Forcing Isomorphism

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If two models of a first order theory are isomorphic then they remain isomorphic in any forcing extension of the universe of sets. In general, however, such a forcing extension may create new isomorphisms. For example, any forcing that collapses cardinals may easily make formerly non-isomorphic models isomorphic. Certain model theoretic constraints on the theory and other constraints on the forcing can prevent this pathology.

A countable first order theory is said to be *classifiable* if it is superstable and does not have either the dimensional order property (DOP) or the omitting types order property (OTOP). Shelah has shown [7] that if a theory T is classifiable then each model of cardinality λ is described by a sentence of

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$L_{\infty,\lambda}$. In fact this sentence can be chosen in the L^*_λ. (L^*_λ is the result of enriching the language $L_{\infty,\beth^+}$ by adding for each $\mu < \lambda$ a quantifier saying the dimension of a dependence structure is greater than μ.) Further work ([3], [2]) shows that $\beth^+$ can be replaced by $\aleph_1$. The truth of such sentences will be preserved by any forcing that does not collapse cardinals $\leq \lambda$ and that adds no new countable subsets of λ, e.g., a λ-complete forcing. That is, if two models of a classifiable theory of power λ are non-isomorphic, they remain non-isomorphic after a λ-complete forcing.

In this paper we show that the hypothesis of the forcing adding no new countable subsets of λ cannot be eliminated. In particular, we show that non-isomorphism of models of a classifiable theory need not be preserved by ccc forcings. The following definition isolates the key issue of this paper.

0.1 Definition. Two structures M and N are *potentially isomorphic* if there is a ccc-notion of forcing $\mathcal{P}$ such that if G is $\mathcal{P}$-generic then $V[G] \models M \approx N$.

In the first section we will show that any theory that is not classifiable has models that are not isomorphic but are potentially isomorphic. In the second, we show that this phenomenon can also occur for classifiable theories. The reader may find it useful to examine first the example discussed in Theorem 2.3.

1 Non-classifiable Theories

We begin by describing a class (which we call amenable) of subtrees of $Q^{\leq\omega}$ that are pairwise potentially isomorphic. Then we use this fact to show that every nonclassifiable theory has a pair of models that are not isomorphic but are potentially isomorphic.

1.1 Notation.

i) We adopt the following notation for relations on subsets of $Q^{\leq\omega}$. $\sqsubset$ denotes the subsequence relation; $<$ denotes lexicographic ordering; for $\alpha \leq \omega$, lev_α is a unary predicate that holds of sequences of length (level) α; $\wedge$ is the operation on two sequences that produces their largest common initial segment. We denote the ordering of the rationals by $<_Q$.

ii) For $\eta \in Q^\omega$, let $D_\eta = \{\sigma \in Q^\omega : \sigma(2n) = \eta(n)\}$ and $S_\eta = \{\sigma \in D_\eta : \sigma(2n+1)$ is 0 for all but finitely many $n\}$. Let $C = \cup_{\eta \in Q^\omega} S_\eta$.

iii) The language L^t (for tree) contains the symbols $\sqsubset$, $<$, lev_α and unary predicates P_η for $\eta \in Q^\omega$.

iv) For any $A \subseteq C$, A^* denotes the L^t-structure with universe $A \cup Q^{<\omega}$ under the natural interpretation of $\sqsubset, <, \mathrm{lev}_\alpha$ and with $P_\eta(A^*) = S_\eta \cap A$.

Note that $\langle C, < \rangle$ is isomorphic to a subordering of the reals. Since C is dense we may assume Q is embedded in C but not necessarily in a natural way.

1.2 Definition. A substructure A^* of C^* is *amenable* if for all $\eta \in Q^\omega$, all $n \in \omega$ and all $s \in Q^n$, if $P_\eta(C^*)$ contains an element extending s then $P_\eta(A^*)$ does also.

1.3 Remark. It is easy to see that a substructure A^* of C^* is *amenable* just if for all even n and all $s \in Q^n$, if $\eta(i) = s(2i)$ for all $i < \frac{n}{2}$ then for every $r \in Q$ there is a $\nu \in A \cap S_\eta$ with $\nu | n+1 = s^\frown r$.

1.4 Main Lemma. If A^* and B^* are amenable substructures of C^* then they are potentially isomorphic.

Proof. Let $\mathcal{P}$ denote the set of finite partial L^t-isomorphisms between A and B under the natural partial order of extension. We can naturally extend any L^t elementary bijection between A and B to an isomorphism of A^* and B^* as L^t-structures.

1.5 Claim 1. $\mathcal{P}$ satisfies the countable chain condition.

In fact, we will show $\mathcal{P} = \cup_{n \in \omega} F_n$ where if $p, q \in F_n$ then $p \cup q \in \mathcal{P}$. Given $p \in \mathcal{P}$, let $\langle a_1 \dots a_n \rangle$ be the lexicographic enumeration of $\operatorname{dom} p$. Let $n(p)$ be the cardinality of $\operatorname{dom} p$ and let $k = k(p)$ be the least integer satisfying the following conditions.

i) If $i \neq j$, $a_i | k \neq a_j | k$.

ii) If $i \neq j$, $p(a_i) | k \neq p(a_j) | k$.

iii) For all n with $2n+1 \geq k$, $a_i(2n+1) = 0$.

iv) For all n with $2n+1 \geq k$, $p(a_i)(2n+1) = 0$.

Now define an equivalence relation on $\mathcal{P}$ by $p \simeq q$ if $n(p) = n(q)$, $k(p) = k(q)$ and letting $\langle a_1 \ldots a_n \rangle$ enumerate (in lexicographic order) $\operatorname{dom} p$, $\langle a'_1 \ldots a'_n \rangle$ enumerate (in lexicographic order) $\operatorname{dom} q$, for each i, $a_i | k(p) = a'_i | k(p)$ and $p(a_i)|k(p) = q(a'_i)|k(p)$. Then since Q is countable and the domains of elements of $\mathcal{P}$ are finite, $\simeq$ has only countably many equivalence classes; we designate these classes as the F_n. We must show that if $p \simeq q$ then $p \cup q \in \mathcal{P}$. It suffices to show that for all i, j if $C \models a_i < a'_j$ then $C \models p(a_i) < q(a'_j)$.

- Case 1: $i \neq j$. By the definitions of $k = k(p)$ and $\simeq$, we must have $j > i$, $a_i|k < a'_j|k$ and $p(a_i)|k < q(a'_j)|k$. This suffices.

- Case 2: $i = j$. Choose the least t such that $a_i(t) \neq a'_i(t)$. Then $a_i(t) < a'_i(t)$. Note that $t > k$ and t must be even since for any odd $t > k$, $a_i(t) = a'_i(t) = 0$.

  Suppose $a_i \in S_\nu$ and $a'_i \in S_\eta$.

  We now claim $p(a_i)|t = q(a'_i)|t$. Fix any $\ell < t$. If $\ell < k$, $p(a_i)(\ell) = q(a'_i)(\ell)$ by the definition of $\simeq$. If $\ell \geq k$ is odd then the fourth condition in the definition of $k(p)$ guarantees that $p(a_i)(\ell) = q(a'_i)(\ell) = 0$. Finally, if $\ell \geq k$ and $\ell = 2u$, since p and q preserve the P_η, $p(a_i)(\ell) = \nu(u) = a_i(\ell)$ and $q(a'_i)(\ell) = \eta(u) = a'_i(\ell)$ but these are equal by the minimality of t.

  It remains to show $p(a_i)(t) <_Q q(a'_i)(t)$. By condition iv) it follows that t is even. But since $a_i(t) <_Q a'_i(t)$, it follows that $\nu(u) < \eta(u)$. As p and q preserve the P_η, $p(a_i)(t) <_Q q(a'_i)(t)$.

To show the generic object is a map defined on all of A, it suffices to show that for any $p \in \mathcal{P}$ and any $a \in A - \operatorname{dom} p$ there is a $q \in \mathcal{P}$ with $p \subseteq q$ and $\operatorname{dom} q = \operatorname{dom} p \cup \{a\}$. Let $\langle a_1 \ldots a_n \rangle$ enumerate $\operatorname{dom} p$ in lexicographic order. Fix $s < n$ with $a_s < a < a_{s+1}$ (the other cases are similar). Let m be least such that $a_s|(m+1), a|(m+1), a_{s+1}|(m+1)$ are distinct and let c denote $a|m$. Suppose $P_\rho(a_s)$, $P_\sigma(a)$, and $P_\tau(a_{s+1})$. Note that since $a_s < a < a_{s+1}$, it is impossible for a_s and a_{s+1} to agree on a larger initial segment than a and a_s do. Thus without loss of generality we may assume that $a_s|m = a|m$. Two cases remain.

- Case 1: $a_s|m = a|m = a_{s+1}|m = c$. Suppose m is odd. Let $b_s = p(a_s)$ and $b_{s+1} = p(a_{s+1})$. Then $b_s|m = b_{s+1}|m$ and $b_s(m) < b_{s+1}(m)$. By the definition of amenability for any r with $b_s(m) < r < b_{s+1}(m)$, there is an $\eta \in B \cap S_\sigma$ with $\eta|(m+1) = b_s{}^\frown r$ so $p \cup \{< a, \eta >\} \in \mathcal{P}$ as required.

  If m is even choose as the image of a, $p(c)^\frown \sigma(m/2)$.

- Case 2: $a_s|m = a|m = c$ but $a_{s+1}|m \neq c$. Again, let $b_s = p(a_s)$ and $b_{s+1} = p(a_{s+1})$ and denote $b_s|m$ by b'. By amenability there is an $\eta \in B \cap S_\sigma$ with $\eta|m = b'$. Any such η is less than b_{s+1}. If m is even $\eta > b_s$ is guaranteed by $\eta(m) > \rho(m)$; if m is odd by Remark 1.3 which recasts the definition of amenability we can choose $\eta(m) > b_s(m)$. In either case η is required image of a.

We deduce three results from this lemma. First we note that there are nonisomorphic but potentially isomorphic suborderings of the reals. Then we will show in two stages that any countable theory that is not classifiable has a pair of models of power $2^{\aleph_0}$ that are not isomorphic but are potentially isomorphic.

1.6 Theorem. *Any two suborderings of $\langle C, < \rangle$ that induce amenable L^t-structures are potentially isomorphic.*

Proof. Since the isomorphism we constructed in proving Lemma 1.4 preserves levels, restricting it to the infinite sequences and reducting to $<$ yields the required isomorphism.

1.7 Definition. Let M be an L-structure. We say that $\langle \overline{a}_\eta \in M : \eta \in Q^{\leq\omega} \rangle$ is a set of L-*tree indiscernibles* if for any two sequences $\overline{\eta}, \overline{\nu}$ from $Q^{\leq\omega}$:

If $\overline{\eta}$ and $\overline{\nu}$ realize the same atomic type in $\langle Q^{\leq\omega}; \sqsubset, <, lev, \bigwedge \rangle$ then $\langle \overline{a}_{\eta_1}, \ldots \overline{a}_{\eta_n} \rangle$ and $\langle \overline{a}_{\nu_1}, \ldots \overline{a}_{\nu_n} \rangle$ satisfy the same L-type.

Note that the isomorphism given in Theorem 1.6 preserves $\wedge$ since $\wedge$ is definable from $\sqsubset$. We have introduced $\wedge$ to the language so that atomic types suffice in Definition 1.7.

1.8 Theorem. *Let T be a complete unsuperstable theory in a language L. Suppose $L \subseteq L_1$ and $T \subseteq T_1$ with $|T_1| \leq 2^\omega$. Then there are L_1-structures $M_1, M_2 \models T_1$ such that each $M_i|L$ is a model of T of cardinality 2^ω, M_1 and M_2 are not L-isomorphic but in a ccc-forcing extension of the universe $M_1 \approx_{L_1} M_2$.*

Proof. We may assume that T_1 is Skolemized. Note there is no assumption that T_1 is stable. Let M be a reasonably saturated model of T_1. By VII.3.5(2) of [4] there are L-formulas $\phi_i(\overline{x}, \overline{y})$ for $i \in \omega$ and a tree of elements $\langle \overline{a}_\eta \in M : \eta \in Q^{\leq\omega} \rangle$ such that for any $n \in \omega$, $\eta \in Q^\omega$, and $\nu \in Q^{n+1}$ if $\nu | n = \eta | n$ then $\phi_{n+1}(\overline{a}_\eta, \overline{a}_\nu)$ if and only if $\nu \sqsubset \eta$. By VII.3.6(3) of [4] (applied in L_1!) we may assume that the index set is a collection of L_1-tree indiscernibles.

Let $Y = \langle \overline{a}_\nu \in M : \nu \in Q^{<\omega} \rangle$. For $\eta \in Q^\omega$, let p_η be the type over Y containing $\phi_{n+1}(\overline{x}; \overline{a}_{\eta|n^\frown \eta(n)}) \wedge \neg\phi_{n+1}(\overline{x}; \overline{a}_{\eta|n^\frown \eta(n)+1})$ for all $n \in \omega$.

Now a direct calculation from the definition of tree indiscernibility (which was implicit in the proof of Theorem VIII.2.6 of [4]) shows:

Claim. For any $\overline{\eta} \in Q^\omega$ and any Skolem term f, if $f(\overline{a}_{\eta_1}, \ldots, \overline{a}_{\eta_n})$ realizes p_ν then some $\eta_i = \nu$.

Let M_2 be the Skolem hull of $C' = Y \cup \{\overline{a}_\eta : \eta \in C\}$ where C is chosen as in 1.1. Since Y is countable there are at most $2^{\aleph_0}$ embeddings of Y into M_2; let f_η for $\eta \in Q^\omega$ enumerate them. For $\eta \in Q^\omega$, define $b_\eta \in S_\eta$ by $b_\eta(2n) = \eta(n)$ and $b_\eta(2n+1) = 0$ for all $n \in \omega$.

Let $A = \cup_{\eta \in Q^\omega} S'_\eta$ where $S'_\eta = S_\eta - \{b_\eta\}$ if M_2 realizes $f_\eta(p_{b_\eta})$ and $S'_\eta = S_\eta$ if M_2 omits $f_\eta(p_{b_\eta})$.

It is easy to check that $A^* \subseteq C^*$ is amenable. Let M_1 be the Skolem Hull of $A' = Y \cup \{\overline{a}_\eta : \eta \in A\}$.

Since A^* and C^* are amenable there is a ccc-forcing notion $\mathcal{P}$ such that $V[G] \models A^* \approx C^*$. Since A' and C' are sets of L_1-tree indiscernibles, the induced map is an L_1-isomorphism. Thus, $V[G] \models M_1 \approx_{L_1} M_2$. Thus we need only show that M_1 and M_2 are not isomorphic in the ground universe. Suppose h were such an isomorphism. Choose $\eta \in Q^\omega$ such that $h|Y = f_\eta$. Now if $b_\eta \in A$ the construction of A guarantees that M_2 omits $f_\eta(p_{b_\eta}) = h(p_{b_\eta})$ but b_η realizes $p_{b_\eta} \in M_1$. On the other hand, if $b_\eta \notin A$ then by the Claim, M_1 omits p_{b_η} but M_2 realizes $f(p_{b_\eta})$.

We now want to show the same result for theories with DOP or OTOP. We introduce some specialized notation to clarify the functioning of DOP.

1.9 Notation. For a structure M elementarily embedded in a sufficiently saturated structure M^*, $\overline{b}$ from M, and $\overline{a}$ from M^*, $\dim(\overline{a}, \overline{b}, M)$ is the minimal cardinality of a maximal, independent over $\overline{b}$, set of realizations of $stp(\overline{a}/\overline{b})$ in M. For models M of superstable theories, if $\dim(\overline{a}, \overline{b}, M)$ is infinite then it is equal to the cardinality of *any* such maximal set. For $p(\overline{x}, \overline{y}) \in S(\emptyset)$ and $\overline{b}$ from M let $d(p(\overline{x}; \overline{b}); M) = \sup\{\dim(\overline{a}', \overline{b}, M) : \mathrm{tp}(\overline{a}'/\overline{b}) = p\}$.

1.10 Lemma. *If a complete, first order, superstable theory T of cardinality λ has DOP then there is a type $p(\overline{v},\overline{u},\overline{x},\overline{y})$ such that for any cardinal κ there is a model M and a sequence $\{\overline{a}_\alpha : \alpha \in \kappa\}$ from M such that for all $\alpha,\beta \in \kappa$ and all $\overline{c}$ from M, $d(p(\overline{v};\overline{c},\overline{a}_\alpha,\overline{a}_\beta);M) \leq \lambda^+$ and*

$$(\exists \overline{u} \in M)\,[d(p(\overline{v};\overline{u},\overline{a}_\alpha,\overline{a}_\beta);M) = \lambda^+] \text{ if and only if } \alpha < \beta. \tag{1}$$

Proof. This is the content of condition (st 1) on page 517 of [7]. (As for any infinite indiscernible $\mathbf{I}$ there is a finite $\mathbf{J} \subseteq \mathbf{I}$ such that if $\overline{d} \in \mathbf{I} \setminus \mathbf{J}$ then $\mathrm{tp}(\overline{d},\cup\mathbf{J})$ is a stationary type and $\mathrm{Av}(\mathbf{I},\cup\mathbf{I})$ is a non-forking extension of it).

1.11 Proposition. *Suppose $|L| = \lambda$ and T is a superstable L-theory with either DOP or OTOP. There is an expansion $T_1 \supseteq T$, $|T_1| = \lambda^+$ such that T_1 is Skolemized, and an L-type p ($p = p(\overline{v},\overline{u},\overline{x},\overline{y})$ if T has DOP, $p = p(\overline{v},\overline{x},\overline{y})$ if T has OTOP) such that $\overline{v},\overline{u},\overline{x},\overline{y}$ are finite, $\lg \overline{x} = \lg \overline{y}$ and for any order type $(I,<)$ there is a model M_I of T_1 and a sequence $\{\overline{a}_i : i \in I\}$ from M_I of L_1-order indiscernibles such that:*

(a) M_I is the Skolem Hull of $\{\overline{a}_i : i \in I\}$;

(b) If T has DOP then for all $i,j \in I$,

$$(\exists \overline{u} \in M_I)\,[d(p(\overline{v};\overline{u},\overline{a}_i,\overline{a}_j);M_I) \geq \lambda^+] \text{ if and only if } i <_I j;$$

(c) If T has OTOP then for all $i,j \in I$, $M_I \models (\exists \overline{v})p(\overline{v},\overline{a}_i,\overline{a}_j)$ iff $i <_I j$.

Proof. Let κ be the Hanf number for omitting types for first-order languages of cardinality λ^+. If T has OTOP then by its definition (see [7] XII §4) there is a model M of T and sequence $\{\overline{a}_\alpha : \alpha \in \kappa\}$ of finite tuples from M and type $p(\overline{v},\overline{x},\overline{y})$ such that $M \models (\exists \overline{v})p(\overline{v},\overline{a}_\alpha,\overline{a}_\beta)$ iff $\alpha < \beta$.

By Lemma 1.10 when T has DOP we can find a model M of T, a sequence $\{\overline{a}_\alpha : \alpha \in \kappa\}$ and a type $p(\overline{v},\overline{u},\overline{x},\overline{y})$ so that $(\exists \overline{u} \in M)\,[d(p(\overline{v};\overline{u},\overline{a}_\alpha,\overline{a}_\beta);M) \geq \lambda^+]$ if and only if $\alpha < \beta$. We may also assume that $d(p(\overline{v};\overline{c},\overline{a}_\alpha,\overline{a}_\beta);M) \leq \lambda^+$ for all $\alpha,\beta \in \kappa$ and $\overline{c}$.

Let L_0 be a minimal Skolem expansion of L. That is, L_0 is a minimal expansion of L such that there is a function symbol $F_\phi(\overline{y}) \in L_0$ for each formula $\phi(x,\overline{y}) \in L_0$. Let M_0 be any expansion of M satisfying the Skolem axioms $\forall \overline{y}[(\exists x)\phi(x,\overline{y}) \to \phi(F_\phi(\overline{y}),\overline{y})]$ and let $T_0 = \mathrm{Th}(M_0)$. Without loss of generality $\lambda^+ + 1 \subseteq M_0$.

¿From now on, assume we are in the DOP case as the OTOP case is similar and does not require a further expansion of the language (i.e., take $L_1 = L_0$ and $T_1 = T_0$.) Expand L_0 to L_0' by adding relation symbols $<, \in, P$, constants for all ordinals less than or equal to λ^+ and a new function symbol $f(w, \overline{u}, \overline{x}, \overline{y})$. Let M_0' be an expansion of M_0 so that $<$ linearly orders the $\overline{a}_\alpha$ and the set of $\overline{a}_\alpha$ is the denotation of P. Interpret the constants and $\in$ in the natural way. For all $\alpha, \beta \in \kappa$ and all realizations $\overline{d}\frown\overline{c}$ of $p(\overline{v}, \overline{u}, \overline{a}_\alpha, \overline{a}_\beta)$ in M_0, let $(\lambda w) f(w, \overline{c}, \overline{a}_\alpha, \overline{a}_\beta)$ be a 1–1 map from an initial segment of λ^+ to a maximal, independent over $\overline{c} \cup \overline{a}_\alpha \cup \overline{a}_\beta$, set of realizations of $stp(\overline{d}/\overline{c}\,\overline{a}_\alpha \overline{a}_\beta)$.

Let L_1 be a minimal Skolem expansion of L_0', let M_1 be a Skolem expansion of M_0' to an L_1-structure and let T_1 denote the theory of M_1. So $|T_1| = \lambda^+$.

Note that if, for some $\overline{c}$, the domain of $(\lambda w) f(\overline{c}, \overline{a}_\alpha, \overline{a}_\beta)$ is λ^+ then $\alpha < \beta$. Also, for all $\alpha, \beta \in \kappa$ and $\overline{c}$ from M_1 the independence of the range of $(\lambda w) f(w, \overline{c}, \overline{a}_\alpha, \overline{a}_\beta)$ is expressed by an L_1-type. Thus M_1 omits the types

$$q(\overline{v}; \overline{u}, \overline{x}, \overline{y}) = p(\overline{v}, \overline{u}, \overline{x}, \overline{y}) \cup \{\overline{v} \underset{\overline{u}\overline{x}\overline{y}}{\perp} \{f(\gamma, \overline{u}, \overline{x}, \overline{y}) : \gamma < \lambda^+\}\} \cup \{P(\overline{x})\} \cup \{P(\overline{y})\} \cup \{\overline{x} \not< \overline{y}\}$$

and $r(v) = \{v \in \lambda^+\} \cup \{v \neq \gamma : \gamma \in \lambda^+\}$.

To complete the proof of the proposition construct an Ehrenfeucht-Mostowski model M_I of T_1 built from a set of L_1-order indiscernibles $\{\overline{a}_i : i \in I\}$ omitting both $q(\overline{v}, \overline{u}, \overline{x}, \overline{y})$ and $r(\overline{v})$. The existence of such a model follows as in the proof of Morley's omitting types theorem (see e.g., [4, VII.5.4]).

Note that in the DOP case of the proposition above the argument shows $d(p(\overline{v}; \overline{c}, \overline{a}_i, \overline{a}_j); M_I) \leq \lambda^+$ for all $i, j \in I$ and $\overline{c}$.

We have included a sketch of the proof of Lemma 1.11 which is essentially Fact X.2.5B+620$_9$ of [7] and Theorem 0.2 of [5] to clarify two points. We would not include this had not experience showed that some readers miss these points. Note that the parameter $\overline{c}$ is needed in the DOP case not only to fix the strong type, but because in general we cannot ensure the existence of a large, independent set of realizations over $\overline{a}_\alpha \cup \overline{a}_\beta$. Also, it is essential that we pass to a Skolemized expansion to carry out the omitting types argument and that the final set of indiscernibles are indiscernible in the Skolem language. We can then reduct to L for the many models argument (if we use III 3.10 of [6]) not just [7] VIII,§3) but for the purposes of this paper we cannot afford to take reducts as the proof of Theorem 1.14 requires that an isomorphism between linear orders I_1, I_2 induces an isomorphism of the corresponding models.

Let us expand on why we quote [6] above. In [6], Theorem III 3.10 it is proved that for all uncountable cardinals λ and all vocabularies τ, if there is a formula $\Phi(\overline{x}, \overline{y})$ such that for every linear order $(J, <)$ of cardinality λ there is a τ-structure M_J of cardinality λ and a subset of elements $\{\overline{a}_s : s \in J\}$ satisfying

i) $M_J \models \Phi(\overline{a}_s, \overline{a}_t)$ if and only if $s <_J t$ and

ii) The sequence $\langle \overline{a}_s : s \in J\rangle$ is skeleton-like in M_J (i.e., any formula of the form $\Phi(\overline{x}, \overline{b})$ or $\Phi(\overline{b}, \overline{x})$ divides $\langle \overline{a}_s : s \in J\rangle$ into finitely many intervals)

then there are 2^λ non-isomorphic M_J's.

The point, compared with earlier many-models proofs, is that we do not demand that the M_J's be constructed from J in any specified way. It is true that the natural example satisfying these conditions is an Ehrenfeucht-Mostowski model built from $\langle \overline{a}_s : s \in J\rangle$ in some expanded language, but this is not required. In particular, our generality allows taking reducts, so long as the formula Φ remains in the vocabulary. Further, there is no requirement that Φ be first-order.

However, in our context we want to introduce an isomorphism between two previously non-isomorphic models. The natural way of doing this is to produce two non-isomorphic but potentially isomorphic orderings J_1 and J_2 and then conclude that M_{J_1} and M_{J_2} become isomorphic. Consequently, it is important for us to know that the models are E.M. models.

We can simplify the statement of the conclusion of Lemma 1.11 if we define the logic with 'dimension quantifiers'. In this logic we demand that in addition to the requirement that 'equality' is a special predicate to be interpreted as identity that another family of predicates also be given a canonical interpretation.

1.12 Notation. Expand the vocabulary L to $\hat{L}$ by adding new predicate symbols $Q_\mu(\overline{x}, \overline{y})$ of each finite arity for all cardinals $\mu \leq \lambda^+$. Now define the logic $\hat{L}_{\lambda^+,\omega}$ by first demanding that each predicate Q_μ is interpreted in an L-structure M by

$$M \models Q_\mu(\overline{a}, \overline{b}) \text{ if and only if } \dim(\overline{a}, \overline{b}, M) = \mu.$$

Then define the quantifiers and connectives as usual. We will only be concerned with the satisfaction of sentences of this logic for models of superstable theories.

1.13 Remarks. i) The property coded in Condition (1) of Lemma 1.11 is expressible by a formula $\Phi(\overline{x}, \overline{y})$ in the logic $\hat{L}_{\lambda^+,\omega}$. Each formula in $\hat{L}_{\lambda^+,\omega}$ and in particular this formula Φ is absolute relative to any extension of the universe that preserves cardinals. More precisely Φ is absolute relative to any extension of the universe that preserves λ^+.

ii) If T has OTOP the formula Φ can be taken in the logic $L_{\lambda^+,\omega}$. So in this case Φ is preserved in any forcing extension.

iii) Alternatively, the property coded in Condition (1) of Lemma 1.11 is also expressible in L_{λ^+,λ^+}. That is, there is a formula $\Psi(\overline{x}, \overline{y}) \in L_{\lambda^+,\lambda^+}$ (in the original vocabulary L) so that

$$M_I \models \Psi(\overline{a}_i, \overline{a}_j) \text{ if and only if } i <_I j.$$

The reader should note that satisfaction of arbitrary sentences of L_{λ^+,λ^+} is, in general, not absolute for cardinal-preserving forcings. However, the particular statements $M_I \models \Psi(\overline{a}_i, \overline{a}_j)$ and $M_I \models \neg\Psi(\overline{a}_i, \overline{a}_j)$ will be preserved under any cardinal-preserving forcing by the first remark.

iv) Note that we could have chosen the type p (in the DOP case) such that $p(\overline{v}; \overline{c}, \overline{a}_\alpha, \overline{a}_\beta)$ is a stationary regular type. Note also that had we followed [7],X2.5B more closely, we could have insisted that $|T_1| = \lambda$. In fact we could have arranged that in M_I, every dimension would be $\leq \aleph_0$ or $\|M_I\|$ (over a countable set). However, neither of these observations improve the statement of 1.14.

1.14 Theorem. *If T is a complete theory in a vocabulary L with $|L| \leq 2^\omega$ and T has either OTOP or DOP then there are models M_1 and M_2 of T with cardinality the continuum that are not isomorphic but are potentially isomorphic.*

Proof. By Theorem 1.8 we may assume that T is superstable. By Proposition 1.11 and Remark 1.13(i) there is a model M of a theory $T_1 \supseteq T$ in a Skolemized language $L_1 \supseteq L$ containing a set of L_1-order indiscernibles

$\{\overline{a}_\eta : \eta \in Q^{\leq\omega}\}$ and an $\hat{L}_{\lambda^+,\omega}$-formula $\Phi(\overline{x},\overline{y})$ so that $\Phi(\overline{a}_\eta,\overline{a}_\nu)$ holds in M if and only if η is lexicographically less than ν. Further, the statements "$M \models \Phi(\overline{a}_\eta,\overline{a}_\nu)$" and "$M \models \neg\Phi(\overline{a}_\eta,\overline{a}_\nu)$" are preserved under any ccc forcing. Note that this L_1-order indiscernibility certainly implies L_1-tree-indiscernibility in the sense of Definition 1.7.

Thus, the construction of potentially isomorphic but not isomorphic models proceeds as in the last few paragraphs of the proof of Theorem 1.8 once we establish the following claim.

Claim. For any $\nu \in Q^\omega$ there is a collection $p_\nu(\overline{x})$ of boolean combinations of $\Phi(\overline{x},\overline{a})$ as $\overline{a}$ ranges over Y such that for any $\overline{\eta} \in Q^\omega$ and any L_1-term f, if $f(\overline{a}_{\eta_1},\dots\overline{a}_{\eta_n})$ realizes p_ν in M then some $\eta_i = \nu$.

Proof. The conjunction of the $\Phi(\overline{x};\overline{a}_{\nu|n^\frown<\nu(n)+1>})$ and $\neg\Phi(\overline{x};\overline{a}_{\nu|n^\frown<\nu(n)+1>})$ that define the 'cut' of $\overline{a}_\nu$ will constitute p_ν. Now if ν is not among the η_i choose any n such that $\eta_1|n,\eta_2|n,\dots\eta_k|n,\nu|n$ are distinct. Then the sequences $\langle\eta_1,\dots\eta_k,\nu|n^\frown\langle\nu(n)+1\rangle\rangle$ and $\langle\eta_1,\dots\eta_k,\nu|n^\frown\langle\nu(n)-1\rangle\rangle$ have the same type in the lexicographic order so

$$M \models \Phi(f(\overline{a}_{\eta_1},\dots\overline{a}_{\eta_k});\overline{a}_{\nu|n^\frown<\nu(n)+1>}) \leftrightarrow \Phi(f(\overline{a}_{\eta_1},\dots\overline{a}_{\eta_k});\overline{a}_{\nu|n^\frown<\nu(n)-1>}).$$

Thus, $f(\overline{a}_{\eta_1},\dots\overline{a}_{\eta_k})$ cannot realize p_ν.

1.15 Remarks. i) Note that in Theorem 1.8 we were able to use any expansion of T as T_1 so the result is actually for PC_Δ-classes. In Theorem 1.14 our choice of T_1 was constrained, so the result is true for only elementary as opposed to pseudoelementary classes. The case of unstable elementary classes could be handled by the second method thus simplifying the combinatorics at the cost of weakening the result.

ii) While we have dealt only with models and theories of cardinality 2^ω, the result extends immediately to models of any larger cardinality and straightforwardly to theories of cardinality κ with $\kappa^\omega = \kappa$.

2 Classifiable examples

We begin by giving an example of a classifiable theory having a pair of non-isomorphic, potentially isomorphic models. We then extend this result to a class of weakly minimal theories.

Let the language L_0 consist of a countable family E_i of binary relation symbols and let the language L_1 contain an additional uncountable set of unary predicates P_η. We first construct an L_0-structure that is rigid but can be forced by a ccc-forcing to be nonrigid. Our example will be in the language L_0 but we will use expansions of the L_0-structures to L_1-structures in the argument.

We now revise the definitions leading up to the notion of an amenable structure in Section 1 by replacing the underlying structure on $Q^{\leq\omega}$ by one with universe 2^ω. In particular, D_η, S_η, and C are now being redefined.

2.1 Notation.

i) For $\eta \in 2^\omega$, let $D_\eta = \{\sigma \in 2^\omega : \sigma(2n) = \eta(n)\}$ and $S_\eta = \{\sigma \in D_\eta : \sigma(2n+1)$ is 0 for all but finitely many $n\} \cup \{b_\eta\}$, where b_η is any element of D_η satisfying $b_\eta(2n+1) = 1$ for infinitely many n. Let $C = \cup_{\eta \in 2^\omega} S_\eta$.

ii) Let M^* be the L_1-structure with universe 2^ω where $E_i(\sigma, \tau)$ holds if $\sigma|i = \tau|i$, and the unary relation symbol P_η holds of the set S_η. Let M_1 be the L_1-substructure of M^* with universe C.

iii) Any subset A of C inherits a natural L_1 structure from M_1 with P_η interpreted as $S_\eta \cap A$.

2.2 Definition. An L_1-substructure M_0 of M_1 is *amenable* if for all $\eta \in 2^\omega$, all $n \in \omega$ and all $s \in 2^n$, if there is a $\nu \in P_\eta(M_1)$ with $\nu|n = s$ then there is a $\nu' \in P_\eta(M_0)$ with $\nu'|n = s$.

Note that any L_1-elementary substructure of M_1 is amenable. Moreover, it easy to see that i) each D_η is a perfect tree, ii) 2^ω is a disjoint union of the D_η and iii) for each $s \in 2^{<\omega}$ there are 2^ω sequences η such that s has an extension $b \in D_\eta$.

2.3 Theorem. *The theory FER_ω of countably many refining equivalence relations with binary splitting has a pair of models of size the continuum which are not isomorphic but are potentially isomorphic.*

This result follows from the next two propositions and the fact that M_1 is not rigid.

2.4 Proposition. *There is an L_1-elementary substructure M_0 of M_1 such that*

i) $|P_\eta(M_1) - P_\eta(M_0)| \leq 1$.

ii) $M_0|L_0$ *is rigid.*

Proof. Note that each automorphism of $M_1|L$ is determined by its restriction to the eventually constant sequences so there are only 2^ω such. Thus we may let $\langle f_i : i < 2^\omega\rangle$ enumerate the nontrivial automorphisms of $M_1|L$. We define by induction disjoint subsets A_i, B_i of M_1 each with cardinality less than the continuum. We denote $\cup_{i<j} A_i$ by $\underline{A}_j$. At stage i, choose $\alpha \in M_1$ such that f_i moves α. Then, by continuity, there is a finite sequence s such that every element of $W_s = \{\tau : s \subseteq \tau\}$ is moved by f_i. Since $|A_i|, |B_i| < 2^\omega$ and by condition iii) of the remark after the definition of amenable there are an $\eta \in 2^\omega$ and a $\beta \in P_\eta \cap (f_i(W_s) - \underline{A}_i)$. Then let $B_i = \{\beta\}$ and $A_i = S_\eta - \{\beta\}$. Finally, let $M_0 = M_1 - \underline{B}_{2^\omega}$.

Since no element is ever removed from an A_i, condition i) is satisfied. It is easy to see that M_0 is rigid, as any nontrivial automorphism h of M_0 would extend in a unique way to an automorphism f_i of M_1 but at step i we ensured that the restriction of f_i to M_0 is not an automorphism.

2.5 Proposition. *If M_0 is an amenable substructure of M_1, M_0 and M_1 are potentially isomorphic.*

Proof. Let $\mathcal{P}$ be the collection of all finite partial L_1-isomorphisms between M_0 and M_1.

We first claim that $\mathcal{P}$ is a ccc set of forcing conditions. In fact, $\mathcal{P} = \cup_{n\in\omega} F_n$ where if $p, q \in F_n$ then $p \cup q \in \mathcal{P}$. Given $p \in \mathcal{P}$, fix an (arbitrary) enumeration $\langle a_1 \ldots a_n\rangle$ of $\operatorname{dom} p$. Let $n(p)$ be the cardinality of $\operatorname{dom} p$ and let $k(p)$ be the least integer satisfying the following conditions.

i) If $i \neq j$, $M_0 \models \neg E_k(a_i, a_j)$.

ii) If $i \neq j$, $M_1 \models \neg E_k(p(a_i), p(a_j))$.

iii) For all n with $2n+1 \geq k$, $a_i(2n+1) = 0$.

iv) For all n with $2n+1 \geq k$, $p(a_i)(2n+1) = 0$.

Now define an equivalence relation on $\mathcal{P}$ by $p \simeq q$ if $n(p) = n(q)$, $k(p) = k(q)$ and letting $\langle a_1 \dots a_n \rangle$ enumerate $\operatorname{dom} p$, $\langle a'_1 \dots a'_n \rangle$ enumerate $\operatorname{dom} q$, for each i, $a_i | k(p) = a'_i | k(p)$ and $p(a_i)|k(p) = q(a'_i)|k(p)$. Then $\simeq$ has only countably many equivalence classes; these classes are the F_n. We must show that if $p \simeq q$ then $p \cup q \in \mathcal{P}$. It suffices to show that for all i, j if $M_0 \models E_n(a_i, a'_j)$ then $M_1 \models E_n(p(a_i), q(a'_j))$.

- Case 1: $i \neq j$. By the definition of $k = k(p)$, $M_0 \models \neg E_k(a_i, a'_j)$. Let ℓ be maximal so that $M_0 \models E_\ell(a_i, a'_j)$. Then $\ell \geq n$ since $M_0 \models E_n(a_i, a'_j)$. Since $a_j|k = a'_j|k$, ℓ is also maximal so that $M_0 \models E_\ell(a_j, a_i)$. As p is an isomorphism, $M_1 \models E_\ell(p(a_i), p(a'_j))$. But $p(a_j)|k = q(a'_j)|k$, so ℓ is also maximal with $M_1 \models E_\ell(q(a'_j), p(a_i))$. Since $n \leq \ell$ we conclude $M_1 \models E_n(q(a'_j), p(a_i))$ as required.

- Case 2: $i = j$. Suppose $a_i \in S_\nu$ and $a'_j \in S_\eta$. We have $M_0 \models E_k(a_i, a'_j)$ by the definition of k and similarly, $M_1 \models E_k(p(a_i), q(a'_i))$. Now we show by induction that for each $m \geq k$, $M_0 \models E_m(a_i, a'_i)$ if and only if $M_1 \models E_m(p(a_i), q(a'_i))$. Assuming this condition for m we show it for $m + 1$. If $m + 1$ is odd, the result is immediate by parts iii) and iv) of the conditions defining $\simeq$. If $m = 2u$ then $a_i(m) = \nu(u)$ and $a'_i(m) = \eta(u)$. Since p and q are L_1-isomorphisms $p(a_i)(m) = \nu(u)$ and $q(a'_i)(m) = \eta(u)$. But $M_0 \models E_m(a_i, a'_i)$ implies $\nu(u) = \eta(u)$ so we have $M_1 \models E_m(p(a_i), q(a'_i))$.

To show that the generic object is a map with domain M_0, it suffices to show that for any $p \in \mathcal{P}$ and any $a \in M_0 - \operatorname{dom} p$ there is a $q \in \mathcal{P}$ with $q \leq p$ and $\operatorname{dom} q = \operatorname{dom} p \cup \{a\}$. Choose n so that the members of $\operatorname{dom} p \cup \{a\}$ are pairwise E_n-inequivalent. Fix any L_1-automorphism g of M^* that extends p. Let $s = g(a)|n$ and $W_s = \{\gamma \in 2^\omega : s \subseteq \gamma\}$. Since there is a $\nu \in C \cap W_s$ and B is amenable, there is a $\nu' \in B \cap W_s$. Choosing ν' for b, $p \cup \langle a, b \rangle$ is the required extension of p.

Since M_0 and M_1 are isomorphic in a generic extension for this forcing, we complete the proof.

2.6 Remark. The notion of a classifiable theory having two non-isomorphic, potentially isomorphic models is not very robust, and in particular can be lost by adding constants. As an example, let FER^*_ω be an expansion of FER_ω

formed by adding constants for the elements of a given countable model of FER_ω. Then every type in this expanded language is stationary and the isomorphism type of any model of FER^*_ω is determined by the number of realizations of each of the 2^ω non-algebraic 1-types. Thus, if two models of FER^*_ω are non-isomorphic then they remain non-isomorphic under any cardinal-preserving forcing.

Similarly, non-isomorphism of models of the theory CEF_ω of countably many crosscutting equivalence relations (i.e., $\mathrm{Th}(2^\omega, E_i)_{i\in\omega}$, where $E_i(\sigma,\tau)$ iff $\sigma(i)=\tau(i)$) is preserved under ccc forcings.

We next want to extend the result from Theorem 2.3 to a larger class of theories. Suppose T is superstable and there is a type q, possibly over a finite set $\overline{e}$ of parameters, and an $\overline{e}$-definable family $\{E_n : n\in\omega\}$ of properly refining equivalence relations, each with finitely many classes that determine the strong types extending q. Let T be such a theory in a language L and let M be a model of T. Let L_0 be a reduct of L containing the E_n's.

We say $\langle a_\eta \in M : \eta \in X \subseteq 2^\omega\rangle$ is a set of *unordered tree L-indiscernibles* if the following holds for any two sequences $\overline{\eta},\overline{\nu}$ from X:

If $\overline{\eta}$ and $\overline{\nu}$ realize the same L_0-type then $\langle a_{\eta_1},\dots a_{\eta_n}\rangle$ and $\langle a_{\nu_1},\dots a_{\nu_n}\rangle$ satisfy the same L-type.

We say that a superstable theory T with a type of infinite multiplicity as above *embeds an unordered tree* if there is a model M of T containing a set of unordered tree L-indiscernibles indexed by 2^ω. We deduce below the existence of potentially isomorphic nonisomorphic models of weakly minimal theories which embed an unordered tree. Every small superstable, non-ω-stable theory has a type of infinite multiplicity with an associated family of $\{E_n : n<\omega\}$ of refining equivalence relations and a set of tree indiscernibles in the sense of [1]. The existence of such a tree of indiscernibles suffices for the many model arguments but does not in itself suffice for this result. Marker has constructed an example of such a theory which does not embed an unordered tree. However, an apparently ad hoc argument shows this example does have potentially isomorphic but not isomorphic models.

2.7 Notation. Given $A=\{a_\eta : \eta\in 2^\omega\}$ a set of unordered tree indiscernibles let $D=\{a_\eta\in A : \eta(n)=0$ for all but finitely many $n\}$. For $\eta\in 2^\omega$ let $p_\eta(x)\in S^1(D)$ be $q(x)\cup\{E_n(x,a_\nu) : a_\nu\in D$ and $\nu|n=\eta|n\}$. Note that D is a dense subset of A, each a_η realizes p_η and each p_η is stationary.

2.8 Lemma. *Let T be a weakly minimal theory that embeds an unordered tree. Fix A and D as described in Notation 2.7. There is a set X satisfying the following conditions:*

i) $X \cup A$ is independent over the empty set;

ii) for any Y with $D \subseteq Y \subseteq A$, and any $\eta \in 2^\omega$, p_η is realized in $\mathrm{acl}(XY)$ *if and only if p_η is realized in Y;*

iii) for any Y with $D \subseteq Y \subseteq A$, $\mathrm{acl}(XY)$ *is a model of T.*

Proof. It is easy to see from the definition of unordered tree indiscernibility that if $X = \emptyset$, then conditions i) and ii) of the Lemma are satisfied for any $Y \subseteq A$. We will show that for any X and Y with $D \subset Y \subseteq A$ with XY satisfying conditions i) and ii) and any consistent formula $\phi(v)$ over $\mathrm{acl}(XY)$ that is not satisfied in $\mathrm{acl}(XY)$ it is possible to adjoin a solution of ϕ to X while preserving the conditions. By iterating this procedure we obtain a model of T.

Now suppose there is a Y with $D \subseteq Y \subseteq A$, such that $\mathrm{acl}(XY)$ is not an elementary submodel of the monster. Choose a formula $\phi(x, \overline{c}, \overline{a})$ with $\overline{c} \in X$ and $\overline{a} \in Y$ such that $\phi(x, \overline{c}, \overline{a})$ has a solution d in $\mathcal{M}$ but not in $\mathrm{acl}(XY)$. If we adjoin d to X we must check that conditions i) and ii) are not violated. Since T is weakly minimal and $d \notin \mathrm{acl}(XY)$, XAd is independent. Suppose for contradiction that for some $\overline{a}' \in Y$, p_ν is not realized in $\overline{a}'$ but p_ν is realized in $\mathrm{acl}(Xd\overline{a}')$ by say e. Since condition ii) holds for XY, $e \notin \mathrm{acl}(X\overline{a}')$. Therefore by the exchange lemma $d \in \mathrm{acl}(Xe\overline{a}')$. Let $\theta(v, \overline{c}', \overline{a},' e)$ with $\overline{c}' \in X$ and $\overline{a}' \in Y$ witness this algebraicity. Then

$$\chi(\overline{c}, \overline{c}', \overline{a}, \overline{a}', z) = (\exists x)[\phi(x, \overline{c}, \overline{a}) \wedge \theta(x, \overline{c}', \overline{a}', z)] \wedge (\exists^{=m} x)\theta(x, \overline{c}', \overline{a}', z)$$

is a formula over $X\overline{a}\overline{a}'$ satisfied by e. Moreover, $e \notin \mathrm{acl}(X\overline{a}\overline{a}')$. For, if so, transitivity would give $d \in \mathrm{acl}(X\overline{a}\overline{a}') \subseteq \mathrm{acl}(XY)$. Now tp(e/X$\overline{\text{a}}\overline{\text{a}}'$) and in particular $\chi(\overline{c}, \overline{c}', \overline{a}, \overline{a}', z)$ is implied by p_ν and the assertion that $z \notin \mathrm{acl}(X\overline{a}\overline{a}')$. Since XA is independent, it follows by compactness that there is $b \in D$ such that $\chi(\overline{c}, \overline{c}', \overline{a}, \overline{a}', b)$ holds. So there is a solution of $\phi(x, \overline{c}, \overline{a})$ in the algebraic closure of XY. This contradicts the original choice of ϕ so we conclude that condition ii) cannot be violated.

2.9 Theorem. *If T is a weakly minimal theory in a language of cardinality at most $2^{\aleph_0}$ that embeds an unordered tree then T has two models that are not isomorphic but are potentially isomorphic (by a ccc-forcing).*

Proof. Let L be the language of T. Assume that the type q is based on a finite set $\overline{e}$. Let T' be the expansion of T formed by adding constants for $\overline{e}$. Let $\mathcal{M}$ be a large saturated model of the theory T' and let the sets A and D be chosen as in Lemma 2.8 applied in L' to T'.

Recall the definition of C from Notation 2.1. For any $W \subseteq C$, let M'_W be the L'-structure with universe $\mathrm{acl}(X \cup \{a_\eta : \eta \in W\})$ and denote $M'_W|L$ by M_W. We will construct an amenable set W such that $M_W \not\approx M_C$. Since both are amenable, there is a forcing extension where $W \approx C$ as L_1-structures. Since $\{a_\eta : \eta \in C\}$ is a set of unordered tree L'-indiscernibles, the induced mapping of $\{a_\eta : \eta \in W\}$ into $\{a_\eta : \eta \in C\}$ is L'-elementary. Thus, $M'_W \approx_{L'} M'_C$ and a fortiori $M_W \approx_L M_C$.

To construct W, let $\{f_\eta : \eta \in 2^\omega\}$ enumerate all L-embeddings of $D\overline{e}$ into M_C. Note that each p_η can be considered as a complete L-type over $D\overline{e}$.

Let $W = \cup_{\eta \in 2^\omega} S'_\eta$ where $S'_\eta = S_\eta - \{b_\eta\}$ if M_C realizes $f_\eta(p_{b_\eta})$ and $S'_\eta = S_\eta$ if M_C omits $f_\eta(p_{b_\eta})$. (see Notation 2.1.)

Suppose for contradiction that g is an L-isomorphism between M_W and M_C. Then for some η, $g|D = f_\eta$. Now if $c_\eta \in W$, the definition of W yields $f_\eta(p_\eta)$ is not realized in M_C. This contradicts the choice of g as an isomorphism. But if c_η is not in W then by the construction of W, $f_\eta(c_\eta) = g(c_\eta)$ does not realize $g(p_\eta)$. But this is impossible since g is a homomorphism.

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