\big((\sum_{i=1}^{l} X_i) - la\varepsilon\big)^2\Big)\\
&= \mathbf{E}\big(\sum_{i=1}^{l} X_i^2 + \sum_{i\neq j} X_i X_j - 2la\varepsilon \sum_{i=1}^{l} X_i + (la\varepsilon)^2\big)\\
&\leq la\varepsilon + l(l-1)b^2\varepsilon^2 - 2(la\varepsilon)^2 + (la\varepsilon)^2\\
&= l\big(a\varepsilon - b^2\varepsilon^2 + l(b^2-a^2)\varepsilon^2\big).
\end{aligned}$$

As $b < a$, the last line here is negative for large l, a contradiction. ■

Let ν_n be the uniform measure on I_n i.e. $\nu(A) = |A| \cdot |I_n|^{-1}$ for $A \subseteq I_n$. Fix $1 < b < a$ and let l be a number from the above lemma chosen for $\varepsilon = 3^{-k}$.

Let $q_1, \ldots, q_l \in \mathcal{A}'$. Find $Y \in \mathcal{F}$ such that for all $i \leq l$ and $n \in Y$

$$\frac{|I_n \cap \bigcap q_i|}{|I_n|} \geq \frac{a}{3^k} .$$

By the lemma for every $n \in Y$ there exist $i \neq j$ such that

$$\frac{|I_n \cap \bigcap (q_i \cup q_j)|}{|I_n|} \geq \frac{b}{9^k} .$$

Since $\mathcal{F}$ is an ultrafilter there exist $i \neq j$ such that

$$\left\{n : \frac{|I_n \cap \bigcap (q_i \cup q_j)|}{|I_n|} \geq \frac{b}{9^k}\right\} \in \mathcal{F} .$$

It follows that $q_i \cup q_j \in \mathcal{Q}_{\mathcal{F}}$.

Note that in fact we proved that for every m there exists l such that for every subset $X \subseteq \mathcal{A}'$ of size l there are $q_1, \ldots, q_m \in X$ such that $q_i \cup q_j \in \mathcal{Q}_{\mathcal{F}}$ for $i, j \leq m$.

Suppose that $n > 1$. For simplicity assume that $n = 2$, the general case is similar.

Let $\mathcal{A}$ be an uncountable subset of $\mathcal{Q}^2_{\mathcal{F}}$. Without loss of generality we can assume there are numbers k_1, k_2 and a such that for every $\langle q^1, q^2\rangle \in \mathcal{A}$, $|q^1| = k_1$, $|q^2| = k_2$ and

$$\left\{n : \frac{|I_n \cap \bigcap q^i|}{|I_n|} \geq \frac{a}{3^{k_i}}\right\} \in \mathcal{F} \text{ for } i = 0, 1.$$

To get two elements of $\mathcal{A}$ which are compatible first apply the above remark to get a large subset $X \subseteq \mathcal{A}$ with first coordinates being pairwise compatible and then apply the case $n = 1$ to the second coordinates of conditions in X. ■

Notice that if G is a $\mathcal{Q}_{\mathcal{F}}$-generic filter over $\mathbf{V}$ then the set

$$\left\{\bigcap q : q \in G\right\} \cup \mathcal{F}_0$$

generates a non-principal filter on ω.

Let $\mathcal{P}_{\mathcal{F}} = \lim_{n\to\infty} \mathcal{Q}^n_{\mathcal{F}}$ be a finite support product of countably many copies of $\mathcal{Q}_{\mathcal{F}}$. By 2.3, $\mathcal{P}_{\mathcal{F}}$ satisfies the countable chain condition.

Lemma 2.5 $\Vdash_{\mathcal{P}_{\mathcal{F}}} A \cap \mathbf{V}$ *is the union of countably many filters.*

Proof Let G be a $\mathcal{P}_{\mathcal{F}}$-generic filter over $\mathbf{V}$.

For $n \in \omega$, let $G_n = \{p(n) : \mathrm{dom}(p) = \{n\}, p \in G\}$. Since G_n is a $\mathcal{Q}_{\mathcal{F}}$-generic filter let $\mathcal{F}_n$ be a filter on ω generated by G_n as above.

We show that $\mathbf{V}[G] \models A \cap \mathbf{V} \subseteq \bigcup_{n\in\omega} \mathcal{F}_n$.

Suppose that $\bar{p} \in \mathcal{P}_{\mathcal{F}}$ and $X \in A$. Find $n \in \omega$ such that $n \notin \mathrm{dom}(\bar{p})$. Let $\bar{q} = \bar{p} \cup \langle n, \{X\}\rangle$. It is clear that $\bar{q} \Vdash X \in \mathcal{F}_n$. ■

Now we can finish the proof of 2.1.

Let $\mathbf{V} \models 2^{\aleph_0} > \aleph_1$. Let $\langle \mathcal{P}_\xi, \dot{\mathcal{Q}}_\xi : \xi < \omega_1\rangle$ be a finite support iteration such that

$$\Vdash_\xi \dot{\mathcal{Q}}_\xi \simeq \mathcal{P}_{\mathcal{F}_\xi} \text{ for some ultrafilter } \mathcal{F}_\xi .$$

Let $\mathcal{P}_{\omega_1} = \lim_{\xi<\omega_1} \mathcal{P}_\xi$.

Let G be a $\mathcal{P}_{\omega_1}$-generic filter over $\mathbf{V}$. We will show that $\mathbf{V}[G]$ is the model we are looking for.

Let $\left\{\mathcal{F}^\xi_n : \xi < \omega_1, n \in \omega\right\}$ be the family of filters added by G. Without loss of generality we can assume that they are ultrafilters. To finish the proof it is enough to show that

$$A \subseteq \bigcup_{\xi<\omega_1} \bigcup_{n\in\omega} \mathcal{F}^\xi_n .$$

Suppose that $X \in A$. There exists $\xi < \omega_1$ such that $X \in \mathbf{V}[G \cap \mathcal{P}_\xi]$. By 2.5 there exists $n \in \omega$ such that $X \in F^{\xi+1}_n$. ■

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