Every null-additive set is meager-additive†

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§1. The basic definitions and the main theorem.

1. Definition. (1) We define addition on ${}^\omega 2$ as addition modulo 2 on each component, i.e., if $x, y, z \in {}^\omega 2$ and $x + y = z$ then for every n we have $z(n) = x(n) + y(n) \pmod 2$.
(2) For $A, B \subseteq {}^\omega 2$ and $x \in {}^\omega 2$ we set $x + A =^{\mathrm{df}} \{x + y : y \in A\}$, and we define $A + B$ similarly.
(3) We denote the Lebesgue measure on ${}^\omega 2$ with μ. We say that $X \subseteq {}^\omega 2$ is *null-additive* if for every $A \subseteq {}^\omega 2$ which is null, i.e. $\mu(A) = 0$, $X + A$ is null too.
(4) We say that $X \subseteq {}^\omega 2$ is *meager-additive* if for every $A \subseteq {}^\omega 2$ which is meager also $X + A$ is meager.

2. Theorem. Every null-additive set is meager-additive.

3. Outline and discussion. Theorem 2 answers a question of Palikowsi. It will be proved in §2. In §3 we shall present direct characterizations of the null-additive sets, and in §4 we shall do the same for the meager-additive sets.

It is obvious that every countable set is both null-additive and meager-additive. Are there uncountable null-additive sets, and even null-additive sets of cardinality $2^{\aleph_0}$? It will be shown in §5 that if the continuum hypothesis holds then there is such a set. Haim Judah has shown that there is a model of ZFC in which all the null-additive sets are countable, but there are in it uncountable meager-additive sets. This is the model obtained by adding to L more than $\aleph_1$ Cohen reals. In this model the Borel conjecture holds, and therefore every null-additive set is strongly meager and hence countable. On the other hand, in this model the uncountable set of all constructible reals is meager-additive.

§2 The proof of Theorem 2.

4. Notation. (1) we shall use variables as follow: i, j, k, l, m, n for natural numbers, f, g, h for functions from ω to ω, $\eta, \zeta, \nu, \sigma, \tau$ for finite sequences of 0's and 1's, x, y, z for members of ${}^\omega 2$, A, B, X, Y for subsets of ${}^\omega 2$, and S, T for trees.
(2) ${}^{\omega>}2 = \bigcup_{n<\omega} {}^n 2$. We shall denote subsets of ${}^{\omega>}2$ with U, V. For $\eta \in {}^{\omega>}2$, $U \subseteq {}^{\omega>}2$ and $x \in {}^\omega 2$ we shall write $\eta + x$ for $\eta + x \restriction \mathrm{length}\,(\eta)$, and $U + x$ for $\{\eta + x : \eta \in U\}$.
(3) For $\eta, \nu \in {}^{\omega>}2$ we write $\eta \trianglelefteq \nu$ if ν is an extension of η.
(4) A tree is a nonempty subset of ${}^{\omega>}2$ such that

(a) If $\eta \trianglelefteq \nu$ and $\nu \in T$ then also $\eta \in T$, and

(b) If $\eta \in T$ and $n > \mathrm{length}\,(\eta)$ then there is a ν of length n such that $\eta \trianglelefteq \nu$ and $\nu \in T$.

(5) For a tree T $\mathrm{Lim}\, T = \{x \in {}^\omega 2 : \text{for every } n < \omega \ \ x \restriction n \in T\}$.
(6) A tree T is said to be *nowhere dense* if for every $\eta \in T$ there is a $\tau \in {}^{\omega>}2$ such that

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$\eta \trianglelefteq \tau$ and $\tau \notin T$. A set $B \subseteq {}^{\omega}2$ is said to be nowhere dense if $B \subseteq \operatorname{Lim} T$ for some nowhere dense tree T.
(7) For every $x, y \in {}^{\omega}2$ we write $x \equiv y$ if $x(n) = y(n)$ for all but finitely many $n < \omega$. For $A \subseteq {}^{\omega}2$ $A^{\text{fin}} =^{\text{df}} \{y \in {}^{\omega}2 : y \equiv x$ for some $x \in A\}$.
(8) $U^{[\nu]} =^{\text{df}} \{\tau \in U : \tau \trianglelefteq \nu$ or $\nu \trianglelefteq \tau\}$ (read: U through ν).
(9) $U^{\langle\nu\rangle} =^{\text{df}} \{\tau \in {}^{\omega>}2 : \nu {}^\frown \tau \in U\}$ (read: U above ν), and for $\eta \in {}^{\omega>}2$
$\eta^{\langle k\rangle} =^{\text{df}} \langle \eta(k+i) : i < \operatorname{length} \eta - k\rangle$.
(10) For $\nu, \eta \in {}^{\omega>}2 \cup {}^{\omega}2$ we write $\nu \sim_n \eta$ if $\operatorname{length}(\nu) = \operatorname{length}(\eta)$ and $\nu(i) = \eta(i)$ for every $n \le i < \operatorname{length}(\nu)$. For $S \subseteq {}^{\omega>}2 \cup {}^{\omega}2$ we define $S^{\sim n} = \{\nu : \nu \sim_n \eta$ for some $\eta \in S\}$.

5. Outline of the proof. Let $X \subseteq {}^{\omega}2$ be null-additive. It clearly suffices to prove that for every $A \subseteq {}^{\omega}2$ which is nowhere dense $X + A$ is meager. Given a nowhere dense tree S we shall prove in Lemma 6 a condition which is sufficient for a tree T to be such that $T + S$ is nowhere dense. Then we shall split X to a union $X = \bigcup_{i=1}^{\infty} X_i$ such that for each i $X_i \subseteq \operatorname{Lim} T_i$ where T_i is a tree which satisfies that condition. Thus for a nowhere dense S each set $X_i + \operatorname{Lim} S \subseteq \operatorname{Lim}(T_i + S)$ is nowhere dense, hence $X + \operatorname{Lim} S \subseteq \bigcup_{i=1}^{\infty} \operatorname{Lim}(T_i + S)$ is meager.

6. Lemma. Let T be a tree such that
(a) T is nowhere dense.
(b) $f = f_T$ is the function from ω to ω given by
$f(n) = \min\{m :$ for every $\eta \in {}^{n}2$ there is a $\tau \in {}^{m}2$ such that $\eta \trianglelefteq \tau$ and $\tau \notin T\}$.
Thus for every sequence η of length n there is a witness of length $\le f(n)$ that T is nowhere dense. Obviously for every $n < \omega$ $f(n) > n$, and if $n < m$ then $f(n) \le f(m)$.

Let g be a function from ω to ω and $\overline{n} = \langle n_i : i < \omega\rangle$, $\overline{n}' = \langle n'_i : i < \omega\rangle$ increasing sequences of natural numbers such that
(c) $f^{g(i)}(n_i) \le n'_i < n_{i+1}$ for every $i < \omega$, where f^m denotes the m-th iteration of f.
Then for every tree S which satisfies
(d) S is of width $(\overline{n}', g)$, i.e., for every $i < \omega$ $|{}^{n'_i}2 \cap S| \le g(i)$,
$T + S$ is nowhere dense.

Proof. Let $\eta \in {}^{n_i}2$. We shall show the existence of an $\eta' \in {}^{n'_i}2$ such that $\eta \trianglelefteq \eta'$ and $\eta' \notin T + S$.

By (c) there is a sequence $m_0, \ldots, m_{g(i)}$ such that $m_0 = n_i$, $f(m_k) \le m_{k+1}$ for $0 \le k \le g(i)$ and $m_{g(i)} = n'_i$. Let $\langle \tau_k : k < k_i\rangle$ enumerate the set ${}^{n'_i}2 \cap S$. $k_i \le g(i)$ by (d). We define $\eta_k \in {}^{m_k}2$ for $0 \le k \le k_i$ by recursion as follows. $\eta_0 = \eta$. Given $\eta_k \in {}^{m_k}2$, for $k < k_i$, we shall define $\eta_{k+1} \in {}^{m_{k+1}}2$ so that for no extension $\eta' \in {}^{n'_i}2$ of η_{k+1} we shall have $\eta' + \tau_k \in T$. We have $\eta_k + \tau_k \restriction m_k \in {}^{m_k}2$ and by the definition of f and by the choice of the m_k's $\eta_k + \tau_k \restriction m_k$ has an extension $\nu \in {}^{m_{k+1}}2$ such that $\nu \notin T$. If we take $\eta_{k+1} = \nu + \tau_k \restriction m_{k+1}$ then $\eta_k + \tau_k \restriction m_k \trianglelefteq \nu$ implies $\eta_k \trianglelefteq \eta_{k+1}$, $\eta_{k+1} \in {}^{m_{k+1}}2$ and $\eta_{k+1} + \tau_k \restriction m_{k+1} = \nu \notin T$, and therefore for every $\eta' \in {}^{n'_i}2$ such that $\eta_k \trianglelefteq \eta'$ we have $\eta' + \tau_k \notin T$. Let $\eta' = \eta_{k_i}$, and assume that $\eta' \in T + S$. Then, for some $k < k_i \le g(i)$ $\eta' + \tau_k \in T$, contradicting our choice of $\eta_{k+1} = \eta' \restriction m_{k+1}$. Thus $\eta' \notin T + S$.

7. Lemma. If S, T_i, $i \in \omega$ are trees and $\operatorname{Lim} S \subseteq \bigcup_{i\in\omega} \operatorname{Lim} T_i$ then for some $\eta \in S$ and $j \in \omega$ $S^{[\eta]} \subseteq T_j$.

Proof. Suppose that this is not the case, i.e., for every $\eta \in S$ and $i < \omega$ there is a ζ

such that $\zeta \in S^{[\eta]}$ and $\zeta \notin T_i$. Once there is such a ζ we can assume that $\eta \trianglelefteq \zeta$ and $\operatorname{length}\zeta > \operatorname{length}\eta$. We define now, by induction on i, η_i and k_i so that $k_i = \operatorname{length}\eta_i$, $k_0 = 0$, $\eta_0 = \langle\rangle$, $\eta_i \trianglelefteq \eta_{i+1}$, $k_i < k_{i+1}$, $\eta_{i+1} \in S$ and $\eta_{i+1} \notin T_i$. Let $y = \bigcup_{i\in\omega} \eta_i$. Since $\eta_i \in S$ for every $i \in \omega$ $y \in \operatorname{Lim} S \subseteq \bigcup_{i\omega} \operatorname{Lim} T_i$, hence for some $j \in \omega$ $y \in \operatorname{Lim} T_j$. However, $y \restriction k_{j+1} = \eta_{j+1} \notin T_j$, contradicting $y \in \operatorname{Lim} T_j$.

8. Lemma. Let S and T be trees such that $\operatorname{Lim} S \subseteq (\operatorname{Lim} T)^{\text{fin}}$. Then there are $k < \omega$, $\eta, \nu \in {}^k 2$, $\eta \in S$ such that $S^{\langle\eta\rangle} \subseteq T^{\langle\nu\rangle}$.

Proof. For $n < \omega$, $\sigma_1, \sigma_2 \in {}^n 2$ and $\sigma_2 \in T$ we define
$T_{\sigma_1,\sigma_2} =^{\text{df}} \{\tau : \tau \trianglelefteq \sigma_1\} \cup \{\sigma_1 \frown \tau : \sigma_2 \frown \tau \in T\}$ (This is the tree $T^{[\sigma_2]}$ with "σ_2 replaced by σ_1"). Clearly

$$(\operatorname{Lim} T)^{\text{fin}} = \bigcup_{n<\omega,\, \sigma_1,\sigma_2 \in {}^n 2,\, \sigma_2 \in T} \operatorname{Lim} T_{\sigma_1,\sigma_2} \tag{1}$$

Since there are only countably many T_{σ_1,σ_2}'s in (1) there are by Lemma 7 a $\zeta \in S$ and $j < \omega$ such that $S^{[\zeta]} \subseteq T_{\sigma_1,\sigma_2}$. Clearly there is an η with $\zeta \trianglelefteq \eta$ and a ν with $\operatorname{length}\nu = \operatorname{length}\eta$ such that $S^{\langle\eta\rangle} \subseteq T^{\langle\nu\rangle}$. (If $\zeta \trianglelefteq \sigma_1$ then $\eta = \sigma_1$ and $\nu = \sigma_2$, else $\sigma_1 \trianglelefteq \zeta$ and then $\eta = \zeta$ and $\nu = \sigma_2 \frown \zeta \restriction [\operatorname{length}\zeta, \operatorname{length}\sigma_2)$).

9. Lemma. Let X be a null-additive set. Let T be a tree such that $\mu(\operatorname{Lim} T) > 0$. There is a tree T^* such that $\mu(\operatorname{Lim} T^*) > 0$, for every $\eta \in T^*$ $\mu(\operatorname{Lim}(T^{*[\eta]})) > 0$, and $(({}^\omega 2 \setminus (\operatorname{Lim} T)^{\text{fin}}) + X) \cap \operatorname{Lim} T^* = \emptyset$, and then $X = \bigcup_{\eta\in T^*,\, \operatorname{length}\zeta=\operatorname{length}\eta} Y_{\eta,\zeta}$ where $Y_{\eta,\zeta} = \{x \in X : \zeta \frown x^{\langle\operatorname{length}\zeta\rangle} + T^{*[\eta]} \subseteq T\}$.

Proof. Since $\mu(\operatorname{Lim} T) > 0$ then, as easily seen, $\mu((\operatorname{Lim} T)^{\text{fin}}) = 1$, hence $\mu({}^\omega 2 \setminus (\operatorname{Lim} T)^{\text{fin}}) = 0$. Since X is null-additive also $\mu(X + ({}^\omega 2 \setminus (\operatorname{Lim} T)^{\text{fin}})) = 0$. Hence there is a tree T^* such that $\mu(\operatorname{Lim} T^*) > 0$ and $(X + ({}^\omega 2 \setminus (\operatorname{Lim} T)^{\text{fin}})) \cap \operatorname{Lim} T^* = \emptyset$. Without loss of generality we can assume that T^* has been pruned so that for $\eta \in T^*$ $\mu(\operatorname{Lim} T^{*[\eta]}) > 0$.

Let $x \in X$ then ${}^\omega 2 \setminus (x + (\operatorname{Lim} T)^{\text{fin}}) = x + ({}^\omega 2 \setminus (\operatorname{Lim} T)^{\text{fin}}) \subseteq X + ({}^\omega 2 \setminus (\operatorname{Lim} T)^{\text{fin}})$. Hence $({}^\omega 2 \setminus (x + (\operatorname{Lim} T)^{\text{fin}})) \cap \operatorname{Lim} T^* \subseteq (X + ({}^\omega 2 \setminus (\operatorname{Lim} T)^{\text{fin}}) \cap \operatorname{Lim} T^* = \emptyset$, i.e., $\operatorname{Lim} T^* \subseteq x + (\operatorname{Lim} T)^{\text{fin}}$, and therefore $\operatorname{Lim}(x + T^*) = x + \operatorname{Lim} T^* \subseteq (\operatorname{Lim} T)^{\text{fin}}$. By Lemma 8 there are $\eta \in T^*$ and $\nu \in {}^{\operatorname{length}\eta}2$ such that $x^{\langle\operatorname{length}\eta\rangle} + T^{*\langle\eta\rangle} \subseteq T^{\langle\nu\rangle}$. Let $\zeta = \eta + \nu$, then $\zeta + \eta = \nu$ and therefore $\zeta \frown x^{\langle\operatorname{length}\eta\rangle} + T^{*[\eta]} \subseteq T^{[\nu]} \subseteq T$, hence $x \in Y_{\eta,\zeta}$.

10. Lemma. Let X be null-additive, and let $\overline{n} = \langle n_i : i < \omega\rangle$, $\overline{n}' = \langle n'_i : i < \omega\rangle$ be such that for every $i < \omega$ $n_i < n'_i$ and $n'_i + i \cdot 2^{n'_i} \leq n_{i+1}$, then we can represent X as $\bigcup_{m<\omega} X_m$ such that for each m, for some real $a_m \in (0,1)$ and S_m of width $(\overline{n}', g_{a_m})$ we have $X_m \subseteq \operatorname{Lim}(S_m)$, where for every real $a \in (0,1)$ g_a is the function on ω given by $g_a(0) = 1$, $g_a(i) = \max(1, \operatorname{int}(\log_2(a)/\log_2(1 - 2^{-i})))$, where for a real d $\operatorname{int}(d)$ is the integral part of d.

Proof. Since $n'_i + i \cdot 2^{n'_i} < n_{i+1}$ we can fix for each $0 < i < \omega$ a sequence $\langle u_{i,\tau} : \tau \in {}^{n'_i}2\rangle$ of pairwise disjoint subsets of the interval $[n'_i, n_{i+1})$ having i members each. Let $B \subseteq {}^\omega 2$ be given by

$$B = \{y \in {}^\omega 2 : (\forall j > 0)(\exists k \in u_{j,y\restriction n'_j})\, y(k) = 1\}.$$

B is clearly a closed subset of ${}^\omega 2$ hence for $T = \{y \restriction n : y \in B \wedge n \in \omega\}$ $B = \operatorname{Lim}(T)$.

The properties of T in which we are interested are
(B0) $T \supseteq {}^{n_1}2$.
(B1) For each $\eta \in T \cap {}^{n'_i}2$ $\left|T^{[\eta]} \cap {}^{n_{i+1}}2\right| = 2^{(n_{i+1}-n'_i)}(1-2^{-i})$.
(B2) If $\eta, \nu_0, \ldots, \nu_{k-1} \in {}^{n'_i}2$, $\nu_0^+, \ldots, \nu_{k-1}^+ \in {}^{n_{i+1}}2$, $\eta + \nu_l \in T$, $\nu_l \trianglelefteq \nu_l^+$ for $l < k$ and $\nu_0, \ldots, \nu_{k-1}$ is with no repetitions then $\left|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n_{i+1}}2,\ (\forall l < k)(\eta^+ + \nu_l^+ \in T)\}\right| \leq 2^{n_{i+1}-n'_i}\left(1-2^{-i}\right)^k$.
(B3) For every $\eta \in {}^{n'_i}2$ we have : $\eta \restriction n_i \in T$ implies $\eta \in T$.

These properties can be established by an obvious counting argument.

By (B0), (B1) and (B3) we have

$$\begin{aligned}\mu(\operatorname{Lim} T) &= \mu\left(\bigcap_{i=1}^{\infty}\{x \in {}^{\omega}2 : x \restriction n_i \in T\}\right)\\ &= \mu\left(\{x \in {}^{\omega}2 : x \restriction n_1 \in T\}\right) \cdot \prod_{i=1}^{\infty} \frac{\mu(\{x \in {}^{\omega}2 : x \restriction n_{i+1} \in T\})}{\mu(\{x \in {}^{\omega}2 : x \restriction n_i \in T\})}\\ &= 1 \cdot \prod_{i=1}^{\infty} \frac{|T \cap {}^{n_{i+1}}2|/2^{n_{i+1}}}{|T \cap {}^{n_{i+1}}2|/2^{n_i}} = \prod_{i=1}^{\infty}(1-2^{-i}) > 0\end{aligned}$$

For the T which we constructed let T^* and $Y_{\eta,\zeta}$ be as in Lemma 9. For $\rho \in {}^{\text{length}\,\eta}2$ let $Y_{\eta,\zeta,\rho} = \{y \in Y_{\eta,\zeta} : y \restriction \text{length}\,\eta = \rho\}$. Clearly

$$X = \bigcup_{\eta \in T^*,\ \text{length}\,\eta = \text{length}\,\zeta = \text{length}\,\rho} Y_{\eta,\zeta,\rho} \tag{2}$$

Since there are only countably many $Y_{\eta,\zeta,\rho}$'s they can be taken to be the X_m's we are looking for, provided we show that every such $Y_{\eta,\zeta,\rho}$ is a subset of $\operatorname{Lim}(S)$ for some tree S of width $\langle \overline{n}', g_a \rangle$ for some real $0 < a < 1$. We shall see that this is indeed the case if we take $S = \{y \restriction m : y \in Y_{\eta,\zeta,\rho},\ m < \omega\}$ and $a = \mu(T^{*[\eta]})$. $a > 0$ by what we assumed about T^*. As, obviously, $Y_{\eta,\zeta,\rho} \subseteq \operatorname{Lim}(S)$ all we have to do is to show that S is of width $\langle \overline{n}', g_a \rangle$.

We can choose a set $W \subseteq S \cap {}^{n_{j+1}}2$ such that the function mapping $\eta \in W$ to $\eta \restriction n'_j$ is one to one and onto $S \cap {}^{n'_j}2$

We fix now η, ζ, ρ and denote $Y_{\eta,\zeta,\rho}$ by Y and the length of η, ζ, ρ by n. Let $z \in {}^{\omega}2$ be such that $z \restriction n = \zeta + \rho$ and $z(i) = 0$ for $i \geq n$. Then for every y such that $y \restriction n = \rho$ we have $y + z = \zeta \frown y^{\langle n \rangle}$. Therefore, by the definition of Y we have

$$Y = \{y \in {}^{\omega}2 : y \restriction n = \rho,\ (\zeta \frown y^{\langle n \rangle}) + T^{*[\eta]} \subseteq T\} = \{y \in {}^{\omega}2 : y \restriction n = \rho,\ y + z + T^{*[\eta]} \subseteq T\} \tag{3}$$

for every $y \in Y$ there is a unique $\tau \in W$ such that $\tau \restriction n'_j = y \restriction n'_j$ (τ may be $y \restriction n_{j+1}$). Clearly $|W| = |S \cap {}^{n'_j}2|$ and we denote $|W|$ with s, so it suffices to prove $s \leq g_a(j)$. If $n'_j \leq n$ then the only member of $S \cap {}^{n'_j}2$ is $\rho \restriction n'_j$ hence $s = 1$, so $s \leq g_a(j)$. We shall now deal with the case where $n'_j > n$. Let $\tau_0, \ldots \tau_{s-1}$ be the members of W. For $m < s$ $\tau_m = y \restriction n_{j+1}$ for some $y \in Y$, hence, by (3), $\tau_m + z + T^{*[\eta]} \subseteq T$ and therefore $(z + T^{*[\eta]}) \cap {}^{n_{j+1}}2 \subseteq \tau_m + T$. Since this holds for every $\tau \in W$ we have

$$z + T^{*[\eta]} \cap {}^{n_{j+1}}2 \subseteq \bigcap_{m<s} \tau_m + T \tag{4}$$

Let us find out the size of $\bigcap_{m<s}(\tau_m+T)$. Let $\sigma\in {}^{n'_j}2$, and we shall ask how many members τ of $\bigcap_{m<s}(\tau_m+T)$ extend σ. Now $\tau\in\tau_m+T$ for each $m<s$ iff $\tau+\tau_m\in T$ for each $m<s$. If for some $m<s$ $\sigma+\tau_m\restriction n'_j\notin T$ then also $\tau+\tau_m\notin T$, hence σ has no extension in $\bigcap_{m<s}(\tau_m+T)$. If for every $m<s$ $\sigma+\tau_m\restriction n'_j\in T$ then by (B2) (where $\eta=\sigma,\ \nu_m=\tau_m\restriction n'_j$ and $\nu^+_m=\tau_m$), since $\tau_m\restriction n'_j\neq\tau_l\restriction n'_j$ for $m\neq l$, the number of τ's such that $\sigma\trianglelefteq\tau\in {}^{n_{j+1}}2$ and $\tau+\tau_m\in T$ for every $m<s$ is $2^{n_{j+1}-n'_j}(1-2^{-j})^s$. Since there are $2^{n'_j}$ different σ's in ${}^{n'_j}2$ we have

$$\left|\bigcap_{m<s}(\tau_m+T)\right|\leq 2^{n_{j+1}}\cdot(1-2^{-j})^s. \tag{5}$$

On the other hand, since $\mu(T^{*[\eta]})=a$ $T^{*[\eta]}\cap {}^{n_{j+1}}2$ has at least $a\cdot 2^{n_{j+1}}$ members, and so has $z+T^{*[\eta]}\cap {}^{n_{j+1}}2$. Comparing (4) with (5) we get $a\cdot 2^{n_{j+1}}\leq 2^{n_{j+1}}(1-2^{-j})^s$, i.e., $a\leq(1-2^{-j})^s$, $\log_2(a)\leq s\cdot\log_2(1-2^{-j})$, $s\leq\log_2(a)/\log_2(1-2^{-j})$.

11. Proof of Theorem 2. Let X be null-additive. As mentioned in **5** it suffices to show that for every nowhere dense tree T $X+\mathrm{Lim}\,(T)$ is meager. Let $f=f_T$ as in Lemma 6. Define by recursion $n_0=0$, $n'_i=f^{g_{1/(i+1)}(i)}(n_i)+1$ and $n_{i+1}=n'_i+i\cdot 2^{n'_i}+1$. By Lemma 10 $X\subseteq\bigcup_{m<\omega}\mathrm{Lim}\,(S_m)$, where for some $a_m\in(0,1)$ S_m is of width $\langle\overline{n}',g_{a_m}\rangle$, hence it suffices to show that if S is of width $\langle\overline{n}',g_a\rangle$ for some $a\in(0,1)$ then $\mathrm{Lim}\,(S)+\mathrm{Lim}\,(T)=\mathrm{Lim}\,(S+T)$ is meager. Let j be such that $\frac{1}{j+1}\leq a$ and let $\eta_1,\dots,\eta_k$ be all the members of S of length n'_j. Then $S=\bigcup_{l=1}^k S^{[\eta_l]}$ and $\mathrm{Lim}\,S=\bigcup_{l=1}^k\mathrm{Lim}\,(S^{[\eta_l]})$. Therefore it suffices to prove that for $1\leq l\leq k$ $\mathrm{Lim}\,(S_l)+\mathrm{Lim}\,(T)$ is meager and this follows once we show that S_l+T is nowhere dense. To prove this we show that the requirements of Lemma 6 hold here for S_l,T. (a) and (b) hold by our choice of T and f. Let g be defined by $g(i)=1$ for $i<j$ and $g(i)=g_a(i)$ for $i\geq j$. Now we shall see that (c) holds. For $i<j$ we have $n'_i=f^{g_{1/(i+1)}(i)}(n_i)+1\geq f(n_i)+1=f^{g(i)}(n_i)+1$, since $f(n)\geq n$ for every n, and for $i\geq j$ we have $n'_i=f^{g_{1/(i+1)}(i)}(n_i)+1\geq f^{g_a(i)}(n_i)+1=f^{g(i)}(n_i)+1$, since $a\geq\frac{1}{j+1}\geq\frac{1}{i+1}$ and $g_a(i)$ is a decreasing function of a. Thus for every $i<\omega$ $f^{g(i)}(n_i)\leq f^{g_{1/(i+1)}(i)}(n_i)\leq n'_i$. (d) of Lemma 6 holds since for $i<j$ $|{}^{n'_i}2\cap S_l|=1=g(i)$ and for $i\geq j$ $|{}^{n'_i}2\cap S_l|\leq|{}^{n'_i}2\cap S|\leq g_a(i)=g(i)$.

§3 Characterization of the null-additive sets

12. Definition. By a *corset* we mean a non decreasing function f from ω to $\omega\setminus\{0\}$ which converges to infinity (i.e., for every $n<\omega$ $f(m)>n$ for all sufficiently large m). For a corset f, we say that a tree T is *of width* f if for every $n<\omega$ $|T\cap {}^n2|\leq f(n)$; and we say that T is *almost of width* f if $|T\cap {}^n2|\leq f(n)$ for all sufficiently large n.

13. Theorem. For every $X\subseteq {}^\omega 2$ the following conditions are equivalent:

a. X is null-additive.

b. For every corset f there is a tree S of width f such that $X\subseteq\mathrm{Lim}\,(S)^{\mathrm{fin}}$.

c. For every corset f there are trees S_m, $m<\omega$, which are almost of width f such that $X\subseteq\bigcup_{m<\omega}\mathrm{Lim}\,(S_m)^{\mathrm{fin}}$.

d. For every corset f there are trees S_m, $m<\omega$, of width f such that $X\subseteq\bigcup_{m<\omega}\mathrm{Lim}\,(S_m)$.

Proof. (b)→(c) is obvious.

(c)→(d). Let S be a tree almost of width f. Then for some k we have $|T \cap {}^n2| \leq f(n)$ for all $n \geq k$. By (1) of Lemma 8 $\mathrm{Lim}\,(S)^{\mathrm{fin}} = \bigcup_{\sigma_1,\sigma_2 \in {}^k2,\ \sigma_2 \in S} \mathrm{Lim}\,(S_{\sigma_1,\sigma_2})$. Each S_{σ_1,σ_2} is of width f since for $n \leq k$ we have $|S_{\sigma_1,\sigma_2} \cap {}^n2| = 1$ and for $n > k$ we have $|S_{\sigma_1,\sigma_2} \cap {}^n2| \leq |S \cap {}^n2| \leq f(n)$. Therefore, if $X \subseteq \bigcup_{m<\omega} \mathrm{Lim}\,(S_n)^{\mathrm{fin}}$ as in (c) then each S_m can be replaced by countably many S_{σ_1,σ_2}'s and (d) holds.

(d)→(b). Let f be a corset. We can easily define by recursion a sequence $0 = n_0 < n_1 < \ldots$ of natural numbers and a corset f^* such that for all $j < \omega$ and $m \geq n_{j+1}$ we have $(j+1) \cdot 2^{n_j} \cdot f^*(m) \leq f(m)$.

For a given corset f, if X satisfies (d) let $S^*_m,\ m < \omega$, be as in (d) for the corset f^*. We construct now a set $S \subseteq {}^{\omega>}2$ by defining $S \cap {}^m2$ by recursion on m. $S \cap {}^02 = \{\langle\rangle\}$. For $n_i < m \leq n_{i+1}$ let

$$S \cap {}^m2 = \{\eta \in {}^m2 : \eta \restriction n_i \in S \cap {}^{n_i}2 \text{ and } \eta \in S_j^{*\sim n_j} \text{ for some } j < i \vee j = 0\}.$$

S can be easily seen to be a tree, and clearly $\mathrm{Lim}\,(S)^{\mathrm{fin}} \supseteq \bigcup_{n<\omega} \mathrm{Lim}\,(S^*_n) \supseteq X$. For $m \leq n_1$ easily , for $n_i \leq m < n_{i+1}, i \geq 1$ we have
$|S \cap {}^m2| \leq \sum_{j\leq i} \left|S_j^{*\sim n_j} \cap {}^m2\right| = \sum_{j\leq i} 2^{n_j} |S^*_j \cap {}^m2| \leq (i+1) \cdot 2^{n_j} \cdot f^*(m) \leq f(m)$,
thus S is of width f.

(d)→(a). Assume now that (d) holds for X, and let $A \subseteq {}^\omega 2,\ \mu(A) = 0$; we shall prove that $\mu(X + A) = 0$. First we shall mention two lemmas of measure theory the proof of which is left to the reader.

Lemma A. For every tree T with $\mu(\mathrm{Lim}\,(T)) = a > 0$ and $\epsilon > 0$ there is an $N \in \omega$ such that for every $n \geq N$ there is a $t \subseteq {}^n2 \cap T$ such that $|t| \geq 2^n(a - \epsilon)$ and for each $\eta \in t$ $\mu(\mathrm{Lim}\,(T^{[\eta]}) > 2^{-n}(1 - \epsilon)$.

Using Lemma A one can prove

Lemma B. For every tree T with $\mu(\mathrm{Lim}\,(T)) > 0$, every $\epsilon > 0$ and every sequence $\langle \epsilon_i : 0 < i < \omega\rangle$ of positive reals there is a subtree T' of T and an increasing sequence $\langle n_i : i < \omega\rangle$ of natural numbers such that $n_0 = 0,\ \mu(\mathrm{Lim}\,(T')) > \mu(\mathrm{Lim}\,(T)) - \epsilon$ and

$$\text{(6)} \qquad \text{for } i > 0 \text{ and every } \eta \in {}^{n_i}2 \cap T' \quad \mu(\mathrm{Lim}\,(T'^{[\eta]})) > 2^{-n_i}(1 - \epsilon_i)$$

By basic mesure theory $\mu(A^{\mathrm{fin}}) = 0$ so there is a tree T such that $\mu(\mathrm{Lim}\,(T)) > 0$ and $\mathrm{Lim}\,(T) \cap A^{\mathrm{fin}} = \emptyset$ hence $\mathrm{Lim}\,(T)^{\mathrm{fin}} \cap A = \emptyset$. Given $\epsilon < \mu(\mathrm{Lim}\,(T))$ and $\langle \epsilon_i : i < \omega\rangle$ as in Lemma B we obtain a subtree T' of T as in that lemma with $\mu(\mathrm{Lim}\,(T')) > 0$. The union of sufficiently many "finite translates" of T', i.e., trees T'_{σ_1,σ_2} as in (1) of Lemma 8 is a tree T'' satisfying (6) with $\mu(\mathrm{Lim}\,(T'')) \geq \frac{1}{2}$. $\mathrm{Lim}\,(T'')^{\mathrm{fin}} = \mathrm{Lim}\,(T')^{\mathrm{fin}} \subseteq \mathrm{Lim}\,(T)^{\mathrm{fin}}$ and hence $\mathrm{Lim}\,(T'') \cap A \subseteq \mathrm{Lim}\,(T)^{\mathrm{fin}} \cap A = \emptyset$. We take now $\epsilon_i = \frac{1}{4(i+1)^3}$ and take T to be T'' and we get $\mu(\mathrm{Lim}\,(T)) \geq \frac{1}{2}$ and

$$\text{(7)} \qquad \text{for } i > 0 \text{ and every } \eta \in {}^{n_i}2 \cap T \quad \mu(\mathrm{Lim}\,(T^{[\eta]})) > 2^{-n_i}\left(1 - \frac{1}{4(i+1)^3}\right)$$

Let f be the corset given by $f(n) = i + 1$ for $n_i \leq n < n_{i+1}$. By (d) there are trees S_m of width f such that $X \subseteq \bigcup_{m<\omega} \mathrm{Lim}\,(S_m)$. To show that $\mu(X + A) = 0$ it clearly suffices to show that for every tree S of width f $\mu(\mathrm{Lim}\,(S) + A) = 0$.

We define

$$T^* = \{\eta \in {}^{\omega>}2 : \nu + \eta \in T \text{ for every } \nu \in S \text{ of the same length as } \eta\}$$

We do not show that T^* is a tree but obviously if $\zeta \trianglelefteq \eta \in T^*$ then $\zeta \in T^*$, thus $\mathrm{Lim}\,(T^*)$ is defined. If $\mu(\mathrm{Lim}\,(T^*)) > 0$ then, by a well-known property of the measure, $\mu(\mathrm{Lim}\,(T^*)^{\mathrm{fin}}) = 1$, hence in order to prove $\mu(\mathrm{Lim}\,(S) + A) = 0$ it suffices to prove $(\mathrm{Lim}\,(S) + A) \cap \mathrm{Lim}\,(T^*)^{\mathrm{fin}} = \emptyset$. Assume $y \in (\mathrm{Lim}\,(S) + A) \cap \mathrm{Lim}\,(T^*)^{\mathrm{fin}}$. Since $y \in \mathrm{Lim}\,(T^*)^{\mathrm{fin}}$ there is a $y' \in {}^{\omega}2$ such that $y'(n) = y(n)$ for all sufficiently big n's and $y' \in \mathrm{Lim}\,(T^*)$. Since $y \in \mathrm{Lim}\,(S) + A$ there is an $x \in \mathrm{Lim}\,(S)$ such that $y + x \in A$, hence $y + x \notin \mathrm{Lim}\,(T)^{\mathrm{fin}}$, hence $y' + x \notin \mathrm{Lim}\,(T)$. Therefore, for some n $y' \restriction n + x \restriction n \notin T$, hence, by the definition of T^*, $y' \restriction n \notin T^*$ contradicting $y' \in \mathrm{Lim}\,(T^*)$.

We still have to prove that $\mu(\mathrm{Lim}\,(T^*)) > 0$. We shall prove, by induction on i, that

$$(8) \qquad n_i \le n \le n_{i+1} \to |(T \setminus T^*) \cap {}^n2| \le 2^n \cdot \sum_{j<i} \frac{1}{4(j+1)^2}.$$

Once we establish (8) we notice that since
$\mathrm{Lim}\,(T) \setminus \mathrm{Lim}\,(T^*) = \bigcup_{n<\omega} \mathrm{Lim}\,(T) \setminus \{x \in {}^{\omega}2 : x \restriction n \in T^*\}$, and the set
$\mathrm{Lim}\,(T) \setminus \{x \in {}^{\omega}2 : x \restriction n \in T^*\}$ is increasing with n hence $\mu(\mathrm{Lim}\,(T) \setminus \mathrm{Lim}\,(T^*))$
$= \lim_{n\to\infty} \mu(\mathrm{Lim}\,(T) \setminus \{x \in {}^{\omega}2 : x \restriction n \in T^*\}) \le \lim_{n\to\infty} 2^{-n}|(T \setminus T^*) \cap {}^n2|$
$\le \lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{4(j+1)^2} = \sum_{j=0}^{\infty} \frac{1}{4(j+1)^2} = \frac{\pi^2}{24} < \frac{1}{2}$ and since $\mu(\mathrm{Lim}\,(T)) \ge \frac{1}{2}$
$\mu(\mathrm{Lim}\,(T^*)) > 0$.

To prove (8), assume now $n_i \le n \le n_{i+1}$. By the definition of T^*
$(T \setminus T^*) \cap {}^n2 = \{\eta \in T \cap {}^n2 : (\exists \rho \in S \cap {}^n2) \rho + \eta \notin T\}$
$= \{\eta \in T \cap {}^n2 : (\exists \rho \in S \cap {}^n2)(\eta \restriction n_i + \rho \restriction n_i \notin T)\} \cup$
$\bigcup_{\rho \in S \cap {}^n2} \{\eta \in T \cap {}^n2 : \eta \restriction n_i + \rho \restriction n_i \in T \,\wedge\, \eta + \rho \notin T\}$
$\subseteq \{\eta \in {}^n2 : \eta \restriction n_i \in T \setminus T^*\} \cup \bigcup_{\rho \in S \cap {}^n2} \{\eta \in {}^n2 : \eta + \rho \in \{\sigma \in {}^n2 : \sigma \restriction n_i \in T \,\wedge\, \sigma \notin T\}\}$.
Therefore $|(T \setminus T^*) \cap {}^n2| \le 2^{n-n_i} |(T \setminus T^*) \cap {}^{n_i}2| + |S \cap {}^n2| |\{\sigma \in {}^n2 : \sigma \restriction n_i \in T \wedge \sigma \notin T\}|$.
For $i > 0$ we have, by the induction hypothesis $|T \setminus T^* \cap {}^{n_i}2| \le 2^{n_i} \sum_{j<i} \frac{1}{4(j+1)^2}$. For $i = 0$ we have $(T \setminus T^*) \cap {}^{n_i}2 = \emptyset$ since $n_0 = 0$ and $\emptyset \in T^*$. $|S \cap {}^n2| \le f(n) = i$ and $|\{\sigma \in {}^n2 : \sigma \restriction n_i \in T \wedge \sigma \notin T\}| \le \frac{2^n}{4(i+1)^3}$, by (7). Thus
$|(T \setminus T^*) \cap {}^n2| \le 2^{n-n_i} \cdot 2^{n_i} \sum_{j<i} \frac{1}{4(j+1)^2} + (i+1) \cdot \frac{2^n}{4(i+1)^3} \le 2^n \sum_{j<i+1} \frac{1}{4(j+1)^2}$ which is what we had to show.

(a)$\to$(c). Most of the proof follows that of Lemma 10. We need also the following Lemma 14, which will be proved later. Let f be a corset.

14. Lemma. There is an infinite sequence $0 = n_0 < n_1 < n_2 < \ldots$ and a tree T such that for every $i \in \omega$ $f(n_{i+1}) > (i+1) \cdot 2^{i+1} + 1$ and
(B1) For each $\eta \in T \cap {}^{n_i}2$ we have $|T^{[\eta]} \cap {}^{n_{i+1}}2| = 2^{(n_{i+1}-n_i)} \cdot (1 - 2^{-(i+1)})$.
(B2) If $\eta, \nu_0, \ldots, \nu_{k-1} \in {}^{n_i}2$, $\nu_0^+, \ldots, \nu_{k-1}^+ \in {}^{n_{i+1}}2$, $\nu_j^+ \neq \nu_l^+$ for $j < l < k$, $\eta + \nu_l \in T$, $\nu_l \trianglelefteq \nu_l^+$ for $l < k$ then
$\left|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n_{i+1}}2,\ (\forall l < k)(\eta^+ + \nu_l^+ \in T)\}\right| \le 2^{n_{i+1}-n_i} \left(1 - 2^{-(i+1)}\right)^{k-1}$.

Let $\langle n_i : i \in \omega\rangle$ and T be as in Lemma 14. As in the proof of Lemma 10 we get $\mu(\mathrm{Lim}\, T) > 0$. Let T^* and $Y_{\eta,\zeta}$ be as in Lemma 9 and let $Y_{\eta,\zeta,\rho}$, S and z be as in the proof of Lemma 10. All we have to do is to show that S is almost of width f. Let us fix η, ζ and ρ. We shall now see that

$$\text{(9)}\qquad \begin{array}{l}\text{If } \eta' \in T^{*[\eta]} \cap {}^{n_i}2 \text{ then}\\ |\{\eta^+ : \eta' \trianglelefteq \eta^+ \in T^* \cap {}^{n_{i+1}}2\}|/2^{(n_{i+1}-n_i)} \leq (1-2^{-(i+1)})^{|S\cap {}^{n_i}2|-1}\end{array}$$

Let $\eta^+ \in T^{*[\eta]} \cap {}^{n_{i+1}}2$, then, by the definition of S (see (3)), if $\rho^+ \in S \cap {}^{n_{i+1}}2$ then $\rho^+ + \eta^+ + z \in T$. Thus
$\{\eta^+ : \eta' \trianglelefteq \eta^+ \in T^* \cap {}^{n_{i+1}}2\} \subseteq \{\eta^+ : \eta' \trianglelefteq \eta^+ \in {}^{n_{i+1}}2,\ (\forall \rho^+ \in S)\rho^+ + \eta^+ + z \in T\}$.
Let us take in (B2) $\eta = \eta'$, $k = |S \cap {}^{n_i}2|$, $\{\tau_l : l < k\} = S \cap {}^{n_i}2$, $\{\tau_l^+ : l < k\} \subseteq S \cap {}^{n_{i+1}}2$, and for $l < k$ $\tau_l^+ \restriction n_i = \tau_l$, $\nu_l = \tau_l + z$, $\nu_l^+ = \tau_l^+ + z$, hence $\nu_l = \nu_l^+ \restriction n_i$ for $l < k$. Since for $l < k$ $\nu_l^+ + z = \tau_l^+ \in S \cap {}^{n_{i+1}}2$ we have

$$\{\eta^+ : \eta' \trianglelefteq \eta^+ \in {}^{n_{i+1}}2,\ (\forall \rho^+ \in S)(\rho^+ + \eta^+ + z \in T\} \subseteq \{\eta^+ : \eta' \trianglelefteq \eta^+ \in {}^{n_{i+1}}2,\ (\forall l < k)(\nu_l^+ + \eta^+ \in T)\},$$

therefore by (B2)
$|\{\eta^+ : \eta' \trianglelefteq \eta^+ \in {}^{n_{i+1}}2,\ (\forall \rho^+ \in S)(\rho^+ + \eta^+ + z \in T\}| \leq 2^{n_{i+1}-n_i}(1-2^{-(i+1)})^{|S\cap {}^{n_i}2|-1}$, which establishes (9).

(9) tells us how T^* grows from the level n_i to the level n_{i+1} and therefore
$|T^* \cap {}^{n_i}2| \cdot 2^{-n_i} \leq \prod_{j<i}(1-2^{-(j+1)})^{|S\cap {}^{n_j}2|-1}$.
Let $c_0 = \mu(\mathrm{Lim}\, T^*)$. We know that $c_0 > 0$ and we can assume $c_0 < 1$. Then
$-\infty < \log c_0 \leq \log(|T^* \cap {}^{n_i}2| \cdot 2^{-n_i}) \leq \sum_{j<i}\big(\log(1-2^{-(j+1)}) \cdot (|S \cap {}^{n_j}2| - 1)\big)$. Since $\log(1-x) \leq -\frac{1}{2}x$ we get $\sum_{j<i} 2^{-(j+2)} \cdot (|S \cap {}^{n_{i+1}}2| - 1) \leq \log \frac{1}{c_0}$. We shall denote $4 \log \frac{1}{c_0}$ by c, so $\sum_{j<i} 2^{-j} \cdot (|S \cap {}^{n_j}2| - 1) \leq c$, and for every j $2^{-j}(|S \cap {}^{n_j}2| - 1) \leq c$, hence $|S \cap {}^{n_j}2| \leq c \cdot 2^j + 1$. For $j > c$ we have, by our choice of the n_i's,
$f(n_j) > j \cdot 2^j + 1 > c \cdot 2^j + 1 \geq |S \cap {}^{n_j}2|$, hence S is almost of width f.

Lemma 14 follows immediately from the following Lemma.

15. Lemma. For every $n \in \omega$ and $0 < p < 1$ there is an $N > n$ such that for every $n' \geq N$ and $t \subseteq {}^n2$ there is a $t' \subseteq {}^{n'}2$ which satisfies the following (i)–(iii).
(i) For each $\zeta \in t'$ $\zeta \restriction n \in t$.
(ii) For each $\eta \in t$ $|t'^{[\eta]}| \geq 2^{n'-n} \cdot p$.
(iii) If $0 < k \leq 2^n$ $\eta, \nu_0, \ldots, \nu_{k-1} \in {}^n2$, $\nu_0^+, \ldots, \nu_{k-1}^+ \in {}^{n'}2$, $\nu_j^+ \neq \nu_l^+$ for $j < l < k$, $\eta + \nu_l \in t$, $\nu_l = \nu_l^+ \restriction n$ for $l < k$ then $|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2, (\forall l < k)\eta^+ + \nu_l^+ \in t'\}| \leq 2^{n'-n}p^{k-1}$.

Proof. We shall prove the lemma by the probabilistic method. Let $n' > n$ and let $A = \{\eta^+ \in {}^{n'}2 : \eta^+ \restriction n \in t\}$. We construct a subset A^* of A as follows. We take a coin which yields heads with probability p. For each $\eta^+ \in A$ we toss this coin and we put η^+ in A^* iff the coin shows heads. We shall see that if we take $t' = A^*$ then, for sufficiently large n', the probability that (ii) holds has a positive lower bound which does not depend on n' while the probability that (iii) holds is arbitrarily close to 1. Hence there is an N and a t' as claimed by the lemma. We prove first two lemmas.

Lemma 16. For $k, \eta, \nu_0, \ldots, \nu_{k-1}, \nu_0^+, \ldots, \nu_{k-1}^+$ as im Lemma 15 there are reals $c_1, c_2 > 0$

which depend only on p, n and k such that

$$\Pr\left(|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ \bigwedge_{l<k} \eta^+ + \nu_l^+ \in A*\}| \geq p^{k-1} 2^{n'-n}\right) < c_1 e^{-c_2 \cdot 2^{n'}}$$

Proof. We denote $2^{n'-n}$ with m. We set $({}^{n'}2)^{[\eta]} = \{\eta_j^+ : j < m\}$. Let G be the graph on m given by

$$iGj \text{ iff } \{\eta_i^+ + \nu_l^+ : l < k\} \cap \{\eta_j^+ + \nu_l^+ : l < k\} \neq \emptyset$$

Obviously each $i < m$ has at most k^2 neighbors in G hence, by a well known theorem, m can be decomposed into $k^2 + 1$ pairwise disjoint sets $B_0, \ldots, B_{k^2}$ such that for every $i \leq k^2$ if $j, l \in B_i$ and $j \neq l$ then jGl does not hold. Let $d < \frac{1}{2}\min\{p^{l-1} - p^l : l \leq 2^n\} = \frac{1}{2} p^{2^n - 1}(1-p) > 0$.

$$\begin{aligned} &\Pr\left(|j < m : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| \geq m \cdot p^{k-1}\right) \\ (10)\quad &\leq \Pr\left(|j < m : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| > m(p^k + d)\right) \qquad \text{since } p^k + d < p^{k-1} \end{aligned}$$

Assume that

$$\begin{aligned} &\text{for every } i \leq k^2 \text{ such that } |B_i| \geq \frac{dm}{2k^2+2} \\ (11)\quad &\text{we have } |\{j \in B_i : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\} \leq |B_i|(p^k + \frac{d}{2}) \end{aligned}$$

then

$$\{j < m : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\} \subseteq \bigcup_{i \leq k^2,\ |B_i| \geq \frac{dm}{2k^2+2}} \{j \in B_i : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\} \bigcup_{i \leq k^2,\ |B_i| < \frac{dm}{2k^2+2}} B_i$$

hence

$$\begin{aligned} &|j < m : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| \\ &\leq \sum_{i \leq k^2,\ |B_i| \geq \frac{dm}{2k^2+2}} |j \in B_i : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| + \sum_{i \leq k^2,\ |B_i| < \frac{dm}{2k^2+2}} |B_i| \\ &\leq \sum_{i \leq k^2,\ |B_i| \geq \frac{dm}{2k^2+2}} |B_i|(p^k + \frac{d}{2}) + \sum_{i \leq k^2,\ |B_i| < \frac{dm}{2k^2+2}} |B_i|, \qquad \text{by (11)} \\ &\leq m(p^k + \frac{d}{2}) + (k^2+1)\frac{dm}{2k^2+2} = m(p^k + d) \end{aligned}$$

Therefore the event $|\{j < m : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| > m(p^k + d)$ is incompatible with (11), so we continue the inequality (10) by

$$\begin{aligned} &\le \Pr\Big(\bigvee_{i \le k^2,\, |B_i| \ge \frac{dm}{2k^2+2}} \big(|\{j \in B_i : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|(p^k + \frac{d}{2})\big)\Big) \\ (12)\quad &\le \sum_{i \le k^2,\, |B_i| \ge \frac{dm}{2k^2+2}} \Pr\Big(|\{j \in B_i : \bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|(p^k + \frac{d}{2})\Big) \end{aligned}$$

For a fixed $j < m$ the events $\eta_j^+ + \nu_l^+ \in A^*$ for different l's are independent hence $\Pr\left(\bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*\right) = p^k$. For a fixed i the events $\bigwedge_{l<k} \eta_j^+ + \nu_l^+ \in A^*$ for different j's in B_i are independent since, by the definition of the B_i's, if $j_1, j_2 \in B_i$ and $j_1 \neq j_2$ then $\eta_{j_1}^+ + \nu_{l_1}^+ \neq \eta_{j_2}^+ + \nu_{l_2}^+$. We have here $|B_i|$ independent events, each with probability p^k. By a formula of probability theory (see, e. g., the formula $\Pr[X > a] < e^{-2a^2/n}$ in Spencer [2], p. 29)

$$\Pr\Big(\{j \in B_i : \bigwedge_{k<l} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|p^k + \epsilon\Big) < e^{-\frac{2\epsilon^2}{|B_i|}}$$

and taking $\epsilon = \frac{1}{2}|B_i|d$ we get

$$\Pr\Big(\{j \in B_i : \bigwedge_{k<l} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|(p^k + \frac{d}{2})\Big) < e^{-\frac{d^2|B_i|}{2}}$$

Continuing (12) we get

$$\le \sum_{i \le k^2,\, |B_i| \ge \frac{dm}{2k^2+2}} e^{-\frac{d^2|B_i|}{2}} \le \sum_{i \le k^2,\, |B_i| \ge \frac{dm}{2k^2+2}} e^{-\frac{d^2}{2}\frac{dm}{2k^2+2}} \le (k^2+1)e^{-\frac{d^3 2^{n'-n}}{4k^2+4}}$$

Combining this with the inequalities (10) and (12) we get

$$\begin{aligned} &\Pr\Big(|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ \bigwedge_{l<k} \eta^+ + \nu_l^+ \in A^*\}| \ge p^{k-1}2^{n'-n}\Big) \\ &< (k^2+1)e^{-\frac{d^3 2^{n'-n}}{4k^2+4}} = (k^2+1)e^{-\frac{d^3 2^{-n} 2^{n'}}{4k^2+4}} \end{aligned}$$

Since $d = \frac{1}{2}p^{2^n-1}(1-p)$ this proves Lemma 16.

17. Lemma. There are c_3, c_4 which depend only on p and n such that

$$\begin{aligned} &\Pr\Big(\bigvee_{k,\eta,\nu_0,\dots,\nu_{k-1},\nu_0^+,\dots,\nu_{k-1}^+} \\ (13)\quad &\qquad |\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ (\forall l < k)\, \eta^+ + \nu_l^+ \in A^*\}| \ge 2^{n'-n}p^{k-1}\Big) \\ &\le c_3(2^{n'})^{2^n} e^{-c_4 2^{n'}} \end{aligned}$$

where $k, \eta, \nu_0, \ldots, \nu_{k-1}, \nu_0^+, \ldots, \nu_{k-1}^+$ are as in (iii) of Lemma 15.

Proof. By our requirements on $k, \eta, \nu_0, \ldots, \nu_{k-1}, \nu_0^+, \ldots, \nu_{k-1}^+$ there are at most 2^n possible k's and η's and $(2^{n'})^{2^n}$ sequences $\langle \nu_0^+, \ldots, \nu_{k-1}^+ \rangle$, while $\nu_0, \ldots, \nu_{k-1}$ are determined by $\nu_0^+, \ldots, \nu_{k-1}^+$ and n. Therefore we get, by Lemma 16,

$$\Pr\Big(\bigvee_{k,\eta,\nu_0,\ldots,\nu_{k-1},\nu_0^+,\ldots,\nu_{k-1}^+} (|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ (\forall l < k)\ \eta^+ + \nu_l^+ \in A^*\}| \geq 2^{n'-n} p^{k-1})\Big)$$

$$\leq \sum_{k,\eta,\nu_0,\ldots,\nu_{k-1},\nu_0^+,\ldots,\nu_{k-1}^+} \Pr\Big(|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ (\forall l < k)\ \eta^+ + \nu_l^+ \in A^*\}| \geq 2^{n'-n} p^{k-1}\Big)$$

$$\leq 2^n \cdot 2^n \cdot (2^{n'})^{(2^n)} \cdot c_1 e^{-c_2 2^{n'}}$$

Proof of Lemma 15 (continued). For each $\eta^+ \in {}^{n'}2$ such that $\eta \trianglelefteq \eta^+$ $\eta^+ \in A^*$ if the coin shows heads and different tosses are independent $|A^{*[\eta]}|$ is a binomial random variable with expectation $2^{n'-n}p$. By the central limit theorem of probability theory (see, e. g., Feller [1, Ch. 7]) the limit, as $n' \to \infty$, of $\Pr\left(|A^{*[\eta]}| \geq 2^{n'-n}p\right)$ is $\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{2}$, hence there is an N such that for every $n' \geq N$ $\Pr\left(|A^{*[\eta]}| \geq 2^{n'-n}p\right) \geq \frac{1}{3}$. For different $\eta \in t$ the random variables $|A^{*[\eta]}|$ are idependent, hence

$$\Pr\big(\bigwedge_{\eta \in t} (|A^{*[\eta]}| \geq 2^{n'-n}p)\big) \geq \frac{1}{3^{|t|}} \geq \frac{1}{3^{2^n}} \tag{14}$$

The right-hand side of (13) clearly vanishes as $n \to \infty$, let us take N to be such that for $n' \geq N$ the right-hand side of (13) is $< 3^{-2^n}$. Therefore we have, by (13) and (14),

$$\begin{aligned} \Pr\Big(& \bigvee_{k,\eta,\nu_0,\ldots,\nu_{k-1},\nu_0^+,\ldots,\nu_{k-1}^+} \\ & (|\{\eta^+ : \eta \trianglelefteq \eta^+ \in {}^{n'}2,\ (\forall l < k)\ \eta^+ + \nu_l^+ \in A^*\}| < 2^{n'-n} p^{k-1}) \\ & \wedge \bigwedge_{\eta \in t} (|A^{*[\eta]}| \geq 2^{n'-n}p)\Big) > 0 \end{aligned} \tag{15}$$

By (15) there is a t' as required by the lemma.

§4 Characterization of the meager-additive sets

18. Theorem. For every $X \subseteq {}^\omega 2$ the following conditions are equivalent:

a. X is meager additive.

b. For every sequence $n_0 < n_1 < n_2 < \ldots$ of natural numbers there is a sequence $i_0 < i_1 < \ldots$ of natural numbers and a $y \in {}^\omega 2$ such that for every $x \in X$ and for every sufficiently big $k < \omega$ there is an $l \in [i_k, i_{k+1})$ such that $x \restriction [n_l, n_{l+1}) = y \restriction [n_l, n_{l+1})$.

Proof. Throughout this proof, if $x \in {}^\omega 2 \cup {}^{\omega>}2$, $k, l \in \omega$ and $k < l$ then $x \restriction [k, l)$ will denote the sequence $\xi \in {}^{l-k}2$ such that $\xi(i) = x(k+i)$ for all $i < l - k$.

(b)→(a). In order to prove (a) it clearly suffices to show that $X + \operatorname{Lim} T$ is meager for every nowhere dense tree T.

For a nowhere dense tree T let $\langle n_i : i < \omega \rangle$ be an ascending sequence of natural numbers such that $n_0 = 0$ and for every $i \in \omega$ there is a sequence $\nu_i \in {}^{n_{i+1}-n_i}2$ such that for every $\tau \in {}^{n_i}2$ $\tau \frown \nu_i \notin T$. Let $\langle i_j : j < \omega \rangle$ and y be as in (b), then, by (b), $X = \bigcup_{k\in\omega} X_k$ where $X_k = \{x \in X : (\forall m \geq k)(\exists l \in [i_m, i_{m+1}))x\restriction[n_l, n_{l+1}) = y\restriction(n_l, n_{l+1})$. It clearly suffices to prove that $X_k + \operatorname{Lim} T$ is nowhere dense.

Let $\tau \in {}^{n_{i_m}}2$ for some $m \geq k$; we shall show that τ has an extension which is not in $X_k + \operatorname{Lim} T$. Let $\nu = \nu_{i_m} \frown \nu_{i_m+1} \frown \ldots \frown \nu_{i_{m+1}-1}$ and let $\rho = y \restriction [n_{i_m}, n_{i_{m+1}}) + \nu$. We show that no extension z of $\tau \frown \rho$ is in $X_k + \operatorname{Lim} T$. Suppose $\tau \frown \rho \trianglelefteq z \in X_k + \operatorname{Lim} T$ then $z = x + w,\ x \in X_k\ \ w \in \operatorname{Lim} T$. Therefore $\tau = \tau_1 + \tau_2$ and $\rho = \rho_1 + \rho_2$ such that $\tau_1 \frown \rho_1 \trianglelefteq x$ and $\tau_2 \frown \rho_2 \trianglelefteq w$, hence $\tau_2 \frown \rho_2 \in T$. Let $\xi \in {}^{n_{i_m}}2$ be such that $\xi(j) = 0$ for every $j < n_{i_m}$, and let $\rho' = \xi \frown \rho,\ \ \rho'_1 = \xi \frown \rho_1,\ \ \rho'_2 = \xi \frown \rho_2$. Clearly $\rho' = \rho'_1 + \rho'_2$. Since $x \in X_k$ there is, by (b), an $l \in [i_m, i_{m+1})$ such that $x \restriction [n_l, n_{l+1}) = y \restriction [n_l, n_{l+1})$. Since $\tau_1 \frown \rho_1 \trianglelefteq x$ we have $\rho'_1 \restriction [n_{i_m}, n_{i_m+1}) = x \restriction [n_{i_m}, n_{i_m+1})$ and hence $\rho_1 \restriction [n_l, n_{l+1}) = x \restriction [n_l, n_{l+1}) = y \restriction [n_l, n_{l+1})$ Therefore, by the definition of ρ and ν

$$\begin{aligned} y \restriction [n_l, n_{l+1}) + \rho'_2 \restriction [n_l, n_{l+1}) = \rho'_1 \restriction [n_l, n_{l+1}) + \rho'_2 \restriction [n_l, n_{l+1}) &= \rho' \restriction [n_l, n_{l+1}) \\ &= y \restriction [n_l, n_{l+1}) + \nu_l \end{aligned}$$

hence $\rho'_2 \restriction [n_l, n_{l+1}) = \nu_l$. By the definition of ν_l $\ \tau_2 \frown \rho_2 \notin T$, contradicting $\tau_2 \frown \rho_2 \in T$.

(a)→(b). Let X be meager-additive. Let $\langle n_i : i < \omega \rangle$ be an ascending sequence of natural numbers. Let $B = \{x \in {}^{\omega}2 : \forall j (\exists k \in [n_j, n_{j+1}))\, x(k) \neq 0\}$ and $T = \{x \restriction n : x \in B,\ n \in \omega\}$. Clearly $B = \operatorname{Lim} T$ is nowhere dense, so $X + \operatorname{Lim} T$ is meager, hence there are nowhere dense trees $S_n,\ \ n \in \omega$ such that for every n $\ S_n \subseteq S_{n+1}$ and $X + \operatorname{Lim} T \subseteq \bigcup_{n\in\omega} S_n$. We define now the sequences $\langle i_l : l < \omega \rangle$, which is an ascending sequence of natural numbers, and $\langle \nu_l : l < \omega \rangle$ by recursion as follows. $i_0 = 0$. Given i_l let ν_l and i_{l+1} be such that $\nu_l \in {}^{n_{i_{l+1}} - n_{i_l}}2$ and for every $\rho \in {}^{n_{i_l}}2$ $\ \rho \frown \nu_l \notin S_l$; there are such ν_l and i_{l+1} since S_l is nowhere dense. Let $y \in {}^{\omega}2$ be given by $y \restriction [n_{i_l}, n_{i_{l+1}}) = \nu_l$ for every $l < \omega$. We shall now prove that $\langle i_l : l < \omega \rangle$ and y are as required by (b).

Let $x \in X$, so $\operatorname{Lim}(x + T) = x + \operatorname{Lim} T \subseteq X + \operatorname{Lim} T \subseteq \bigcup_{n\in\omega} S_n$. Therefore, by Lemma 7 (where we take $x + T$ for S) there is an $\eta \in T$ and $n \in \omega$ such that $x + T^{[\eta]} \subseteq S_n$. Let k be such that $k \geq n$ and $i_k \geq \operatorname{length} \eta$. By $x + T^{[\eta]} \subseteq S_n$ we have $x \restriction n_{i_{k+1}} + (T^{[\eta]} \cap {}^{n_{i_{k+1}}}2) \subseteq S_n \subseteq S_k$. Thus for every $\rho \in T^{[\eta]} \cap {}^{n_{i_{k+1}}}2$ $\ x \restriction n_{i_{k+1}} + \rho \in S_k$, hence, by the definition of ν_k and y, $\ x\restriction[n_{i_k}, n_{i_{k+1}}) + \rho\restriction[n_{i_k}, n_{i_{k+1}}) \neq \nu_k = y\restriction[n_{i_k}, n_{i_{k+1}})$ and therefore $x\restriction[n_{i_k}, n_{i_{k+1}}) - y\restriction[n_{i_k}, n_{i_{k+1}}) \neq \rho\restriction[n_{i_k}, n_{i_{k+1}})$, i.e., $x\restriction[n_{i_k}, n_{i_{k+1}}) - y\restriction[n_{i_k}, n_{i_{k+1}}) \notin \{\rho \restriction [n_{i_k}, n_{i_{k+1}}) : \rho \in T^{[\eta]}\}$. Since $i_k > \operatorname{length} \eta$ this can happen, by the definition of T, only if for some $i_k \leq j < i_{k+1}$ $\ x \restriction [n_j, n_{j+1}) - y \restriction [n_j, n_{j+1})$ is identically zero, and this is what we had to prove.

§5. An uncountable null-additive set.

19. Theorem. If the continuum hypothesis holds then there is an uncountable null-additive set.

Proof. Let $\langle f_\alpha : \alpha < \omega_1 \rangle$ be a sequence containing all corsets and let $\langle T_\alpha : \alpha < \omega_1 \rangle$ be a sequence containing all perfect trees. Let E be the set of all limit ordinals $\delta < \omega_1$ such that for every $\alpha, \beta < \delta$ and $n < \omega$ there is a $\gamma < \delta$ such that

$$T_\gamma \subseteq T_\alpha,\ \ T_\gamma \cap {}^n2 = T_\alpha \cap {}^n2 \ \text{ and for all } m \quad |T_\gamma \cap {}^m2| \leq \max\left(|T_\alpha \cap {}^m2|, f_\beta(m)\right)$$

Clearly E is closed. For every $\alpha, \beta < \omega_1$ there is a perfect tree T such that $T \subseteq T_\alpha$, $T \cap {}^n2 = T_\alpha \cap {}^n2$ and for all $m < \omega$ $|T \cap {}^m2| \leq \max\left(|T_\alpha \cap {}^n2|, f_\beta(m)\right)$. This tree T is T_γ for some $\gamma < \omega_1$. By a simple closure argument this implies that E is unbounded.

We need now the following lemma which will be proved later.

20. Lemma. There is an increasing and continuous sequence $\langle \delta_\zeta : \zeta < \omega_1 \rangle$ of ordinals in E such that for every $\zeta < \omega_1$, $k < \omega$ and $\alpha < \delta_\zeta$ there is an ordinal γ which is good for (ζ, α, k), where by γ *is good for* (ζ, α, k) we mean that

(16)
- (i) $\gamma < \delta_{\zeta+1}$
- (ii) $T_\gamma \subseteq T_\alpha$, $T_\gamma \cap {}^k2 = T_\alpha \cap {}^k2$
- (iii) for all $\xi \leq \zeta$ such that $\delta_\xi > \alpha$ and for every $\epsilon < \delta_\zeta$, there is a $\beta < \delta_\xi$ such that $T_\gamma \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ

For $\xi < \omega_1$ let γ_ξ be the γ which is good for $(\xi, 0, 0)$. We choose $\eta_\xi \in \operatorname{Lim} T_{\gamma_\xi} \setminus \{\eta_\beta : \beta < \xi\}$, and let $X = \{\eta_\xi : \xi < \omega_1\}$. X is clearly uncountable. We shall prove that X is null-additive by proving that X satisfies condition (c) of Theorem 13. For a given corset f $f = f_\epsilon$ for some $\epsilon < \omega_1$. Let $\xi < \omega_1$ be such that $\delta_\xi > \epsilon$. Let $Z = \{\beta < \delta_{\xi+1} : T_\beta \text{ is almost of width } f_\epsilon\}$. We shall see that
$X \subseteq \{\eta_\zeta : \zeta \leq \xi\} \cup \bigcup_{\beta \in Z} \operatorname{Lim} T_\beta$. Since Z and ξ are countable condition (c) of Theorem 13 holds.

Let $\zeta > \xi$, it suffices to prove that $\eta_\zeta \in \operatorname{Lim} T_\beta$ for some $\beta \in Z$. $\epsilon < \delta_\xi$ and since γ_ζ is good for $\alpha = k = 0$ hence there is a $\beta < \delta_\xi$ such that $T_{\gamma_\zeta} \subseteq T_\beta$, and T_β is of width f_ϵ. Thus $\beta \in Z$ and $\eta_\zeta \in \operatorname{Lim} T_{\gamma_\zeta} \subseteq \operatorname{Lim} T_\beta$.

Proof of Lemma 20. We define $\langle \delta_\zeta : \zeta < \omega_1 \rangle$ as follows. δ_0 is the least member of E. For a limit ordinal ζ $\delta_\zeta = \bigcup_{\xi < \zeta} \delta_\xi$. Since $\delta_\xi \in E$ for $\xi < \zeta$ also $\delta_\zeta \in E$. We shall now define $\delta_{\zeta+1}$. We shall assume, as an induction hypothesis, that for each $\xi < \zeta$ the lemma holds. For each $\alpha < \delta_\zeta$ and $k < \omega$ we shall find a $\gamma(\alpha, k)$ which is good for (ζ, α, k) and we shall choose $\delta_{\zeta+1}$ to be the least member of E greater than all these $\gamma(\alpha, k)$'s.

First we shall show that what the lemma claims holds for the case where ζ is a successor or 0. Whenever we shall write $\zeta - 1$ we shall assume that ζ is a successor. Let $\alpha < \delta_\zeta$ and $k < \omega$ be given, and let $\{\epsilon_n : n < \omega\} = \{\epsilon : \epsilon < \delta_\zeta\}$. We define sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle k_n : n < \omega \rangle$ so that

(a) $k_0 = k$. If $\zeta = 0$ or $\alpha < \delta_{\zeta-1}$ then $\alpha_0 = \alpha$. If $\alpha \geq \delta_{\zeta-1}$ then α_0 is an ordinal which is good for $(\zeta - 1, \alpha, k)$. In any case $\alpha_0 < \delta_\zeta$, $T_{\alpha_0} \subseteq T_\alpha$ and $T_{\alpha_0} \cap {}^k2 = T_\alpha \cap {}^k2$.
(b) $\alpha_{n+1} < \delta_\zeta$.
(c) $T_{\alpha_{n+1}} \subseteq T_{\alpha_n}$.
(d) $T_{\alpha_{n+1}} \cap {}^{k_n}2 = T_{\alpha_n} \cap {}^{k_n}2$.
(e) $T_{\alpha_{n+1}}$ is almost of width f_{ϵ_n}.

(f) $k_{n+1} > k_n$ and every $\eta \in T_{\alpha_{n+1}} \cap {}^{k_n}2$ has at least two extensions in $T_{\alpha_{n+1}} \cap {}^{k_{n+1}}2$.
There are indeed such sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle k_n : n < \omega \rangle$. (a) determines k_0 and α_0; if $\alpha < \delta_{\zeta-1}$ then there is an α_0 as in (a) by the induction hypothesis. δ_ζ in E and let us take $\alpha_n, \epsilon_n, k_n, \alpha_{n+1}$ for α, β, n, γ in the definition of E, then $\delta_\zeta \in E$ says that there is an α_{n+1} which satisfies (b)–(e). Since $T_{\alpha_{n+1}}$ is perfect there is a k_{n+1} as in (f).

Let $T = \bigcap_{n \in \omega} T_{\alpha_n}$. By (c),(d),(f) T is a perfect tree, hence it is T_γ for some $\gamma < \omega_1$. Since T, and therefore also γ, depend on α and k we denote γ with $\gamma(\alpha, k)$. As is easily seen $T_{\gamma(\alpha,k)} \subseteq T_\alpha$, $T_{\gamma(\alpha,k)} \cap {}^k2 = T_\alpha \cap {}^k2$, and for every $\epsilon < \delta_\zeta$ $T_{\gamma(\alpha,k)} \subseteq T_{\alpha_{l+1}} \subseteq T_\alpha$, where l is such that $\epsilon = \epsilon_l$. This means that (iii) of (16) holds for $\xi = \zeta$. We shall have to show that (iii) holds for $\xi < \zeta$ and to deal with the case where ζ is a limit ordinal.

If ζ is a limit ordinal let $\langle \zeta_n : n < \omega \rangle$ be an increasing sequence such that $\delta_{\zeta_0} > \alpha$ and $\bigcup_{n<\omega} \zeta_n = \zeta$. We construct the sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle k_n : n < \omega \rangle$ as in the case where ζ is a successor, except that (a),(b),(e) are replaced by

(a′) $k_0 = k$, $\alpha_0 = \alpha$.

(b′) $\alpha_n < \delta_{\zeta_n}$.

(e′) α_{n+1} is good for (ζ_n, α, k).

By the induction hypothesis that the lemma holds for the ζ_n's there are indeed such sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle k_n : n < \omega \rangle$. Let $T = \bigcap_{n<\omega} T_{\alpha_n}$. As above, T is a perfect tree and $T = T_{\gamma(\alpha,k)}$, $T_{\gamma(\alpha,k)} \subseteq T_\alpha$ and $T_{\gamma(\alpha,k)} \cap {}^k2 = T_\alpha \cap {}^k2$.

We shall now see that for both cases of ζ with which we are dealing (iii) holds for $\xi < \zeta$. If ζ is a successor then $\xi \leq \zeta - 1$ and since α_o is, by (a), good for $(\zeta - 1, \alpha, k)$ there is a $\beta < \delta_\zeta$ such that $T_{\alpha_0} \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ. Note that if $\alpha < \delta_{\zeta-1}$ then, by the induction hypothesis, we have a $\gamma < \delta_\zeta$ which is good for (ζ, α, k), and if $\zeta = 0$ then (iii) holds vacuously, hence we may assume that $\zeta > 0$ and $\alpha \in [\delta_{\zeta-1}, \delta_\zeta)$. Since $T_{\gamma(\alpha,k)} \subseteq T_{\alpha_0}$ β is as required by (iii). If ζ is a limit ordinal then $\xi \leq \zeta_n$ for some $n < \omega$. Since α_{n+1} is good for ζ_n then there is a $\beta < \delta_\xi$ such that $T_{\alpha_{n+1}} \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ. Since $T_{\gamma(\alpha,k)} \subseteq T_{\alpha_{n+1}}$ β is as required by (iii).

The only case left is that where ζ is a limit ordinal and $\xi = \zeta$ in (iii). Since $\alpha, \epsilon < \zeta$ also $\alpha, \epsilon < \zeta_n$ for some $n < \omega$. α_{n+1} is good for ζ_n hence there is a $\beta < \delta_{\zeta_n}$ such that $T_{\alpha_{n+1}} \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ. Since $T_{\gamma(\alpha,k)} \subseteq T_{\alpha_{n+1}}$ and $\zeta_n < \zeta$ β is as required by (iii).

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