PEANO ARITHMETIC MAY NOT BE INTERPRETABLE IN THE MONADIC THEORY OF ORDER

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ABSTRACT

Gurevich and Shelah have shown that Peano Arithmetic cannot be interpreted in the monadic second-order theory of short chains (hence, in the monadic second-order theory of the real line). We will show here that it is consistent that there is no interpretation even in the monadic second-order theory of all chains.

0. Introduction

A *reduction* of a theory T to a theory T^* is an algorithm, associating a sentence φ^* in the language of T^*, to each sentence φ in the language of T, in such a way that: $T \vdash \varphi$ if and only if $T^* \vdash \varphi^*$.

Although reduction is a powerful method of proving undecidability results, it lacks in establishing any semantic relation between the theories.

A *(semantic) interpretation* of a theory T in a theory T' is a special case of reduction in which models of T are defined inside models of T'.

It is known (via reduction) that the monadic theory of order and the monadic theory of the real line are complicated at least as Peano Arithmetic, (In [Sh] this was proven from ZFC+MA and in [GuSh1] from ZFC), and even as second order logic ([GuSh2], [Sh1], for the monadic theory of order). Moreover, second order logic was shown to be interpretable in the monadic theory of order ([GuSh3]) but this was done by using a weaker, non–standard form of interpretation: into a Boolean valued model. Using standard

interpretation ([GMS]) it was shown that it is consistent that the second–order theory of ω_2 is interpretable in the monadic theory of ω_2.

On the other hand, by [GuSh], Peano Arithmetic is not interpretable in the monadic theory of short chains, and in particular in the monadic theory of the real line.

More details and Historical background can be found in [Gu].

The previous results leave a gap concerning the question whether it is provable from ZFC that Peano Arithmetic is interpretable in the monadic theory of order. In this paper we fill the gap and show that the previous results are the best possible, by proving:

Theorem. *There is a forcing notion P such that in V^P, Peano Arithmetic (in fact a much weaker theory) is not interpretable and even not weakly interpretable in the monadic second-order theory of chains.*

From another point of view the theorem may be construed as presenting the strength of the interpretation method by showing that although Peano Arithmetic is recursive in the monadic theory of order, it is not interpretable in it.

In the proof we use notations and definitions from [Sh] and [GuSh] but although we omit some proofs, it is self contained.

We start by defining in §1 the notion of interpretation. Althogh this notion is not uniform in the literature, our notion of weak interpretation seems to follow from every reasonable definition.

In §2 we define partial theories and present the relevant results about them from [Sh].

In §3 we define a theory T, easily interpretable in Peano Arithmetic, with the following axioms:

(a) $\forall x \exists y \forall z [p(z, y) \leftrightarrow z = x]$

(b) $\forall x \forall y \exists u \forall z [p(z, u) \leftrightarrow (p(z, x) \vee p(z, y))]$

(c) $\exists x \forall y [\neg p(y, x)]$

Assuming there is a chain C that interprets T, we show that the interpretation 'concentrates' on an initial segment $D \subseteq C$.

The main idea in the proof is that of shuffling subsets $X, Y \subseteq C$: Given a partition of C, $\langle S_j : j \in J \rangle$ and a subset $a \subseteq J$, the shuffling of X and Y with respect to J and a is the set: $\bigcup_{j \in a}(X \cap S_j) \cup \bigcup_{j \notin a}(Y \cap S_j)$. We show in §4 and §5 that under suitable circumstances (in particular, if a is a 'semi–club'), partial theories are preserved under shufflings.

We use a simple class forcing P, defined in §5, to obtain a universe V^P in which generic semi–clubs are added to every suitable partition.

The contradiction to the assumption that an interpretation exists in V^P can be roughly described as follows: We start with an interpreting chain C. The interpretation defines an equivalence relation between subsets of C, and we choose a large enough number of

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nonequivalent subsets. We fix a partition of C and after some manipulations we are left with 3 ordered pairs of nonequivalent subsets of C. We shuffle each pair U, V with respect to a generic semi–club a, added by the forcing, and get a new subset which is equivalent to U. (This uses the preservation of partial theories undershufflings). But a condition $p \in P$ that forces these equivalences determines only a bounded subset of a. We show that we could have got the same results if we had shuffled the pairs with respect to the complement of a. Thus for each pair U, V, p forces that the 'inverse' shuffling is also equivalent to U. We conclude by showing that one of the shufflings is equivalent to V as well, and get a contradiction since U and V were not equivalent.

1. The notion of interpretation

The notion of semantic interpretation of a theory T in a theory T' is not uniform. Usually it means that models of T are defined inside models of T' but the definitions vary with context. Here we will define the notion of interpretation of one first order theory in another following the definitions and notatins of [GuSh].

Remark. The idea of our definition is that in every model of T' (or maybe of some extension T'' if T' is not complete) we can define a model of T. An alternative definition could demand that *every* model of T is interpretable in a model of T' (As in [BaSh]). Actually we need a weaker notion than the one we define and this seems to follow from every reasonable definition of semantic interpretation. We will show that it is consistent that no chain C interprets Peano arithmetic. We even allow parameters from C in the interpreting formulas. Thus, our notion is: "A model of T' defines (with parameters) a model of T ". We call this notion "*Weak Interpretation*"

Definition 1.1. Let σ be a signature $\langle P_1, P_2, \ldots \rangle$ where each P_i is a predicate symbol of some arity r_i, in the language $L = L(\sigma)$. An *interpretation* of σ in a first order language L' is a sequence
$I = \langle d, U(\bar{v}_1, \bar{u}), E(\bar{v}_1, \bar{v}_2, \bar{u}), P_1'(\bar{v}_1, \ldots \bar{v}_{r_1}, \bar{u}), P_2'(\bar{v}_1, \ldots \bar{v}_{r_2}, \bar{u}), \ldots \rangle$ where:
    (a) d is a positive integer (the *dimension*);
    (b) $U(\bar{v}_1, \bar{u})$ and $E(\bar{v}_1, \bar{v}_2, \bar{u})$ are L'-formulas (the *universe* and the *equality formulas*);
    (c) each $P_i'(\bar{v}_1, \ldots \bar{v}_{r_i}, \bar{u})$ is an L'-formula (the *interpretation* of P_i);
    (d) $\bar{v}_1, \bar{v}_2 \ldots$ are disjoint d-tuples of distinct variables of L';
    (e) $\bar{u}$ is a finite sequence (standing for the parameters of the interpretation).

Definition 1.2. Let σ, L' and I be as in 1.1. Fix a function that associates each L variable v with a d-tuple v' of distinct L' variables in such a way that if u and v are different L-variables then the tuples u' and v' are disjoint.

We define, by induction, the *I-translation* φ' of an arbitrary L-formula φ:

(a) $(x = y)' = E(x', y')$.

(b) If P is a predicate symbol of arity r in L, then $\mathrm{P}(x_1 \ldots x_r)' = \mathrm{P}'(x_1' \ldots x_r')$.

(c) $(\neg\varphi)' = \neg(\varphi')$, and $(\varphi \wedge \psi)' = (\varphi' \wedge \psi')$.

(d) $(\forall x)\varphi(x)' = (\forall x')[U(x') \to \varphi'(x')]$, and $(\exists x)\varphi(x)' = (\exists x')[U(x') \wedge \varphi'(x')]$.

Definition 1.3. Let T and T' be first order theories such that the signature of T consists of predicate symbols, and T' is consistent and complete. Let I be an interpretation of the signature of T in $L(T')$, and let $U(x)$ be the universe formula of I.

I is an *interpretation* of T in T' if:

(a) the formula $\exists x U(x)$ is a theorem of T', and

(b) the I translation of every closed theorem of T is a theorem of T'.

T is *interpretable* in T' if there is an interpretation of T in T'.

Definition 1.4. Let T, T' and $U(x)$ as in 1.3. except T' may be incomplete. Let T'' be the extension of T' by an additional axiom $\exists x U(x)$.

I is an *interpretation* of T in T' if:

(a) T'' is consistent, and

(b) the I translation of every closed theorem of T is a theorem of T''.

T is *interpretable* in T' if there is an interpretation of T in T'.

Remark 1.5. 1) The definitions are easily generalized to the case that $\sigma(T)$ consists also of function symbols, see [GuSh].

2) Definitions 1.3 and 1.4 make sense in case there are no parameters in the interpretation.

Definition 1.6. Let $\sim$ be an equivalence relation on a non empty set A, and let R be a relation of some arity r on A. We say that $\sim$ *respects* R if for all elements $a_1, \ldots, a_r, \ b_1, \ldots, b_r$ of A,

$[R(a_1, \ldots, a_r) \ \& \ (a_1 \sim b_1) \ \& \ \ldots \ \& \ (a_r \sim b_r)]$ implies $R(b_1, \ldots, b_r)$.

Definition 1.7. Let σ, I and L' be as in def. 1.1. Let M be a model for L' and

(a) $U^* = \{x : x$ is a d-tuple of elements of M and $U(x)$ holds in $M\}$;

(b) $E^* = \{(x, y) : \ x, y \in U^*$ and $E(x, y)$ holds in $M\}$; and

(c) if P is a predicate symbol of arity r in σ, then

$\mathrm{P}^* = \{(x_1, \ldots, x_r) : $ each x_i belongs to U^* and $\mathrm{P}'((x_1, \ldots, x_r)$ holds in $M\}$.

The interpretation I *respects* the structure M if U^* is not empty, E^* is an equivalence relation, and E^* respects every P^*. (The definition is easily generalized when we allow parameters in I).

Lemma 1.8. *Any interpretation of a first-order theory T in a consistent complete first order theory T' respects every model of T.*

Proof. Easy

♡

Definition 1.9. Let σ, I and L' be as in Definition 1.1. and let M, U^*, E^* and P* be as in Definition 1.7. We suppose that I respects M and define a Model for L which will be called the *I-image* of M and will be denoted $I(M)$.

Elements of $I(M)$ are equivalence classes $x/E^* = \{y \in U^* : xE^*y\}$ of E^* (where x ranges over U^*). If P is a predicate symbol of arity r in σ then P is interpreted in $I(M)$ as the relation $\{(x_1/E^*, \ldots, x_r/E^*) : (x_1, \ldots, x_r) \in P^*\}$. Again, we may allow parameters in I and slightly modify this definition.

Lemma 1.10. *Let $I = (d, U(v_1), E(v_1, v_2), \ldots)$ be an interpretation of a signature σ in the first order language of a structure M. Suppose that I respects M. Let:*

  $\varphi(v_1, \ldots, v_l)$ be an arbitrary $L(\sigma)$-formula, $\varphi'(v'_1, \ldots, v'_l)$ its I-translation,
  $U^* = \{x : x$ is a d-tuple of elements of M and $U(x)$ holds in $M\}$,
  $E^* = \{(x, y) : x, y \in U^*$ and $E(x, y)$ holds in $M\}$,
  $x_1, \ldots, x_l$ belong to U^*.

Then, $\varphi'(x_1, \ldots, x_l)$ holds in M if and only if $\varphi(x_1/E^*, \ldots, x_l/E^*)$ holds in $I(M)$.

Proof. By induction on φ.

♡

So we can conclude:

Theorem 1.11. *If I is an interpretation of a first-order theory T in the first-order theory of a structure M, then the I-image of M is a model for T.*

Proof. Let φ be any closed theorem of T. Since I interpretes T in the theory of M, the I-translation φ' of φ holds in M. By Lemma 1.10, φ holds in $I(M)$.

♡

Remark 1.12. The notion of interpretation presents a connection between theories: It implies that models of a theory T are defined inside models of the interpretating theory T'. (Assuming $T' \vdash (\exists x)U(x)$, for every $M \models T$, $I(M)$ is a model of T). But rephrasing a previous remark we demand less: In our world V^P we will show that there is no model M of (actually a weaker theory than) Peano Arithmetic, and no chain C (= a model of the monadic theory of order), and an interpretation I, such that the I-image of C is isomorphic to M. This will hold even if we allow parameters in the interpreting formulas in I. This leads to the following definition:

Definition 1.13. T' *weakly interprets* T if there is a model M of T' and an interpretation I of the signature of T, respecting M, maybe with parameters from M appearing in I, such that $I(M)$ is a model of T.

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From now on, whenever we write 'interpretation' we will mean weak interpretation in the sense of the previous definition.

2. Partial Theories

In this section we will define 3 kinds of partial theories following [Sh]: Th^n (definition 2.3) which is the theory of formulas with monadic quantifier depth n, ATh^n (definition 2.11) which is the n-theory of segments (and by 2.10 'many' segments have the same theory), and WTh^n which gives information about stationary subsets of the chain. The last two theories are naturally defined for well ordered chains only, but by embeding a club in the chain we can modify them so that they can be applied also to general chains.
The main result of this section states roughly that for every n there is an m such that WTh^m and ATh^m determine Th^n (theorem 2.15).

Definition 2.1. The *monadic second-order theory* of a chain C is the theory of C in the language of order enriched by adding variables for sets of elements, atomic formulas of the form "$x \in Y$" and the quantifier $(\exists Y)$ ranging over subsets. Call this language L.

Remark. We can identify the monadic theory of C with the first order theory of the associated structure

$$C' = \langle \mathcal{P}(C), \subset, <, \emptyset \rangle$$

where $\mathcal{P}(C)$ is the power set of C, and $<$ is the binary relation $\{(\{x\}, \{y\}) : x, y \text{ are elements of } C \text{ and } x < y \text{ in } C\}$.

Notation. The universe of a model M will be denoted $|M|$. Let x, y, z be individual variables; X, Y, Z set variables; a, b, c elements; A, B, C sets. Bar denotes a finite sequence, like $\bar{a}$, and $l(\bar{a})$ it's length. We write e.g. $\bar{a} \in M$ and $\bar{A} \subseteq M$ instead of $\bar{a} \in |M|^{l(\bar{a})}$, or $\bar{A} \in \mathcal{P}(|M|)^{l(\bar{A})}$

Definition 2.2. For any L-model M, $\bar{A} \in \mathcal{P}(M)$, $\bar{a} \in |M|$, and a natural number n define

$$t = th^n(M, \bar{A}, \bar{a})$$

by induction on n:
for $n = 0$: $t = \{\varphi(X_{l_1}, \ldots, x_{j_1}, \ldots) : \varphi(X_{l_1}, \ldots, x_{j_1}, \ldots) \text{ is an atomic formula in } L, M \models \varphi[A_{l_1}, \ldots, a_{j_1}, \ldots]\}$.
for $n = m + 1$: $t = \{th^m(M, \bar{A}, \bar{a}^\frown b) : \bar{b} \in |M|\}$.

Definition 2.3. For any L-model M, $\bar{A} \in \mathcal{P}(M)$, and a natural number n define

$$T = Th^n(M, \bar{A})$$

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by induction on n:

for $n = 0$: $T = th^2(M, \bar{A})$.

for $n = m + 1$: $T = \{Th^m(M, \bar{A}^\wedge B) : B \in \mathcal{P}(M)\}$.

Remark.   By $Th^0(M, \bar{A})$ we can tell which subset is a singleton, so we can proceed to quantify only over subsets.

Lemma 2.4.   (A) For every formula $\psi(\bar{X}) \in L$ there is an n such that from $Th^n(M, \bar{A})$ we can find effectively whether $M \models \psi(\bar{X})$.

(B) For every n and m there is a set $\Psi = \{\psi_l(\bar{X}) : l < l_0 (< \omega), l(\bar{X}) = m\} \subset L$ such that for any L-models M, N and $\bar{A} \in \mathcal{P}(M)^m, \bar{B} \in \mathcal{P}(N)^m$ the following hold:

    (1) $Th^n(N, \bar{B})$ can be computed from $\{l < l_0 : N \models \psi_l[\bar{B}]\}$

    (2) $Th^n(M, \bar{A}) = Th^n(N, \bar{B})$ if and only if for any $l < l_0, \quad M \models \psi_l[\bar{A}] \leftrightarrow N \models \psi_l[\bar{B}]$.

Proof.   In [Sh], Lemma 2.1 (Note that our language L is finite).

<div align="right">♡</div>

Lemma 2.5.   For given n, m, each $Th^n(M, \bar{A})$ is hereditarily finite, (where $l(\bar{A}) = m$, M is an L-model), and we can effectively compute the set of formally possible $Th^n(M, \bar{A})$.

Proof.   In [Sh], Lemma 2.2

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Definition 2.6.   If C, D are chains then $C + D$ is any chain that can be split into an initial segment isomorphic to C and a final segment isomorphic to D.

If $\langle C_i : i < \alpha \rangle$ is a sequence of chains then $\sum_{i < \alpha} C_i$ is any chain D that is the concatenation of segments D_i, such that each D_i is isomorphic to C_i.

Theorem 2.7 (composition theorem).

    (1) If $l(\bar{A}) = l(\bar{B}) = l$, and

$$Th^m(C, \bar{A}) = Th^m(C', \bar{A}')$$

and

$$Th^m(D, \bar{B}) = Th^m(D', \bar{B}')$$

then

$$Th^m(C + D, A_0 \cup B_0, \ldots, A_{l-1} \cup B_{l-1}) = Th^m(C' + D', A'_0 \cup B'_0, \ldots, A'_{l-1} \cup B'_{l-1}).$$

(2) If $Th^m(C_i, \bar{A}_i) = Th^m(D_i, \bar{B}_i)$ for each $i < \alpha$, then

$$Th^m\left(\sum_{i < \alpha} C_i, \ \cup_i A_{1,i}, \ldots, \cup_i A_{l-1,i}\right) = Th^m\left(\sum_{i < \alpha} D_i, \ \cup_i B_{1,i}, \ldots, \cup_i B_{l-1,i}\right).$$

Proof.   By [Sh] Theorem 2.4 (where a more general theorem is proved), or directly by induction on m.

<div align="right">♡</div>

Notation 2.8. (1) $Th^m(C, A_0, \ldots, A_{l-1}) + Th^m(D, B_0, \ldots, B_{l-1})$ is $Th^m(C + D, A_0 \cup B_0, \ldots, A_{l-1} \cup B_{l-1})$.

(2) $\sum_{i<\alpha} Th^m(C_i, \bar{A}_i)$ is $Th^m(\sum_{i<\alpha} C_i, \cup_{i<\alpha} A_{1,i}, \ldots, \cup_{i<\alpha} A_{l-1,i})$.

(3) If D is a subchain of C and $X_1, \ldots, X_{l-1}$ are subsets of C then $Th^m(D, X_0, \ldots, X_{l-1})$ abbreviates $Th^m(D, X_0 \cap D, \ldots, X_{l-1} \cap D)$.

The following definitions and results apply to well ordered chains (i.e. ordinals), later we will modify them.

Definition 2.9. For $a \in (M, \bar{A})$ let

$$th(a, \bar{A}) = \{x \in X_i : a \in A_i\} \cup \{x \notin X_i : a \notin A_i\}.$$

So it is a finite set of formulas.

For α an ordinal with $cf(\alpha) > \omega$, let D_α denote the filter generated by the closed unbounded subsets of α.

Lemma 2.10. *If the cofinality of α is $> \omega$, then for every $\bar{A} \in \mathcal{P}(\alpha)^m$ there is a closed unbounded subset J of α such that: for each $\beta < \alpha$, all the models*

$$\{(\alpha, \bar{A})|_{[\beta,\gamma)} : \gamma \in J, \ cf(\gamma) = \omega, \ \gamma > \beta\}$$

have the same monadic theory.

Proof. *In [Sh] Lemma 4.1.*

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Definition 2.11. $ATh^n(\beta, (\alpha, \bar{A}))$ for $\beta < \alpha, \alpha$ a limit ordinal of cofinality $> \omega$ is $Th^n(\alpha, \bar{A})|_{[\beta,\gamma)})$ for every $\gamma \in J, \ \gamma > \beta, \ cf(\gamma) = \omega$; Where J is from Lemma 2.10.

Remark. As D_α is a filter, the definition does not depend on the choice of J.

Definition 2.12. We define $WTh^n(\alpha, \bar{A})$:

(1) if α is a successor or has cofinality ω, it is $\emptyset$;

(2) otherwise we define it by induction on n:

for $n = 0$: $WTh^0(\alpha, \bar{A}) = \{t : \{\beta < \alpha : th(\beta, \bar{A}) = t\}$ is a stationary subset of $\alpha \}$;

for $n + 1$: $WTh^{n+1}(\alpha, \bar{A}) = \{\langle S_1^{\bar{A}}(B), S_2^{\bar{A}}(B)\rangle : B \in \mathcal{P}(\alpha)\}$

Where:

$S_1^{\bar{A}}(B) = WTh^n(\alpha, \bar{A}, B)$,

$S_2^{\bar{A}}(\bar{B}) = \{\langle t, s\rangle : \{\beta < \alpha : WTh^n((\alpha, \bar{A}, B)|_\beta) = t, \ th(\beta, \bar{A}^\wedge B) = s\}$ is a stationary subset of $\alpha\}$.

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Remark. Clearly, if we replace $(\alpha, \bar{A})$ by a submodel whose universe is a club subset of α, $WTh^n(\alpha, \bar{A})$ will not change.

Definition 2.13. Let $cf(\alpha) > \omega$, $M = (\alpha, \bar{A})$ and we define the model $g^n(M) = (\alpha, g^n(\bar{A}))$.

Let: $\left(g^n(\bar{A})\right)_s = \{\beta < \alpha : s = ATh^n(\beta, (\alpha, \bar{A}))\}$

let $m = l(\bar{A})$ and $T(n, m) :=$ the set of formally possible $Th^n(M, \bar{B})$, where $l(\bar{B}) = m$. We define:

$$g^n(\bar{A}) := \langle \ldots, \left(g^n(\bar{A})\right)_s, \ldots \rangle_{s \in T(n,m)}.$$

Lemma 2.14. (A) $g^n(\alpha, \bar{A})$ is a partition of α.
(B) $g^n(\alpha, \bar{A}^\wedge \bar{B})$ is a refinement of $g^n(\alpha, \bar{A})$ and we can effectively correlate the parts.
(C) $g^{n+1}(\alpha, \bar{A})$ is a refinement of $g^n(\alpha, \bar{A})$ and we can effectively correlate the parts.

Proof. Easy.

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The next theorem shows that the (partial) monadic theories can be computed from ATh and WTh and is the main tool for showing that the monadic theories are preserved under shufflings of subsets.

Theorem 2.15. If $cf(\alpha) > \omega$, then for each n there is an $m = m(n)$ such that if:
$t_1 = WTh^m\left(\alpha, g^m(\alpha, \bar{A})\right)$, $t_2 = ATh^m\left(0, (\alpha, \bar{A})\right)$
then we can effectively compute $Th^n(\alpha, \bar{A})$ from t_1, t_2.

Proof. By [Sh], Thm. 4.4.

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Notation 2.16. We will denote $\langle t_1, t_2 \rangle$ from Thm. 2.15 by $WA^{m(n)}$.

In [Sh] the partial theories ATh and WTh were defined only to well ordered chains. We will show now how we can modify our definitions and apply them to general chains of cofinality $> \omega$. The only loss of generality is that we assume that we can find in every chain C a closed cofinal sequence. This does not hurt us because if a chain C interprets a theory T, then there is a chain C^c that interprets T, with this property and all we have to pay is maybe adding an additional parameter to the interpreting formulas. The proofs of the results are easy generalizations of the original proofs.

Notation 2.17. Let C be a chain of cofinality $\lambda > \omega$, and $J^* = \langle \beta_i : i < \lambda \rangle$ be a closed cofinal subchain of C. Fix a club subset of λ, $J = \langle \alpha_i : i < \lambda \rangle$ such that $\alpha_0 = 0$ and for

simplicity $cf(\alpha_{i+1}) = \omega$, and let $h\colon J^* \to J$ be an isomorphism, $h(\beta_i) = \alpha_i$. For a fixed n and $\bar{A} \subseteq \mathcal{P}(C)^d$, denote by s_i the theory $Th^n\big((C,\bar{A})|_{[\beta_i,\beta_{i+1})}\big)$. Using these notations we can generalize the definitions and facts concerning ATh and WTh:

Lemma 2.10*. If the cofinality of C is $> \omega$, then for every $\bar{A} \in \mathcal{P}(C)^d$ there is a subchain $J^{**} \subseteq J^*$ such that $h''(J^{**}) = J' \subseteq J$ is a club subset of λ, with $0 \in J'$, and such that for each $i < \lambda$, all the models

$$\{(C,\bar{A})|_{[\beta_i,\beta_j)} : j > i, \beta_j \in J^{**}, cf(h(\beta_j)) = \omega\}$$

have the same monadic theory.

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Remark. We could have chosen J to be all λ. The definitions and the results do not depend on the particular choice of J.

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Definition 2.11*. $ATh^n(\beta_i, (C,\bar{A}))$ for $\beta_i \in J^*$ is: $Th^n(C,\bar{A})|_{[\beta_i,\gamma)})$ for every $\gamma \in J^{**}$ $\gamma > \beta_i$, $cf(\gamma) = \omega$; Where J^{**} is from Lemma 2.10*. (Actually this is s_i from notation 2.17).

Remark. Again, fixing J^* and h it is easily seen that the definition does not depend on the choice of J^{**}.

Definition 2.13*. Let $cf(C) > \omega$, $M = (C,\bar{A})$ and we define the model $g^n(M) = \big(C, g^n(\bar{A})\big)$.
Let: $\big(g^n(\bar{A})\big)_s = \{\alpha_i < \lambda : s = ATh^n(\beta_i, (C,\bar{A})), \beta_i \in J^*, h(\beta_i) = \alpha_i\}$ (so this is a subset of λ).
let $d = l(\bar{A})$ and $T(n,d) :=$ the set of formally possible $Th^n(M,\bar{B})$, where $l(\bar{B}) = d$.
    We define a finite sequence of subsets of λ:

$$g^n(\bar{A}) := \langle \ldots, \big(g^n(\bar{A})\big)_s, \ldots \rangle_{s \in T(n,d)}$$

Lemma 2.14*. The analogs of lemma 2.14 hold for $g^n(C,\bar{A})$

Theorem 2.15*. If $cf(C) > \omega$, then for each n there is an $m = m(n)$ such that if:
$t_0 = Th^m(C,\bar{A})|_{\beta_0}$, $t_1 = WTh^m\big(C, g^m(C,\bar{A})\big)$, $t_2 = ATh^m\big(\beta_0, (C,\bar{A})\big)$
    then we can effectively compute $Th^n(C,\bar{A})$ from t_0, t_1, t_2. (If C has a first element δ, set $\beta_0 = \delta$ and we don't need t_0).

Remark. Following our notations, $Th^n(C,\bar{A})$ is equal to $t_0 + \sum_{i<\lambda} s_i$. By 2.10* we get for example (if $J^* = J^{**}$ from 2.10): $\sum_{i \leq k < j} s_i = s_i$ for $cf(j) \leq \omega$.
What we say in 2.13* is that if we know t_0 and s_0 and we know, roughly speaking, 'how many' theories of every kind appear in the sum (this information is given by t_1), then we can compute the sum of the theories exactly as in the case of well ordered chains.

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Notation 2.16*.   We will denote $\langle t_0, t_1, t_2 \rangle$ from Thm. 2.15* by $WA^{m(n)}$.

3. Major segments

In this section we define a theory T which is interpretable in Peano arithmetic and reduce a supposed interpretation of T in a chain C to an interpretation of even a simpler theory in a chain D having some favorable properties which will lead us to a contradiction.

Definition 3.0.   Let T be a first order theory with a signature consisting of one binary predicate p. The axioms of T are as follows:

(a) $\forall x \exists y \forall z [p(z, y) \leftrightarrow z = x]$

(b) $\forall x \forall y \exists u \forall z [p(z, u) \leftrightarrow (p(z, x) \vee p(z, y))]$

(c) $\exists x \forall y [\neg p(y, x)]$

Intuitively (a) means that for every set x there exists the set $\{x\}$, (b) means that for every set x, y there exists the set $x \cup y$ and (c) means that the empty set (or an atom) exists. Now, Peano arithmetic easily interprets T in the sense of definition 1.4 (let $d = 1$, $U(x) := x = x$, $E(x, y) := x = y$ and $p'(x, y) :=$ "there exists a prime number p such that p^x divides y but p^{x+1} does not"), so it suffices to show that no chain C interprets T.

So Suppose C is a chain that interprets T by:

$$I = \langle d, U(\bar{X}_1, \bar{W}), E(\bar{X}_1, \bar{X}_2, \bar{W}), P(\bar{X}_1, \bar{X}_2, \bar{W}) \rangle.$$

We may assume, by changing E, that the interpretation is universal, i.e. $C \models (\forall \bar{X}) U(\bar{X})$, and that the relation P satisfies extensionality. $\bar{W} \subseteq C$ is a finite sequence of parameters and we will usually forget to write them. Remember, for later stages, that we may assume that there is a closed cofinal subchain in C, if not add the completion of some cofinal subchain to C and to the parameters and, if necessary, modify I.

Hence, the interpretation defines a model of T:

$$\mathcal{M} = \langle\ (\mathcal{P}(C)^d / E), P\ \rangle$$

Notation.   We will refer to (d-tuples of) subsets of C as 'elements'. If not otherwise mentioned, all the sequences appearing in the formulas have length d (= the dimension of the interpretation).

We write $\bar{X} \sim \bar{Y}$ when $\mathcal{M} \models E(\bar{X}, \bar{Y})$

We write, for example, $\bar{A} \cap \bar{B}$ meaning $\langle A_0 \cap B_0, \ldots, A_{lg(\bar{A})-1} \cap B_{lg(\bar{B})-1} \rangle$ and assuming $lg(\bar{A}) = lg(\bar{B})$, $\bar{A} = \langle A_0 \ldots, A_{lg(\bar{A})-1} \rangle$ etc.

We also write $\bar{A} \subseteq C$ when $\bar{A} \in \mathcal{P}(C)^{lg \bar{A}}$.

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Definition 3.1.

1) A subchain $D \subseteq C$ is a *segment* if it is convex (i.e. $x < y < z$ & $x, z \in D \Rightarrow y \in D$).

2) We will write $\bar{A} \sim \bar{B}$ when $\bar{A}, \bar{B} \subseteq C$ and $C' \models E(\bar{A}, \bar{B})$.

3) Let $\bar{A}, \bar{B} \subseteq C$. We will say that $\bar{A}, \bar{B}$ *coincide on* (resp. *outside*) a segment $D \subseteq C$, if $\bar{A} \cap D = \bar{B} \cap D$ (resp. $\bar{A} \cap (C - D) = \bar{B} \cap (C - D)$).

4) The *bouquet size* of a segment $D \subseteq C$ is the supremum of cardinals $|S|$ where S ranges over collections of nonequivalent elements coinciding outside D.

5) A *Dedekind cut* of C is a pair (L, R) where L is an initial segment of C, R is a final segment of C and $L \cap R = \emptyset, L \cup R = C$.

Our next step is to show that the bouquet size of every initial segment is either infinite or a-priory bounded.

Lemma 3.2. There are monadic formulas $\theta_1(\bar{X}, \bar{Z})$, and $\theta_2(\bar{X}, \bar{Y}, \bar{Z})$ such that:

1) For every finite, nonempty collection S of elements, there is an element $\bar{W}$ such that for an arbitrary element $\bar{A}$, $C \models \theta_1(\bar{A}, \bar{W})$ if and only if there is an element $\bar{B} \in S$ such that $\bar{B} \sim \bar{A}$.

2) For every finite, nonempty collection S of pairs of elements, there is an element $\bar{W}$ such that for an arbitrary pair of elements $\langle \bar{A}_1, \bar{A}_2 \rangle$, $C \models \theta_2(\bar{A}_1, \bar{A}_2, \bar{W})$ if and only if there is a pair $\langle \bar{B}_1, \bar{B}_2 \rangle \in S$ such that $\bar{B}_1 \sim \bar{A}_1, \bar{B}_2 \sim \bar{A}_2$.

proof. Easy (T allows coding of finite sets and C interprets T).

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Thinking of P as the ϵ relation, we will sometimes denote by something like $\{\{\bar{X}\}, \{\bar{Y}\}, \dots \}$ the set that codes $\bar{X}, \bar{Y}, \dots$.

Proposition 3.3. Fix a large enough $m < \omega$, (e.g. such that from $Th^m(C, \bar{A}_1, \bar{A}_2, \bar{W})$ we can compute whether $C \models \theta_2(\bar{A}_1, \bar{A}_2, \bar{W})$). Let: $N_1 = \left|\{Th^m(D, \bar{X}, \bar{Y}, \bar{Z}) : D \text{ is a chain}, \bar{X}, \bar{Y}, \bar{Z} \subseteq D\}\right|$

Then, for every Dedekind cut (L, R) of C, either the bouquet size of L is at most N_1 and the bouquet size of R is infinite, or, the bouquet size of R is at most N_1 and the bouquet size of L is infinite.

Proof. See [GuSh] Thm. 6.1 and Lemma 8.1.

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Definition 3.4.

1) A segment $D \subseteq C$ is called *minor* if it's bouquet size is at most N_1.

2) A segment $D \subseteq C$ is called *major* if it's bouquet size is infinite.

Conclusion 3.5. C is major and for every Dedekind cut (L, R) of C, either L is minor and R is major, or vice versa.

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Proof. By Prop. 3.3. (and note that T has only infinite models so C has an infinite number of E-equivalence classes).

♡

Definition 3.6. An initial (final) segment D is called a *minimal major segment* if D is major and for every initial (final) segment $D' \subset D$, D' is minor.

Lemma 3.7. *There is a chain C^* that interprets T and an initial segment $D \subseteq C^*$ (possibly $D = C^*$) such that D is a minimal major segment.*

Proof. (By [GuSh] lemma 8.2). Let L be the union of all the minor initial segments (note that if L is minor and $L' \subseteq L$ then L' is minor). If L is major then set $L = D$ and we are done. Otherwise, let $D = C - L$, and by conclusion 3.5 D is major. If there is a final segment $D' \subseteq D$ which is major then $C - D'$ is minor. But, $C - D' \supset L$, a contradiction. So D is a minimal major (final) segment. Now take C^{INV} to be the inverse chain of C. By virtue of symmetry C^{INV} interprets T and D is a minimal major initial segment of C^{INV}.

♡

Notation. Let $D \subseteq C$ be the minimal major initial segment we found in the previous lemma.

Discussion. It is clear that D is definable in C. (It's the shortest initial segment such that there at most N_1 nonequivalent elements coinciding outside it). What about $cf(D)$? It's easy to see that D does not have a last point. On the other hand, it was proven in [GuSh] that T is not interpretable in the monadic theory of short chains (where a chain C is short if every well ordered subchain of C or C^{INV} is countable). But we don't need to assume that the interpreting chain is short in order to apply [GuSh]'s argument. All we have to assume, to get a contradiction is that $cf(D) = \omega$ (which is of course the only possible case when C is short). So, if C interprets T and $cf(D) = \omega$, we can repeat the argument from [GuSh] to get a contradiction. Therefore, we can conclude:

Proposition 3.9. $cf(D) > \omega$

♡

Notation 3.10. T_k will denote the theory of a family of k sets and the codings of every subfamily.

Discussion (continued). Now, fix an element $\bar{R} \subseteq (C - D)$ witnessing the fact that D is major, and define:
$$S = \{\ \bar{A} \subseteq C :\ \bar{A} \cap (C - D) = \bar{R}\ \}$$

So $S/_\sim$ is infinite by the choice of $\bar{R}$ (and of course definable in C with an additional parameter $\bar{R}$). For the moment let $k = 2$ and fix a finite subset of 6 nonequivalent

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elements in S, $\langle \bar{A}_1, \bar{A}_1 \ldots \bar{A}_6 \rangle$. We want to define in D a structure that contains 2 'atoms' and 4 codings by using the $\bar{A}_i$'s.

Since $\mathcal{M} \models T$ we have an element $\bar{W} \subseteq C$ (not necessarily in S) which can be identified with the set:

$$\{\{\bar{A}_1\}, \{\bar{A}_2\}, \{\{\bar{A}_3\}\}, \{\{\bar{A}_4, \bar{A}_1\}\}, \{\{\bar{A}_5, \bar{A}_2\}\}, \{\{\bar{A}_6, \bar{A}_1\}\{\bar{A}_6, \bar{A}_2\}\}\}.$$

Look at the following formulas:

$$\text{Atom}(\bar{X}, \bar{W}) := P(\{\bar{X}\}, \bar{W})$$

$$\text{Set}(\bar{Y}, \bar{W}) := \neg\text{Atom}(\bar{Y}, \bar{W}) \ \& \ \exists \bar{Z} \exists \bar{V}(P(\bar{Z}, \bar{W}) \& P(\bar{V}, \bar{Z}) \& (P(\bar{Y}, \bar{V}))$$

$$\text{Code}(\bar{X}, \bar{Y}, \bar{W}) := \text{Atom}(\bar{X}, \bar{W}) \ \& \ \text{Set}(\bar{Y}, \bar{W}) \ \& \ (\exists \bar{Z}(P(\bar{Z}, \bar{W}) \& P(\{\bar{X}, \bar{Y}\}, \bar{Z}))$$

Using these formulas we can easily define in C a structure which satisfies T_2, where $\bar{A}_1$ and $\bar{A}_2$ are the atoms $\bar{A}_3$ codes the empty subfamily, $\bar{A}_4$ codes $\bar{A}_1$ etc. But for every natural number k we can define a structure for T_k by picking $k + 2^k$ elements from S and a suitable $\bar{W}$, and note that the above formulas do not depend on k.

Now we claim that we can interpret T_k even in D and not in all C. To see that, look at the formula $\text{Code}(\bar{X}, \bar{Y}, \bar{W})$. There is an $n < \omega$ such that we can decide from $Th^n(C, \bar{X}, \bar{Y}, \bar{W}, \bar{R})$ if $\text{Code}(\bar{X}, \bar{Y}, \bar{W})$ holds. By the composition theorem it suffices to look at $Th^n(D, \bar{X} \cap D, \bar{Y} \cap D, \bar{W} \cap D, \bar{R} \cap D)$ and $Th^n(C - D, \bar{X} \cap (C - D), \bar{Y} \cap (C - D), \bar{W} \cap (C - D), \bar{R} \cap (C - D))$. But, since we restrict ourselves only to elements in S, the second theory is constant for every $\bar{X}, \bar{Y}$ in S. It is: $Th^n(C - D, \bar{R}, \bar{R}, \bar{W} \cap (C - D), \bar{R})$. So it suffices to know only $Th^n(D, \bar{X} \cap D, \bar{Y} \cap D, \bar{W} \cap D), (\bar{R} \cap D = \emptyset)$. Now use Lemma 2.4 to get a formula $\text{Code}^*(\bar{X}, \bar{Y}, \bar{W} \cap D)$ that implies $\text{Code}(\bar{X} \cup \bar{R}, \bar{Y} \cup \bar{R}, \bar{W})$, and the same holds for the other formulas (including the equality formula for members of S).

We get an interpretation of T_k on D with an additional parameter $\bar{W}$. Remember that we allowed parameters $\bar{V}$ in the original interpretation of T in C and we can assume that $\bar{W}$ is a sequence that contains the coding set and the old parameters (all intersected with D).

The universe formula of the interpretation is $\text{Atom}^*(\bar{X}, \bar{W}) \vee \text{Set}^*(\bar{X}, \bar{Y}, \bar{W})$, the coding formula is $\text{Code}^*(\bar{X}, \bar{Y}, \bar{W})$ and the equality formula is $E^*(\bar{X}, \bar{Y}, \bar{W})$. And for different k's and even different choices of members of S, the formulas (and their quantifier depth) are unchanged except for the parameters $\bar{W}$.

It is easy to see that, since D is minimal major, for every proper initial segment $D' \subset D$ there are no more then N_1 (from definition 3.4) E^* nonequivalent members of S coinciding outside D'. We will say, by abuse of definition, that D is still a minimal major initial segment with respect to E^*. To sum up, we have proven:

Theorem 3.11. *If there is an interpretation of T in the monadic theory of a chain C then, there is a chain D such that $cf(D) > \omega$, and such that for every $k < \omega$ there is an interpretation of T_k in the monadic theory of D such that the interpretation does not*

"concentrate" on any proper initial segment of D (i.e. D itself is the minimal major initial segment of D). Furthermore, there is an $n < \omega$ which does not depend on k, such that all the interpreting formulas have quantifier depth $< n$.

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4.Preservation of theories under shufflings

We will define here shufflings of subchains and show that the partial theories defined in §2 are preserved under them.

Convention: 1. Throughout this section, δ will denote an ordinal with $cf(\delta) = \lambda > \omega$.
2. Unless otherwise said, all the chains mentioned in this section are well ordered chains (i.e. ordinals). We will deal with general chains in the next section.

Definition 4.1.
1) Let $a \subseteq \lambda$. We say that a is a *semi–club subset of* λ if for every $\alpha < \lambda$ with $cf(\alpha) > \omega$:
if $\alpha \in a$ then there is a club subset of α, C_α such that $C_\alpha \subseteq a$ and if $\alpha \notin a$ then there is a club subset of α, C_α such that $C_\alpha \cap a = \emptyset$.
Note that λ and $\emptyset$ are semi-clubs and that a club $J \subseteq \lambda$ is a semi–club provided that the first and the successor points of J are of cofinality $\leq \omega$.
2) Let $X, Y \subseteq \delta$, $J = \{\alpha_i : i < \lambda\}$ a club subset of δ, and let $a \subseteq \lambda$ be a semi–club of λ. We will define the *shuffling of X and Y with respect to a and J*, denoted by $[X, Y]_a^J$, as:

$$[X, Y]_a^J = \bigcup_{i \in a} \left(X \cap [\alpha_i, \alpha_{i+1}) \right) \cup \bigcup_{i \notin a} \left(Y \cap [\alpha_i, \alpha_{i+1}) \right)$$

3) When J is fixed (which is usually the case), we will denote the shuffling of X and Y with respect to a and J, by $[X, Y]_a$.
4) When $\bar{X}, \bar{Y} \subseteq \delta$ are of the same length, we define $[\bar{X}, \bar{Y}]_a$ naturally.
5) We can define shufflings naturally when $J \subset \delta$ is a club, and $a \subseteq otp(J)$ is a semi–club.

Notation 4.2.
1) Let $\bar{P}_0 \subseteq \delta$ and $J \subseteq \delta$ a club subset of δ witnessing $ATh(\delta, \bar{P}_0)$ as in lemma 2.10. For $n < \omega$, and $\beta < \gamma$ with $\gamma \in J, cf(\gamma) = \omega$, we denote $Th^n \left((\delta, \bar{P}_0)|_{[\beta,\gamma)} \right) = ATh^n \left(\beta, (\delta, \bar{P}_0) \right)$ by $s_{\bar{P}_0}^n (\beta)$ or just $s_0^n(\beta)$. (Of course, this does not depend on the choice of J and γ).
2) When n is fixed we denote this theory by $s_{\bar{P}_0}(\beta)$ or $s_0(\beta)$.
3) Remember: $g^n(\bar{P}_0)_s$ is the set $\{\beta < \delta : s_{\bar{P}_0}^n(\beta) = s\}$. (See def. 2.13.)
4) S_0^δ is the set $\{\gamma < \delta : cf(\gamma) = \omega\}$.

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Definition 4.3. Let $\bar{P}_0, \bar{P}_1 \subseteq \delta$ be of the same length and $J \subseteq \delta$ be a club. We will say that J is *n-suitable for* $\bar{P}_0, \bar{P}_1$ if the following hold:

a) J witnesses $ATh(\delta, \bar{P}_l)$ for $l = 0, 1$.

b) $J = \{\alpha_i : i < \lambda\}, \alpha_0 = 0$ and $cf(\alpha_{i+1}) = \omega$.

c) $J \cap g^n(\bar{P}_l)_s \cap S_0^\delta$ is either a stationary subset of δ or is empty.

When $n \geq 1$ and $WA^n(\delta, \bar{P}_0) = WA^n(\delta, \bar{P}_1)$ (see notation 2.15) we require also that:

d) If $\alpha_j \in J \, cf(\alpha_j) \leq \omega$ and $s_l(\alpha_j) = s$ then there are $k_1, k_2 < \omega$ such that $s_l(\alpha_{j+k_1}) = s$, and $s_{1-l}(\alpha_{j+k_2}) = s$.

Remark. It is easy to see that for every finite sequence

$$\langle \bar{P}_0, \bar{P}_1, \ldots, \bar{P}_n \rangle \subseteq \delta$$ with equal lengths, there is a club $J \subseteq \delta$ which is n-suitable for every pair of the $\bar{P}_i$'s.

We will show now that ATh is preserved under 'suitable' shufflings.

Theorem 4.5. *Suppose that* $\bar{P}_0, \bar{P}_1 \subseteq \delta$ *are of the same length,* $n \geq 1$ *and* $WA^n(\delta, \bar{P}_0) = WA^n(\delta, \bar{P}_1)$. *(In particular,* $ATh^n(0, (\delta, \bar{P}_0)) = ATh^n(0, (\delta, \bar{P}_1)) := t)$. *Let* $J \subset \delta$ *be n-suitable for* $\bar{P}_0, \bar{P}_1$ *of order type* λ *and* $a \subseteq \lambda$ *a semi–club. Then,* $ATh^n(0, (\delta, [\bar{P}_0, \bar{P}_1]_a^J)) = t$

Proof. Denote $\bar{X} := [\bar{P}_0, \bar{P}_1]_a^J$.

We will prove the following facts by induction on $0 < j < \lambda$:

(∗) For every $i < j < \lambda$ with $cf(j) \leq \omega$:

$$i \in a \Rightarrow Th^n([\alpha_i, \alpha_j), \bar{X}) = Th^n([\alpha_i, \alpha_j), \bar{P}_0) = s_0(\alpha_i).$$
$$i \notin a \Rightarrow Th^n([\alpha_i, \alpha_j), \bar{X}) = Th^n([\alpha_i, \alpha_j), \bar{P}_1) = s_1(\alpha_i).$$

(∗∗) For every $i < j < \lambda$ with $cf(j) > \omega$:

$$i, j \in a \Rightarrow Th^n([\alpha_i, \alpha_j), \bar{X}) = Th^n([\alpha_i, \alpha_j), \bar{P}_0).$$
$$i, j \notin a \Rightarrow Th^n([\alpha_i, \alpha_j), \bar{X}) = Th^n([\alpha_i, \alpha_j), \bar{P}_1).$$

In particular, by choosing $i = 0$ we get (remember $\alpha_0 = 0$), $Th^n([0, \alpha_j), \bar{X}) = t$ whenever $cf(\alpha_j) = \omega$.

$\underline{j = 1}$ (so $i = 0$): Let $l = 0$ if $i \in a$ and $l = 1$ if $i \notin a$. So $\bar{X} \cap [0, \alpha_j) = \bar{P}_l \cap [0, \alpha_j)$ and so $Th^n([0, \alpha_j), \bar{X}) = Th^n([0, \alpha_j), \bar{P}_l) = t$

$\underline{j = k + 1 < \omega}$: There are 4 cases. Let us check for example the case $i \in a$, $j - 1 = k \notin a$. By the composition theorem (2.7) and the induction hypothesis we have:

$$Th^n([\alpha_i, \alpha_j), \bar{X}) = Th^n([\alpha_i, \alpha_k), \bar{X}) + Th^n([\alpha_k, \alpha_{k+1}), \bar{X}) = s_0(\alpha_i) + Th^n([\alpha_k, \alpha_j), \bar{P}_1)$$
$$= s_0(\alpha_i) + s_1(\alpha_k).$$ So we have to prove $s_0(\alpha_i) + s_1(\alpha_k) = s_0(\alpha_i)$.

Since J is n-suitable there is an $m < \omega$ such that $s_0(\alpha_{i+m}) = s_1(\alpha_k)$ and so,

$$s_0(\alpha_i) = Th^n([\alpha_i, \alpha_{i+m+1}), \bar{P}_0) = Th^n([\alpha_i, \alpha_{i+m}), \bar{P}_0) + Th^n([\alpha_{i+m}, \alpha_{i+m+1}), \bar{P}_0) =$$
$$s_0(\alpha_i) + s_0(\alpha_{i+m}) = s_0(\alpha_i) + s_1(\alpha_k).$$

So $s_0(\alpha_i) + s_1(\alpha_k) = s_0(\alpha_i)$ as required.

The other cases are proven similarilly.

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$\underline{j = \omega}$: Suppose $i < \omega, i \in a$. We have to prove that $Th^n([\alpha_i, \alpha_\omega), \bar{X}) = s_0(i)$. Now either $(\lambda \setminus a) \cap \omega$ is unbounded or $a \cap \omega$ is unbounded and suppose the first case holds. Let $i < i_0 < i_1 \ldots$ be a strictly increasing sequence in $(\lambda \setminus a) \cap \omega$. By the induction hypothesis we have:

$Th^n([\alpha_i, \alpha_\omega), \bar{X}) = Th^n([\alpha_i, \alpha_{i_1}), \bar{X}) + \sum_{0<m<\omega} Th^n([\alpha_{i_m}, \alpha_{i_{m+1}}), \bar{X}) = s_0(\alpha_i) + \sum_{0<m<\omega} s_1(\alpha_{i_m})$.

Now choose (using the suitability of J), a strictly increasing sequence $\beta_{i_0} < \beta_{i_1} \ldots \subseteq \lambda$ such that $\beta_{i_m} = \alpha_{j_m+1}$ for some $j_m < \lambda$, $\beta_{i_1} > \alpha_i$ and such that for every $0 < m < \omega$, $s_0(\beta_{i_m}) = s_1(\alpha_{i_m})$. We will get:

$s_0(\alpha_i) = Th^n([\alpha_i, \alpha_\omega), \bar{P}_0) = Th^n([\alpha_i, \beta_{i_1}), \bar{P}_0) + \sum_{0<m<\omega} Th^n([\beta_{i_m}, \beta_{i_{m+1}}), \bar{P}_0) = s_0(\alpha_i) + \sum_{0<m<\omega} s_0(\beta_{i_m}) = s_0(\alpha_i) + \sum_{0<m<\omega} s_1(\alpha_{i_m})$.

So we have $s_0(\alpha_i) = Th^n([\alpha_i, \alpha_\omega), \bar{X})$ as required.

When only the other case holds (i.e. only $a \cap \omega$ is unbounded) the proof is easier.

When $i \notin a$ we prove similarly that $Th^n([\alpha_i, \alpha_\omega), \bar{X}) = s_1(\alpha_i)$

$\underline{cf(j) = \omega}$: Choose a sequence (in a or $\lambda \setminus a$), $i < i_0 < i_1 \ldots \sup_m i_m = j$, i_m non limit, and continue as in the case $j = \omega$.

$\underline{cf(j) > \omega}$: Now we have to check $(**)$.

So suppose $i, j \in a$ and we have to show $Th^n([\alpha_i, \alpha_j), \bar{X}) = Th^n([\alpha_i, \alpha_j), \bar{P}_0)$.

Let $\{\beta_\gamma : \gamma < cf(j)\} \subseteq a$ be a club subset of j with $\beta_0 = i$. By the induction hypothesis we have: $Th^n([\alpha_i, \alpha_j), \bar{X}) = \sum_{\gamma<cf(j)} Th^n([\beta_\gamma, \beta_{\gamma+1}), \bar{X}) = \sum_{\gamma<cf(j)} s_0(\beta_\gamma) = \sum_{\gamma<cf(j)} Th^n([\beta_\gamma, \beta_{\gamma+1}), \bar{P}_0) = Th^n([\alpha_i, \alpha_j), \bar{P}_0)$ as required.

The case $i, j \notin a$ is similar.

$\underline{j = k + 2}$: Easy.

$\underline{j = k + 1, cf(k) = \omega}$: Easy.

$\underline{j = k + 1, cf(k) > \omega}$: There are 8 cases. We will check for example the case: $0 < i$, $i \in a$, $k \notin a$.

Choose $\{i_\gamma : \gamma < cf(k)\} \subseteq \lambda \setminus a$ a club such that $i < i_0$ and $s_0(\alpha_i) = s_1(\alpha_{i_0})$.

Note that $(i \neq 0)$ $s_0(\alpha_i) + s_0(\alpha_i) = s_0(\alpha_i) = s_0(\alpha_i) + s_1(\alpha_{i_0})$.

So we get $Th^n([\alpha_i, \alpha_j), \bar{X}) =$
$Th^n([\alpha_i, \alpha_{i_0}), \bar{X}) + \sum_{\gamma<cf(k)} Th^n([\alpha_{i_\gamma}, \alpha_{i_{\gamma+1}}), \bar{X}) + Th^n([\alpha_k, \alpha_j), \bar{X}) =$
$s_0(\alpha_i) + \sum_{\gamma<cf(k)} s_1(\alpha_{i_\gamma}) + s_1(\alpha_k) = s_0(\alpha_i) + s_1(\alpha_{i_0}) + \sum_{0<\gamma<cf(k)} s_1(\alpha_{i_\gamma}) + s_1(\alpha_k) =$
$s_0(\alpha_i) + s_0(\alpha_i) + \sum_{0<\gamma<cf(k)} s_1(\alpha_{i_\gamma}) + s_1(\alpha_k) = s_0(\alpha_i) + \sum_{0<\gamma<cf(k)} s_1(\alpha_{i_\gamma}) + s_1(\alpha_k) =$
$s_1(\alpha_{i_0}) + \sum_{0<\gamma<cf(k)} s_1(\alpha_{i_\gamma}) + s_1(\alpha_k) = \sum_{\gamma<cf(k)} s_1(\alpha_{i_\gamma}) + s_1(\alpha_k) = Th^n([\alpha_i, \alpha_{i_0}), \bar{P}_1) =$
$Th^n([\alpha_i, \alpha_j), \bar{P}_0) = s_0(\alpha_i)$. So $Th^n([\alpha_i, \alpha_j), \bar{X}) = s_0(\alpha_i)$ as required.

Check the other cases: when $i = 0$ use the fact $s_0(0) = s_1(0)$.

So we have gone through all the cases and proven $(*)$ and $(**)$.

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Conclusion 4.6. Let $\bar{P}_0, \bar{P}_1 \subseteq \delta$, $WA^n(\delta, \bar{P}_0) = WA^n(\delta, \bar{P}_1)$, $a \subseteq \lambda$ a semi–club and $J \subseteq \delta$ an n-suitable club subset for $\bar{P}_0, \bar{P}_1$. Then:

1) J is an n-suitable club subset for the pair $\bar{P}_0$, $[\bar{P}_0, \bar{P}_1]_a^J$.

2) $[g^n(\bar{P}_0), g^n(\bar{P}_1)]_a^J \cap J = g^n([\bar{P}_0, \bar{P}_1]_a^J) \cap J$ (so $[g^n(\bar{P}_0), g^n(\bar{P}_1)]_a^J$ and $g^n([\bar{P}_0, \bar{P}_1]_a^J)$ have the same WTh).

Proof. Use $(*), (**)$ from the last theorem.

$\heartsuit$

Our next aim is to show that WTh (hence, by 2.15 and 4.6(2), also Th) is preserved under shufflings.

Definition 4.7. Let $a, \bar{P} \subseteq \lambda$. We define $a - WTh^n(\lambda, \bar{P})$ by induction on n:

for $n = 0$: $a - WTh^0(\lambda, \bar{P}) = \{t : th(\lambda, (\bar{P}, a)) \text{ is stationary in } \lambda\}$ (see def. 2.9)

for $n + 1$: $a - WTh^{n+1}(\lambda, \bar{P}) = \{\langle S_1^{\bar{P},a}(Q), S_2^{\bar{P},a}(Q), S_3^{\bar{P},a}(Q)\rangle : Q \subseteq \lambda\}$ Where:

$S_1^{\bar{P},a}(Q) = a - WTh^n(\lambda, \bar{P}, Q)$

$S_2^{\bar{P},a}(Q) = \{\langle t, s \rangle : \{\beta \in a : WTh^n(\lambda, \bar{P}, Q)|_\beta = t, \ th(\beta, \bar{P}, Q) = s\} \text{ is stationary in } \lambda\}$

$S_3^{\bar{P},a}(Q) = \{\langle t, s \rangle : \{\beta \in \lambda \setminus a : WTh^n(\lambda, \bar{P}, Q)|_\beta = t, \ th(\beta, \bar{P}, Q) = s\} \text{ is stationary in } \lambda\}$

Remark 4.8. 0) Remember that if $\bar{P} \subseteq \delta$ and $J \subseteq \delta$ is a club, then $WTh^n(\delta, \bar{P} \cap J) = WTh^n(\delta, \bar{P})$. Moreover, if $J \subseteq \delta$ club of order type λ and $h \colon J \to \lambda$ is the isomorphism between J and λ, then for every $\bar{P} \subseteq \delta$, $WTh^n(\delta, \bar{P}) = WTh^n(\lambda, h(\bar{P} \cap J))$.

1) $WTh^n(\lambda, \bar{P})$ tells us if certain sets are stationary. $a - WTh^n(\lambda, \bar{P})$ tells us if their intersection with λ and $\lambda \setminus a$ are stationary.

2) We could have defined $a - WTh^n(\lambda, \bar{P})$ by $WTh^n(\lambda, \bar{P}, a)$, which gives us the same information. We prefared the original definition because it seems to be easier to see the preservation under shufflings using it.

Fact 4.9. For any $a \subseteq \lambda$, $WTh^n(\lambda, \bar{P})$ is effectively computable from $a - WTh^n(\lambda, \bar{P})$, so if $\bar{P}, \bar{Q} \subseteq \lambda$ and $a - WTh^n(\lambda, \bar{P}) = a - WTh^n(\lambda, \bar{Q})$ then: $WTh^n(\lambda, \bar{P}) = WTh^n(\lambda, \bar{Q})$.

Proof. Trivial.

$\heartsuit$

Theorem 4.10. Suppose $a, J, \bar{P}_0, \bar{P}_1 \subseteq \lambda$, a semi–club, J club, $\bar{X} := [\bar{P}_0, \bar{P}_1]_a^J$ and $a - WTh^n(\lambda, \bar{P}_0) = a - WTh^n(\lambda, \bar{P}_1)$.

Then: $a - WTh^n(\lambda, \bar{P}_0) = a - WTh^n(\lambda, \bar{X})$. (It follows $WTh^n(\lambda, \bar{X}) = WTh^n(\lambda, \bar{P}_0) = WTh^n(\lambda, \bar{P}_1)$).

Proof. by induction on n (for every $a', J', \bar{X}', \bar{Y}'$):

$n = 0$: Check.

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$n + 1$: Suppose $Q_0 \subseteq \lambda$ and $\{\langle S_1^{\bar{P}_0,a}(Q_0), S_2^{\bar{P}_0,a}(Q_0), S_3^{\bar{P}_0,a}(Q_0)\rangle\} \in a - WTh^{n+1}(\lambda, \bar{P}_0)$. Choose (using the equality of the theories) $Q_1 \subseteq \lambda$ such that $\{\langle S_1^{\bar{P}_1,a}(Q_1), S_2^{\bar{P}_1,a}(Q_1), S_3^{\bar{P}_1,a}(Q_1)\rangle\} \in a - WTh^{n+1}(\lambda, \bar{P}_1)$, and such that the two triples are equal. Define $Q_X := [\bar{Q}_0, \bar{Q}_1]_a^J$.

By the induction hypothesis $a - WTh^n(\lambda, \bar{P}_0, Q_0) = a - WTh^n(\lambda, \bar{X}, Q_X)$ so $S_1^{\bar{P}_0,a}(Q_0) = S_1^{\bar{X},a}(Q_X)$. Now suppose $\langle t, s\rangle \in S_2^{\bar{P}_0,a}(Q_0), t \neq \emptyset$.

Let $B_{t,s}^{\bar{P}_0} := \{\beta \in a : WTh^n(\lambda, \bar{P}_0, Q_0)|_\beta = t, \ th(\beta, \bar{P}_0, Q_0) = s\}$ and this is a stationary subset of λ. But for each such β, since $t \neq \emptyset \Rightarrow cf(\beta) > \omega$, a contains a club $C_\beta \subseteq \beta$ and, remembering a previous remark, we can restrict ourselves to $(\bar{P}_0, Q_0) \cap C_\beta$.

Now suppose: $a = \langle i_\gamma : \gamma < \lambda\rangle$ (note that a has to be stationary otherwise S_2 is empty) and $J = \langle \alpha_{i_\gamma} : \gamma < \lambda\rangle$. Look at the club $J' = \langle \alpha_{i_\gamma} : \alpha_{i_\gamma} = i_\gamma\rangle$ and let $J'' :=$ the accumulation points of J'. Now $B_{t,s}^{\bar{P}_0} \cap J''$ is also stationary, and choose β in this set, and a club $C_\beta \subseteq a \cap J'$. By the choice of C_β we get: $(\bar{P}_0, Q_0) \cap C_\beta = (\bar{X}, Q_X) \cap C_\beta$, and this implies: $WTh^n(\lambda, \bar{X}, Q_X)|_\beta = t$, and $th(\beta, \bar{X}, Q_X) = s$. So, (since β was random) $B_{t,s}^{\bar{X}}$ is also stationary.

The case $t = \emptyset$ is left to the reader. We deal with S_3 symmetrically, replacing a with $\lambda \setminus a$. So we have proven that $a - WTh^{n+1}(\lambda, \bar{P}_0) \subseteq a - WTh^{n+1}(\lambda, \bar{X})$

Now, for the inverse inclusion suppose $Q_X \subseteq \lambda$ and: $\langle S_1^{\bar{X},a}(Q_X), S_2^{\bar{X},a}(Q_X), S_3^{\bar{X},a}(Q_X)\rangle \in a - WTh^{n+1}(\lambda, \bar{X})$. Choose R_0 such that $R_0 \cap a = Q_X \cap a$ and R_1 such that $R_1 \cap (\lambda \setminus a) = Q_X \cap (\lambda \setminus a)$. Now choose T_0 such that $S_3^{\bar{P}_0,a}(T_0) = S_3^{\bar{P}_1,a}(R_1)$ and T_1 such that $S_2^{\bar{P}_0,a}(R_0) = S_2^{\bar{P}_1,a}(T_1)$. Let Q_0 be equal to R_0 on a and to T_0 on $\lambda \setminus a$. Let Q_1 be equal to T_1 on a and to R_1 on $\lambda \setminus a$. It can be easily checked that: $\langle S_1^{\bar{P}_0,a}(Q_0), S_2^{\bar{P}_0,a}(Q_0), S_3^{\bar{P}_0,a}(Q_0)\rangle = \langle S_1^{\bar{P}_1,a}(Q_1), S_2^{\bar{P}_1,a}(Q_1), S_3^{\bar{P}_1,a}(Q_1)\rangle$. But $Q_X = [Q_0, Q_1]_a^J$, hence this triples are, by the same arguments as in first part of the proof, equal to $\langle S_1^{\bar{X},a}(Q_X), S_2^{\bar{X},a}(Q_X), S_3^{\bar{X},a}(Q_X)\rangle$ This proves the inverse inclusion: $a - WTh^{n+1}(\lambda, \bar{P}_0) \supseteq a - WTh^{n+1}(\lambda, \bar{X})$, hence the equality $a - WTh^{n+1}(\lambda, \bar{P}_0) = a - WTh^{n+1}(\lambda, \bar{P}_1) = a - WTh^{n+1}(\lambda, \bar{X})$

$\heartsuit$

Notation 4.11. Suppose $\bar{P}, J \subseteq \delta$, J club of order type λ and $a \subseteq \lambda$ a semi–club. Let $t_1 := ATh^m(0, (\delta, \bar{P}))$ and (keeping in mind remark 4.8.(0)), let $h: J \to \lambda$ be the isomorphism between J and λ and let $t_2 := a - WTh^m(\lambda, h(g^m(\delta, \bar{P}) \cap J))$ We denote $\langle t_1, t_2\rangle$ by $a - WA^m(\delta, \bar{P})$ (assuming J is fixed).

Collecting the last results we can conclude:

Theorem 4.12. Let $J, \bar{P}_0, \bar{P}_1 \subseteq \delta$, $lg(\bar{P}_0) = lg(\bar{P}_1)$, J an n-suitable club for P_0, P_1 of order type λ and $a \subseteq \lambda$ a semi–club and set $\bar{X} := [\bar{P}_0, \bar{P}_1]_a^J$. Then: $a - WA^m(\delta, \bar{P}_0) = a - WA^m(\delta, \bar{P}_1) \Rightarrow a - WA^m(\delta, \bar{P}_0) = a - WA^m(\delta, \bar{X})$, and in particular, if $m = m(n)$ then: $Th^n(\delta, \bar{P}_0) = Th^n(\delta, \bar{P}_1) = Th^n(\delta, \bar{X})$.

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Proof. The first statement follows directly from 4.5 and 4.10.

For the second, by the definition of $a - WA$, and by 4.8(0), 4.9, equality of $a - WA^{m(n)}$ implies equality of $WA^{m(n)}$ from definition 2.16. But by 2.15 this implies the equality of Th^n.

$\heartsuit$

5. Formal shufflings

In the previous section we showed how to shuffle subsets of well ordered chains and preserve their theories. Here we present the notion of formal shufflings in order to overcome two difficulties:

1. It could happen that the interpreting chain is of cofinality λ but of a larger cardinality. Still, we want to shuffle objects of cardinality $\leq \lambda$. The reason for that is that the contradiction we want to reach depends on shufflings of elements along a generic semi–club added by the forcing, and a semi–club of cardinality λ will be generic only with respect to objects of cardinality $\leq \lambda$. So we want to show now that we can shuffle theories, rather than subsets of our given chain.

2. We want to generalize the previous results, which were proven for well ordered chains, to the case of a general chain.

Discussion. Suppose we are given a chain C and a finite sequence of subsets $\bar{A} \subseteq C$ and we want to compute $Th^n(C, \bar{A})$. As before we can choose an n-suitable club $J = \langle \alpha_i : i < \lambda \rangle$ witnessing $ATh^n(C, \bar{A})$ and letting $s_i := Th^n(C, \bar{A})|_{[\alpha_i, \alpha_{i+1})}$ we have: $Th^n(C, \bar{A}) = \sum_{i<\lambda} s_i$. Theorem 2.15 says that (for a large enough $m = m(n)$) $WA^m(C, \bar{A})$ which is s_0 and $WTh^m(\lambda, g^n(C, \bar{A}))$, determines $Th^n(C, \bar{A})$. ($g^n(C, \bar{A})$ is a sequence of subsets of λ of the form $g_s = \{i : s_i = s\}$).

Moreover, since we have only finitely many possibilities for $WA^m(C, \bar{A})$, we can decide whether $\sum_{i<\lambda} s_i = t$ inside $H(\lambda^+) := \{x : x$ is hereditarilly of cardinality smaller than $\lambda^+\}$ even if the s_i's are theories of objects of cardinality greater than λ. This motivates our next definitions:

Definition 5.1. fix an $l < \omega$

1) $S = \langle s_i : i < \lambda \rangle$ is an *n-formally possible set of theories* if each s_i is a formally possible member of $\{Th^n(D, \bar{B}) : D$ is a chain, $\bar{B} \subseteq D$, $lg(\bar{B}) = l\}$, and for every $i < j < \lambda$ with $cf(j) \leq \omega$ we have $s_i = \sum_{i \leq k < j} s_k$.

2) The *n*-formally possible set of theories S is *realized* in a model N if there are $J, C, \bar{A}$ as usual in N, and $s_i := Th^n(C, \bar{A})|_{[\alpha_i, \alpha_{i+1})}$.

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3) Let $S = \langle s_i : i < \lambda \rangle$, $T = \langle t_i : i < \lambda \rangle$ be n-formally possible sets of theories, $a \subseteq \lambda$ a semi–club. We define the *formal shuffling of S and T with respect to a* as: $[S,T]_a := \langle u_i : i < \lambda \rangle$ where

$$u_i = \begin{cases} s_i & \text{if } i \in a \\ t_i & \text{if } i \notin a \end{cases}$$

Fact 5.2. *1. Let $\bar{A}, \bar{B} \subseteq C$ of length l, $J = \langle \alpha_i : i < \lambda \rangle$ an n-suitable club and $a \subseteq \lambda$ a semi–club. Let $s_i := Th^n(C, \bar{A})|_{[\alpha_i, \alpha_{i+1})}$, $S = \langle s_i : i < \lambda \rangle$, $t_i := Th^n(C, \bar{B})|_{[\alpha_i, \alpha_{i+1})}$, $T = \langle t_i : i < \lambda \rangle$. Then: S and T are n-formally possible sets of theories, and $[S,T]_a = \langle Th^n(C, [\bar{A}, \bar{B}]_a^J)|_{[\alpha_i, \alpha_{i+1})} : i < \lambda \rangle$.*
2. If in addition $WA^{m(n)}(C, \bar{A}) = WA^{m(n)}(C, \bar{B})$, then $[S,T]_a$ is an n-formally possible set of theories.
3. If in addition $a - WA^{m(n)}(C, \bar{A}) = a - WA^{m(n)}(C, \bar{B})$, then $\sum_{i<\lambda} s_i = \sum_{i<\lambda} t_i = \sum_{i<\lambda} u_i$.

Proof. Part 1 is obvious, part 2 follows from theorem 4.5 and part 3 from 4.12.

♡

We can define in a natural way the partial theories WTh^m and $a - WTh^m$.

Definition 5.3. For $S = \langle s_i : i < \lambda \rangle$ an n-formally possible set of theories, denote $g^n(S)_s := \langle i < \lambda : s_i = s \rangle$ and $g^n(S) := \langle g^n(S)_s : s$ is a formally possible $n-$theory$\rangle$. We define $WTh^m(S)$ to be $WTh^m(\lambda, g^n(S))$, and for $a \subseteq \lambda$ a semi–club, $a - WTh^m(S)$ is $a - WTh^m(\lambda, g^n(S))$.
Finally we define $a - WA^m(S)$ to be the pair $\langle s_0, a - WTh^m(S) \rangle$.

Theorem 5.4. *If $C, \bar{A}, J, S$ are as usual then we can compute $Th^n(C, \bar{A})$ from $WA^{m(n)}(S)$, moreover, the computation can be done in $H(\lambda^+)$ even if $|C| > \lambda$.*

Proof. The first claim is exactly 2.15. The second follows from the fact that S and $WA^{m(n)}(S)$ are elements of $H(\lambda^+)$ and so is the correspondence between the (finite) set of formally possible WA^m's and the formally possible Th^n's which are determined by them.

♡

6. The forcing

To contradict the existence of an interpretation we will need generic semi–clubs in every regular cardinal. To obtain that we use a simple class forcing.

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Definition 6.1. Let $\lambda > \aleph_0$ be a regular cardinal

1) $SC_\lambda := \{f : \quad f : \alpha \to \{0,1\}, \ \alpha < \lambda, \ cf(\alpha) \leq \omega \}$ where each f, considered to be a subset of α (or λ), is a semi–club. The order is inclusion. (So SC_λ adds a generic semi–club to λ).

2) Q_λ will be an iteration of the forcing SC_λ with length λ^+ and with support $\leq \lambda$.

3) $P := \langle P_\mu, \underset{\sim}{Q}_\mu \colon \mu$ a cardinal $> \aleph_0 \rangle$ where $\underset{\sim}{Q}_\mu$ is forced to be Q_μ if μ is regular, otherwise it is $\emptyset$. The support of P is sets: each condition in P is a function from the class of cardinals to names of conditions where the names are non-trivial only for a set of cardinals.

4) $P_{<\lambda}, P_{>\lambda}, P_{\leq\lambda}$ are defined naturally. For example $P_{<\lambda}$ is $\langle P_\mu, \underset{\sim}{Q}_\mu \colon \aleph_0 < \mu < \lambda \rangle$.

Remark 6.2. Note that (if G.C.H holds) Q_λ and $P_{\geq\lambda}$ do not add subsets of λ with cardinality $< \lambda$. Hence, P does not collapse cardinals and does not change cofinalities, so V and V^P have the same regular cardinals. Moreover, for a regular $\lambda > \aleph_0$ we can split the forcing into 3 parts, $P = P_0 * P_1 * P_2$ where P_0 is $P_{<\lambda}$, P_1 is a P_0-name of the forcing Q_λ and P_2 is a $P_0 * P_1$-name of the forcing $P_{>\lambda}$ such that V^P and $V^{P_0*P_1}$ have the same $H(\lambda^+)$.

In the next section, when we restrict ourselves to $H(\lambda^+)$ it will suffice to look only in $V^{P_0*P_1}$.

7. The contradiction

Collecting the results from the previous sections we will reach a contradiction from the assumption that there is, in V^P, an interpretation of T in the monadic theory of a chain C. For the moment we will assume that the minimal major initial segment D is regular (i.e. isomorphic to a regular cardinal), later we will dispose of this by using formal shufflings. So we may assume the following:

Assumptions.

1. $C \in V^P$ interprets T by $\langle d, U_C(\bar{X}, \bar{V}), E_C(\bar{X}, \bar{Y}, \bar{V}), P(\bar{X}, \bar{Y}, \bar{V}) \rangle$.

2. $D = \lambda$ is a minimal major initial segment of C, $cf(\lambda) = \lambda > \omega$.

3. $\bar{R} \subseteq (C - D)$ and $S := \{\bar{A} \subseteq C : \bar{A} \cap (C - D) = \bar{R}\}$ contains an infinite number of nonequivalent representatives of E_C-equivalence classes.

4. There are formulas $U(\bar{X}, \bar{Z}), E(\bar{X}, \bar{Y}, \bar{Z}), Atom(\bar{X}, \bar{Z}), Set(\bar{Y}, \bar{Z})$ and $Code(\bar{X}, \bar{Y}, \bar{Z})$ in the language of the monadic theory of order such that for every $k < \omega$ there is a sequence $\bar{W} \subseteq D$ such that

$$I = \langle d, U(\bar{X}, \bar{W}), E(\bar{X}, \bar{Y}, \bar{W}), Atom(\bar{X}, \bar{W}), Set(\bar{Y}, \bar{W}), Code(\bar{X}, \bar{Y}, \bar{W}) \rangle$$

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is an interpretation of T_k in D.

5. There is an $n < \omega$ such that for every k and $\bar{W}$ as above, $Th^n(D, \bar{U}_{i_1}, \bar{U}_{i_2}, \bar{U}_{i_3}, \bar{W})$ determines the truth value of all the interpreting formulas when we replace the variables with elements from $\{\bar{U}_{i_1}, \bar{U}_{i_2}, \bar{U}_{i_3}\}$.

6. m is such that for every $\bar{U}_{i_1}, \bar{U}_{i_2}, \bar{U}_{i_3}$, from $WA^m(D, \bar{U}_{i_1}, \bar{U}_{i_2}, \bar{U}_{i_3}, \bar{W})$ we can compute $Th^{n+d}(D, \bar{U}_{i_1}, \bar{U}_{i_2}, \bar{U}_{i_3}, \bar{W})$ and in particular, the truth value of the interpreting formulas.

7. Let $N_1 := \left| \{ Th^n(C, \bar{X}, \bar{Y}, \bar{Z}) : C \text{ is a chain}, \bar{X}, \bar{Y}, \bar{Z} \subseteq C \} \right|$. Then (by proposition 3.3 and theorem 3.11), for every proper initial segment $D' \subset D$ there are less than N_1 E_C-nonequivalent (hence E-nonequivalent) elements, coinciding outside D'.

Definition 7.1. The *vicinity* $[\bar{X}]$ of an element $\bar{X}$ is the collection $\{\bar{Y} : \text{some element } \bar{Z} \sim \bar{Y} \text{ coincides with } \bar{X} \text{ outside some proper (hence minor) initial segment of } D \}$.

Lemma 7.2. *Every vicinity $[\bar{X}]$ is the union of at most N_1 different equivalence classes.*

Proof. See [GuSh] lemma 9.1.

$\heartsuit$

Next we use Ramsey theorem for defining the following functions.

Notation 7.3.

1. Given $k < \omega$, let $t(k)$ be such that for every sequence $\bar{W} \subseteq D$ of a prefixed length and $a \subseteq \lambda$ and for every sequences of elements $\langle \bar{B}_i : i < t(k) \rangle$ and $\langle \bar{B}_s : s \subseteq t(k) \rangle$ there are subsequences $s, s' \subseteq t(k)$ with $|s'| \geq k$ and $s' \subseteq s$ such that $a - WA^m(D, \bar{B}_i, \bar{B}_j, \bar{B}_s, \bar{W})$ is constant for every $i < j \in s'$.

2. Given $k < \omega$, let $h(k)$ be such that for every coloring of $\{(i, j, l) : i < j < l < h(k)\}$ into 32 colors, there is a subset I of $\{0, 1, \ldots, h(k) - 1\}$ such that $|I| > k$ and all the triplets $\{(i, j, l) : i < j < l, \ i, j, l \in I\}$ have the same color.

We are ready now to prove the main theorem:

Theorem 7.4. *Assuming the above assumptions we reach a contradiction*

Proof. The proof will be splitted into several steps.

STEP 1: Let $K_1 := h(t(3N_1))$ and $K := h(t(2K_1 + 2N_1))$. Let $\bar{R} \subseteq (C - D)$ be such that $S := \{\bar{A} \subseteq C : \bar{A} \cap (C - D) = \bar{R}\}$ contains an infinite number of nonequivalent representatives. Choose sequences of nonequivalent elements from S, $B := \langle \bar{U}_i : i < K \rangle$, and $B_1 := \langle \bar{V}_s : s \subseteq \{0, 1, \ldots, K - 1\} \rangle$ and an appropriate $\bar{W} \subseteq D$ and interpret T_K on D such that B is the family of "atoms" of the interpretation and B_1 the family of "sets" of the interpretation.

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STEP 2: Choose $J := \{\alpha_j : j < \lambda\} \subseteq \lambda$ an $(n+d)$-suitable club witnessing ATh^{n+d} for every combination you can think of from the U_i's, the $\bar{V}_s$'s and $\bar{W}$.

Now, everything mentioned happens in $H(\lambda^+)^{V^P}$ and, using a previous remark and notations, it is the same thing as $H(\lambda^+)^{V^{P_0 * P_1}}$. P_1 is an iteration of length λ^+ and it follows that all the mentioned subsets of λ are added to $H(\lambda^+)^{V^{P_0 * P_1}}$ after a proper initial segment of the forcing which we denote by $P_0 * (P_1|_\beta)$. So there is a semi–club $a \subseteq \lambda$ in $H(\lambda^+)^{V^{P_0 * P_1}}$ which is added after all the mentioned sets, say at stage β of P_1.

STEP 3: We will begin now to shuffle the elements with respect to a and J. Let, for $i < j < K$, $k(i,j) := \mathrm{Min}\{k : [\bar{U}_i, \bar{U}_j]_a^J \sim \bar{U}_k, \text{ or } k = K\}$. By the definitions of h and K there is a subset $s \subseteq \{0, 1, \ldots, K-1\}$ of cardinality at least $K_2 := t(2K_1 + 2N_1)$ such that for every $\bar{U}_i, \bar{U}_j, \bar{U}_l$ with $i < j < l$, $i, j, l \in s$ the following five statements have the same truth value:

$k(j,k) = i$, $k(i,k) = j$, $k(i,j) = i$, $k(i,j) = j$, $k(i,j) = k$. Moreover, by [GuSh] lemma 10.2, if there is a pair $i < j$ in s such that $k(i,j) \in s$ then, either for every pair $i < j$ in s, $k(i,j) = i$ or for every $i < j$ in s, $k(i,j) = j$.

STEP 4: Let $\bar{V}_s$ be the set that codes $\langle \bar{U}_i : i \in s \rangle$. By the definitions of t and K_2, there is a set $s' \subseteq s$ with at least $K_3 := 2K_1 + 2N_1$ elements and a sequence $\langle \bar{U}_i : i \in s' \rangle$ such that for every $r < l$ in s', $a - WA^m(D, \bar{U}_r, \bar{U}_l, \bar{V}_s, \bar{W})$ is constant.

It follows that for every $r < l$ in s', $a - WA^m(D, \bar{U}_r, \bar{V}_s, \bar{W}) = a - WA^m(D, \bar{U}_l, \bar{V}_s, \bar{W})$, and by the preservation theorem 4.12 they are equal to $a - WA^m(D, [\bar{U}_r, \bar{U}_l]_a^J, \bar{V}_s, \bar{W})$. But $\bar{V}_s$ codes s so $D \models Code(\bar{U}_r, \bar{V}_s, \bar{W})$, and since we can decide from $a - WA^m$ if $Code$ holds, the equality of the theories implies that $D \models Code([\bar{U}_r, \bar{U}_l]_a^J, \bar{V}_s, \bar{W})$. But by the definition of $Code$ there is $k \in s$ such that $[\bar{U}_r, \bar{U}_l]_a^J \sim U_k$. So there are r, l in s with $k(r,l) \in s$ and by step 3 we can conclude that, without loss of generality, for every $i < j$ in s, $[\bar{U}_i, \bar{U}_j]_a^J \sim \bar{U}_i$.

STEP 5: Note that if a is a semi–club then $\lambda \setminus a$ is also a semi–club. We will use the fact that a is generic with respect to the other sets for finding a pair $i < j \in s'$ such that $[\bar{U}_i, \bar{U}_j]_{\lambda \setminus a}^J \sim \bar{U}_i$ holds as well. Let $p \in P_0 * P_1$ be a condition that forces the value of all the theories $a - WA^m(D, \bar{U}_r, \bar{U}_l, \bar{V}_s, \bar{W})$ for $r < l \in s'$. The condition p is a pair (q, r) where $q \in P_0$ and r is a P_0-name of a function from λ^+ to conditions in the forcing SC_λ. $r(\beta)$ is forced by p to be an initial segment of a of height $\gamma < \lambda$ and w.l.o.g. we can assume that $\gamma = \alpha_{j+1} \in J$. (So $cf(\gamma) = \omega$). As $\gamma < \lambda = D$, γ is a minor segment. Remember that $|s'| \geq K_3 = 2K_1 + 2N_1$ and define $s'' \subseteq s'$ to be $\{i \in s' : |\{j \in s' : j < i\}| > N_1$, and $|\{j \in s' : j > i\}| > N_1\}$. So $|s''| > K_1$. Denote by $\bar{A}^\frown \bar{B}$ the element $(\bar{A} \cap \gamma) \cup (\bar{B} \cap (D - \gamma))$. We claim that for every i, j, k in s'', $\bar{U}_k \sim [\bar{U}_i, \bar{U}_j]_a^\frown \bar{U}_k$.

To see that note that by the definition of s' and the preservation theorem for ATh, p forces: "$Th^{n+d}(D, [\bar{U}_i, \bar{U}_j]_a^\frown \bar{U}_k, \bar{V}_S, \bar{W}) =$

$Th^{n+d}(D, [\bar{U}_i, \bar{U}_j]_a, \bar{V}_S, \bar{W})|_\gamma + Th^{n+d}(D, \bar{U}_k, \bar{V}_S, \bar{W})|_{[\gamma, \lambda)} =$

(by $\gamma \in J$ and the equality of the a-WA's and the preservation theorem)

$Th^{n+d}(D, \bar{U}_i, \bar{V}_S, \bar{W})|_\gamma + Th^{n+d}(D, \bar{U}_k, \bar{V}_S, \bar{W})|_{[\gamma, \lambda)} =$

(by $\gamma \in J$ and the equality of the ATh's)

$Th^{n+d}(D, \bar{U}_i, \bar{V}_S, \bar{W})$".

Hence, since $\bar{V}_s$ codes s, $[\bar{U}_i, \bar{U}_j]_a {}^\frown \bar{U}_k \sim \bar{U}_l$ for some $l \in s$. If $l = k$ we are done so assume w.l.o.g that $l < k$. Now $U_l \in [U_k]$ and we will show that for every $m < k$, in s, $U_m \in [U_k]$. Contradiction follows from the choice of s'' and lemma 7.2 (1).

Now $Th^{n+d}(D, \bar{U}_m, [\bar{U}_i, \bar{U}_j]_a {}^\frown \bar{U}_k, \bar{W}) =$

$Th^{n+d}(D, \bar{U}_m, [\bar{U}_i, \bar{U}_j]_a, \bar{W})|_\gamma + Th^{n+d}(D, \bar{U}_m, \bar{U}_k, \bar{W})|_{[\gamma, \lambda)}$.

But $ATh^{n+d}(D, \bar{U}_m, \bar{W}) = ATh^{n+d}(D, \bar{U}_l, \bar{W})$. So there is $\bar{Y} \subseteq D$ such that

$Th^n(D, \bar{U}_m, \bar{Y}, \bar{W})|_\gamma = Th^n(D, \bar{U}_l, [\bar{U}_i, \bar{U}_j]_a, \bar{W})|_\gamma$.

We get $Th^n(D, U_m, \bar{Y} {}^\frown \bar{U}_k, \bar{W}) = Th^n(D, U_l, [\bar{U}_i, \bar{U}_j]_a {}^\frown \bar{U}_k, \bar{W})$, and since $[\bar{U}_i, \bar{U}_j]_a {}^\frown \bar{U}_k \sim \bar{U}_l$ the equality of the theories implies: $\bar{Y} {}^\frown \bar{U}_k \sim U_m$, so $\bar{U}_m \in [\bar{U}_k]$.

But by 7.2 (1), $|[U_k]| \leq N_1$ and by the choice of s'' there are more than N_1 nonequivalent U_m's with the same property and this is a contradiction.

So we have proven that it is possible to replace an initial segment of an element with a shuffling of two other elements without changing it's equivalence class. (Actually there are $|s''|$ elements like that).

STEP 6: We are ready to prove that for every $i < j$ in s'', $[\bar{U}_i, \bar{U}_j]_a \sim [\bar{U}_i, \bar{U}_j]_{\lambda \setminus a}$.

By step 4 $p \Vdash [\bar{U}_i, \bar{U}_j]_a \sim \bar{U}_i$ (because it forces equality of theories for a large number of elements). Remember that p 'knows' only an initial segment of a, namely only $a \cap (j + 1)$ where $\gamma = \alpha_{j+1}$. Since our forcing is homogeneous $b := (a \cap [0, j+1)) \cup ((\lambda \setminus a) \cap [j+1, \lambda))$ is also generic for all the mentioned sets and parameters, and everything p forces for a it forces for b. So $p \Vdash$ "$[\bar{U}_i, \bar{U}_j]_b \sim \bar{U}_i''$.

Note that by the preservation theorem $Th^n(D, [\bar{U}_i, \bar{U}_j]_{\lambda \setminus a}, \bar{W})|_\gamma = Th^n(D, [\bar{U}_j, \bar{U}_i]_a, \bar{W})|_\gamma = Th^n(D, [\bar{U}_i, \bar{U}_j]_a, \bar{W})|_\gamma = Th^n(D, \bar{U}_i, \bar{W})|_\gamma = Th^n(D, \bar{U}_j, \bar{W})|_\gamma$.

It follows that $Th^n(D, [\bar{U}_i, \bar{U}_j]_a, [\bar{U}_i, \bar{U}_j]_a, \bar{W})|_\gamma = Th^n(D, [\bar{U}_i, \bar{U}_j]_{\lambda \setminus a}, [\bar{U}_i, \bar{U}_j]_{\lambda \setminus a}, \bar{W})|_\gamma$.

By step 5 (Where we used only the fact that $i, j \in s''$), $[\bar{U}_i, \bar{U}_j]_{\lambda \setminus a} {}^\frown \bar{U}_i \sim \bar{U}_i \sim [\bar{U}_i, \bar{U}_j]_b$. But

$Th^n(D, [\bar{U}_i, \bar{U}_j]_{\lambda \setminus a} {}^\frown \bar{U}_i, [\bar{U}_i, \bar{U}_j]_{\lambda \setminus a}, \bar{W}) =$

$Th^n(D, [\bar{U}_i, \bar{U}_j]_{\lambda \setminus a}, [\bar{U}_i, \bar{U}_j]_{\lambda \setminus a}, \bar{W})|_\gamma + Th^n(D, \bar{U}_i, [\bar{U}_i, \bar{U}_j]_{\lambda \setminus a}, \bar{W})|_{[\gamma, \lambda)} =$

$Th^n(D, [\bar{U}_i, \bar{U}_j]_a, [\bar{U}_i, \bar{U}_j)]_a, \bar{W})|_\gamma + Th^n(D, \bar{U}_i, [\bar{U}_i, \bar{U}_j]_{\lambda \setminus a}, \bar{W})|_{[\gamma, \lambda)} =$

$Th^n(D, [\bar{U}_i, \bar{U}_j]_a {}^\frown \bar{U}_i, [\bar{U}_i, \bar{U}_j]_b, \bar{W})$.

But $[\bar{U}_i, \bar{U}_j]_a {}^\frown \bar{U}_i \sim \bar{U}_i \sim [\bar{U}_i, \bar{U}_j]_b$. So it follows by the equality of the theories that $[\bar{U}_i, \bar{U}_j]_{\lambda \setminus a} \sim [\bar{U}_i, \bar{U}_j]_a \sim \bar{U}_i$ as required.

STEP 7: Rename a subsequence of $\langle \bar{U}_i : i \in s'' \rangle$ by $\langle \bar{A}_i : i < 2K_1 \rangle$ such that for every $i < j < 2K_1, r < l < 2K_1$ we have:

(i) $a - WA^m(D, \bar{A}_i, \bar{A}_j, \bar{V}_s, \bar{W}) = a - WA^m(D, \bar{A}_r, \bar{A}_l, \bar{V}_s, \bar{W})$.

(ii) $[\bar{A}_i, \bar{A}_j]_a \sim [\bar{A}_i, \bar{A}_j]_{\lambda \backslash a} \sim \bar{A}_i$.

For $i < K_1$ denote by $\bar{B}_i$ the element that codes $\bar{A}_i, \bar{A}_{2K_1-i-1}$ and look at the sequence $\langle \bar{B}_i : i < K_1 \rangle$. K_1 is large enough so that repeating steps 1,2 and 3 we are left with $i < j < K_1$ such that :

(iii) $a - WA^m(D, \bar{A}_i, \bar{A}_{2K_1-i-1}, \bar{B}_i, \bar{W}) = a - WA^m(D, \bar{A}_j, \bar{A}_{2K_1-j-1}, \bar{B}_j, \bar{W})$.

(iv) $[\bar{B}_i, \bar{B}_j]_a \sim \bar{B}_i$ or $[\bar{B}_i, \bar{B}_j]_a \sim \bar{B}_j$.

Now let's shuffle with respect to a and J using clause (iii):

$Th^n(D, \bar{A}_i, \bar{A}_{2K_1-i-1}, \bar{B}_i, \bar{W}) = Th^n(D, [\bar{A}_i, \bar{A}_j]_a, [\bar{A}_{2K_1-i-1}, \bar{A}_{2K_1-j-1}]_a, [\bar{B}_i, \bar{B}_j]_a, \bar{W}) = Th^n(D, [\bar{A}_i, \bar{A}_j]_a, [\bar{A}_{2K_1-j-1}, \bar{A}_{2K_1-i-1}]_{\lambda \backslash a}, [\bar{B}_i, \bar{B}_j]_a, \bar{W})$.

But $[\bar{A}_i, \bar{A}_j]_a \sim \bar{A}_i$, and by step 6, $[\bar{A}_{2K_1-j-1}, \bar{A}_{2K_1-i-1}]_{\lambda \backslash a} \sim \bar{A}_{2K_1-j-1}$ and by clause (iv) $[\bar{B}_i, \bar{B}_j]_a \sim \bar{B}_i$ or $\bar{B}_j$.

So we have, as implied by the equality of Th^n either

$\models Code(\bar{A}_i, \bar{B}_i, \bar{W}) \& Code(\bar{A}_{2K_1-j-1}, \bar{B}_i, \bar{W})$

or

$\models Code(\bar{A}_i, \bar{B}_j, \bar{W}) \& Code(\bar{A}_{2K_1-j-1}, \bar{B}_j, \bar{W})$

and both cases are impossible!

We have reached a contradiction assuming, in V^P, that a well ordered chain C interprets T with a minimal major initial segment D which is a regular cardinal.

♡

We still have to prove that there is no interpretation in the case D is not a regular cardinal. For that we will use formal shufflings as in section 5.

Lemma 7.5. *The assumption "D is a regular cardinal" is not necessary.*

Proof. Assume first that $D = \delta > cf(\delta) = \lambda > \omega$. The main point is to find 2 elements $\bar{A}, \bar{B}$ and a semi–club a such that $[\bar{A}, \bar{B}]_a \sim [\bar{B}, \bar{A}]_a$ and since $|a| < |A|$, a will be generic not with respect to A, B but with respect to sequences of theories of length λ. We will repeat steps 1 to 7 from the previous proof modifying and translating them to the language of formal shufflings.

STEP 1: We assume D iterprets T_K, and choose $\bar{W}$, K atoms $\langle \bar{U}_i : i < K \rangle$ and codings V_s as before.

STEP 2: Use notation 2.17*: fix a cofinal sequence in D, $J^* := \langle \beta_i : i < \lambda \rangle$, a club $J \subseteq \lambda$, $J := \langle \alpha_i : i < \lambda \rangle$ ($\alpha_0 = \beta_0 = 0$), and $h: J^* \to J$. W.l.o.g J is an $(n+d)$-suitable club for

all the combinations of elements we need. (Look at lemma 2.10* and definition 2.11* for the exact meaning).

For $\bar{k} \subseteq \left\{ \{0, 1, \ldots, K-1\} \cup \{(i,j) : i < j < K\} \cup \{s : s \subseteq \{K-1\}\} \right\}$ of length ≤ 3, let $s_{\bar{k}}^i$ be the theory $Th^{n+d}(\bar{U}_{\bar{k}(0)}, \ldots, \bar{W})|_{[\beta_i, \beta_{i+1})}$. So $Th^{n+d}(\bar{U}_{\bar{k}(0)}, \ldots, \bar{W}) = \sum_{i < \lambda} s_{\bar{k}}^i$.

Now let $\mathcal{T}$ denote the set $\{s_{\bar{k}}^i : \bar{k}\} \cup \{\sum_{i < \lambda} s_{\bar{k}}^i : \bar{k}, i\}$. $\mathcal{T}$ belongs to $H(\lambda^+)^{V^{P_0 * P_1}} = H(\lambda^+)^{V^P}$. Call such a $\mathcal{T}$ a *system of theories*. In $H(\lambda^+)^{V^P}$ we don't know the U_i's nor the actual $\mathcal{T}$ but we have a set of all the possible systems which must satisfy two sets of restrictions:

a) formal restrictions (as in definition 5.1).

b) material restrictions that reflect the fact that we are dealing with an interpretation of T_K. (For example for $\bar{k} = \langle i, j, \{i, j\}\rangle$ the theory $\sum_{i < \lambda} s_{\bar{k}}^i$ must imply $Code(\bar{X}_i, \bar{X}_{i,j}, \bar{W}) \& Code(\bar{X}_j, \bar{X}_{i,j}, \bar{W})$).

So in $H(\lambda^+)^{V^P}$ we only know that somewhere, (in $H(\delta^+)^{V^P}$) there are elements that interpret T_K with a system of theories $\mathcal{T}$. We scan all the possible systems (they all belong to $H(\lambda^+)^{V^P}$) and show that every one of them leads to a contradiction.

Fixing a system $\mathcal{T}$, let $a \in H(\lambda^+)^{V^P}$, $a \subseteq \lambda$, be a generic semi–club for all the members of $\mathcal{T}$, which is added at stage β of P_1.

STEPS 3-5: We shuffle the elements with respect to J and a as in definition 5.1.(3). The operations are basically the same, but we have to translate all the statements to a 'formal' language. Just for an example, the 'formal' meaning of $[\bar{U}_i, \bar{U}_j]_a^J \sim \bar{U}_k$ is: "if $s_i = Th^n(\bar{U}_i, \bar{U}_k, \bar{W})|_{[\beta_i, \beta i+1)}$ and $t_i = Th^n(\bar{U}_j, \bar{U}_k, \bar{W})|_{[\beta_i, \beta i+1)}$ then $Th^n([\bar{U}_i, \bar{U}_j]_a^J, \bar{U}_k, \bar{W}) = \sum_{i < \lambda} u_i$ where $i \in a \Rightarrow u_i = s_i$ and $i \notin a \Rightarrow u_i = t_i$". So $[\bar{U}_i, \bar{U}_j]_a^J \sim \bar{U}_k$ is formally: $\sum_{i < \lambda} u_i$ implies $E(\bar{X}, \bar{Y}, \bar{W})$. From this we can easily define formally the number $k(i, j)$ as in step 3 in the previous proof.

For choosing a condition p as in step 5, we simply choose a condition in $P_0 * P_1$ which forces all the 'formal' statements we have made. This is possible since we are talking about objects of cardinality $\leq \lambda$ only. It should be clear that after all the operations we are left with a large enough set of elements with some desired properties. Actually if you look at the achievements so far, you can note that we didn't use the formal theories. s'' as in the previous proof can be obtained for any semi–club a so we could have worked in the entire V^P or in $H(\delta^+)^{V^P}$. But for the next step we need a to be generic.

STEP 6: We have to prove the existence of some $\bar{U}_i, \bar{U}_j$ such that $i < j$ and $[\bar{U}_i, \bar{U}_j]_a \sim [\bar{U}_j, \bar{U}_i]_a \sim \bar{U}_i$. Formally we have to prove: "if $s_i = Th^n(\bar{U}_i, \bar{U}_i, \bar{W})|_{[\beta_i, \beta i+1)}$ and $t_i = Th^n(\bar{U}_j, \bar{U}_i, \bar{W})|_{[\beta_i, \beta i+1)}$ then $\sum_{i < \lambda} u_i$ and $\sum_{i < \lambda} u^*_i$ imply $E(\bar{X}, \bar{Y}, \bar{W})$ where $i \in a \Rightarrow u_i = s_i, u_i^* = t_i$ and $i \notin a \Rightarrow u_i = t_i, u_i^* = s_i$". This follows from the fact that a is generic as in step 7 in the previous proof. (Of course, here we can not avoid some translation

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work).

STEP 7: We found a semi–club a and enough elements (at least K_1) such that it does not matter if we shuffle them with respect to a or with respect to $\lambda \setminus a$. Carry them back to V^P or to $H(\delta^+)^{V^P}$ and proceed as before, (We don't need the forcing anymore).

The contradiction we have reached proves that $\mathcal{T}$ can not be realized as an interpretation to T_K, but since we have chosen it arbitrarily, it proves that there is no interpretation at all.

STEP 8: We still have to take care of the case "D is not a well ordered chain". The only problem is that there may be no first element in D, but we can fix a $\beta_0 \in D$ and take into our consideration also theories of the form $Th^{n+d}(\bar{U}_{\bar{k}(0)}, \ldots, \bar{W})|_{\beta_0}$, but this is taken care of in the modified definition of WA^m (look at notation 2.16*). Of course all the K's should be computed from the modified definition.

$\heartsuit$

Combining 7.4 and 7.5 we get the desired theorem

Theorem 7.6.   *There is a forcing notion P such that in V^P, Peano arithmetic is not interpretable in the monadic second-order theory of chains.*

$\heartsuit$

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