

CONSTRUCTING STRONGLY EQUIVALENT NONISOMORPHIC MODELS FOR UNSUPERSTABLE THEORIES, PART A

Tapani Hyttinen and Saharon Shelah *

Abstract

We study how equivalent nonisomorphic models an unstable theory can have. We measure the equivalence by Ehrenfeucht-Fraïssé games. This paper continues the work started in [HT].

1. Introduction

In [HT] we looked how equivalent nonisomorphic models first-order theories can have i.e. we tried to strengthen S.Shelah's nonstructure theorems. We used Ehrenfeucht-Fraïssé games to measure the equivalence (see Definition 2.2 below). If the theory is unstable or it has OTOP or it is superstable with DOP then we were able to prove maximal results by assuming strong cardinal assumptions. We showed that if $\lambda^{<\lambda} = \lambda$ then there is a model \mathcal{A} of the theory such that $|\mathcal{A}| = \lambda$ and for all λ^+ , λ -trees t there is a model \mathcal{B} such that $|\mathcal{B}| = \lambda$, $\mathcal{A} \not\cong \mathcal{B}$ and \exists has a winning strategy in the Ehrenfeucht-Fraïssé game $G_t^2(\mathcal{A}, \mathcal{B})$.

By assuming only that the theory is unstable we were not able to say much if we tried to measure the equivalence by the length of Ehrenfeucht-Fraïssé games in which \exists has a winning strategy. But if instead, we measured the equivalence by the length of Ehrenfeucht-Fraïssé games in which \forall does not have a winning strategy, then we were able to get rather strong results.

In this paper we look the unstable case again. We measure the equivalence by the length of Ehrenfeucht-Fraïssé games in which \exists has a winning strategy. We study λ^+ , $\kappa + 1$ -trees (see Definition 2.1) and give a rather complete answer to the question: how equivalent nonisomorphic λ^+ , $\kappa + 1$ -trees can there be? In Chapter 3 we show that if $\lambda = \mu^+$, $cf(\mu) = \mu$, $\kappa = cf(\kappa) \leq \mu$ and $\lambda^{<\kappa} = \lambda$ then there are

* partially supported by the United States Israel binational science foundation, publ. 474

$\lambda^+, \kappa + 1$ -trees I_0 and I_1 such that $|I_0| \cup |I_1| \leq \lambda^\kappa$, $I_0 \not\cong I_1$ and

$$I_0 \equiv_{\mu \times \kappa}^\lambda I_1$$

(see Definition 2.2 and Definition 2.4 (iii)). Instead of two such trees it is possible to get 2^λ such trees.

In chapter 4 we show that if in addition $\lambda \in I[\lambda]$ then the result of Chapter 3 is best possible.

As in [HT], this implies that essentially the same is true also for the models of the canonical example of unsuperstable theories.

In [HS] we will prove the results of chapter 3 for unsuperstable theories in general.

This paper was born during the first author's visit to the second author at Rutgers University. The first author wishes to express his gratitude to Rutgers University for the hospitality shown to him during the visit.

2. Basic definitions

In this chapter we define the basic concepts we shall use.

2.1 Definition. *Let λ be a cardinal and α an ordinal. Let t be a tree (i.e. for all $x \in t$, the set $\{y \in t \mid y < x\}$ is well-ordered by the ordering of t). If $x, y \in t$ and $\{z \in t \mid z < x\} = \{z \in t \mid z < y\}$, then we denote $x \sim y$, and the equivalence class of x for \sim we denote $[x]$. By a λ, α -tree t we mean a tree which satisfies:*

- (i) $|[x]| < \lambda$ for every $x \in t$;
- (ii) there are no branches of length $\geq \alpha$ in t ;
- (iii) t has a unique root;
- (iv) if $x, y \in t$, x and y have no immediate predecessors and $x \sim y$, then $x = y$.

If t satisfies only (i), (ii) and (iii) above, we say that t is a wide λ, α -tree.

Note that in a λ, α -tree each ascending sequence of a limit length has at most one supremum, but in a wide λ, α -tree an ascending sequence may have more than one supremum.

2.2 Definition. *Let t be a tree and κ a cardinal. The Ehrenfeucht-Fraisse game of length t between models \mathcal{A} and \mathcal{B} , $G_t^\kappa(\mathcal{A}, \mathcal{B})$, is the following. At each move α :*

- (i) player \forall chooses $x_\alpha \in t$, $\kappa_\alpha < \kappa$ and either $a_\alpha^\beta \in \mathcal{A}$, $\beta < \kappa_\alpha$ or $b_\alpha^\beta \in \mathcal{B}$, $\beta < \kappa_\alpha$, we will denote this sequence of elements of \mathcal{A} or \mathcal{B} by X_α ;
- (ii) if \forall chose from \mathcal{A} then \exists chooses $b_\alpha^\beta \in \mathcal{B}$, $\beta < \kappa_\alpha$, else \exists chooses $a_\alpha^\beta \in \mathcal{A}$, $\beta < \kappa_\alpha$, we will denote this sequence by Y_α .

\forall must move so that $(x_\beta)_{\beta \leq \alpha}$ form a strictly increasing sequence in t . \exists must move so that $\{(a_\gamma^\beta, b_\gamma^\beta) \mid \gamma \leq \alpha, \beta < \kappa_\gamma\}$ is a partial isomorphism from \mathcal{A} to \mathcal{B} . The player who first has to break the rules loses.

We write $\mathcal{A} \equiv_t^\kappa \mathcal{B}$ if \exists has a winning strategy for $G_t^\kappa(\mathcal{A}, \mathcal{B})$.

2.3 Remark. *Notice that the Ehrenfeucht-Fraisse game $G_t^\kappa(\mathcal{A}, \mathcal{B})$ need not be determined, i.e. it may happen that neither \exists nor \forall has a winning strategy for $G_t^\kappa(\mathcal{A}, \mathcal{B})$ (see [MSV]).*

2.4 Definition. Let t and t' be trees.

(i) If $x \in t$, then $pred(x)$ denotes the sequence $(x_\alpha)_{\alpha < \beta}$ of the predecessors of x , excluding x itself, ordered by $<$. Alternatively, we consider $pred(x)$ as a set. The notation $succ(x)$ denotes the set of immediate successors of x . If $x, y \in t$ and there is z , such that $x, y \in succ(z)$, then we say that x and y are brothers.

(ii) By $t^{<\alpha}$ we mean the set

$$\{x \in t \mid \text{the order type of } pred(x) \text{ is } < \alpha\}.$$

Similarly we define $t^{\leq \alpha}$.

(iii) If α and β are ordinals then by $\alpha + \beta$ and $\alpha \times \beta$ we mean ordinal sum and product (see [Je]). Notice that ordinals are also trees.

3. On nonstructure of trees of fixed height

In this chapter we will assume that $\lambda = \mu^+$, $cf(\mu) = \mu$, $\kappa = cf(\kappa) \leq \mu$ and $\lambda^{<\kappa} = \lambda$.

Let $I_n^+ = \{\eta \in {}^{<\kappa}\lambda \mid \eta(0) = n\} - \{()\}$ and $I_n^- = \{\eta \in {}^{<\kappa}\lambda \mid \eta(0) = n\} - \{()\}$, $n = 0, 1$. We consider these as trees ordered by initial segment relation. Because for all $\delta \leq \kappa$, $(I_n^+)^{<\delta} = (I_n^-)^{<\delta}$ (see Definition 2.4), we denote this set by $I_n^{<\delta}$ and similarly we define $I_n^{\leq \delta} = (I_n^+)^{\leq \delta}$ for all $\delta < \kappa$.

If $\eta \in I_0^+$ and $\xi \in I_1^+$ then we write $\eta R^- \xi$ and $\xi R^- \eta$ iff $\eta(j) = \xi(j)$ for all $0 < j < \min\{\text{length}(\eta), \text{length}(\xi)\}$ even. For all $i < \kappa$ odd, we define P_i to be the set of all $\eta \in I_0^-$ such that $\text{length}(\eta) = i$. Let $P = \bigcup \{P_i \mid i < \kappa, i \text{ odd}\}$

3.1 Lemma. There is a partition $\{S_\eta \mid \eta \in P\}$ of λ such that for all $\eta \in P$

- (i) $\{\delta \in S_\eta \mid cf(\delta) = \mu\}$ is stationary;
- (ii) if $\delta \in S_\eta$ and $cf(\delta) = \mu$ then $\delta = \sup(\delta \cap S_\eta)$.

Proof. Because $|P| = \lambda$ we can find a partition of $\{\alpha < \lambda \mid cf(\alpha) = \mu\}$ which satisfies (i). Let this partition be $\{S'_{\eta_\gamma} \mid \gamma < \lambda\}$, where $\{\eta_\gamma \mid \gamma < \lambda\}$ is an enumeration of P . Let $\{\sigma_\gamma \mid \gamma < \lambda\}$ be an enumeration of $\{\alpha < \lambda \mid cf(\alpha) = \mu\}$ so that if $\sigma_\gamma > \sigma_{\gamma'}$ then $\gamma > \gamma'$. We may assume that if $\delta \in S'_{\eta_\gamma}$, $\gamma \neq 0$, then $\delta > \sigma_\gamma$. By induction on $\alpha \leq \lambda$ we define sets $S_{\eta_\gamma}^\alpha$. Let $S_{\eta_0}^0 = S'_{\eta_0} \cup \sigma_0$ and for all $\gamma > 0$, $S_{\eta_\gamma}^0 = S'_{\eta_\gamma}$. If α is limit ordinal and $cf(\alpha) \geq \mu$, then we define $S_{\eta_\gamma}^\alpha = \bigcup_{\beta < \alpha} S_{\eta_\gamma}^\beta$ for all $\gamma < \lambda$. Assume α is successor or limit ordinal with $cf(\alpha) < \mu$. Let $\sigma'_\alpha = \bigcup_{\delta < \alpha} \sigma_\delta$. Then we choose $S_{\eta_\gamma}^\alpha$ so that (a)-(f) below are satisfied:

- (a) $\bigcup_{\delta < \alpha} S_{\eta_\gamma}^\delta \subseteq S_{\eta_\gamma}^\alpha$,
- (b) $S_{\eta_\gamma}^\alpha \cap S_{\eta_{\gamma'}}^\alpha = \emptyset$ if $\gamma \neq \gamma'$,
- (c) $\sigma_\alpha \subseteq \bigcup_{\gamma < \lambda} S_{\eta_\gamma}^\alpha$,
- (d) $S_{\eta_\gamma}^\alpha - \sigma_\alpha = S_{\eta_\gamma}^0 - \sigma_\alpha$ for all $\gamma < \lambda$,
- (e) if $\sigma_\alpha \in S'_{\eta_\gamma}$ then $\sigma_\alpha = \sup(\sigma_\alpha \cap S_{\eta_\gamma}^\alpha)$,
- (f) if $\gamma \leq \alpha$ then $(\sigma_\alpha - \sigma'_\alpha) \cap S_{\eta_\gamma}^\alpha \neq \emptyset$.

Then clearly $S_{\eta_\gamma} = S_{\eta_\gamma}^\lambda$, $\gamma < \lambda$, is a partition of λ and (i) is satisfied. We show that also (ii) is satisfied: If $\sigma_\delta \in S_{\eta_\gamma}$ and δ is successor or limit with $cf(\delta) < \mu$ then by (e) $\sigma_\delta = \sup(\sigma_\delta \cap S_{\eta_\gamma})$. Otherwise we know that $\sigma_\delta > \sigma_\gamma$ i.e. $\delta > \gamma$ and $\sup\{\sigma_\beta \mid \beta < \delta\} = \sigma_\delta$. By (f) this implies that $\sigma_\delta = \sup(\sigma_\delta \cap S_{\eta_\gamma})$. \square

3.2 Definition. We define a relation $R \subseteq (I_0^+ - I_0^-) \times (I_1^+ - I_1^-)$. Let $\eta \in I_0^+ - I_0^-$ and $\xi \in I_1^+ - I_1^-$. Then $(\eta, \xi) \in R$ iff

- (i) $\eta R^- \xi$;
- (ii) for every $j < \kappa$ odd, η and ξ satisfy the following: for all $\rho \in P$, $\eta(j) \in S_\rho$ iff $\xi(j) \in S_\rho$ and if $\eta(j) \notin S_{\eta \upharpoonright j}$, then $\eta(j) = \xi(j)$;
- (iii) the set $W_{\eta, \xi}^\kappa$ is bounded in κ , where $W_{\eta, \xi}^\kappa$ is defined in the following way: Let $\delta \leq \kappa$, $\eta \in I_0^+ - I_0^{<\delta}$ and $\xi \in I_1^+ - I_1^{<\delta}$ then

$$W_{\eta, \xi}^\delta = \{j < \delta \mid j \text{ odd and } \eta(j) \in S_{\eta \upharpoonright j} \text{ and}$$

$$cf(\eta(j)) = \mu \text{ and } \xi(j) \geq \eta(j)\}.$$

In order to simplify the notation we write $\eta R \xi$ and $\xi R \eta$ for $(\eta, \xi) \in R$. Notice that by this we do not try to claim that the relation is symmetric, in fact it is antisymmetric, if $(\eta, \xi) \in R$ then always $\eta \in I_0^+ - I_0^-$ and $\xi \in I_1^+ - I_1^-$. We also take liberty to write $W_{\xi, \eta}^\delta$ for $W_{\eta, \xi}^\delta$ when it is convenient.

Our first goal in this chapter is to prove the following theorem. We will prove it in a sequence of lemmas.

3.3 Theorem. If I_0 and I_1 are such that

- (i) $I_n^- \subseteq I_n \subseteq I_n^+$, $n = 0, 1$

and

- (ii) if $\eta R \xi$, $\eta \in I_0^+$ and $\xi \in I_1^+$ then $\eta \in I_0$ iff $\xi \in I_1$,
- then $I_0 \equiv_{\mu \times \kappa}^\lambda I_1$.

From now on in this chapter we assume that I_0 and I_1 satisfy (i) and (ii) above.

3.4 Definition. Let $\alpha < \kappa$.

- (i) G_α is the family of all partial functions f satisfying:
 - (a) f is a partial isomorphism from I_0 to I_1 ;
 - (b) $dom(f)$ and $rng(f)$ are closed under initial segments and for some $\beta < \lambda$ they are included in $\{\eta \in I_0^+ \mid \text{for all } j < \kappa, \eta(j) < \beta\}$ and $\{\xi \in I_1^+ \mid \text{for all } j < \kappa, \xi(j) < \beta\}$, respectively;
 - (c) if $f(\eta) = \xi$ then $\eta R^- \xi$;
 - (d) if $\eta \in I_0$, $\xi \in I_1$, $f(\eta) = \xi$ and $length(\eta) = j + 1$, j odd, then η and ξ satisfy the following: for all $\rho \in P$, $\eta(i) \in S_\rho$ iff $\xi(i) \in S_\rho$ and if $\eta(j) \notin S_{\eta \upharpoonright j}$, then $\eta(j) = \xi(j)$;
 - (e) assume $\eta \in I_0^+ - I_0^{<\delta}$ and $\{\eta \upharpoonright \gamma \mid \gamma < \delta\} \subseteq dom(f)$ and let $\xi = \bigcup_{\gamma < \delta} f(\eta \upharpoonright \gamma)$, then $W_{\eta, \xi}^\delta$ has order type $\leq \alpha$;
 - (f) if $\eta \in dom(f)$ then $\{\gamma < \lambda \mid \eta \frown (\gamma) \in dom(f)\} = \{\gamma < \lambda \mid f(\eta) \frown (\gamma) \in rng(f)\}$ is an ordinal.
- (ii) We define $F_\alpha \subseteq G_\alpha$ by replacing (f) above by
 - (f') if $\eta \in dom(f)$ then $\{\gamma < \lambda \mid \eta \frown (\gamma) \in dom(f)\} = \{\gamma < \lambda \mid f(\eta) \frown (\gamma) \in rng(f)\}$ is an ordinal of cofinality $< \mu$.

3.5 Definition. For $f, g \in G_\alpha$ we write $f \leq g$ if $f \subseteq g$ and if $\gamma < \delta \leq \kappa$, $\eta \in I_0^+ - I_0^{<\delta}$, $\eta \upharpoonright \gamma \in dom(f)$, $\eta \upharpoonright (\gamma + 1) \notin dom(f)$, $\eta \upharpoonright j \in dom(g)$ for all $j < \delta$ and $\xi = \bigcup_{j < \delta} g(\eta \upharpoonright j)$, then $W_{\eta, \xi}^\gamma = W_{\eta, \xi}^\delta$.

Notice that $f \leq g$ is a transitive relation.

3.6 Remark. Let $f \in G_\alpha$.

(i) We define \bar{f} by

$$\text{dom}(\bar{f}) = \text{dom}(f) \cup \{\eta \in I_0 \mid \eta \upharpoonright \gamma \in \text{dom}(f) \text{ for all } \gamma < \text{length}(\eta) \\ \text{and } \text{length}(\eta) \text{ is limit}\}$$

and if $\eta \in \text{dom}(\bar{f}) - \text{dom}(f)$ then

$$\bar{f}(\eta) = \bigcup_{\gamma < \text{length}(\eta)} f(\eta \upharpoonright \gamma).$$

(ii) If $f \in F_\alpha$ then $\bar{f} \in F_\alpha$ and if $f \in G_\alpha$ then $\bar{f} \in G_\alpha$.

3.7 Lemma. Assume $\alpha < \kappa$, $\delta \leq \mu$, $f_i \in F_\alpha$ for all $i < \delta$ and $f_i \leq f_j$ for all $i < j < \delta$.

(i) $\bigcup_{i < \delta} f_i \in G_\alpha$.

(ii) If $\delta < \mu$ then $\bigcup_{i < \delta} f_i \in F_\alpha$ and $f_j \leq \bigcup_{i < \delta} f_i$ for all $j < \delta$.

Proof. Follows immediately from the definitions. \square

3.8 Lemma. If $\delta < \kappa$, $f_i \in G_i$ for all $i < \delta$ and $f_i \subseteq f_j$ for all $i < j < \delta$ then

$$\bigcup_{i < \delta} f_i \in G_\delta.$$

Proof. Follows immediately from the definitions. \square

3.9 Lemma. If $f \in F_\alpha$ and $A \subseteq I_0 \cup I_1$, $|A| < \lambda$, then there is $g \in F_\alpha$ such that $f \leq g$ and $A \subseteq \text{dom}(g) \cup \text{rng}(g)$.

Proof. Let $\eta \in \text{dom}(f)$ and let

$$\{i < \lambda \mid \eta \frown (i) \in \text{dom}(f)\} = \{i < \lambda \mid f(\eta) \frown (i) \in \text{rng}(f)\} = \delta,$$

$cf(\delta) < \mu$, and let $\beta > \delta$. We show first that there are $f^{\eta\beta} \in F_\alpha$ and $\gamma \geq \beta$ such that $f^{\eta\beta} \geq f$, $cf(\gamma) < \mu$ and

$$\{i < \lambda \mid \eta \frown (i) \in \text{dom}(f^{\eta\beta})\} = \{i < \lambda \mid f(\eta) \frown (i) \in \text{rng}(f^{\eta\beta})\} = \gamma.$$

Let $\text{length}(\eta) = j$. If j is even it is trivial to find $f^{\eta\beta}$ and γ . So we assume that j is odd. We choose $\gamma \geq \beta$ so that $cf(\gamma) < \mu$. For any $i \in \gamma - \delta$ satisfying:

(i) $cf(i) = \mu$

and

(ii) $i \in S_\eta$,

we choose $j_i \in i - \delta$ so that $j_i \in S_\eta$, $cf(j_i) < \mu$ and if $i \neq i'$ then $j_i \neq j_{i'}$. These j_i exist because $\sup i \cap S_\eta = i$ and $i \neq \delta$.

Then we define $f^{\eta\beta}(\eta \frown (i)) = f(\eta) \frown (j_i)$ and $f^{\eta\beta}(\eta \frown (j_i)) = f(\eta) \frown (i)$. For all other $i \in \gamma - \delta$ we let $f^{\eta\beta}(\eta \frown (i)) = f(\eta) \frown (i)$. It is easy to see that $f^{\eta\beta} \in F_\alpha$ and $f^{\eta\beta} \geq f$.

It is easy to see that we can choose $\eta_i \in I_0$ and $\beta_i < \lambda$, $i < \mu$, so that the following functions are well-defined:

- (i) $g_o = f$;
- (ii) $g_{i+1} = \frac{(g_i)^{\eta_i \beta_i}}{\quad}$;
- (iii) $g_i = (\bigcup_{j < i} g_j)$, if i is limit;

and $A \subseteq \text{dom}(\bigcup_{i < \mu} g_i) \cup \text{rng}(\bigcup_{i < \mu} g_i)$. Furthermore we can choose η_i and β_i so that if $i \neq i'$ then $\eta_i \neq \eta_{i'}$. Then $g = \bigcup_{i < \mu} g_i$ is as wanted. \square

3.10 Lemma. *If $f \in G_\alpha$, then there is $g \in F_{\alpha+1}$ such that $f \subseteq g$.*

Proof. Essentially as the proof of Lemma 3.9. \square

Theorem 3.3 follows now easily from the lemmas above.

In the rest of this chapter we prove that there are trees I_0 and I_1 which satisfy the assumptions of Theorem 3.3 and are not isomorphic. For this we use the following Black Box. We define $H_{<\kappa^+}(\lambda)$ to be the smallest set H such that

- (i) $\lambda \subseteq H$

and

- (ii) if $x \subseteq H$ and $|x| \leq \kappa$ then $x \in H$.

3.11 Theorem. *([Sh3] Lemma 6.5) There is $W = \{(\overline{M}^\alpha, \eta^\alpha) \mid \alpha < \alpha(*)\}$ such that:*

(i) $\overline{M}^\alpha = (M_i^\alpha \mid i \leq \kappa)$ is an increasing continuous elementary chain of models belonging to $H_{<\kappa^+}(\lambda)$ and $\eta^\alpha \in {}^\kappa \lambda$ is increasing;

(ii) $M_i^\alpha \cap \kappa^+$ is an ordinal, $\kappa + 1 \subseteq M_i^\alpha$, $M_i^\alpha \in H_{<\kappa^+}(\eta^\alpha(i))$, $(M_j^\alpha \mid j \leq i) \in M_{i+1}^\alpha$ and $\eta^\alpha \upharpoonright i \in M_{i+1}^\alpha$;

(iii) In the following game, $G(\kappa, \lambda, W)$, player \forall does not have winning strategy: The play lasts κ moves, in the i -th move \forall chooses a model $M_i \in H_{<\kappa^+}(\lambda)$ and then \exists chooses $\gamma_i < \lambda$. \forall must choose models M_i , $i < \kappa$, so that $(M_i \mid i \leq \kappa)$ is an increasing continuous elementary chain of models, $M_i \cap \kappa^+$ is an ordinal, $\kappa + 1 \subseteq M_i$ and $(M_j \mid j \leq i) \in M_{i+1}$. In the end \exists wins the play if for some $\alpha < \alpha(*)$, $\eta^\alpha = (\gamma_i \mid i < \kappa)$ and $M_i = M_i^\alpha$ for all $i < \kappa$;

(iv) $\eta^\alpha \neq \eta^\beta$ for $\alpha \neq \beta$.

Notice that in the game above \forall can choose the similarity type of models freely as long as other requirements are satisfied.

We define I_0 and I_1 with help of W . We do this by defining J_α , $\neg J_\alpha$, K_α and $\neg K_\alpha$ by induction on $\alpha < \alpha(*)$ so that $J_\alpha \cap \neg J_\alpha = \emptyset$ and $K_\alpha \cap \neg K_\alpha = \emptyset$ and then letting $I_0 = I_0^- \cup \bigcup_{\alpha < \alpha(*)} J_\alpha$ and $I_1 = I_1^- \cup \bigcup_{\alpha < \alpha(*)} K_\alpha$. We assume that we have well-ordered $I_0^+ - I_0^-$.

We say that $\alpha < \alpha(*)$ is active, if there is $\eta \in I_0^+ - I_0^-$ such that α and η satisfy (i)-(vii) or (i)-(v), (vi') and (vii') below.

(i) For all $i \leq \kappa$, the similarity type of M_i^α is $\{\in, I_0^-, I_1^-, g\}$ where \in and g are two-ary relation symbols and I_0^- and I_1^- are unary relation symbols;

(ii) for all $i \leq \kappa$,

$$M_i^\alpha \upharpoonright \{\in, I_0^-, I_1^-\} \prec (H_{<\kappa^+}(\lambda), \in, I_0^-, I_1^-);$$

(iii) for all $i < \kappa$, $\eta \upharpoonright i \in M_{i+1}^\alpha$;

- (iv) for all $i \leq \kappa$, $M_i^\alpha \models$ " g is an isomorphism from I_0^- to I_1^- ";
 - (v) for all $\omega \leq i < \kappa$, if $i = \gamma + 2k$ for some γ limit and $k < \omega$ then $\eta(i) = \eta^\alpha(\gamma + k)$, and for all $i < \omega$, if $i = 2k + 2$ then $\eta(i) = \eta^\alpha(k)$;
- let

$$\xi = \bigcup_{i < \kappa} g_\alpha(\eta \upharpoonright i),$$

where g_α is the interpretation of g in M_κ^α ,

- (vi) $\eta R^- \xi$
- (vi') $\eta \bar{R}^- \xi$
- (vii) for all $i < \kappa$ odd, $\eta(i)$ satisfies:
 - (a) $cf(\eta(i)) = \mu$ and $\eta(i) \in S_{\eta \upharpoonright i}$;
 - (b) $M_\kappa^\alpha \models$ "the set $\{\eta \upharpoonright i \frown (j) \mid j < \eta(i)\} \cup \{g(\eta \upharpoonright i) \frown (j) \mid j < \eta(i)\}$ is closed under g and g^{-1} "
 - (vii') there is $j_\eta < \kappa$ such that for all $i > j_\eta$ odd the following holds:
 - (a) if $i = \gamma + 4n + 1$ for some limit ordinal γ and $n \in \omega$ then $\xi(i) \in S_{\eta \upharpoonright i}$
 - (b) if $i = \gamma + 4n + 3$ for some limit ordinal γ and $n \in \omega$ then $\eta(i) \in S_{\eta \upharpoonright i}$, $cf(\eta(i)) = \mu$ and $\xi(i) \geq \eta(i)$.

If α is active and there exists such η that α and η satisfy (i)-(vii) above, then we define η_α to be the least such $\eta \in I_0^+ - I_0^-$ in the well-ordering of $I_0^+ - I_0^-$. Otherwise we let η_α to be the least $\eta \in I_0^+ - I_0^-$ in the well-ordering of $I_0^+ - I_0^-$ such that α and η satisfy (i)-(v), (vi') and (vii') above. Let

$$\xi_\alpha = \bigcup_{i < \kappa} g_\alpha(\eta_\alpha \upharpoonright i),$$

where g_α is the interpretation of g in M_κ^α . If α is active and $\eta_\alpha \bar{R}^- \xi_\alpha$ then let $j_\alpha = j_{\eta_\alpha}$.

Let \bar{R} be the transitive and reflexive closure of R .

3.12 Lemma. *If γ is active then $\eta_\gamma \bar{R} \xi_\gamma$.*

Proof. Clearly we may assume that $\eta_\gamma R^- \xi_\gamma$. For a contradiction assume, that there are ρ_0, \dots, ρ_n such that $\rho_0 = \eta_\gamma$, $\rho_n = \xi_\gamma$, for all $m < n$, $\rho_m R \rho_{m+1}$ and for all $k < m \leq n$, $\rho_k \neq \rho_m$. We choose $i < \kappa$ so that

- (α) i is odd;
- (β) for all $k < m \leq n$, $\rho_k \upharpoonright i \neq \rho_m \upharpoonright i$;
- (γ) for all $m < n$, $W_{\rho_m, \rho_{m+1}}^\kappa \subseteq i$.

Because $\eta_\gamma(i) \in S_{\eta_\gamma \upharpoonright i}$ and $cf(\eta_\gamma(i)) = \mu$, $\rho_1(i) < \rho_0(i)$ and $\rho_1(i) \in S_{\eta_\gamma \upharpoonright i}$. By the definition of R , $\rho_2(i) \in S_{\eta_\gamma \upharpoonright i}$. By (β) above $\rho_2(i) = \rho_1(i)$ and $\rho_3(i) = \rho_2(i)$. We can continue this and get $\rho_n(i) = \dots = \rho_1(i)$. So $\eta_\gamma(i) > g(\eta_\gamma \upharpoonright (i+1))(i)$ which contradicts with (vii)(b) in the definition of active. \square

3.13 Lemma. *Let α and β be active, $\alpha \neq \beta$, $\xi_\alpha \bar{R} \xi_\beta$ and $\eta_\alpha \bar{R}^- \xi_\alpha$ then $\eta_\beta R^- \xi_\beta$.*

Proof. For a contradiction assume $\eta_\beta \bar{R}^- \xi_\beta$. By (vii') (a) in the definition of active we can find $i < \kappa$ odd such that $\xi_\alpha(i) \in S_{\eta_\alpha \upharpoonright i}$ and $\xi_\beta(i) \in S_{\eta_\beta \upharpoonright i}$. By Definition 3.2 (ii) this implies $\xi_\alpha \bar{R} \xi_\beta$, a contradiction. \square

3.14 Lemma. *Let α and β be active.*

- (i) *If $\alpha \neq \beta$ then $\eta_\alpha \overline{R} \eta_\beta$.*
- (ii) *If $\eta_\alpha R^- \xi_\alpha$ then for all active γ , $\eta_\alpha \overline{R} \xi_\gamma$.*

Proof. (i) By (vii) (a) and (or) (vii') (b) in the definition of active there is $i < \kappa$ odd such that $\eta_\alpha(i) \in S_{\eta_\alpha \upharpoonright i}$, $\eta_\beta(i) \in S_{\eta_\beta \upharpoonright i}$ and $\eta_\alpha \upharpoonright i \neq \eta_\beta \upharpoonright i$. By Definition 3.2 (ii) this implies $\eta_\alpha \overline{R} \eta_\beta$.

(ii) If $\gamma = \alpha$ the claim follows immediately from Lemma 3.12. So assume $\gamma \neq \alpha$. We may also assume $\eta_\alpha R^- \xi_\gamma$, because otherwise we have proved the claim. Then $\eta_\gamma \overline{R}^- \xi_\gamma$. By (vii) (a) and (vii') (a) in the definition of active we can find $i < \kappa$ odd such that $\eta_\alpha(i) \in S_{\eta_\alpha \upharpoonright i}$, $\xi_\gamma(i) \in S_{\eta_\gamma \upharpoonright i}$ and $\eta_\alpha \upharpoonright i \neq \eta_\gamma \upharpoonright i$. As above this implies $\eta_\alpha \overline{R} \xi_\gamma$. \square

3.15 Lemma. *Let α and β be active. If $\eta_\alpha \overline{R} \xi_\beta$ then there is $l_{\alpha\beta} < \kappa$ such that for all $i > l_{\alpha\beta}$, $i = \gamma + 4k + 3$, γ limit and $k \in \omega$, $\eta_\alpha(i) > \xi_\beta(i)$.*

Proof. By Lemma 3.14 (ii) we may assume $\eta_\alpha \overline{R}^- \xi_\alpha$. For a contradiction assume, that there are ρ_0, \dots, ρ_n such that $\rho_0 = \eta_\alpha$, $\rho_n = \xi_\beta$, for all $m < n$, $\rho_m R \rho_{m+1}$ and for all $k < m \leq n$, $\rho_k \neq \rho_m$. We choose $l_{\alpha\beta} < \kappa$ so that

- (α) $j_{\eta_\alpha} < l_{\alpha\beta}$;
- (β) for all $k < m \leq n$, $\rho_k \upharpoonright i \neq \rho_m \upharpoonright i$;
- (γ) for all $m < n$, $W_{\rho_m, \rho_{m+1}}^\kappa \subseteq i$.

Let $i > l_{\alpha\beta}$, $i = \gamma + 4k + 3$, γ limit and $k \in \omega$. Because $\eta_\alpha(i) \in S_{\eta_\alpha \upharpoonright i}$ and $cf(\eta_\alpha(i)) = \mu$, $\rho_1(i) < \rho_0(i)$ and $\rho_1(i) \in S_{\eta_\alpha \upharpoonright i}$. By the definition of R , $\rho_2(i) \in S_{\eta_\alpha \upharpoonright i}$. By (β) above $\rho_2(i) = \rho_1(i)$ and $\rho_3(i) = \rho_2(i)$. We can continue this and get $\rho_n(i) = \dots = \rho_1(i)$. So $\eta_\alpha(i) > \xi_\beta(i)$. \square

3.16 Lemma. *There does not exist a sequence (τ_0, \dots, τ_n) , $n \in \omega$, $n \geq 3$, such that*

- (i) *for all $m \leq n$ there is active α such that $\tau_m = \eta_\alpha$ or $\tau_m = \xi_\alpha$,*
- (ii) *for all $m < n$ either*

(a) $\tau_m \overline{R} \tau_{m+1}$

or

- (b) *there is active α such that $\tau_m = \eta_\alpha$ and $\tau_{m+1} = \xi_\alpha$ or $\tau_m = \xi_\alpha$ and*

$\tau_{m+1} = \eta_\alpha$

and at least case (b) exist in the sequence,

- (iii) $\tau_0 = \tau_n$,
- (iv) *for all $m, m' < n$ if $m \neq m'$ then $\tau_m \neq \tau_{m'}$*

Proof. For a contradiction assume that such sequence exists. By (ii) (b) we may choose the sequence so that for some α , $\tau_0 = \xi_\alpha$ and $\tau_1 = \eta_\alpha$. Then by (iv) and because $n \geq 3$, $\tau_1 \overline{R} \tau_2$. By Lemma 3.12 $\eta_\alpha \overline{R} \xi_\alpha$ and so we may drop elements from the sequence so that (i)-(iv) remain true, there are still at least 4 elements in the sequence and

(*) if $m < n - 1$ and $\tau_m \overline{R} \tau_{m+1}$ then $\tau_{m+1} \overline{R} \tau_{m+2}$.

By induction on $m < n$ we show that if $\tau_m \overline{R} \tau_{m+1}$ then $\tau_{m+1} \overline{R} \tau_{m+2}$ and if $\tau_m = \eta_\beta$ or $\tau_m = \xi_\beta$ for some β then $\eta_\beta \overline{R}^- \xi_\beta$. Above we showed that $\eta_\alpha \overline{R} \tau_2$. By Lemma 3.14 (i) $\tau_2 = \xi_\beta$ for some active β . By Lemma 3.14 (ii) $\eta_\alpha \overline{R}^- \xi_\alpha$.

Then by (*) above $\tau_3 = \eta_\beta$. By (iv) and Lemma 3.14 (i) $\tau_4 = \xi_\gamma$ for some active γ , $\gamma \neq \beta$ and $\eta_\beta \bar{R} \xi_\gamma$. By Lemma 3.14 (ii) $\eta_\beta \not\bar{R} \xi_\beta$. We can continue this and get the claim.

So there are active $\alpha_0, \dots, \alpha_m$ such that the sequence is of the following form:

$$(\xi_{\alpha_0}, \eta_{\alpha_0}, \xi_{\alpha_1}, \eta_{\alpha_1}, \dots, \eta_{\alpha_m}, \xi_{\alpha_0}).$$

We choose $i < \kappa$ so that for all $k \leq m$, $i > j_{\alpha_k}$, for all $k < m$, $i > l_{\alpha_k \alpha_{k+1}}$, $i > l_{\alpha_m \alpha_0}$ and $i = \gamma + 4p + 3$ for some limit γ and $p \in \omega$. By (vii')(b) $\xi_{\alpha_0}(i) \geq \eta_{\alpha_0}(i)$. By Lemma 3.15 $\eta_{\alpha_0}(i) > \xi_{\alpha_1}(i)$. We can continue this and finally we get $\eta_{\alpha_m}(i) > \xi_{\alpha_0}(i)$. So $\xi_{\alpha_0}(i) > \xi_{\alpha_0}(i)$, a contradiction. \square

We define now J_α , $\neg J_\alpha$, K_α and $\neg K_\alpha$ by induction on $\alpha < \alpha(*)$. We say that $(J_\alpha, \neg J_\alpha, K_\alpha, \neg K_\alpha)$ is closed if

- (i) $J_\alpha \cup K_\alpha$ and $\neg J_\alpha \cup \neg K_\alpha$ are closed under \bar{R} ,
- (ii) if β is active then $\eta_\beta \in J_\alpha$ iff $\xi_\beta \in \neg K_\alpha$ and $\eta_\beta \in \neg J_\alpha$ iff $\xi_\beta \in K_\alpha$,
- (iii) $J_\alpha \cap \neg J_\alpha = \emptyset$ and $K_\alpha \cap \neg K_\alpha = \emptyset$.

We assume that for all $\beta < \alpha$ we have defined J_β , $\neg J_\beta$, K_β and $\neg K_\beta$ so that $(J_\beta, \neg J_\beta, K_\beta, \neg K_\beta)$ is closed.

If α is not active or for some $\beta < \alpha$, $\eta_\alpha \in J_\beta \cup \neg J_\beta$ then we let $J_\alpha = \bigcup_{\beta < \alpha} J_\beta$, $\neg J_\alpha = \bigcup_{\beta < \alpha} \neg J_\beta$, $K_\alpha = \bigcup_{\beta < \alpha} K_\beta$ and $\neg K_\alpha = \bigcup_{\beta < \alpha} \neg K_\beta$.

If α is active and for all $\beta < \alpha$, $\eta_\alpha \notin J_\beta \cup \neg J_\beta$ then we let $(J_\alpha, \neg J_\alpha, K_\alpha, \neg K_\alpha)$ be such that it is closed and $J_\alpha \supseteq \{\eta_\alpha\} \cup \bigcup_{\beta < \alpha} J_\beta$, $\neg J_\alpha \supseteq \bigcup_{\beta < \alpha} \neg J_\beta$, $K_\alpha \supseteq \bigcup_{\beta < \alpha} K_\beta$ and $\neg K_\alpha \supseteq \bigcup_{\beta < \alpha} \neg K_\beta$. We prove the existence of these set by defining sets J_α^i , $\neg J_\alpha^i$, K_α^i and $\neg K_\alpha^i$ by induction on $i < |\alpha(*)|^+$.

We let $J_\alpha^0 = \{\eta_\alpha\} \cup \bigcup_{\beta < \alpha} J_\beta$, $\neg J_\alpha^0 = \bigcup_{\beta < \alpha} \neg J_\beta$, $K_\alpha^0 = \bigcup_{\beta < \alpha} K_\beta$ and $\neg K_\alpha^0 = \bigcup_{\beta < \alpha} \neg K_\beta$. If $i < |\alpha(*)|^+$ is limit we let $J_\alpha^i = \bigcup_{j < i} J_\alpha^j$ and similarly for the other sets. If $i = j + 1$ and odd then we let the sets J_β^i , $\neg J_\beta^i$, K_β^i and $\neg K_\beta^i$ be the least sets so that $J_\alpha^i \supseteq J_\alpha^j$, $\neg J_\alpha^i \supseteq \neg J_\alpha^j$, $K_\alpha^i \supseteq K_\alpha^j$, $\neg K_\alpha^i \supseteq \neg K_\alpha^j$ and $J_\alpha^i \cup K_\alpha^i$ and $\neg J_\alpha^i \cup \neg K_\alpha^i$ are closed under \bar{R} . If $i = j + 1$ and even then if there is not active γ such that

- (1) $\eta_\gamma \in J_\alpha^j$ and $\xi_\gamma \notin \neg K_\alpha^j$ or
- (2) $\eta_\gamma \in \neg J_\alpha^j$ and $\xi_\gamma \notin K_\alpha^j$ or
- (3) $\xi_\gamma \in K_\alpha^j$ and $\eta_\gamma \notin \neg J_\alpha^j$ or
- (4) $\xi_\gamma \in \neg K_\alpha^j$ and $\eta_\gamma \notin J_\alpha^j$

then we let $J_\alpha^i = J_\alpha^j$ and similarly for the other sets. Otherwise we let γ be the least such ordinal and define

- case (1): $J_\alpha^i = J_\alpha^j$, $\neg J_\alpha^i = \neg J_\alpha^j$, $K_\alpha^i = K_\alpha^j$ and $\neg K_\alpha^i = \neg K_\alpha^j \cup \{\xi_\gamma\}$;
- case (2): $J_\alpha^i = J_\alpha^j$, $\neg J_\alpha^i = \neg J_\alpha^j$, $K_\alpha^i = K_\alpha^j \cup \{\xi_\gamma\}$ and $\neg K_\alpha^i = \neg K_\alpha^j$;
- case (3): $J_\alpha^i = J_\alpha^j$, $\neg J_\alpha^i = \neg J_\alpha^j \cup \{\eta_\gamma\}$, $K_\alpha^i = K_\alpha^j$ and $\neg K_\alpha^i = \neg K_\alpha^j$;
- case (4): $J_\alpha^i = J_\alpha^j \cup \{\eta_\gamma\}$, $\neg J_\alpha^i = \neg J_\alpha^j$, $K_\alpha^i = K_\alpha^j$ and $\neg K_\alpha^i = \neg K_\alpha^j$.

Finally we define $J_\alpha = \bigcup_{i < |\alpha(*)|^+} J_\alpha^i$ and similarly for the other sets. If these sets are not as required then for some $i = j + 1 < |\alpha(*)|^+$ even we have defined f.ex. $\neg K_\alpha^i = \neg K_\alpha^j \cup \{\xi_\gamma\}$ while ξ_γ belongs already to K_α^j . If i is the least such ordinal then we can easily find a circle such that it contradicts Lemma 3.16.

So the sets J_α , $\neg J_\alpha$, K_α and $\neg K_\alpha$ exist.

We define $I_0 = I_0^- \cup \bigcup_{\alpha < \alpha(*)} J_\alpha$ and $I_1 = I_1^- \cup \bigcup_{\alpha < \alpha(*)} K_\alpha$.

3.17 Lemma. $I_0 \not\cong I_1$.

Proof. For a contradiction assume $g : I_0 \rightarrow I_1$ is an isomorphism. By Theorem 3.11 (iii) there exists an active $\alpha < \alpha^*$ such that for all $i \leq \kappa$,

$$M_i^\alpha \prec (H_{<\kappa^+}(\lambda), \in, I_0^-, I_1^-, g).$$

But then $\eta_\alpha \in I_0$ iff $\xi_\alpha \notin I_1$ and $g(\eta_\alpha) = \xi_\alpha$, which contradicts the assumption that g is an isomorphism. \square

3.18 Conclusion. Assume $\lambda = \mu^+$, $cf(\mu) = \mu$, $\kappa = cf(\kappa) \leq \mu$ and $\lambda^{<\kappa} = \lambda$. Then there are $\lambda^+, \kappa + 1$ -trees I_0 and I_1 such that $I_0 \not\cong I_1$ and

$$I_0 \equiv_{\mu \times \kappa}^\lambda I_1.$$

If $\lambda^\kappa = \lambda$ then I_0 and I_1 are of cardinality λ .

Notice that if we replace Theorem 3.11 with a slightly stronger black box (see [Sh3]), we can, instead of two λ^+, κ -trees, get 2^λ λ^+, κ -trees such that any two of them satisfy Conclusion 3.18.

4. On structure of trees of fixed height

In this chapter we will show that trees of fixed height are isomorphic if they are equivalent up to some relatively small tree. This implies that essentially the same is true for the models of the canonical example of unsuperstable theories (see [HT]).

4.1 Definition. ([Sh1]) Let λ be a regular cardinal. We define $I[\lambda]$ to be the set of $A \subseteq \lambda$ such that there exist a cub $E \subseteq \lambda$ and $\mathcal{P} = \{P_\alpha \mid \alpha < \lambda\}$ satisfying

- (i) P_α is a set of subsets of α and $|P_\alpha| < \lambda$;
- (ii) for all limit $\delta \in A \cap E$ such that $cf(\delta) < \delta$, there exists $C \subseteq \delta$ such that
 - (a) the order type of C is $< \delta$ and $\sup C = \delta$;
 - (b) $C \cap \alpha \in \bigcup_{\beta < \delta} P_\beta$ for all $\alpha < \delta$.

Notice that for example $\omega_1 \in I[\omega_1]$: Let $E \subset \omega_1$ be the set of all limit ordinals $< \omega_1$ and $\mathcal{P} = \{P_\alpha \mid \alpha < \lambda\}$ such that $P_\alpha = \{B \subseteq \alpha \mid |B| < \omega\}$. Then (i) and (ii) above are satisfied. For further properties of $I[\lambda]$ see [Sh1].

4.2 Definition. Let λ be a regular cardinal and t a λ^+, λ -tree of cardinality λ . Let $\{x_i \mid i < \lambda\}$ be an enumeration of t and let t' be a subtree of t . Then $S[t']$ is the set of those limit ordinals $\delta < \lambda$ which satisfy the following condition (*):

- (*) $\{x_i \in t' \mid i < \delta\}$ contains a branch of length δ .

From now on we assume that when ever we talk about a tree t , we have fixed an enumeration $\{x_i \mid i < |t|\}$ for it. We assume that the enumeration is such that if $x_i < x_j$ then $i < j$.

4.3 Definition. Let λ and κ be regular cardinals, $\kappa < \lambda$ and t a λ^+, λ -tree of cardinality λ . Let $\{x_i \mid i < \lambda\}$ be the enumeration of t . We say that t is λ, κ -large if t satisfies the following condition: There are sets E_ξ , $\xi \leq \kappa$, such that

- (i) $E_\xi \subseteq t$ and if $\xi \neq \xi'$ then $E_\xi \cap E_{\xi'} = \emptyset$;

- (ii) for $\xi < \delta$ and $x \in E_\delta$ there is a unique $y \in E_\xi$ such that $y < x$;
- (iii) if $\delta \leq \kappa$ is limit, $x_\xi \in E_\xi$ for all $\xi < \delta$ and $(x_\xi)_{\xi < \delta}$ is increasing then there is $y \in E_\delta$ such that $x_\xi < y$ for all $\xi < \delta$;
- (iv) if $\xi < \kappa$, $x \in E_\xi$ then we write

$$t_x = \{y \in t \mid x \leq y \text{ and there is } z \in E_{\xi+1} \text{ such that } y < z\}$$

and require than there exists a set Θ of regular cardinals $< \lambda$ such that

- (a) $S[t_x] \cup \{\delta < \lambda \mid cf(\delta) < \delta, cf(\delta) \in \Theta\}$ contains a cub set (in λ);
- (b) $\{\delta < \lambda \mid cf(\delta) < \delta, cf(\delta) \in \Theta, \delta \notin S[t_x]\} \in I[\lambda]$;
- (c) for $\delta \in \Theta$ there is $y \in t_x$ such that the order type of $\{z \mid x \leq z < y\}$ is δ ;
- (v) if $\gamma = \beta + 1 < \kappa$, $(x_\xi)_{\xi < \delta}$ is an increasing sequence in t , $x_0 \in E_\beta$ and for all $\xi < \delta$ there is $y_\xi \in E_\gamma$ such that $x_\xi < y_\xi$, then there is $y \in E_\gamma$ such that $x_\xi < y$ for all $\xi < \delta$.

Notice that if $\lambda = \mu^+$, $\lambda \in I[\lambda]$ and $\kappa < \lambda$ is regular then $\mu \times \kappa + 1$ is a λ, κ -large λ^+, λ -tree. If λ is weakly compact then there is no λ, κ -large λ^+, λ -trees.

The proof of the theorem below is a modification of the proof of related result in [HT]. The most conspicuous difference is the use of elementary submodels of $H(\lambda^*)$. They are used only to make it easier to define the closures needed in the proof.

4.4 Theorem. *Let λ and κ be regular cardinals, $\kappa < \lambda$ and I_0 and I_1 be $\lambda^+, \kappa + 1$ -trees. Assume t is a λ, κ -large λ^+, λ -tree of cardinality λ . Then*

$$I_0 \equiv_t^\lambda I_1 \iff I_0 \cong I_1.$$

Proof. Without loss of generality we may assume that I_0 and I_1 are such that if $x, y \in I_0$ ($\in I_1$), they have no immediate predecessors, $x \sim y$ and $pred(x)$ is of power $< \kappa$ then $x = y$.

Let ρ be a winning strategy of \exists in $G_t^\lambda(I_0, I_1)$. We define by induction on $\alpha \leq \kappa$ the following:

- (i) an isomorphism f_α from $I_0^{\leq \alpha}$ onto $I_1^{\leq \alpha}$;
- (ii) for each $x \in I_0^{\leq \alpha} \cup I_1^{\leq \alpha}$ we define an initial segment $R_x = ((a_i, X_i, Y_i))_{i \leq \beta}$ of a play in $G_t^\lambda(I_0, I_1)$, such that $x \in \bigcup_{i \leq \beta} (rng(X_i) \cup rng(Y_i))$, $rng(X_i) \cup rng(Y_i) \subseteq I_0^{\leq \alpha} \cup I_1^{\leq \alpha}$ for all $i < \beta$, \exists has used ρ and if x is not a leaf then for some $\delta < \kappa$ there is $a_x \in E_\delta$ such that $a_i \leq a_x$ for all $i < \beta$. Furthermore we require that if $x < x'$ then R_x is an initial segment of $R_{x'}$ and for each $x \in I_0^{\leq \alpha}$ $f_\alpha(x)$ is the element \exists has chosen to be the image of x in R_x .

If we can do this we have clearly proved the theorem. The cases $\alpha = 0$ and α is limit are trivial. So we assume that $\alpha = \gamma + 1$.

Let $z \in I_0^{\leq \gamma} - \bigcup_{\delta < \gamma} I_0^{\leq \delta}$. Clearly it is enough to define $f_\alpha \upharpoonright succ(z)$ and R_x for all $x \in succ(z)$ so that $f_\alpha \upharpoonright succ(z)$ is onto $succ(f_\gamma(z))$. Let $y = f_\gamma(z)$ and let $n : \lambda \rightarrow t$ be the function that gives the enumeration of t , $t = \{n(i) \mid i < \lambda\}$ (see the assumption after Definition 4.2). Let $R_z = ((a_i, X_i, Y_i))_{i \leq \beta}$. By induction assumption there is $a_z \in E_\delta$, $\delta < \kappa$, such that $a_i < a_z$ for all $i \leq \beta$. Let E and $\mathcal{P} = \{P_i \mid i < \lambda\}$ be the sets which show that

$$\{\delta < \lambda \mid cf(\delta) < \delta, cf(\delta) \in \Theta, \delta \notin S[t_{a_z}]\} \in I[\lambda].$$

Let λ^* be large enough, say $(\beth_{10}(\lambda))^+$. We choose \mathcal{A}_i , $i < \lambda$, so that

- (a) $|\mathcal{A}_i| < \lambda$ and $\mathcal{A}_i \prec (H(\lambda^*), \in, I_0, I_1, t, <_0, <_1, <)$, where $<_0$ denotes the ordering of I_0 , $<_1$ denotes the ordering of I_1 and $<$ denotes the ordering of t ;
- (b) $\rho, n, (E_\xi \mid \xi \leq \kappa), E, (P_i \mid i < \lambda), R_z, \lambda, \beta, a_z \in \mathcal{A}_0$, $\kappa + 1 \subseteq \mathcal{A}_0$ and $i \subseteq \mathcal{A}_i$;
- (c) $\mathcal{A}_i \prec \mathcal{A}_j$ if $i < j$ and $\mathcal{A}_i = \bigcup_{j < i} \mathcal{A}_j$ if i limit;
- (d) for all $i \leq \beta$, $\text{dom}(X_i) \in \mathcal{A}_0$ (see Definition 2.2);
- (e) $\mathcal{A}_i \cap \lambda$ is ordinal, $\mathcal{A}_i \in \mathcal{A}_{i+1}$ and $\mathcal{A}_i \cap \lambda \in \mathcal{A}_{i+1}$;
- (f) $\text{succ}(z) \cup \text{succ}(y) \subseteq \bigcup_{i < \lambda} \mathcal{A}_i$;
- (g) if $x \in t \cap \mathcal{A}_i$, $y \in t$ and $y < x$, then $y \in \mathcal{A}_i$.

Let

$$C \subseteq S[t_{a_z}] \cup \{\delta < \lambda \mid cf(\delta) < \delta \text{ and } cf(\delta) \in \Theta\}$$

be cub. We may assume that for all $c \in C$, $\mathcal{A}_c \cap \lambda = c$ and $c \in E$.

For all $i < \lambda$ we define by induction $c_i \in C$ and $f_\alpha \upharpoonright (\text{succ}(z) \cap \mathcal{A}_{c_i})$. If i is limit then $c_i = \bigcup_{j < i} c_j$ and $f_\alpha \upharpoonright (\text{succ}(z) \cap \mathcal{A}_{c_i})$ is already defined.

Assume that we have defined c_i and $f_\alpha \upharpoonright (\text{succ}(z) \cap \mathcal{A}_{c_i})$ as wanted and

$$\text{rng}(f_\alpha \upharpoonright (\text{succ}(z) \cap \mathcal{A}_{c_i})) = \text{succ}(y) \cap \mathcal{A}_{c_i}.$$

Let us define c_{i+1} and

$$f_\alpha \upharpoonright (\text{succ}(z) \cap (\mathcal{A}_{c_{i+1}} - \mathcal{A}_{c_i})).$$

Now either $c_i \in S[t_{a_z}]$ or $c_i \in \{\delta < \lambda \mid cf(\delta) < \delta \text{ and } cf(\delta) \in \Theta\}$.

(1) $c_i \in S[t_{a_z}]$: Let $B \in \mathcal{A}_{c_{i+1}}$ be a branch in

$$S[t_{a_z}] \cap \mathcal{A}_{c_i} = \{n(j) \mid j < c_i\}$$

of length c_i . Let $h \in \mathcal{A}_{c_i}$ be a one-one function from $(\text{succ}(z) \cup \text{succ}(y)) \cap \mathcal{A}_{c_i}$ to $\mathcal{A}_{c_i} \cap \lambda$. We let the players continue the play R_z so that in the next c_i moves \forall chooses the sets $\{h^{-1}(\delta)\}$, $\delta < c_i$, from $I_0 \cup I_1$ and from t he chooses elements of B . We let \exists follow ρ . If B' is an initial segment of B then $B' = \{y \in t \mid a_z \leq y < x\}$ for some $x \in B$. So $B' \in \mathcal{A}_{c_i}$, which implies that every initial segment of the play belongs to \mathcal{A}_{c_i} . Because \mathcal{A}_{c_i} is closed under ρ , all the elements \exists chooses are from \mathcal{A}_{c_i} . It is also easy to see that this play belongs to \mathcal{A}_γ for all $\gamma > c_i$.

By Definition 4.3 (v) we can find $a \in E_{\delta+1} \cap \mathcal{A}_{c_{i+1}}$, such that a is larger than any element $b \in t$ chosen by \forall in the play above. Let

$$C' \subseteq S[t_a] \cup \{\delta < \lambda \mid cf(\delta) < \delta \text{ and } cf(\delta) \in \Theta\}$$

be cub. Let $c_{i+1} \in C \cap C'$ be such that $c_{i+1} > c_i$. Then $a \in \mathcal{A}_{c_{i+1}}$. Now either $c_{i+1} \in S[t_a]$ or $c_{i+1} \in \{\delta < \lambda \mid cf(\delta) < \delta \text{ and } cf(\delta) \in \Theta\}$. In the first case we let \forall play the elements $(\text{succ}(z) \cup \text{succ}(y)) \cap \mathcal{A}_{c_{i+1}}$ as above. So let us assume that $c_{i+1} \notin S[t_a]$ and $c_{i+1} \in \{\delta < \lambda \mid cf(\delta) < \delta \text{ and } cf(\delta) \in \Theta\}$. Especially then

$$(*) \quad c_{i+1} \in E \cap \{\delta < \lambda \mid cf(\delta) < \delta, cf(\delta) \in \Theta, \delta \notin S[t_a]\}.$$

Let $h' \in \mathcal{A}_{c_{i+1}}$ be a one-one function from $(succ(z) \cup succ(y)) \cap \mathcal{A}_{c_{i+1}}$ to $c_{i+1} = \mathcal{A}_{c_{i+1}} \cap \lambda$. Let

$$D' \subseteq c_{i+1}$$

be a set such that for all $\xi < c_{i+1}$, $\xi \cap D' \in \bigcup_{j < c_{i+1}} P_j$, $sup D' = c_{i+1}$ and the order type of D' is $cf(c_{i+1})$. The existence of this set follows from (*) above. Let $D = \{d_j \mid j < cf(c_{i+1})\}$ be the closure of D' in c_{i+1} . Because $cf(c_{i+1}) \in \mathcal{A}_{c_{i+1}}$, it is easy to see that in $t_a \cap \mathcal{A}_{c_{i+1}}$ there is a branch B of length $cf(c_{i+1})$. We let the players continue the play above so that in the next $cf(c_{i+1})$ moves \forall chooses the sets $\{h'^{-1}(k) \mid k < d_j\}$ from $I_0 \cup I_1$, $j < cf(c_{i+1})$, and from t he chooses elements of B . We let \exists follow ρ .

Because $\bigcup_{i < c_{i+1}} P_i \subseteq \mathcal{A}_{c_{i+1}}$, every initial segment of this play is in $\mathcal{A}_{c_{i+1}}$ and so all elements chosen by \exists from $I_0 \cup I_1$ are from $\mathcal{A}_{c_{i+1}}$. Then by using the moves of \exists we can define

$$f_\alpha \upharpoonright (succ(z) \cap (\mathcal{A}_{c_{i+1}} - \mathcal{A}_{c_i})).$$

For each $x \in succ(z) \cap (\mathcal{A}_{c_{i+1}} - \mathcal{A}_{c_i})$, R_x will be the play defined above.

(2) $c_i \notin S[t_{a_z}]$: Now we first let \forall play the elements of $(succ(z) \cup succ(y)) \cap \mathcal{A}_{c_i}$ as in the second half of the case (1) and then continue as above. Notice that in this case (also) we have to define the first $cf(c_i)$ moves so that the play belongs to $\mathcal{A}_{c_{i+1}}$. We can guarantee this by choosing $D' \subseteq c_i$ so that $D' \in \mathcal{A}_{c_{i+1}}$. \square

4.5 Remark. Let $\lambda = \mu^+$ and $\kappa < \lambda$ regular. Let I_0 and I_1 be $\lambda^+, \kappa + 1$ -trees. Assume $\lambda \in I[\lambda]$. Above we proved that if $\alpha = \mu \times \kappa + 1$ then

$$(*) \quad I_0 \equiv_\alpha^\lambda I_1 \Leftrightarrow I_0 \cong I_1.$$

In Chapter 3 we showed that if μ is regular then this is best possible. But if μ is not regular then we can get better results.

If $\kappa < cf(\mu) < \mu$ then (*) is true if $\alpha = \mu$ and if $\kappa = cf(\mu) < \mu$ then (*) is true if $\alpha = \mu + 1$. This can be proved as Theorem 4.4.

References

- [HS] T.Hyttinen and S.Shelah, Constructing strongly equivalent nonisomorphic models for unsuperstable theories, part B, to appear.
- [HT] T.Hyttinen and H.Tuuri, Constructing strongly equivalent nonisomorphic models for unstable theories, *APAL* 52, 1991, 203–248.
- [Je] T.Jech, *Set Theory*, Academic Press, New York, 1978.
- [MSV] A. Mekler, S. Shelah and J. Vninen, The Ehrenfeucht-Fraïssé-game of length ω_1 , to appear.
- [Sh1] S.Shelah, Nonelementary classes: Part II, in: J.T.Baldwin (ed.), *Classification Theory*, Springer Lecture Notes 1292, (Springer-Verlag, 1987) 419–497.
- [Sh2] S.Shelah, *Classification Theory*, Stud. Logic Found. Math. 92 (North-Holland, Amsterdam, 2nd rev. ed., 1990).
- [Sh3] S.Shelah, *Non-structure Theory*, to appear.

Tapani Hyttinen
Department of Mathematics
P. O. Box 4
00014 University of Helsinki
Finland

Saharon Shelah
Institute of Mathematics
The Hebrew University
Jerusalem
Israel

Rutgers University
Hill Ctr-Busch
New Brunswick
NJ 08903
USA