ADDENDUM TO "MAXIMAL CHAINS IN ${}^{\omega}\omega$ AND ULTRAPOWERS OF THE INTEGERS"

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This note is intended as a supplement and clarification to the proof of Theorem 3.3 of [1]; namely, it is consistent that $\mathfrak{b} = \aleph_1$ yet for every ultrafilter U on ω there is a $\leq^*$ chain $\{f_\xi : \xi \in \omega_2\}$ such that $\{f_\xi / U : \xi \in \omega_2\}$ is cofinal in ω/U.

The general outline of the the proof remains the same. In other words, a ground model is taken which satisfies $2^{\aleph_0} = \aleph_1$ and in which there is a $\diamondsuit_{\omega_2}$ sequence $\{D_\xi : \xi \in \omega_2\}$ such that for every $X \subseteq \omega_2$ there is a stationary set of ordinals, μ, such that $\mathrm{cof}(\mu) = \omega_1$ and such that $X \cap \mu = D_\mu$. Actually, a coding will be used to associate with subsets of ω_2, names for subsets on ω in certain partial orders. The details of this coding will be ignored except to state that $c(D_\eta)$ will denote the coded set and that if $\mathbb{P}_{\omega_2} = \lim\{\mathbb{P}_\xi : \xi \in \omega_2\}$ is the finite support iteration of ccc partial orders of size no greater than ω_1 and $1 \Vdash_{\mathbb{P}_{\omega_2}}$ "$X \subseteq [\omega]^{\aleph_0}_0$" then there is a stationary set $S_X \subseteq \omega_2$, consisting of ordinals of uncountable cofinality, such that $1 \Vdash_{\mathbb{P}_\xi}$ "$X \restriction \mathbb{P}_\xi = c(D_\xi)$" for each $\xi \in S_X$. Here $X \restriction \mathbb{P}_\xi$ denotes the $\mathbb{P}_\xi$-name obtained by considering only those parts of X that mention conditions in $\mathbb{P}_\xi$; to be more precise here would requires providing the details of a specific development of names in the theory of forcing, and so this will not be done.

The partial order $\mathbb{P}_{\omega_2}$ is defined by induction using the $\diamondsuit_{\omega_2}$ sequence. Simultaneously, a partial ordering $\prec$ will be defined on ω_2 by $\eta \prec \zeta$ if and only if

- $1 \Vdash_{\mathbb{P}_\eta}$ "$c(D_\eta)$ is an ultrafilter on ω"
- $1 \Vdash_{\mathbb{P}_\zeta}$ "$c(D_\zeta)$ is an ultrafilter on ω"
- $1 \Vdash_{\mathbb{P}\zeta}$ "$c(D_\eta) = c(D_\zeta) \cap V^{\mathbb{P}_\eta}$"

If $\alpha \in \omega_2$ then the order type of $\{\beta \in \alpha : \beta \prec \alpha\}$ will be denoted by $o(\alpha)$. Furthermore, an enumeration $\{g_\xi : \xi \in \omega_2\}$ will be constructed by induction along with $\mathbb{P}_{\omega_2}$ which will list all $\mathbb{P}_{\omega_2}$-names for functions from ω to ω.

If $\mathbb{P}_\xi$ has been defined and $1 \Vdash_{\mathbb{P}_\xi}$ "$c(D_\xi)$ is an ultrafilter on ω" then $\mathbb{P}_{\xi+1}$ is defined to be $\mathbb{P}_\xi * \mathbb{C}_\xi * \mathbb{Q}_\xi$ where $\mathbb{C}_\xi$ is simply Cohen forcing which adds a single generic function $A_\xi : \omega \to 2$ and $\mathbb{Q}_\xi$ adds a function $F_\xi : \omega \to \omega$ such that $F_\xi \geq^* F_\mu$ and $F_\xi \restriction A_\mu^{-1}\{k\} \geq^* g_{o(\mu)} \restriction A_\mu^{-1}\{k\}$, for a certain $k \in 2$, for all μ such that $\mu \prec \xi$. To be more precise, $\mathbb{Q}_\xi$ is defined, in the forcing extension by $\mathbb{P}_\xi$, to consist of all pairs (f, Δ) such that f is a finite partial function from ω to ω and $\Delta \in [\xi]^{<\aleph_0}$, and the ordering is defined by $(f, \Delta) \leq (f', \Delta')$ if

- $f \subseteq f'$
- $\Delta \subseteq \Delta'$
- if $\mu \in \Delta$ and $\mu \prec \xi$ and $m \in \mathrm{dom}(f' \setminus f)$ then $f'(m) \geq F_\mu(m)$
- if $\mu \in \Delta$, $\mu \prec \xi$, $m \in \mathrm{dom}(f' \setminus f)$, $1 \Vdash_{\mathbb{P}_\xi}$ "$A_\mu^{-1}\{k\} \in c(D_\xi)$" and $A_\mu^{-1}(m) = k$ then $f'(m) \geq g_{o(\mu)}(m)$

If $1 \Vdash_{\mathbb{P}_\xi}$ "$c(D_\xi)$ is an ultrafilter on ω" fails to be true then $\mathbb{Q}_\xi$ is defined to be empty. At limits the iteration is with finite support.

To see that for every ultrafilter on ω there is an increasing $\leq^*$ chain which is cofinal in the ultrapower, let G be $\mathbb{P}_{\omega_2}$ generic over V and let $\mathcal{U}$ be an ultrafilter on ω in $V[G]$. There must be some name U such that $1 \Vdash_{\mathbb{P}_{\omega_2}}$ "U is an ultrafilter on ω" and $\mathcal{U}$ is the interpretation U in $V[G]$. It is well known that there is a set which is closed under increasing ω_1 sequences, C such that $1 \Vdash_{\mathbb{P}_\xi}$ "U is an ultrafilter on ω" for each $\xi \in C$. It follows that if $\alpha \in \beta$ and $\{\alpha, \beta\} \subseteq C \cap S_U$ then $\alpha \prec \beta$. It is now easy to verify that $\{F_\xi : \xi \in C \cap S_U\}$ is an $\leq^*$-increasing sequence. Moreover, because $C \cap S_U$ is cofinal in ω_2 it follows that $\{o(\xi) : \xi \in C \cap S_U\} = \omega_2$ and hence $1 \Vdash_{\mathbb{P}_{\omega_2}}$ "$\{g_{o(\xi)} : \xi \in C \cap S_U\} = {}^\omega\omega$". Therefore, if $1 \Vdash_{\mathbb{P}_{\omega_2}}$ "$g : \omega \to \omega$" there is some $\xi \in C \cap S_U$ such that $1 \Vdash_{\mathbb{P}_{o(\xi)}}$ "$(g = g_{o(\xi)})$" and so it follows that

$$1 \Vdash_{\mathbb{P}_{\omega_2}} \text{“}(\forall^\infty n \in A_\xi^{-1}\{k\}) F_\eta(n) \geq g_{o(\xi)}(n) \text{ and } A_\xi^{-1}\{k\} \in c(D_\eta) \subseteq U\text{”}$$

for any $\eta \in C \cap S_U \setminus \xi$ It follows immediately that $\{F_\xi : \xi \in C \cap S_U\}$ is cofinal in the ultrapower by $\mathcal{U}$.

The only thing which now has to be proved is that $\mathbb{P}_{\omega_2}$ is locally Cohen since this immediately implies that $\mathfrak{b} = \aleph_1$. A condition $p \in \mathbb{P}_{\omega_2}$ will be said to be determined if there is some $\Sigma_p \in [\omega_2]^{<\aleph_0}$ such that Σ_p is the support of p and for each $\sigma \in \Sigma_p$ there is a quadruple $(a_p^\sigma, f_p^\sigma, \Delta_p^\sigma, g_p^\sigma)$ such that:

- $p \restriction \sigma \Vdash_{\mathbb{P}_\sigma}$ "$p(\sigma) = a_p^\sigma * (f_p^\sigma, \Delta_p^\sigma)$" for each $\sigma \in \Sigma_p$
- $\Delta_p^\sigma \subseteq \Sigma_p \cap \sigma$ for each $\sigma \in \Sigma_p$
- $p \restriction \sigma \Vdash_{\mathbb{P}_\sigma}$ "$g_{o(\sigma)} \restriction \mathrm{dom}(a_p^\sigma) = g_p^\sigma$" for each $\sigma \in \Sigma_p$
- for each $\{\sigma, \tau\} \in [\Sigma_p]^2$ such that $\sigma \prec \tau$ there is some $k_p(\sigma, \tau) \in 2$ such that $p \restriction \tau \Vdash_{\mathbb{P}_\tau}$ "$A_\sigma^{-1}\{k_p(\sigma, \tau)\} \in D_\tau$"
- $\mathrm{dom}(f_p^\sigma) \supseteq \mathrm{dom}(a_p^\sigma)$ for each $\sigma \in \Sigma_p$
- $\mathrm{dom}(f_p^\tau) \subseteq \mathrm{dom}(f_p^\sigma)$ for each $\{\sigma, \tau\} \in [\Sigma_p]^2$ such that $\sigma \prec \tau$

This definition of determined differs in a substantial way from the definition of *somewhat determined* in [1]. The next lemma shows that every condition can be extended to a determined condition; this is problematic for the somewhat determined conditions.

Lemma 0.1. *The set of determined conditions is dense in $\mathbb{P}_{\omega_2}$.*

Proof: Induction on $\alpha \in \omega_2 + 1$ will be used to prove the following stronger statment: For each $m \in \omega$ and each $p \in \mathbb{P}_\alpha$ there is a determined condition $q \geq p$ such that if σ is the maximal element of Σ_q then $m \subseteq a_q^\sigma$ and σ is the maximal element of the support of p. Note that a_q^σ has the smallest domain of any function appearing in q so the requirement that $m \subseteq a_q^\sigma$ implies that m is in the domain of any function appearing in q.

To prove this, suppose the statement is true for all $\alpha \in \beta$. If β is a limit ordinal the result follows from the finite support of the iteration; therefore suppose that $\beta = \gamma + 1$. Then extend p so that $p \Vdash_{\mathbb{P}_\gamma}$ "$p(\gamma) = a * (f, \Delta)$". By extending, it may be assumed that $m \subseteq \mathrm{dom}(a) \subseteq \mathrm{dom}(f)$. Let $\bar{m}$ be the maximal element of $\mathrm{dom}(f)$. Let $p' \geq p \restriction \gamma$ be such that Δ is contained in the support of p'.

There are now two cases to consider: Either β is a successor in $\prec$ or it is a limit. If it is a successor then let β^* be the predecessor of β in $\prec$. Otherwise, let β^* be such that β^* is greater then the support of of p' and $\beta^* \prec \beta$ and β^* is the successor of β^{**} in the ordering $\prec$. In the first case, let $p'' \geq p'$ be such that $p'' \Vdash_{\mathbb{P}_\gamma}$ "$A_{\beta^*}^{-1} k \in D_\beta$". In the second case, choose p'' such that $p'' \Vdash_{\mathbb{P}_{\beta^*}}$ "$A_{\beta^{**}}^{-1} k \in D_{\beta^*}$" and such that β^{**} belongs to the support of p''.

Now use the induction hypothesis to find a determined condition q such that if σ is the maximal element of Σ_q then $\bar{m} \in \mathrm{dom}(a_q^\sigma)$. Moreover, in the case that β is a limit of $\prec$, then the induction hypothesis can be used to ensure that $\sigma < \beta^*$. It will be shown that the transitivity of $\prec$ guarantees that $q * p(\gamma) = r$ is a determined condition satisfying the extra induction requirements. Let $\Sigma_r = \Sigma_q \cup \{\beta\}$ and let f_r^σ, a_r^σ and Δ_r^σ have the values inherited from q and $p(\beta)$. Furthermore, $k_r(\alpha, \tau)$ can be defined to be $k_q(\alpha, \tau)$ unless $\beta = \tau$. Here the choice of p'' helps.

In the case that β is the successor of β^*, then p'' decides that $A_{\beta^*}^{-1} k \in D_{\beta^*}$ so $k_r(\beta^*, \beta)$ can be defined to be k and, moreover $k_r(\mu, \beta)$ can be defined to be k for each $\mu \in \Sigma_q$ such that $\mu \prec \beta^*$. Since β is the successor of β^* in $\prec$ there are no new instances with which to deal. In the case that β is a limit in the partial order $\prec$, it is possibe to define $k_r(\beta^{**}, \beta) = k$ because of the transitivity of $\prec$. For the same reason it is possible to define $k_r(\mu, \beta)$ to be k for each $\mu \in \Sigma_q$ such that $\mu \prec \beta^{**}$. Since the support of q is contained in β^* and β^* is the successor of β^{**} in the partial order $\prec$, it follows that there are no new instances to consider in this case as well. ■

Lemma 0.2. *The partial order $\mathbb{P}_{\omega_2}$ is locally Cohen.*

Proof: Let $X \in [\mathbb{P}_{\omega_2}]^{\aleph_0}$. Let $\mathfrak{M}$ be a countable elementary submodel of $H(\omega_3)$ which contains X and the $\diamondsuit$-sequence $\{D_\xi : \xi \in \omega_2\}$ as well as $\mathbb{P}_{\omega_2}$. It suffices to show that if $p \in \mathbb{P}_{\omega_2}$ and $D \subseteq \mathfrak{M} \cap \mathbb{P}_{\omega_2}$ is a dense subset of the partial order $\mathfrak{M} \cap \mathbb{P}_{\omega_2}$ then there is $q \in D$ and $r \in \mathbb{P}_{\omega_2}$ such that $r \geq p$ and $r \geq q$.

Given $p \in \mathbb{P}_{\omega_2}$, by using Lemma 0.1, it may, without loss of generality, be assumed that p is determined. Using the elementarity of $\mathfrak{M}$ it follows that there is some determined condition p' which is isomorphic to p. In particular, there is an order preserving bijection $I : \Sigma_p \to \Sigma_{p'}$ such that I is the identity on $\Sigma p \cap \mathfrak{M}$, $a_p^\sigma = a_{p'}^{I(\sigma)}$, $f_p^\sigma = f_{p'}^{I(\sigma)}$, $g_p^\sigma = g^{I(\sigma)}$ and I preserve the partial ordering $\prec$. It is not required that $\Delta_p^\sigma = \Delta_{p'}^{I(\sigma)}$ because $\Delta_{p'}^\sigma$ will be defined to be $\Sigma_{p'} \cap \sigma$.

Now let $q \in D$ be a condition extending p'. Using Lemma 0.1 it may again be assumed that q is determined. It must be shown how to define $r \in \mathbb{P}_{\omega_2}$ extending both q and p. In order to do this, define $s(\alpha)$ to be the unique, minimal ordinal $\delta \in \mathfrak{M}$ such that $\alpha \prec \delta$ if such a unique ordinal exists. Notice that if $\alpha \notin \mathfrak{M}$ and there is some $\delta \in \mathfrak{M}$ such that $\alpha \prec \delta$ then $s(\alpha)$ exists. The reason for this is that the only way that $s(\alpha)$ can fail to exist in this context is that there are two minimal ordinals $\delta \in \mathfrak{M}$ and $\delta' \in \mathfrak{M}$ such that $\alpha \prec \delta$ and $\alpha \prec \delta'$. However, this means that the supremum of $\{\gamma : \gamma \prec \delta \text{ and } \gamma \prec \delta'\}$ belongs to $\mathfrak{M}$ and hence there is some $\alpha' \in \mathfrak{M} \setminus \alpha$ such that $\alpha' \prec \delta$ and $\alpha' \prec \delta'$. From the easily verified fact that $\prec$ is a tree ordering it follows $\alpha \prec \alpha'$ contradicting the minimality assumption on δ and δ'.

Now define r as follows:

- the domain of r is the union of the domains of q and p
- if $\alpha \in \mathfrak{M}$ then $r(\alpha) = a_q^\alpha * (f_q^\alpha, \Delta_q^\alpha \cup \Delta_p^\alpha)$
- if $\alpha \notin \mathfrak{M}$ and there does not exist $\delta \in \mathrm{dom}(q)$ such that $\alpha \prec \delta$ then $r(\alpha) = p(\alpha)$
- if $\alpha \notin \mathfrak{M}$ and there exists $\delta \in \mathrm{dom}(q)$ such that $\alpha \prec \delta$ then recall that $s(\alpha)$ is defined and define $r(\alpha) = a_r^\alpha * (f_r^\alpha, \Delta_p^\alpha)$ where the function a_r^α is defined

by

$$a^\alpha_r(n) = \begin{cases} a^\alpha_p(n) & \text{if } n \in \text{dom}(a^\alpha_p) \\ k_p(\alpha, s(\alpha)) + 1 \mod 2 & \text{if } n \notin \text{dom}(a^\alpha_p) \end{cases}$$

(note that in this case $k_p(\alpha, s(\alpha))$ has a natural definition because $\prec$ is a tree ordering) and the function f^α_r is defined by

$$f^\alpha_r(n) = \begin{cases} f^\alpha_p(n) & \text{if } n \in \text{dom}(f^\alpha_p) \\ \min\{f^\beta_q(n) : \beta \in \Sigma_p \cap \mathfrak{M} \text{ and } \alpha \prec \beta\} & \text{if } n \notin \text{dom}(a^\alpha_p) \end{cases}$$

The fact that $r \geq q$ is immediate because $a^\mu_r = a^\mu_q$ and $f^\mu_r = f^\mu_q$ for each $\mu \in \Sigma_q$ and, moreover, if $\alpha \in \text{dom}(q)$ then $\Delta^\alpha_q \subseteq \mathfrak{M}$; so there is no restriction on the points in the domain of r not in the domain of q.

It will be shown that $r \geq p$ by inductively proving that $r \restriction \rho \geq p \restriction \rho$ for each $\rho \in \omega_2$. If $\rho = 0$ there is nothing to do and at limits the finite support of the iteration makes the task easy. So suppose that $r \restriction \rho \geq p \restriction \rho$. Tt suffices to show that the following *Key Condition* is satisfied: If

- $\alpha \prec \beta \leq \rho$
- $\beta \in \Sigma_p$
- $\alpha \in \Delta^\beta_p$
- n is in the domain of $f^\beta_r \setminus f^\beta_p$

then $f^\beta_r(n) \geq f^\alpha_r(n)$ and, in addition, if $a^\alpha_r(n) = k_r(\alpha, \beta)$ then $r \restriction (\rho + 1) \Vdash_{\mathbb{P}_{\rho+1}}$ "$f^\beta_r(n) \geq g_{o(\alpha)}(n)$". This will be established by considering various cases.

Case 1

Suppose that α and β both belong to $\mathfrak{M}$. Since $q \geq p'$, from the definition of p' and the partial order $\mathbb{Q}_\beta$ it easily follows that the Key Condition is satisfied. There is no need to use the induction hypothesis in this case.

Case 2

Suppose now that β belongs to $\mathfrak{M}$ but α does not. First it will be shown that $f^\beta_r(n) \geq f^\alpha_r(n)$. There are two subcases to consider; either n belongs to the domain of f^α_p or it does not. If it does, then $f^{I(\alpha)}_{p'}(n) = f^\alpha_p(n) = f^\alpha_r(n)$ and $I(\alpha) \in \Delta^\beta_{p'}$. Because $I(\alpha) \prec \beta$, it follows from the fact that $q \geq p'$ that $f^\beta_q(n) \geq f^{I(\alpha)}_{p'}(n) = f^\alpha_r(n)$. The other possibility is that n does not belong to the domain of f^α_p. In this case, the definition of r asserts that

$$f^\alpha_r(n) = \min\{f^\gamma_q(n) : \alpha \prec \gamma \text{ and } \gamma \in \mathfrak{M} \cap \text{dom}(p)\}$$

and, since β is in the support of p and $\alpha \prec \beta$, it follows that $f^\alpha_r(n) \leq f^\beta_q(n)$.

It must now be shown that, if $k_r(\alpha, \beta) = a_r(n)$ then $r \restriction (\rho + 1) \Vdash_{\mathbb{P}_{\rho+1}}$ "$f^\beta_r(n) \geq g_{o(\alpha)}(n)$". There are again two subcases to consider; either n belongs to the domain of a^α_p or it does not. If it does, then $g^{I(\alpha)}_{p'}(n) = g^\alpha_p(n)$ and $I(\alpha) \in \Delta^\beta_{p'}$. Because $I(\alpha) \prec \beta$, it follows from the fact that $q \geq p'$ that $q \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$f^\beta_q(n) \geq g^{I(\alpha)}_{p'}(n)$" while, on the other hand, $p \restriction \alpha \Vdash_{\mathbb{P}_\alpha}$ "$g_{o(\alpha)}(n) = g^\alpha_p(n)$". Since the induction hypothesis implies that $r \restriction \alpha \geq p \restriction \alpha$ and it has already been noted that $r \geq q$ it follows that $r \Vdash_{\rho+1}$ "$f^\beta_q(n) \geq g^{I(\alpha)}_{p'}(n) = g^\alpha_p(n) = g_{o(\alpha)}(n)$". The other possibility is that n does not belong to the domain of a^α_p. In this case, the definition of r guarantees that $a^\alpha_r(n) \neq k_r(\alpha, s(\alpha))$ and, the minimality of $s(\alpha)$ guarantees that $s(\alpha) \prec \beta$ because $\alpha \prec \beta$ and $\beta \in \mathfrak{M}$. The transitivity of $\prec$, now guarantees that $k_r(\alpha, \beta) \neq a^\alpha_r(n)$ and so the Key Condition is vacuously satsified.

Case 3
Suppose now that $\alpha \in \mathfrak{M}$ but $\beta \notin \mathfrak{M}$. It will first be shown that $f_r^\beta(n) \geq f_r^\alpha(n)$. To see this, recall that $f_r^\beta(n) = f_q^\gamma(n)$ for some γ such that $\beta \prec \gamma$, $\gamma \in \mathfrak{M}$ and γ belongs to the support of p — recall that it is being assumed that n is not in the domain of f_p^β and this function was only extended in the case that there was an appropriate γ. Recall also that this implies that $\Delta_{p'}^\gamma = \Sigma_{p'} \cap \gamma \cap \mathfrak{M}$ and hence $\alpha \in \Delta_{p'}^\gamma$. Because $\alpha \prec \beta \prec \gamma$ it follows from the fact that $q \geq p'$ that $f_q^\alpha(n) \leq f_q^\gamma(n) = f_r^\beta(n)$.

Now consider $g_{o(\alpha)}(n)$. Since $\alpha \in \Delta_{p'}^\gamma$ and $\alpha \prec \gamma$ it follows that

$$q \Vdash \text{"}g_{o(\alpha)} \leq f_q^\gamma(n)\text{"}$$

and, because it has already been noted that $r \geq q$ it follows that

$$r \restriction (\rho+1) \Vdash_{\mathbb{P}_{\rho+1}} \text{"}g_{o(\alpha)} \leq f_q^\gamma(n)\text{"}$$

Since $f_q^\gamma(n) = f_r^\beta(n)$ it follows that the Key Condition has been satisfied.

Case 4
Finally, suppose that neither α nor β belongs to $\mathfrak{M}$. To show that $f_r^\beta(n) \geq f_r^\alpha(n)$ two cases must again be considered; either n belongs to the domain of f_p^α or it does not. If it does, then $f_r^\beta(n) = f_q^\gamma(n)$ for some γ such that $\beta \prec \gamma$, $\gamma \in \mathfrak{M}$ and γ belongs to the support of p. Since $\alpha \prec \beta \prec \gamma$ it follows that $I(\alpha) \prec \gamma$ and so $f_q^\gamma(n) \geq f_{p'}^{I(\alpha)}(n) = f_p^\alpha(n) = f_r^\alpha(n)$. On the other hand, if n does not belong to the domain of f_p^α then

$$f_r^\alpha(n) = \min\{f_q^\gamma(n) : \alpha \prec \gamma \text{ and } \gamma \in \mathfrak{M} \cap \operatorname{dom}(\mathfrak{p})\}$$

and, since this minimum is taken over a set which includes γ, it follows that $f_r^\alpha(n) \leq f_q^\gamma(n) = f_r^\beta(n)$.

To show that $r \restriction (\rho+1) \Vdash_{\mathbb{P}_{\rho+1}} \text{"}g_{o(\alpha)}(n) \leq f_r^\beta(n)\text{"}$ there are, once again, two cases to consider; either n belongs to the domain of a_p^α or it does not. If it does, then $g_{p'}^{I(\alpha)}(n) = g_r^\alpha(n)$ and $I(\alpha) \in \Delta_{p'}^\gamma$. Because $I(\alpha) \prec \gamma$, it follows from the fact that $q \geq p'$ that $q \Vdash \text{"}f_q^\gamma(n) \geq g_{p'}^{I(\alpha)}(n)\text{"}$. On the other hand, $p \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \text{"}g_{o(\alpha)}(n) = g_r^\alpha(n)\text{"}$ and so the, because the induction hypothesis yields that $r \restriction \alpha \geq p \restriction \alpha$ it follows that $r \restriction (\rho+1) \Vdash_{\mathbb{P}_{\rho+1}} \text{"}f_r^\beta(n) = f_q^\gamma(n) \geq g_{p'}^{I(\alpha)}(n) = g_r^\alpha(n) = go(\alpha)(n)\text{"}$. The other possibility is that n does not belong to the domain of a_p^α. In this case, the definition of r guarantees that $a_r^\alpha(n) \neq k_r(\alpha, s(\alpha))$. The fact that $\alpha \prec \beta$, together with the uniquenness of $s(\alpha)$ guarantees that $s(\alpha) = s(\beta)$. The transitivity of $\prec$, now guarantees that $k_r(\alpha,\beta) = k_r(\alpha, s(\beta)) = k_r(\alpha, s(\alpha)) \neq a_r^\alpha(n)$ and so the Key Condition is vacuously satisified. The use of $k_r(\alpha, s(\beta))$ and $k_r(\alpha, s(\alpha))$ here is a slight abuse of notation because there is no guarantee that $s(\alpha)$ belongs to the domain of r. Nevertheless, because $k_r(\alpha,\gamma)$ is defined for some γ such that $\alpha \prec s(\alpha) \prec \gamma$ there is no harm in this abuse. ■

References

1. S. Shelah and J. Steprāns, *Maximal Chains in ${}^{\omega}\omega$ and Ultrapowers of the Integers*, to appear in Arch. für Math. Log.

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