Combinatorial properties of Hechler forcing

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ABSTRACT

Using a notion of rank for Hechler forcing we show: 1) assuming $\omega_1^V = \omega_1^L$, there is no real in $V[d]$ which is eventually different from the reals in $L[d]$, where d is Hechler over V; 2) adding one Hechler real makes the invariants on the left-hand side of Cichoń's diagram equal ω_1 and those on the right-hand side equal 2^ω and produces a maximal almost disjoint family of subsets of ω of size ω_1; 3) there is no perfect set of random reals over V in $V[r][d]$, where r is random over V and d Hechler over $V[r]$, thus answering a question of the first and second authors.

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Introduction

In this work we use a notion of rank first introduced by James Baumgartner and Peter Dordal in [BD, § 2] and later developed independently by the third author in [GS, § 4] to show that *adding a Hechler real* has strong combinatorial consequences. Recall that the *Hechler p. o.* $\mathbb{D}$ is defined as follows.

$$(s,f) \in \mathbb{D} \Longleftrightarrow s \in \omega^{<\omega} \ \wedge \ f \in \omega^\omega \ \wedge \ s \subseteq f \ \wedge \ f \text{ strictly increasing}$$

$$(s,f) \leq (t,g) \Longleftrightarrow s \supseteq t \ \wedge \ \forall n \in \omega \ (f(n) \geq g(n))$$

We note here that our definition differs from the usual one in that it generically adds a strictly increasing function from ω to ω. This is, however, a minor point making the definition of the rank in section 1 easier. We indicate at the end of § 1 how it can be changed to get the corresponding results in §§ 2 and 4 for *classical* Hechler forcing.

The theorems of section 2 are all consequences of one technical result which is expounded in 2.1. We shall sketch how some changes in the latter's argument prove that adding one Hechler real produces a maximal almost disjoint family of subsets of ω of size ω_1 (2.2.). Recall that $A, B \subseteq \omega$ are said to be *almost disjoint* (*a. d.* for short) iff $|A \cap B| < \omega$; $\mathcal{A} \subseteq [\omega]^\omega$ is an *a. d. family* iff the members of $\mathcal{A}$ are pairwise a. d.; and $\mathcal{A}$ is a *m. a. d. family* (*maximal almost disjoint family*) iff it is a. d. and maximal with this property. — We shall then show that assuming $\omega_1^V = \omega_1^L$, there is no real in $V[d]$ which is eventually different from the reals in $L[d]$, where d is Hechler over V (2.4.). Here, we say that given models $M \subseteq N$ of ZFC, a real $f \in \omega^\omega \cap N$ is *eventually different* from the reals in M iff $\forall g \in \omega^\omega \cap M \ \forall^\infty n \ (g(n) \neq f(n))$, where $\forall^\infty n$ abbreviates *for all but finitely many* n. (Similarly, $\exists^\infty n$ will stand for *there are infinitely any* n.) — Next we will prove that adding one Hechler real makes the invariants on the left-hand side of Cichoń's diagram equal ω_1 and those on the right-hand side equal 2^ω (2.5.). These invariants (which describe combinatorial properties of measure and category on the real line, and of the eventually dominating order on ω^ω) will be defined, and the shape of Cichoń's diagram explained, in the discussion preceding the result in § 2. Theorem 2.5. should be seen as a continuation of research started by Cichoń and Pawlikowski in [CP] and [Pa]. They investigated the effect of *adding a Cohen or a random real* on the invariants in Cichoń's diagram. — We close section 2 with an application concerning absoluteness in the projective hierarchy (2.6.);

namely we show that $\Sigma^1_4 - \mathbb{D}$-*absoluteness* (which means that V and $V[d]$, where d is Hechler over V, satisfy the same Σ^1_4-sentences with parameters in V) implies that $\omega_1^V > \omega_1^{L[r]}$ for any real r; in particular ω_1^V is inaccessible in L. So, for projective statements, Hechler forcing is much stronger than Cohen or random forcing for Σ^1_n-Cohen-absoluteness (Σ^1_n-random-absoluteness) is true in any model gotten by adding ω_1 Cohen (random) reals [Ju, § 2].

In § 3 we leave Hechler forcing for a while to deal with *perfect sets of random reals* instead, and to continue a discussion initiated in [BaJ] and [BrJ]. Recall that given two models $M \subseteq N$ of ZFC, we say that $g \in \omega^\omega \cap N$ is a *dominating real over* M iff $\forall f \in \omega^\omega \cap M \; \forall^\infty n \; (g(n) > f(n))$; and $r \in 2^\omega \cap N$ is *random over* M iff r avoids all Borel null sets coded in M iff r is the real determined by some filter which is $\mathbb{B}$-generic over M (where $\mathbb{B}$ is the algebra of Borel sets of 2^ω modulo the null sets (*random algebra*) – see [Je, section 42] for details). — A tree $T \subseteq 2^{<\omega}$ is *perfect* iff $\forall t \in T \; \exists s \supseteq t \; (s\hat{\ }\langle 0\rangle \in T \; \wedge s\hat{\ }\langle 1\rangle \in T)$. For a perfect tree T we let $[T] := \{f \in 2^\omega;\; \forall n \; (f{\restriction}n \in T)\}$ denote the set of its branches. Then $[T]$ is a perfect set (in the topology of 2^ω). Conversely, given a perfect set $S \subseteq 2^\omega$ there is perfect tree $T \subseteq 2^{<\omega}$ such that $[T] = S$. This allows us to confuse perfect sets and perfect trees in the sequel; in particular, we shall use the symbol T for both the tree and the set of its branches. — We will show in 3.1. that given models $M \subseteq N$ of ZFC such that there is a perfect set of random reals in N over M, either there is a dominating real in N over M or $\mu(2^\omega \cap M) = 0$ in N. This result is sharp and has some consequences concerning the relationship between cardinals related to measure and to the eventually dominating order on ω^ω (cf [BrJ, 1.9] and the discussion preceding 3.2. for details).

The argument for theorem 3.1. together with the techniques of § 1 yield the main result of section 4; namely, there is no perfect set of random reals over M in $M[r][d]$, where r is random over M, and d Hechler over $M[r]$ (4.2.). This answers questions 2 and 2' in [BrJ].

Notation. Our notation is fairly standard. We refer the reader to [Je] and [Ku] for set theory in general and forcing in particular.

Given a finite sequence s (i.e. either $s \in 2^{<\omega}$ or $s \in \omega^{<\omega}$), we let $lh(s) := dom(s)$ denote the length of s; for $\ell \in lh(s)$, $s{\restriction}\ell$ is the restriction of s to ℓ. $\hat{\ }$ is used for concatenation of sequences; and $\langle\rangle$ is the empty sequence. Given a perfect tree $T \subseteq 2^{<\omega}$ and $s \in T$, we let $T_s := \{t \in T;\; t \subseteq s \text{ or } s \subseteq t\}$. — Given a p.o. $\mathbb{P} \in V$, we shall denote $\mathbb{P}$-names by symbols like τ, $\breve{f}$, $\breve{T}$, ... and their interpretation in $V[G]$ (where G is $\mathbb{P}$-generic over V) by $\tau[G]$, $\breve{f}[G]$, $\breve{T}[G]$, ...

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§ 1. Prelude — a notion of rank for Hechler forcing

1.1. Main Definition (Shelah, see [GS, § 4] — cf also [BD, § 2]). Given $t \in \omega^{<\omega}$ strictly increasing and $A \subseteq \omega^{<\omega}$, we define by induction when the *rank* $rk(t, A)$ is α.

(a) $rk(t, A) = 0$ iff $t \in A$.

(b) $rk(t, A) = \alpha$ iff for no $\beta < \alpha$ we have $rk(t, A) = \beta$, but there are $m \in \omega$ and $\langle t_k;\ k \in \omega\rangle$ such that $\forall k \in \omega$: $t \subseteq t_k$, $t_k \in \omega^m$, $t_k(lh(t)) \geq k$, and $rk(t_k, A) < \alpha$. $\square$

Clearly, the rank is either $< \omega_1$ or undefined (in which case we say $rk = \infty$). We repeat the proof of the following result for it is the main tool for §§ 2 and 4.

1.2. Main Lemma (Baumgartner–Dordal [BD, § 2] and Shelah [GS, § 4]). *Let $I \subseteq \mathbb{D}$ be dense. Set $A := \{t;\ \exists f \in \omega^\omega$ such that $(t, f) \in I\}$. Then $rk(t^*, A) < \omega_1$ for any $t^* \in \omega^{<\omega}$.*

Proof. Suppose $rk(t^*, A) = \infty$ for some $t^* \in \omega^{<\omega}$. Let $S := \{s \in \omega^{<\omega}$ strictly increasing; $t^* \subseteq s$ and for all s^* with $t^* \subseteq s^*$ and with $\forall i \in dom(s^*) \backslash dom(t^*)$ $(s^*(i) \geq s(i))$, we have $rk(s^*, A) = \infty\}$. $S \subseteq \omega^{<\omega}$ is a tree with stem t^*.

Suppose S has an infinite branch $\langle s_i;\ i \in \omega\rangle$ (i.e. $s_0 = t^*$, $lh(s_i) = lh(t^*) + i$, and $s_i \subseteq s_{i+1}$). Let g be the function defined by this branch: $g = \bigcup_{i \in \omega} s_i$. Then $(t^*, g) \in \mathbb{D}$. Choose $(t, f) \leq (t^*, g)$ such that $(t, f) \in I$. Then $t \in A$, i.e. $rk(t, A) = 0$; but also $t \in S$, i.e. $rk(t, A) = \infty$, a contradiction.

So suppose S has no infinite branches, and let s^* be a maximal point in S. Then we have a sequence $\langle t_k;\ k \in \omega\rangle$ such that $lh(t_k) = lh(s^*) + 1$, $t_k(lh(s^*)) \geq k$, $t^* \subseteq t_k$, $\forall i \in dom(s^*) \backslash dom(t^*)$ $(t_k(i) \geq s^*(i))$, and $rk(t_k, A) < \infty$. Now we can find a subset $B \subseteq \omega$ and $lh(t^*) \leq m \leq lh(s^*)$ and $t \in \omega^m$ such that $\forall k \in B$ $(t_k \restriction m = t)$ and $k < \ell$, $k, \ell \in B$, implies $t_k(lh(t)) < t_\ell(lh(t))$. Hence the sequence $\langle t_k;\ k \in B\rangle$ witnesses $rk(t, A) < \infty$. On the other hand $t \in S$; i.e. $rk(t, A) = \infty$, again a contradiction. $\square$

Usually Hechler forcing $\mathbb{D}'$ is defined as follows.

$$(s, f) \in \mathbb{D}' \iff s \in \omega^{<\omega}\ \wedge\ f \in \omega^\omega\ \wedge\ s \subseteq f$$
$$(s, f) \leq (t, g) \iff s \supseteq t\ \wedge\ \forall n \in \omega\ (f(n) \geq g(n))$$

We sketch how to introduce a rank on $\mathbb{D}'$ having the same consequences as the one on $\mathbb{D}$ defined above. Let $\Omega = \{t;\ dom(t) \subseteq \omega\ \wedge\ |t| < \omega\ \wedge\ rng(t) \subseteq \omega\}$. Given $t \in \Omega$ and $A \subseteq \omega^{<\omega}$ we define by induction when the *rank* $rk(t, A)$ is α.

(a) $rk(t, A) = 0$ iff $t \in A$.

(b) $rk(t, A) = \alpha$ iff for no $\beta < \alpha$ we have $rk(t, A) = \beta$, but there are $M \in [\omega]^{<\omega}$ and $\langle t_k;\ k \in \omega\rangle$ such that $dom(t) \subset M$ and $\forall k \in \omega$: $t \subseteq t_k$, $t_k \in \omega^M$, $rk(t_k, A) < \alpha$ and $\forall i \in M \setminus dom(t)\ \forall k_1 \neq k_2\ (t_{k_1}(i) \neq t_{k_2}(i))$.

We leave it to the reader to verify that the result corresponding to 1.2. is true for this rank on $\mathbb{D}'$, and that the theorems of §§ 2 and 4 can be proved for $\mathbb{D}'$ in the same way as they are proved for $\mathbb{D}$.

§ 2. Application I — the effect of adding one Hechler real on the invariants in Cichoń's diagram

Before being able to state the main result of this section (the consequences of which will be 1) and 2) in the abstract) we have to set up some notation.

Let $\mathcal{A} \subseteq [\omega]^\omega$ be an a. d. family. We will produce a set of $\mathbb{D}$-names $\{\tau_A;\ A \in \mathcal{A}\}$ for functions in ω^ω as follows. For each $A \in \mathcal{A}$ fix $f_A : A \to \omega$ onto with $\forall n\ \exists^\infty m \in A\ (f_A(m) = n)$. Now, if $r \in \omega^\omega$ is a real having the property that $\{n \in \omega;\ r(n) \in A\}$ is infinite, let $g_r : \omega \to \omega$ be an enumeration of this set (i.e. $g_r(0) :=$ the least n such that $r(n) \in A$; $g_r(1) :=$ the least $n > g_r(0)$ such that $r(n) \in A$; etc.). In this case we let $\tau_A(r) : \omega \to \omega$ be defined as follows.

$$\tau_A(r)(n) := f_A(r(g_r(n))).$$

As A is infinite, we have $\Vdash_{\mathbb{D}}$"$|rng(\check{d}) \cap A| = \omega$", where $\check{d}$ is the name for the Hechler real; in particular $\tau_A(d)$ will be defined in the generic extension. Thus we can think of $\langle \tau_A(\check{d});\ A \in \mathcal{A} \rangle$ as a sequence of names in Hechler forcing for objects in ω^ω.

2.1. MAIN THEOREM. *Whenever $\mathcal{A} \subseteq [\omega]^\omega$ is an a. d. family in the ground model V, d is Hechler over V, and $f \in \omega^\omega$ is any real in $V[d]$, then $\{A \in \mathcal{A};\ \forall^\infty n\ (f(n) \neq \tau_A(d)(n))\}$ is at most countable (in $V[d]$).*

Remark. Slight changes in the proof show that, in fact, $\{\tau_A;\ A \in \mathcal{A}\}$ is a Luzin set in $V[d]$ for uncountable $\mathcal{A}$. (Recall that an uncountable set of reals is called *Luzin* iff for all meager sets M, $M \cap S$ is at most countable.)

Proof. The proof uses the main lemma (1.2.) as principal tool. Let $\check{f}$ be a $\mathbb{D}$-name for a real (for an element of ω^ω). Let I_n be the set of conditions deciding $\check{f}\restriction(n+1)$ $(n \in \omega)$. All I_n are dense. Let $D_n := \{t;\ \exists f \in \omega^\omega$ such that $(t, f) \in I_n\}$ (cf the main lemma). We want to define when a set $A \in \mathcal{A}$ is *n-bad.*

For each $t \in \omega^{<\omega} \setminus D_n$ strictly increasing we can find (according to the main lemma for D_n) an $m \in \omega$ and $\langle t_k;\ k \in \omega \rangle$ such that for all $k \in \omega$: t_k is strictly increasing, $t \subseteq t_k$, $t_k \in \omega^m$, $t_k(lh(t)) \geq k$, and $rk(t_k, D_n) < rk(t, D_n)$. Let $m_t := m - lh(t)$. We define by induction on $i < m_t$ when $A \in \mathcal{A}$ is *$t - i - n$-bad.* Along the way we also construct sets B_i $(i < m_t)$.

$i = 0$. Let $B_0 = \omega$. If there is $A \in \mathcal{A}$ such that $A \cap \{t_k(lh(t));\ k \in B_0\}$ is infinite, choose such an A_0 and let A_0 be $t - 0 - n$-bad. Now let $B_1 = \{k \in \omega;\ t_k(lh(t)) \in A_0\}$. If there is no such A, let $B_1 = B_0 = \omega$.

$i \to i+1$ $(i+1 < m_t)$. We assume that B_{i+1} is defined and infinite. If there is $A \in \mathcal{A}$ such that $A \cap \{t_k(lh(t)+i+1);\ k \in B_{i+1}\}$ is infinite, choose such an A_{i+1} and let A_{i+1} be $t - (i+1) - n$-bad. Now let $B_{i+2} = \{k \in B_{i+1};\ t_k(lh(t)+i+1) \in A_{i+1}\}$. If there is no such A, let $B_{i+2} = B_{i+1}$.

In the end, we set $B_t := B_{m_t}$. We say that $A \in \mathcal{A}$ is *n-bad* iff it is $t - i - n$-bad for some strictly increasing $t \in \omega^{<\omega} \setminus D_n$ and $i < m_t$. Finally $A \in \mathcal{A}$ is *bad* iff it is n-bad for some $n \in \omega$. Let $\mathcal{A}_{\check{f}} = \{A \in \mathcal{A};\ A$ bad $\}$. Since for $n \in \omega$, $t \in \omega^{<\omega}$ and $i < m_t$ at most one $A \in \mathcal{A}$ is $t - i - n$-bad, $\mathcal{A}_{\check{f}}$ is countable.

CLAIM. *If $A \in \mathcal{A} \setminus \mathcal{A}_{\check{f}}$, then $\Vdash_{\mathbb{D}} \exists^\infty n\ (\check{f}(n) = \tau_A(\check{d})(n))$.*

Remark. Clearly this claim finishes the proof of the main theorem.

Proof. Suppose not, and choose $(s,g) \in \mathbb{D}$, $k \in \omega$, and $A \in \mathcal{A} \setminus \mathcal{A}_{\breve{f}}$ such that

$$(s,g) \Vdash_{\mathbb{D}} \forall n \geq k\ (\breve{f}(n) \neq \tau_A(\breve{d})(n)).$$

Let $\ell \geq k$ be such that $|rng(s) \cap A| \leq \ell$; i.e. *s does not decide* the value of $\tau_A(\breve{d})(\ell)$. By increasing s, if necessary, we can assume that $|rng(s) \cap A| = \ell$. Let $Y := \{t \in \omega^{<\omega};\ t$ strictly increasing, $s \subseteq t$, $\forall i \in dom(t) \setminus dom(s)\ (t(i) \geq g(i))$, and $|rng(t) \cap A| = \ell\}$. Choose $t \in Y$ such that $rk(t, D_\ell)$ is minimal.

Subclaim. $rk(t, D_\ell) = 0$.

Proof. Suppose not. Then choose by the main lemma (1.2.) $m \in \omega$ and $\langle t_k;\ k \in \omega\rangle$ (i.e. all t_k are strictly increasing, $t \subseteq t_k$, $t_k \in \omega^m$, $t_k(lh(t)) \geq k$, and $rk(t_k, D_\ell) < rk(t, D_\ell)$). In fact, we require that m and $\langle t_k;\ k \in \omega\rangle$ are the same as the ones chosen for ℓ, t in the definition of ℓ-badness. Let $m_t = m - lh(t)$ as above, and look at B_t. By construction (as A is not $t-i-\ell$-bad for any $i < m_t$) and almost-disjointness, $A \cap \{t_k(lh(t)+i);\ k \in B_{i+1}\}$ is finite for all $i < m_t$. So there is $k \in B_t$ such that $rng(t_k) \cap A = rng(t) \cap A$, i.e. $|rng(t_k) \cap A| = \ell$, and $t_k(i) \geq g(i)$ for all $i \in dom(t_k) \setminus dom(s)$. Hence $t_k \in Y$ and $rk(t_k, D_\ell) < rk(t, D_\ell)$, contradicting the minimality of $rk(t, D_\ell)$. This proves the subclaim. $\square$

Continuation of the proof of the claim. As $rk(t, D_\ell) = 0$ we have an $h \in \omega^\omega$ such that $(t,h) \in I_\ell$. Then $(t, max(h,g)) \leq (s,g)$, and this condition decides the value of $\breve{f}$ at ℓ without deciding the value of $\tau_A(\breve{d})$ at ℓ. Suppose that $(t, max(h,g)) \Vdash_{\mathbb{D}}$"$\breve{f}(\ell) = j$". Now choose $i \geq max(h,g)(lh(t))$ such that $i \in A$ and $f_A(i) = j$ (this exists by the choice of the function f_A). Then

$$(t\hat{\ }\langle i\rangle, max(h,g)) \Vdash_{\mathbb{D}} \breve{f}(\ell) = j = f_A(i) = f_A(\breve{d}(g_{\breve{d}}(\ell))) = \tau_A(\breve{d})(\ell).$$

This final contradiction ends the proof of the claim and of the main theorem. $\square\,\square$

We will sketch how a modification of this argument gives the following result.

2.2. Theorem. *After adding one Hechler real d to V, there is a maximal almost disjoint family of subsets of ω of size ω_1 in $V[d]$.*

Sketch of proof. We start with an observation which will relate Luzin sets and maximal almost disjoint families.

Observation. *Let $\langle N_\alpha;\ \omega \leq \alpha < \omega_1\rangle$, $\langle h_\alpha;\ \omega \leq \alpha < \omega_1\rangle$ and $\langle r_\alpha;\ \omega \leq \alpha < \omega_1\rangle$ be sequences such that $N_\alpha \prec H(\kappa)$ is countable and $N_\alpha \prec N_\beta$ for $\alpha < \beta$, $h_\alpha \in \alpha^\omega \cap N_\alpha$ is* *one-to-one and onto, $r_\alpha \in \omega^\omega$ is Cohen over N_α and $\langle r_\alpha;\ \alpha < \beta\rangle \in N_\beta$. Define recursively sets C_α for $\alpha < \omega_1$. $\langle C_n;\ n \in \omega\rangle$ is a partition of ω into countable pieces lying in N_ω. For $\alpha \geq \omega$, $C_\alpha := \{r_\alpha(n);\ n \in \omega\ \wedge\ \forall m < n\ (r_\alpha(n) \notin C_{h_\alpha(m)})\}$. Then $\{C_\alpha;\ \alpha \in \omega_1\}$ is an a. d. family.*

Proof. The construction gives almost–disjointness. So it suffices to show that each C_α is infinite. But this follows from the fact that each r_α is Cohen over N_α and that the union of finitely many C_β's (for $\beta < \alpha$) is coinfinite. $\square$

Now let $\mathcal{A} = \langle A_\alpha;\ \alpha < \omega_1\rangle \in V$ be an a. d. family. As $\langle \tau_{A_\alpha}(d);\ \alpha < \omega_1\rangle$ is Luzin in $V[d]$ (see the remark following the statement of theorem 2.1.) we can find a strictly increasing function $\phi : \omega_1 \setminus \omega \to \omega_1$ and sequences $\langle N_\alpha;\ \omega \leq \alpha < \omega_1\rangle$, $\langle h_\alpha;\ \omega \leq \alpha < \omega_1\rangle$ such that for $r_\alpha := \tau_{A_{\phi(\alpha)}}(d)$ the requirements of the above observation are satisfied. By *ccc*–ness of $\mathbb{D}$, we may assume that $\phi \in V$; and hence, that $\phi = id$, thinning $\mathcal{A}$ out if necessary. We want to show that the resulting family $\langle C_\alpha;\ \alpha < \omega_1\rangle$ is a m. a. d. family.

For suppose not. Then there is a $\mathbb{D}$-name $\breve{C}$ such that

$$\Vdash_{\mathbb{D}} \forall \alpha < \omega_1\ (|\breve{C}_\alpha \cap \breve{C}| < \omega).$$

Let $\breve{f}$ be the $\mathbb{D}$-name for the strictly increasing enumeration of $\breve{C}$. As in the proof of 2.1. we let I_n be the set of conditions deciding $\breve{f}\restriction(n+1)$, $D_n := \{t;\ \exists f \in \omega^\omega\ ((t,f) \in I_n)\}$, and we define when a set $A \in \mathcal{A}$ is n-bad (so that at most countably many sets will be n-bad).

Furthermore, for each $\alpha < \omega_1$ we let σ_α be the $\mathbb{D}$-name for a natural number such that

$$\Vdash_{\mathbb{D}} \breve{C}_\alpha \cap \breve{C} \subseteq \sigma_\alpha.$$

We let I'_α be the set of conditions deciding σ_α, $D'_\alpha := \{t;\ \exists f \in \omega^\omega\ ((t,f) \in I'_\alpha)\}$; analogously to the proof of theorem 2.1. we define when a set $A \in \mathcal{A}$ is α-bad (so that at most countably many sets will be α-bad).

Next choose $\alpha < \omega_1$ such that

1) if A_β is n-bad for some n, then $\beta < \alpha$;

2) if $\beta < \alpha$ and A_γ is β-bad, then $\gamma < \alpha$.

Claim. $\Vdash_{\mathbb{D}} |\breve{C}_\alpha \cap \breve{C}| = \omega$.

Proof. Suppose not, and choose $(s,g) \in \mathbb{D}$ and $k \in \omega$ such that

$$(s,g) \Vdash_{\mathbb{D}} \breve{C}_\alpha \cap \breve{C} \subseteq k.$$

Let $\ell \geq k$ be such that $|rng(s) \cap A_\alpha| \leq \ell$; without loss $|rng(s) \cap A_\alpha| = \ell$. Let $Y := \{t \in \omega^{<\omega};\ t$ strictly increasing, $s \subseteq t$, $\forall i \in dom(t) \setminus dom(s)\ (t(i) \geq g(i))$, and $|rng(t) \cap A_\alpha| = \ell\}$. By the argument of the subclaim in the proof of 2.1. there is a $t \in Y$ such that $\forall m < \ell\ (rk(t, D'_{h_\alpha(m)}) = 0)$. Hence there is an $h \in \omega^\omega$ such that $(t,h) \in \bigcap_{m<\ell} I'_{h_\alpha(m)}$. Without loss $h \geq g$. Then $(t,h) \leq (s,g)$, and this condition decides the values of $\sigma_{h_\alpha(m)}$ $(m < \ell)$; suppose that $(t,h) \Vdash_{\mathbb{D}}$"$\forall m < \ell\ (\sigma_{h_\alpha(m)} = s_m)$". Choose ℓ' larger that the maximum of the s_m $(m < \ell)$ and k. Again using the argument of the subclaim (2.1.) find $t' \supseteq t$ such that $\forall i \in dom(t') \setminus dom(t)\ (t'(i) \geq h(i))$, $|rng(t') \cap A_\alpha| = \ell$, and $rk(t', D_{\ell'}) = 0$. Thus there exists an $h' \in \omega^\omega$ such that

$$(t', h') \Vdash_{\mathbb{D}} \text{"}\breve{f}(\ell') = j\text{" for some } j.$$

Without loss $h' \geq h$. Then $(t',h') \leq (t,h)$. As $\Vdash_{\mathbb{D}}$"$\breve{f}$ is strictly increasing", $j \geq \ell' \geq k$; by construction we have in particular that $(t',h') \Vdash_{\mathbb{D}}$"$\forall m < \ell\ (j \notin \check{C}_{h_\alpha(m)})$". Choose $i \geq h'(lh(t'))$ such that $i \in A_\alpha$ and $f_{A_\alpha}(i) = j$. Then

$$(t'\hat{\ }\langle i\rangle, h') \Vdash_{\mathbb{D}} \breve{f}(\ell') = j = f_{A_\alpha}(i) = \tau_{A_\alpha}(\breve{d})(\ell) = \breve{r}_\alpha(\ell) \in \check{C}_\alpha.$$

This final contradiction proves the claim, and the theorem as well. $\square\,\square$

In our proof we constructed a m. a. d. family of size ω_1 from a Luzin set in $V[d]$. We do not know whether this can be done in ZFC.

2.3. Question (Fleissner, see [Mi, 4.7.]) *Does the existence of a Luzin set imply the existence of a m. a. d. family of size ω_1?*

Remark. It is consistent that there is a m. a. d. family of size ω_1, but no Luzin set. This is known to be true in the model obtained by adding at least ω_2 random reals to a model of $ZFC + CH$.

We next turn to consequences of theorem 2.1.

2.4. Theorem. *Let $V \subseteq W$ be universes of set theory, $\omega_1^V = \omega_1^W$. Then no real in $W[d]$ is eventually different from the reals in $V[d]$, where d is Hechler over V.*

Remark. Remember that Hechler forcing has an absolute definition. So d will be Hechler over V as well.

Proof. Let $\mathcal{A} \subseteq [\omega]^\omega$ be an almost disjoint family in V of size ω_1. Assume that the functions f_A for $A \in \mathcal{A}$ (defined at the beginning of this section) are also in V. Then each real in $W[d]$ can only be eventually different from countably many of the reals in $\{\tau_A(d);\ A \in \mathcal{A}\} \in V[d]$, by the main theorem. $\square$

To be able to explain our next corollary to the main theorem, we need to introduce a few cardinals. Given a σ-ideal $\mathcal{I} \subseteq P(2^\omega)$, we let

$$\begin{aligned} add(\mathcal{I}) &:= \text{ the least } \kappa \text{ such that } \exists \mathcal{F} \in [\mathcal{I}]^\kappa \ (\bigcup \mathcal{F} \notin \mathcal{I}); \\ cov(\mathcal{I}) &:= \text{ the least } \kappa \text{ such that } \exists \mathcal{F} \in [\mathcal{I}]^\kappa \ (\bigcup \mathcal{F} = 2^\omega); \\ unif(\mathcal{I}) &:= \text{ the least } \kappa \text{ such that } [2^\omega]^\kappa \setminus \mathcal{I} \neq \emptyset; \\ cof(\mathcal{I}) &:= \text{ the least } \kappa \text{ such that } \exists \mathcal{F} \in [\mathcal{I}]^\kappa \ \forall A \in \mathcal{I} \ \exists B \in \mathcal{F} \ (A \subseteq B). \end{aligned}$$

We also define

$$\begin{aligned} b &:= \text{ the least } \kappa \text{ such that } \exists \mathcal{F} \in [\omega^\omega]^\kappa \ \forall f \in \omega^\omega \ \exists g \in \mathcal{F} \ \exists^\infty n \ (g(n) > f(n)); \\ d &:= \text{ the least } \kappa \text{ such that } \exists \mathcal{F} \in [\omega^\omega]^\kappa \ \forall f \in \omega^\omega \ \exists g \in \mathcal{F} \ \forall^\infty n \ (g(n) > f(n)). \end{aligned}$$

If $\mathcal{M}$ is the ideal of meager sets, and $\mathcal{N}$ is the ideal of null sets, then we can arrange these cardinals in the following diagram (called Cichoń's diagram).

$$\begin{array}{cccccc} & cov(\mathcal{N}) & unif(\mathcal{M}) & cof(\mathcal{M}) & cof(\mathcal{N}) & 2^\omega \\ & & & & & \\ & & b & d & & \\ & & & & & \\ \omega_1 & add(\mathcal{N}) & add(\mathcal{M}) & cov(\mathcal{M}) & unif(\mathcal{N}) & \end{array}$$

(Here, the invariants grow larger, as one moves up and to the right in the diagram.) The dotted line says that $add(\mathcal{M}) = min\{b, cov(\mathcal{M})\}$ and $cof(\mathcal{M}) = max\{d, unif(\mathcal{M})\}$. For the results which determine the shape of this diagram, we refer the reader to [Fr]. A survey on independence proofs showing that no other relations can be proved between these cardinals can be found in [BJS]. We shall need the following characterizations of the cardinals $unif(\mathcal{M})$ and $cov(\mathcal{M})$, which are due to Bartoszyński [Ba].

$$\begin{aligned} unif(\mathcal{M}) &= \text{ the least } \kappa \text{ such that } \exists \mathcal{F} \in [\omega^\omega]^\kappa \ \forall g \in \omega^\omega \ \exists f \in \mathcal{F} \ \exists^\infty n \ (f(n) = g(n)); \\ cov(\mathcal{M}) &= \text{ the least } \kappa \text{ such that } \exists \mathcal{F} \in [\omega^\omega]^\kappa \ \forall g \in \omega^\omega \ \exists f \in \mathcal{F} \ \forall^\infty n \ (f(n) \neq g(n)). \end{aligned}$$

We are ready to give our next result, which says essentially that after adding one Hechler real, the invariants on the left-hand side of the above diagram all equal ω_1, whereas those on the right-hand side are all equal to 2^ω.

2.5. THEOREM. *After adding one Hechler real d to V, $unif(\mathcal{M}) = \omega_1$ and $cov(\mathcal{M}) = 2^\omega$ in $V[d]$.*

Proof. (i) Let $\mathcal{A} \subseteq [\omega]^\omega$ be an a. d. family of size ω_1 in V. Then by the main theorem no real is eventually different from $\{\tau_A(d);\ A \in \mathcal{A}\}$, giving $unif(\mathcal{M}) = \omega_1$ (by Bartoszyński's characterization).

(ii) Let $\mathcal{A} \subseteq [\omega]^\omega$ be an a. d. family of size 2^ω in V (such a family exists, see e.g. [Ku, chapter II, theorem 1.3]). Suppose $\kappa = cov(\mathcal{M}) < 2^\omega$, and let $\{g_\alpha;\ \alpha < \kappa\}$ be a family of functions such that $\forall g \in V[d] \cap \omega^\omega\ \exists \alpha < \kappa\ \forall^\infty n\ (g(n) \neq g_\alpha(n))$, using Bartoszyński's characterization. As $|\mathcal{A}| = 2^\omega > \kappa$, there is $\mathcal{A}' \subseteq \mathcal{A}$, $|\mathcal{A}'| \geq \omega_1$, and $\alpha < \kappa$ such that $\forall A \in \mathcal{A}'\ \forall^\infty n\ (\tau_A(d)(n) \neq g_\alpha(n))$. This contradicts the main theorem. $\square$

Remark. Instead of Bartoszyński's characterization we could have used the fact that $\{\tau_A(d);\ A \in \mathcal{A}\}$ is a Luzin set (see the remark after 2.1.). We leave it to the reader to verify that the existence of a Luzin set implies $unif(\mathcal{M}) = \omega_1$; and that the existence of a Luzin set of size 2^ω implies $cov(\mathcal{M}) = 2^\omega$.

We close with an application concerning absoluteness in the projective hierarchy. We first recall a notion due to the second author [Ju, § 2]. Given a universe of set theory V and a forcing notion $\mathbb{P} \in V$ we say that V is $\Sigma^1_n - \mathbb{P}$*-absolute* iff for every Σ^1_n-sentence ϕ with parameters in V we have $V \models \phi$ iff $V^{\mathbb{P}} \models \phi$. So this is equivalent to saying that $\mathbb{R}^V \prec_{\Sigma^1_n} \mathbb{R}^{V^{\mathbb{P}}}$. Note that Shoenfield's Absoluteness Lemma [Je, theorem 98] says that V is alway $\Sigma^1_2 - \mathbb{P}$-absolute. Furthermore, $\Sigma^1_3 - \mathbb{D}$- absoluteness is equivalent to *all Σ^1_2-sets have the property of Baire* [Ju, § 2]. This is a consequence of Solovay's classical characterization of the latter statement which says that it is equivalent to: *for all reals a, the set of reals Cohen over $L[a]$ is comeager.*

2.6. THEOREM. *$\Sigma^1_4 - \mathbb{D}$-absoluteness implies that $\omega_1 > \omega_1^{L[r]}$ for any real r.*

Proof. Suppose there is an $a \in \mathbb{R}$ such that $\omega_1^{L[a]} = \omega_1^V$. By $\Sigma^1_3 - \mathbb{D}$-absoluteness we have that all Σ^1_2-sets have the property of Baire (see above); i.e. $\forall b \in \mathbb{R}$ ($Co(L[b])$ is comeager) ($Co(M)$ denotes the set of reals Cohen over some model M of ZFC). Note that $x \in Co(L[b])$ is equivalent to

$$\forall c\ (c \notin L[b] \cap BC\ \vee\ \hat{c} \text{ is not meager} \vee x \notin \hat{c}),$$

where BC is the set of Borel codes which is Π^1_1 [Je, lemma 42.1], and for $c \in BC$, $\hat{c}$ is the set coded by c. As $L[b]$ is Σ^1_2 [Je, lemma 41.1], $Co(L[b])$ is a Π^1_2-set. Hence $\forall b \in \mathbb{R}$ ($Co(L[b])$ is comeager) which is equivalent to

$$\forall b \exists c \, (c \in BC \;\wedge\; \hat{c} \text{ is meager} \wedge \forall x \, (x \in \hat{c} \;\vee\; x \in Co(L[b])))$$

is a Π^1_4-sentence. So it is true in $V^{\mathbb{D}}$ by $\Sigma^1_4 - \mathbb{D}$-absoluteness; in particular $Co(L[a][d])$ is comeager in $V[d]$ which implies that there is a dominating real in $V[d]$ over $L[a][d]$, contradicting theorem 2.4. □

2.7. Question. *Are there results similar to theorems 2.4., 2.5., and 2.6. for Amoeba forcing or Amoeba-meager forcing?*

We conjecture that the answer is yes because both the Amoeba algebra and the Amoeba-meager algebra contain $\mathbb{D}$ as a complete subalgebra (see [Tr, § 6]; a definition of the algebras can also be found there). But there doesn't seem to be a way to introduce a rank on these algebras (as in § 1).

§ 3. Interlude — perfect sets of random reals

3.1. Theorem. *Let $V \subseteq W$ be models of ZFC. Suppose there is a perfect set of random reals in W over V. Then either*

1) there is a dominating real in W over V; or

2) $\mu(2^\omega \cap V) = 0$ in W.

Proof. Suppose not, and let $T \in W$ be a perfect set of random reals. Define $f \in \omega^\omega \cap W$ as follows.

$$f(i) = min\{k; \; \forall \sigma \in T \cap 2^i \; (|T_\sigma \cap 2^k| > 4^i)\}$$

Let $g \in \omega^\omega \cap V$ be such that $\exists^\infty i \; (g(i) \geq f(i))$. Let U be the family of all $u \in \prod_{i \in \omega} P(2^{g(i)})$ such that $u(i) \subseteq 2^{g(i)}$ and $\frac{|u(i)|}{2^{g(i)}} = 2^{-i}$. U can be thought of as a measure space (namely, for $u \subseteq 2^{g(i)}$ with $\frac{|u|}{2^{g(i)}} = 2^{-i}$ let $\mu_i(u) = \frac{1}{\binom{2^{g(i)}}{2^{g(i)-i}}}$; and let μ be the product measure of the μ_i).

Let $N \prec \langle H(\kappa)^W, ...\rangle$ be countable with $g, T \in N$. As $\mu(2^\omega \cap V) \neq 0$ in W, we cannot have that $2^\omega \cap V \subseteq \cup\{B;\ \mu(B) = 0,\ B \in N,\ B \text{ Borel }\}$; i.e. there are reals in V which are random over N. Let $u^* \in U$ be such a real. Using u^* we can define a measure zero set B in V as follows.

$$B = \{h \in 2^\omega;\ \exists^\infty i\ (h\restriction g(i) \in u^*(i))\}$$

Let (for $k \in \omega$) $B_k = \{h \in 2^\omega;\ \forall i \geq k\ (h\restriction g(i) \notin u^*(i))\}$. Clearly $2^\omega \setminus B = \cup_{k\in\omega} B_k$; and the B_k form an increasing chain of perfect sets of positive measure.

As all reals in T are random over V we must have $T \subseteq \cup_{k\in\omega} B_k$. This gives us $\sigma \in T$ and $k \in \omega$ such that $T_\sigma \subseteq B_k$ (otherwise choose $\sigma_0 \in T$ such that $\sigma_0 \notin B_0$, $\sigma_1 \in T_{\sigma_0}$ such that $\sigma_1 \notin B_1$, etc. This way we construct a branch in T which does not lie in $\cup_{k\in\omega} B_k$, a contradiction).

By construction, we know that for infinitely many i, we have $|T_\sigma \cap 2^{g(i)}| > 4^i$ and $u^*(i) \cap (T_\sigma \cap 2^{g(i)}) = \emptyset$. For each such i and $u \subseteq 2^{g(i)}$ with $\frac{|u|}{2^{g(i)}} = 2^{-i}$, the probability that $u \cap (T_\sigma \cap 2^{g(i)}) = \emptyset$ (in the sense of the measure μ_i defined above) is

$$\leq (\frac{2^{g(i)} - 4^i}{2^{g(i)}})^{2^{g(i)-i}} \leq (e^{-\frac{4^i}{2^{g(i)}}})^{2^{g(i)-i}} = e^{-2^i}.$$

So the probability that this happens infinitely often is zero. But u^* is random over N, a contradiction. $\square$

Corollary (Cichoń [BaJ, § 2]). *If r is random over V, then there is no perfect set of random reals in $V[r]$ over V.* $\square$

Remark. Theorem 3.1. is best possible in the following sense.

1) It is consistent that there are $V \subseteq W$ and a perfect tree T of random reals in W over V and $\mu^*(2^\omega \cap V) > 0$ in W (μ^* denotes outer measure). To see this add a Laver real ℓ to V and then a random real r to $V[\ell]$; set $W = V[\ell][r]$. By [BaJ, theorem 2.7] there is a perfect tree of random reals in W over V; and by [JS, § 1] $\mu^*(2^\omega \cap V) > 0$ in $V[\ell]$ and hence in W.
2) It is consistent that there are $V \subseteq W$ and a perfect tree T of random reals in W over V and no dominating real in W over V (see [BrJ, theorem 1]).

Before being able to state some consequences of this result, we need to introduce two further cardinals.

$wcov(\mathcal{N}) :=$ the least κ such that $\exists \mathcal{F} \in [\mathcal{N}]^\kappa$ $(2^\omega \setminus \bigcup \mathcal{F}$ does not contain a perfect set);

$wunif(\mathcal{N}) :=$ the least κ such that there is a family $\mathcal{F} \in [[2^{<\omega}]^\omega]^\kappa$ of perfect sets with $\forall N \in \mathcal{N}\ \exists T \in \mathcal{F}\ (N \cap T = \emptyset)$.

We can arrange these cardinals and some of those of the preceding section in the following diagram.

$$\begin{array}{ccc}
 & 2^\omega & \\
 & cof(\mathcal{N}) & \\
cov(\mathcal{N}) & d & wunif(\mathcal{N}) \\
wcov(\mathcal{N}) & b & unif(\mathcal{N}) \\
 & add(\mathcal{N}) & \\
 & \omega_1 &
\end{array}$$

(Here the invariants get larger as one moves up in the diagram.) The dotted line says that $wcov(\mathcal{N}) \geq min\{cov(\mathcal{N}), b\}$ (and dually, $wunif(\mathcal{N}) \leq max\{unif(\mathcal{N}), d\}$) (see [BaJ, § 2] or [BrJ, 1.9]). Using the above result we get

3.2. Theorem. *(i)* $wcov(\mathcal{N}) \leq max\{b, unif(\mathcal{N})\}$;

(ii) $wunif(\mathcal{N}) \geq min\{d, cov(\mathcal{N})\}$ *— In fact, given* $V \subseteq W$ *models of ZFC such that in* W *there is a real which is random over a real which is unbounded over* V*, there exists a null set* $N \in W$ *such that for all perfect sets* $T \in V$, $T \cap N \neq \emptyset$.

Proof. (i) follows immediately from theorem 3.1; and the first sentence of (ii) follows from the last sentence of (ii). The latter is proved by an argument which closely follows the lines of the proof of theorem 3.1, and is therefore left to the reader. □

The most interesting question concerning the relationship of the cardinals in the above diagram is the following (question 3' of [BrJ]).

3.3. Question. *Is it consistent that* $wcov(\mathcal{N}) > d$*? Dually, is it consistent that* $wunif(\mathcal{N}) < b$*?*

§ 4. Application II — adding a Hechler real over a random real does not produce a perfect set of random reals

4.1. Theorem. *Let $V \subseteq W$ be models of ZFC such that*

1) there is no dominating real in W over V;

2) $2^\omega \cap V$ is non-measurable in W.

Then there is no perfect set of random reals in $W[d]$, where d is Hechler over W.

Remark. This result clearly contains theorem 3.1. as a special case; still we decided to bring the latter as a separate result because it has consequences for the cardinals involved (see above, 3.2.). Also, the proof of theorem 4.1. can be seen as a combination of the argument for 3.1. and the techniques developed in § 1.

4.2. Corollary. *There is no perfect set of random reals in $V[r][d]$, where r is random over V, and d is Hechler over $W = V[r]$.* $\square$

Proof of theorem 4.1. We work in W. Let $\breve{T}$ be a $\mathbb{D}$-name for a perfect tree. We want to show that $T = \breve{T}[G]$ (G $\mathbb{D}$-generic over W) contains reals which are not random over V. We say that $A \subseteq \omega^{<\omega}$ is *large* iff $\forall (s,f) \in \mathbb{D}\ \exists s' \in A$ with $(s',f) \leq (s,f)$ (By (s',f) we mean here and in the sequel the condition (s',f') where $f' \restriction dom(s') = s'$ and $f'(n) = f(n)$ for $n \geq dom(s')$).

Claim. *The following set A is large: $s \in A \Longleftrightarrow$ for some $k < \omega$ and $\langle t_\ell, f^1_\ell, f^2_\ell;\ \ell \in \omega\rangle$ we have $s \subseteq t_\ell$, $t_\ell \in \omega^k$, $t_\ell(lh(s)) \geq \ell$, $f^1_\ell \neq f^2_\ell \in 2^\omega$, $f^1_\ell \restriction \ell = f^2_\ell \restriction \ell$, and $\forall f \in \omega^\omega$ (with $t_\ell \subseteq f$) $\forall m \in \omega\ \forall i \in \{1,2\}$ $((t_\ell, f) \not\Vdash f^i_\ell \restriction m \notin \breve{T})$.*

Proof. Let $sp\,\breve{T}$ be the $\mathbb{D}$-name for the subset of ω which describes the levels at which there is a splitting node in $\breve{T}$. By thinning out T (in the generic extension) if necessary, we may assume that

$$\Vdash_{\mathbb{D}} \text{ the } j\text{-th member of } sp\,\breve{T} \text{ (denoted by } \tau_j) \text{ is } > \breve{d}(j),$$

where $\breve{d}$ is (as always) the $\mathbb{D}$-name for the Hechler real. Let $(s^*, f^*) \in \mathbb{D}$, $lh(s^*) = j^*$. So (s^*, f^*) forces no bound on τ_{j^*} — even no (s^*, f') does (*). We assume there is no $s \in A$ with $(s, f^*) \leq (s^*, f^*)$ and reach a contradiction.

Let I be the dense set of conditions forcing a value to τ_{j^*}; and let $B = \{s \in \omega^{<\omega};\ \exists f \in \omega^\omega\ ((s,f) \in I)\}$. By the main lemma 1.2. we have $rk(s^*, B) < \omega_1$. We prove by induction on the ordinal $\beta < \omega_1$

(**) *if* $s \in \omega^{<\omega}$ *is such that* $(s, f^*) \le (s^*, f^*)$ *and* $rk(s, B) = \beta$, *then* $\exists m < \omega\ \forall f \in \omega^\omega$ *(with* $s \subseteq f$*)* $((s, f) \not\Vdash \tau_{j^*} \ne m)$.

If we succeed for $s = s^*$ then we get a contradiction to (*).

$\beta = 0$. So $s \in B$. Thus for some $f' \ge f^*$, (s, f') forces a value to τ_{j^*}: $(s, f') \Vdash \tau_{j^*} = m$, for some $m \in \omega$, giving (**).

$\beta > 0$. By the definition of rank there are $k \in \omega$, $t_\ell \in \omega^k$ $(\ell \in \omega)$ such that $s \subseteq t_\ell$, $t_\ell(lh(s)) \ge \ell$, and $rk(t_\ell, B) = \beta_\ell < \beta$. (We consider only ℓ with $\ell \ge max(rng(f^* \restriction k))$.) By induction hypothesis there are $m_\ell \in \omega$ such that $\forall f \in \omega^\omega$ (with $t_\ell \subseteq f$) $((t_\ell, f) \not\Vdash \tau_{j^*} \ne m_\ell)$. We consider two subcases.

Case 1. For some m we have infinitely many ℓ such that $m_\ell = m$. Then we can use this m for s and get (**).

Case 2. $\langle m_\ell;\ \ell \in \omega\rangle$ converges to ∞. Replacing it by a subsequence, if necessary, we may assume that it is strictly increasing. We show that $\langle t_\ell;\ \ell \in \omega\rangle$ witnesses $s \in A$, contradicting our initial assumption.

For each ℓ let $T_\ell = \{\rho \in 2^{<\omega};\ \text{for no } f \in \omega^\omega \text{ does } (t_\ell, f) \Vdash \rho \notin \check{T}\}$. Clearly $T_\ell \subseteq 2^{<\omega}$, $\langle\rangle \in T_\ell$, and T_ℓ is closed under initial segments. Also we have that $\rho \in T_\ell$ implies either $\rho\hat{\ }\langle 0\rangle \in T_\ell$ or $\rho\hat{\ }\langle 1\rangle \in T_\ell$ (otherwise we can find $f_0, f_1 \in \omega^\omega$ such that $(t_\ell, f_0) \Vdash \rho\hat{\ }\langle 0\rangle \notin \check{T}$ and $(t_\ell, f_1) \Vdash \rho\hat{\ }\langle 1\rangle \notin \check{T}$; let $f = max\{f_0, f_1\}$; choose $p \le (t_\ell, f)$ such that $p \Vdash \rho \in \check{T}$ (by assumption on ρ); but then there exists $q \le p$ such that either $q \Vdash \rho\hat{\ }\langle 0\rangle \in \check{T}$ or $q \Vdash \rho\hat{\ }\langle 1\rangle \in \check{T}$, a contradiction).

Finally, T_ℓ has a splitting node at level m_ℓ; i.e. for some $\rho = \rho_\ell \in T_\ell \cap 2^{m_\ell}$, we have $\rho\hat{\ }\langle 0\rangle \in T_\ell$ and $\rho\hat{\ }\langle 1\rangle \in T_\ell$ (if not, for each $\rho \in 2^{m_\ell}$ $\exists f_\rho \in \omega^\omega$ such that $(t_\ell, f_\rho) \Vdash$"$\rho\hat{\ }\langle 0\rangle \notin \check{T}$ or $\rho\hat{\ }\langle 1\rangle \notin \check{T}$"; let $f = max\{f_\rho;\ \rho \in 2^{m_\ell}\}$. We know that $(t_\ell, f) \not\Vdash m_\ell \ne \tau_{j^*}$; so there is $p \le (t_\ell, f)$ such that $p \Vdash m_\ell = \tau_{j^*}$; i.e. $p \Vdash m_\ell \in sp\ \check{T}$; we now get a contradiction as before).

Hence we can find $f_\ell^1, f_\ell^2 \in [T_\ell]$ such that $f_\ell^1 \restriction (m_\ell + 1) = \rho_\ell\hat{\ }\langle 0\rangle$ and $f_\ell^2 \restriction (m_\ell + 1) = \rho_\ell\hat{\ }\langle 1\rangle$. Thus $\langle t_\ell;\ \ell \in \omega\rangle$, $\langle f_\ell^1, f_\ell^2;\ \ell \in \omega\rangle$ witness $s \in A$. This final contradiction proves the claim. $\square$

Continuation of the proof of the theorem. We assume that $\Vdash_{\mathbb{D}} \check{T} = \{\tau_j;\ j \in \omega\}$; i.e. $\tau_j[G]$ $(j \in \omega)$ will enumerate the tree $T = \check{T}[G]$ in the generic extension. We also let $\check{T}_j$ be the name for the tree $T_{\tau_j[G]}$; i.e. $\Vdash_{\mathbb{D}} \check{T}_j = \{\nu \in \check{T};\ \nu \subseteq \tau_j \text{ or } \tau_j \subseteq \nu\}$. For each $j \in \omega$

there is — according to the claim for $\breve{T}_j$ instead of $\breve{T}$ — a large set $A_j \subseteq \omega^{<\omega}$; and for $s \in A_j$ there is a sequence $\langle t_\ell^{s,j}, f_\ell^{1,s,j}, f_\ell^{2,s,j};\ \ell \in \omega\rangle$ that witnesses $s \in A_j$. For every $j \in \omega$, $s \in A_j$ and $m \in \omega$ we define $S_{j,s,m} = \{f_\ell^{i,s,j} \upharpoonright k;\ k \in \omega,\ i \in \{1,2\},\ m \le \ell \in \omega\}$. By construction the function $f_{j,s,m}$ defined by $f_{j,s,m}(k) = |S_{j,s,m} \cap 2^k|$ converges to ∞. By assumption 1) we can choose $g \in \omega^\omega \cap M$ such that $\forall j, s, m\ \exists^\infty i\ (|S_{j,s,m} \cap 2^{g(i)}| > 4^i)$.

Now let U be as in the proof of theorem 3.1.; and choose $u^* \in U$ as there (i.e. u^* is random over a countable model N containing g and all $S_{j,s,m}$ — using assumption 2)). We also define B and B_k ($k \in \omega$) as in the proof of theorem 3.1.

We assume that $\Vdash_{\mathbb{D}}$"$\breve{T}$ is a perfect set of reals random over V"; in particular $\Vdash_{\mathbb{D}} \breve{T} \subseteq \bigcup_{k\in\omega} B_k$. So there are $(s^*, f^*) \in \mathbb{D}$, $j \in \omega$ and $k \in \omega$ such that

$$(s^*, f^*) \Vdash_{\mathbb{D}} \breve{T}_j \subseteq B_k$$

(cf the corresponding argument in the proof of theorem 3.1.). Without loss $s^* \in A_j$ (otherwise increase the condition using the claim). Let $m > max(rng(f^* \upharpoonright k_{j,s^*}))$ where k_{j,s^*} is such that for all $\ell \in \omega$, $t_\ell^{s^*,j} \in \omega^{k_{j,s^*}}$. Then $\forall \ell \ge m$, $(t_\ell^{s^*,j}, f^*)$ is an extension of (s^*, f^*). So we must have $S_{j,s^*,m} \subseteq B_k$ (because for any element of the former set we have an extension of (s^*, f^*) forcing this element into $\breve{T}_j$).

The rest of the proof is again as in the proof of theorem 3.1. For infinitely many i we have $|S_{j,s^*,m} \cap 2^{g(i)}| > 4^i$; for each such i, the probability that $u^*(i) \cap (S_{j,s^*,m} \cap 2^{g(i)}) = \emptyset$ is $\le e^{-2^i}$; the probability that this happens infinitely often is zero, contradicting the fact that u^* is random over N. $\square$

References

[Ba] T. Bartoszyński, *Combinatorial aspects of measure and category,* Fundamenta Mathematicae, vol. 127 (1987), pp. 225-239.

[BaJ] T. Bartoszyński and H. Judah, *Jumping with random reals,* Annals of Pure and Applied Logic, vol. 48 (1990), pp. 197-213.

[BJS] T. Bartoszyński, H. Judah and S. Shelah, *The Cichoń diagram,* submitted to Journal of Symbolic Logic.

[BD] J. Baumgartner and P. Dordal, *Adjoining dominating functions,* Journal of Symbolic Logic, vol. 50 (1985), pp. 94-101.

[BrJ] J. Brendle and H. Judah, *Perfect sets of random reals,* submitted to Israel Journal of Mathematics.

[CP] J. Cichoń and J. Pawlikowski, *On ideals of subsets of the plane and on Cohen reals,* Journal of Symbolic Logic, vol. 51 (1986), pp. 560-569.

[Fr] D. Fremlin, *Cichoń's diagram,* Séminaire Initiation à l'Analyse (G. Choquet, M. Rogalski, J. Saint Raymond), Publications Mathématiques de l'Université Pierre et Marie Curie, Paris, 1984, pp. 5-01 - 5-13.

[GS] M. Gitik and S. Shelah, *More on ideals with simple forcing notions,* to appear in Annals of Pure and Applied Logic.

[Je] T. Jech, *Set theory,* Academic Press, San Diego, 1978.

[Ju] H. Judah, *Absoluteness for projective sets,* to appear in Logic Colloquium 1990.

[JS] H. Judah and S. Shelah, *The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing),* Journal of Symbolic Logic, vol. 55 (1990), pp. 909-927.

[Ku] K. Kunen, *Set theory,* North-Holland, Amsterdam, 1980.

[Mi] A. Miller, *Arnie Miller's problem list,* to appear in Proccedings of the Bar-Ilan conference on set theory of the reals, 1991.

[Pa] J. Pawlikowski, *Why Solovay real produces Cohen real,* Journal of Symbolic Logic, vol. 51 (1986), pp. 957-968.

[Tr] J. Truss, *Sets having calibre $\aleph_1$,* Logic Colloquium 76, North-Holland, Amsterdam, 1977, pp. 595-612.