

# Cofinalities of Elementary Substructures of Structures on $\aleph_\omega$

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## 1. INTRODUCTION.

Let  $0 < n^* < \omega$  and  $f : X \rightarrow n^* + 1$  be a function where  $X \subseteq \omega \setminus (n^* + 1)$  is infinite. Let's consider the following set

$$S_f = \{x \subset \aleph_\omega : |x| \leq \aleph_{n^*} \wedge (\forall n \in X) cf(x \cap \alpha_n) = \aleph_{f(n)}\}.$$

The question is whether  $S_f$  is stationary in  $[\alpha_\omega]^{<\aleph_{n^*+1}}$ . The question was first posed by Baumgartner in [B7]. By a standard result, the above question can also be rephrased as certain transfer property. Namely,  $S_f$  is stationary iff for any structure  $A = \langle \aleph_\omega, \dots \rangle$  there's a  $B \prec A$  such that  $|B| = \aleph_{n^*}$  and for all  $n \in X$  we have  $cf(B \cap \aleph_n) = \aleph_{f(n)}$ .

Note that if  $f$  is eventually constant then  $S_f$  is always stationary. (see [B7]). Also, any reasonable "finite variation" of  $f$  will preserve the property, i.e., if  $n_1^* > n^*$  and  $f_1 : X \rightarrow n^* + 1$  agrees with  $f$  on  $\text{dom}(f) \setminus (n_1^* + 1)$ , then  $S_{f_1}$  is stationary provided that  $S_f$  is. So we are interested in the case that  $f$  is not eventually constant. You may wonder how strong the statement that  $S_f$  is stationary is. Magidor (unpublished, but close to [Mg3]) has shown that if  $f : \omega \setminus 2 \rightarrow 2$  assumes the values 0 and 1 alternatively and  $S_f$

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is stationary (here  $n^* = 1$ ), then there's an inner model with infinitely many measurable cardinals.

Liu has proven (in [Liu]) that under the existence of huge cardinals, it is consistent that  $S_f$  is stationary together with GCH for many  $f$ 's assuming (any) two different values. In this paper, we are going to prove a few results concerning the above question, which are due to the second author except for Proposition 3.1 and a small remark of the first author improving Theorem 4.2.

Theorems here (3.1, 4.2) are the first results for function  $f$  with more than 2 values gotten infinitely many times. Also the present results have relatively small consistent strength. A version of 4.1 was proved in the mid eighties and forgotten and we thank M. Gitik for reminding. The main results are as follows:

**Theorem 3.1.** *Assume  $\sup(\text{pcf}(\{\aleph_n : n < \omega\})) = \aleph_{\omega+n^*}$ . Let  $1 < m^* < \omega$ . Let  $I$  be the ideal of finite subsets of  $\omega$ . Let  $\langle A_i : 1 \leq i \leq n^* \rangle$  be pairwise disjoint subsets of  $\omega \setminus (m^* + 1)$  such that  $\prod_{k \in A_i} \aleph_k / I$  has true cofinality  $\aleph_{\omega+i}$  for  $1 \leq i \leq n^*$ . Let  $\langle \kappa_i : 1 \leq i \leq n^* \rangle$  be a sequence of uncountable cardinals below  $\aleph_{m^*+1}$ . Then the set*

$$S = \{x \subseteq \aleph_\omega : |x| \leq \aleph_{m^*} \wedge (\forall i)[1 \leq i \leq n^* \rightarrow (\forall k \in A_i) \text{cf}(x \cap \aleph_k) = \kappa_i]\}$$

is stationary in  $[\aleph_\omega]^{<\aleph_{m^*+1}}$ .

**Theorem 4.1.** *Assume  $A \subseteq \omega$ ,  $0 < n^* < \min(A)$  and for each  $n \in A$  there is an  $\aleph_n$ -complete filter  $F_n$  on  $\aleph_n$  such that  $F_n$  contains the cobounded subsets of  $\aleph_n$  and the second player has a winning strategy in the game  $GM_{\aleph_{n^*}}(F_n)$ .*

(See Definition 4.1.)

Then the set

$$S = \{x \subseteq \aleph_\omega : |x| \leq \aleph_{n^*} \text{ and } (\forall n \in A)[\text{cf}(x \cap \aleph_n) = \aleph_0] \text{ and } (\forall n)[n < \omega \wedge n^* < n \notin A \rightarrow \text{cf}(x \cap \aleph_n) = \aleph_{n^*}]\}$$

is stationary in  $[\aleph_\omega]^{<\aleph_{n^*+1}}$ .

**Theorem 4.2.** *Assume GCH. Let  $0 \leq m < n^* < \omega$  and  $E \subseteq \omega \setminus n^*$  be such that for all  $i \in E$ ,  $i + 1 \notin E$ . Let  $\langle n_i : i \in E \rangle$  with each  $n_i < n^*$ . Suppose that for each  $i \in E$ , there is an  $\aleph_i$ -complete filter  $F_i$  on  $\aleph_i$  containing all clubs of  $\aleph_i$  such that  $W_i = \{\alpha < \aleph_i : \text{cf}(\alpha) = \aleph_{n_i}\} \in F_i^+$  and the second player has a winning strategy in the game  $GM'_{\aleph_m}(F_i)$  (see Definition 4.1).*

Let  $f : \omega \rightarrow n^*$  be the function defined by

$$f(i) = \begin{cases} n_i & \text{if } i \in E, \\ m & \text{if } n^* < i \notin E. \end{cases}$$

Then the set  $S' = \{x \subseteq \aleph_\omega : |x| \leq \aleph_{n^*} \text{ and } (\forall i > n^*)[\text{cf}(x \cap \aleph_i) = \aleph_{f(i)}]\}$  is stationary in  $[\aleph_\omega]^{<\aleph_{n^*+1}}$ .

In this paper, we concentrate on  $\{\aleph_n : n \in \omega\}$ , but we can deal with other sets with natural changes (by the referee request an explanation was added in 95 in the end of

the paper). We implicitly assume that all models under consideration have a countable language.

## 2. PRELIMINARIES.

Let's start with a standard result. The proof will be omitted.

**Proposition 2.1.** *Let  $n^* < \omega$ ,  $X \subseteq \omega \setminus (n^* + 1)$  and  $f : X \rightarrow n^* + 1$ . Consider the set  $S = S_f$  as defined in §1. Let  $\theta > (\aleph_\omega^{\aleph_{n^*}})^+$  be a regular cardinal. Let  $M \prec \langle \mathcal{H}_\theta, \in, S, \triangleleft, \dots \rangle$ ,  $M \supseteq \aleph_\omega + 1$  and  $|M| = \aleph_\omega$ , where  $\triangleleft$  is a well-ordering of  $\mathcal{H}_\theta$ . Then the following are equivalent:*

1.  $S$  is stationary in  $[\aleph_\omega]^{<\aleph_{n^*+1}}$ .
2. For any structure  $A = \langle \aleph_\omega, \dots \rangle$  with a countable language, there is a  $B \prec A$  such that  $|B| = \aleph_{n^*}$  and  $B \in S$ .
3. There is  $N \prec M$  such that  $|N| = \aleph_{n^*}$  and  $\forall n \in X$ ,  $cf(N \cap \aleph_n) = \aleph_{f(n)}$ .

**Lemma 2.1.** *Let  $\kappa < \mu < \lambda$  be regular cardinals. Let  $A = \langle \mathcal{H}_\lambda, \in, \triangleleft, \kappa, \mu, \dots \rangle$  be a structure of a countable language on  $\mathcal{H}_\lambda$  with skolem functions closed under composition and  $\triangleleft$  a well-ordering of  $\mathcal{H}_\lambda$ . If  $B \prec A$  and  $X \subseteq \kappa$  and  $B' = sk^A(B \cup X)$ , then  $\sup(B' \cap \mu) = \sup(B \cap \mu)$ .*

**Proof.** It's clear that the lemma holds if  $\sup(B \cap \mu) = \mu$ . So we assume  $\sup(B \cap \mu) < \mu$ .

It's clear that  $\sup(B \cap \mu) \leq \sup(B' \cap \mu)$ . Now suppose  $\alpha \in B' \cap \mu$ . WLOG, assume  $\alpha = \tau(b, x_0)$  for some  $b \in B$ ,  $x_0 \in X$  and some skolem function  $\tau$ . Define  $f : \kappa \rightarrow \mu$  by letting  $f(x) = \tau(b, x)$  if  $\tau(b, x) < \mu$  and  $f(x) = 0$  otherwise. Then  $f$  is definable from  $b$  in  $B$ . So  $f \in B$ . Let  $\delta = \sup(f''\kappa)$ . Then  $\delta \in B \cap \mu$  and  $B' \models \delta = \sup(f''\kappa)$ . So  $B' \models \alpha = f(x_0) \leq \delta$ . Hence  $\alpha \leq \delta < \sup(B \cap \mu)$ . Therefore,  $\sup(B \cap \mu) \geq \sup(B' \cap \mu)$ . This completes the proof of the lemma.  $\square$

**Lemma 2.2.** *Let  $n \in \omega$  and  $X_0, S_1, \dots, X_{n-1}$  be disjoint subsets of  $\omega$ . Let  $X = \bigcup_{i < n} X_i$  and  $f$  be a function from  $X$  to  $\omega$  such that  $f$  is constant on  $X_i$  for each  $i < n$  and the constant values of  $f$  on different  $X_i$ 's are distinct. Let  $A = \langle \aleph_\omega, \dots \rangle$  be an algebra on  $\aleph_\omega$ . Let  $B \prec A$  be such that  $(\forall i < n) (\forall m \in X_i) [cf(B \cap \aleph_m) = \aleph_{f(m)}]$ . Let  $k = \max(f''X)$ ,  $A_{i_0} = f^{-1}\{k\}$  and  $\ell < k$ . Then for any  $n^*$  such that  $n^* \geq \ell$  and  $n^* > \max(f''X \setminus \{k\})$ , and  $j < \omega$  such that  $|B \cap \aleph_j| = \aleph_{n^*}$ , there is  $B' \prec B$  such that*

- (1)  $|B'| = \aleph_{n^*}$  and  $B' \cap \aleph_m = B \cap \aleph_m$  for  $m \leq j$ ;
- (2)  $(\forall m \in X_{i_0}) [m > j \rightarrow cf(B' \cap \aleph_m) = \aleph_\ell]$ ;
- (3)  $(\forall m \in X \setminus X_{i_0}) [m > j \rightarrow cf(B' \cap \aleph_m) = cf(B \cap \aleph_m)]$ .

**Proof.** For each  $i \neq i_0$ , for each  $m \in X_i$ , let  $a_m$  be a cofinal subset of  $B \cap \aleph_m$  with order type  $\aleph_{f(i)}$ . Now we can build a sequence  $\langle B_\alpha : \alpha < \aleph_\ell \rangle$  such that

- (1)  $\cup\{a_m : m \in X \setminus X_{i_0}\} \cup (B \cap \aleph_j) \subseteq B_0$ ;
- (2)  $\forall m \in X_{i_0} [\sup(B_\alpha \cap \aleph_m) < \sup(B_{\alpha+1} \cap \aleph_m)]$ ;
- (3)  $B_\alpha \prec B_{\alpha+1} \prec B$  and  $|B_\alpha| = \aleph_{n^*}$ .

The construction is obvious. Now let  $B' = \bigcup_{\alpha < \aleph_\ell} B_\alpha$ . It is clear that  $B'$  is as required.  $\square$

## 3. AN APPLICATION OF PCF THEORY.

We are going to prove the following theorem using pcf theory (see [Sh:g]).

**Theorem 3.1.** *Assume  $\max(\text{pcf}(\{\aleph_n : n < \omega\})) = \aleph_{\omega+n^*}$ . Let  $1 < m^* < \omega$ . Let  $I$  be the ideal of finite subsets of  $\omega$ . Let  $\langle A_i : 1 \leq i \leq n^* \rangle$  be a sequence of pairwise disjoint subsets of  $\omega \setminus (m^* + 1)$  such that  $\prod_{k \in A_i} \aleph_k / I$  has true cofinality  $\aleph_{\omega+i}$  for  $1 \leq i \leq n^*$ . Let  $\langle \kappa_i : 1 \leq i \leq n^* \rangle$  be a sequence of uncountable cardinals below  $\aleph_{m^*+1}$ . Then the set*

$$S = \{x \subseteq \aleph_\omega : |x| \leq \aleph_{m^*} \wedge (\forall i)(1 \leq i \leq n^* \rightarrow (\forall k \in A_i)[cf(x \cap \aleph_k) = \kappa_i])\}$$

is stationary in  $[\aleph_\omega]^{<\aleph_{m^*+1}}$ .

**Remarks.**

1. Using Lemma 2.2, Lemma 2.1 and this theorem, we can show that for any given  $1 \leq n \in \omega$  and  $\langle \kappa_i : 1 \leq i \leq n^* \rangle$  with cardinals  $\kappa_i \leq \aleph_n$  for  $1 \leq i \leq n^*$ , we have that the set

$$S' = \{x \subseteq \aleph_\omega : |x| \leq \aleph_n \text{ and } (\forall i)(1 \leq i \leq n^* \rightarrow \forall k \in A_i[cf(x \cap \aleph_k) = \kappa_i])\}$$

is stationary in  $[\aleph_\omega]^{<\aleph_{n+1}}$ .

2. Notice that in order for the theorem not to be trivial, we assume  $n^* > 1$  and therefore GCH fails at  $\aleph_\omega$ .
3. If  $pp\aleph_\omega = \sup_{\text{def}}(\text{pcf}(\{\aleph_m : m < \omega\})) > \aleph_{\omega+n^*}$ , no harm is done since we can use Levy collapse to collapse  $pp\aleph_\omega$  to  $\aleph_{\omega+n^*}$  and no new subset of  $\aleph_\omega$  is added.
4. The theorem can be generalized to other singular cardinals. Also, we can use other regular cardinals in  $(\aleph_\omega, pp\aleph_\omega)$  in the proof of the theorem.
5. Consistency results giving the assumptions are well know, starting with [Mg]; see history and references on this in [Sh:g].

The proof of Theorem 3.1 uses the following lemma:

**Lemma 3.1.** *For each  $1 \leq i \leq n^*$ , there is a sequence  $\vec{C}^i = \langle C_\alpha^i : \alpha < \aleph_{\omega+1} \rangle$  such that*

(1)  $\forall \alpha C_\alpha^i \subseteq \alpha$  and o.t.  $(C_\alpha^i) \leq \kappa_i$

(2)  $\beta \in C_\alpha^i$  implies  $C_\beta^i = C_\alpha^i \cap \beta$

(3)  $S_i =_{\text{def}} \{\alpha < \aleph_{\omega+i} : cf(\alpha) = \kappa_i \text{ and } \alpha = \sup(C_\alpha^i)\}$  is stationary in  $\aleph_{\omega+i}$ .

**Remarks.** Note that the  $C_\alpha^i$ 's are not necessarily closed.

For a proof of Lemma 3.1, see [Sh 351, 4.4] for successor of regular cardinals and in general [Sh 420, 1.5] which rely on [Sh 351, 4.4].

For each  $1 \leq i \leq n^*$ , let  $S_i$  and  $\vec{C}^i = \langle C_\alpha^i : \alpha < \aleph_{\omega+i} \rangle$  be as in Lemma 3.1. We now proceed to prove Theorem 3.1.

**Proof of Theorem 3.1.** For  $1 \leq i \leq n^*$ , let  $\vec{f}^i = \langle f_\alpha^i : \alpha < \aleph_{\omega+i} \rangle \subseteq \prod_{k \in A_i} \aleph_k$  be increasing and cofinal in  $\prod_{k \in A_i} \aleph_k / I$ . Let  $\lambda \gg pp\aleph_\omega$  be a regular cardinal. Let's consider the structure  $A = \langle \mathcal{H}_\lambda, \in, \triangleleft, \dots \rangle$  with skolem functions closed under composition,

where  $\triangleleft$  is a well-ordering on  $\mathcal{H}_\lambda$ . We define  $X_{\vec{\alpha}}, N_{\vec{\alpha}}$  by induction on  $\vec{\alpha}_{n^*}$  as follows. Let  $x = \{\vec{f}^i, A_i, \vec{C}^i\}_{1 \leq i \leq n^*}$ . For each  $\vec{\alpha} = \langle \alpha_i : 1 \leq i \leq n^* \rangle \in \prod_{1 \leq i \leq n^*} \aleph_{\omega+i}$ , let

$$X_{\vec{\alpha}} = \{\gamma : \gamma < \aleph_{n^*}\} \cup \left( \bigcup_{1 \leq i \leq n^*} C_{\alpha_i}^i \right) \cup \{N_{\vec{\beta}} : \vec{\beta} \in \prod_{i=1}^n C_{\alpha_i}^i\}$$

(hence  $C_{\xi}^i \subseteq X_{\vec{\alpha}}$  if:  $1 \leq i \leq n^* \wedge \xi \in C_{\alpha_i}^i$ )

and let  $N_{\vec{\alpha}} = sk^A(X_{\vec{\alpha}})$ . Note that  $|N_{\vec{\alpha}}| = \aleph_n$  and  $\vec{\alpha} \in \prod_{i=1}^{n^*} C_{\beta_i}^i \Rightarrow N_{\vec{\alpha}} \prec N_{\vec{\beta}}$  and  $N_{\vec{\alpha}} \in N_{\vec{\beta}}$ .

**Claim.** There is  $\vec{\delta} = \langle \delta_i : 1 \leq i \leq n^* \rangle \in \prod_{1 \leq i \leq n^*} S_i$  such that all  $1 \leq i \leq n^*$

- 1) For all  $\vec{\alpha} \in \prod_{1 \leq i \leq n^*} C_{\delta_i}^i$ , we have  $\sup(N_{\vec{\alpha}} \cap \aleph_{\omega+i}) < \delta_i$  for all  $1 \leq i < n^*$ .
- 2)  $\sup(N_{\vec{\delta}} \cap \aleph_{\omega+i}) = \delta_i$ .
- 3) For some  $n_i < \omega$ ,  $\{f_{\delta_i}^i(k) : \alpha \in C_{\delta_i}^i\}$  is cofinal in  $N_{\vec{\delta}} \cap \aleph_k$  for all  $k \in A_i \setminus n_i$ .
- 4) For some  $m_i \geq n_i$ ,  $cf(N_{\vec{\delta}} \cap \aleph_k) = cf(\delta_i) = \kappa_i$  for all  $k \in A_i \setminus m_i$ .

**Proof of Claim.** We first construct  $\vec{\delta}$  as required by part 1) of the claim. The construction is as follows: Let  $E_{n^*} = \{\delta < \aleph_{\omega+n^*} : \forall \vec{\alpha} \in \prod_{1 \leq i \leq n^*} \aleph_{\omega+i}, \text{ if } \alpha_{n^*} < \delta, \text{ then } \sup(N_{\vec{\alpha}} \cap \aleph_{\omega+n^*}) < \delta\}$ . Then  $E_{n^*}$  is clearly closed unbounded in  $\aleph_{\omega+n^*}$ . Since  $S_{n^*}$  is stationary in  $\aleph_{\omega+n^*}$  we have  $S_{n^*} \cap E_{n^*} \neq \emptyset$ . Pick some  $\delta_{n^*} \in S_{n^*} \cap E_{n^*}$ .

Suppose we have defined  $\delta_j$  for  $n^* \geq j > i$ . We now define  $\delta_i$ . Let  $E_i = \{\delta < \aleph_{\omega+i} : \forall \vec{\alpha} \in \prod_{1 \leq \ell \leq n^*} \aleph_{\omega+\ell} \text{ if } \alpha_i < \delta \text{ and } \alpha_j = \delta_j \text{ for all } i < j \leq n^* \text{ then } \sup(N_{\vec{\alpha}} \cap \aleph_{\omega+i}) < \delta\}$ . It's easy to see that  $E_i$  is closed unbounded in  $\aleph_{\omega+i}$ . So we can find  $\delta_i \in S_i \cap E_i$  since  $S_i$  is stationary in  $\aleph_{\omega+i}$ .

We now show  $\vec{\delta}$  satisfies clause (1). Let  $\vec{\alpha} \in \prod_{1 \leq i \leq n^*} C_{\delta_i}^i$ . Let  $1 \leq i \leq n^*$  be fixed, and we want to show that  $\sup(N_{\vec{\alpha}} \cap \aleph_{\omega+i}) < \delta_i$ . Consider  $\vec{\beta} = \langle \alpha_1, \dots, \alpha_i, \delta_{i+1}, \dots, \delta_{n^*} \rangle$ . By the choice of  $\delta_i$ , we have that  $\sup(N_{\vec{\beta}} \cap \aleph_{\omega+i}) < \delta_i$ . Since  $\vec{\alpha} \in \prod_{1 \leq k \leq n^*} C_{\delta_k}^k$  clearly  $X_{\vec{\alpha}} \subseteq X_{\vec{\beta}}$ . So  $\sup(N_{\vec{\alpha}} \cap \aleph_{\omega+i}) \leq \sup(N_{\vec{\beta}} \cap \aleph_{\omega+i}) < \delta_i$ .

Let's now prove clause (2) of the claim. Fix  $1 \leq i \leq n^*$ . Let  $\beta \in N_{\vec{\delta}} \cap \aleph_{\omega+i}$ . Then  $\beta = \tau(\vec{y})$  for some  $\vec{y} \in [X_{\vec{\delta}}]^{<\omega}$  and some skolem function  $\tau$ . We need to show  $\beta < \delta_i$ . Since  $\delta_j = \sup(C_{\delta_j}^j)$  and  $C_{\delta_j}^j$  has no last element for  $1 \leq j \leq n^*$ , there is  $\vec{\alpha} \in \prod_{1 \leq j \leq n^*} C_{\delta_j}^j$  such that  $\vec{y} \in [X_{\vec{\alpha}}]^{<\omega}$ . But then  $\beta \in N_{\vec{\alpha}} \cap \aleph_{\omega+i}$ . By clause (1) we have  $\beta < \delta_i$  so  $\sup(N_{\vec{\alpha}} \cap \aleph_{\omega+i}) \leq \delta_i$  and by the previous paragraph we get equality.

We show clause (3) by contradiction. Assume that (3) fails. So there is an unbounded set  $b \subseteq A_i$  such that  $\forall k \in b \exists \beta_k \in N_{\vec{\delta}} \cap \aleph_k [\sup(\{f_{\xi}^i(k) : \xi \in C_{\delta_i}^i\}) < \beta_k]$ . Fix such  $\beta_k$  for each  $k \in b$ . Since for each  $1 \leq j \leq n^*$  the set  $C_{\delta_j}^j$  has order whose cofinality is uncountable, there is  $\vec{\alpha} \in \prod_{1 \leq \ell \leq n^*} C_{\delta_\ell}^\ell$  such that  $(\forall k \in b) \beta_k \in N_{\vec{\alpha}}$ . Clearly  $k \in B \Rightarrow \beta_k < \sup(N_{\vec{\alpha}} \cap \aleph_k)$  hence  $\langle \beta_k : k \in b \rangle < \langle \sup(N_{\vec{\alpha}} \cap \aleph_k) : k \in b \rangle$ . The later belong to  $\prod_{k \in b} \aleph_k$  hence for some  $\xi < \aleph_{\omega+i}$  we have  $\langle \sup(N_{\vec{\alpha}} \cap \aleph_k) : k \in b \rangle \leq^* f_{\xi}^i$ , where  $f \leq^* g$  means  $f(k) \leq g(k)$  for all but finitely many  $k$ 's. Since  $N_{\vec{\alpha}} \in N_{\vec{\delta}}$ , clearly  $\langle \sup(N_{\vec{\alpha}} \cap \aleph_k) : k \in b \rangle$  belongs to  $N_{\vec{\delta}}$  and also  $\langle f_{\xi}^i : \xi < \aleph_{\omega+i} \rangle$  belongs to  $X_{\vec{\delta}}$  hence to  $N_{\vec{\delta}}$ , so wlog  $\xi \in N_{\vec{\delta}} \cap \aleph_{\omega+i}$ . Now we can replace  $\xi$  by any  $\xi' \in (\xi, \aleph_{\omega+i})$  and  $C_{\delta_i}^i$  is unbounded in  $N_{\vec{\delta}} \cap \aleph_{\omega+i}$ , so wlog  $\xi \in C_{\delta_i}^i$  hence  $k \in b \Rightarrow f_{\xi}^i(k) \in N_{\vec{\delta}} \cap \aleph_{\omega+i}$  hence is  $< \beta_k$ . This is clearly absurd.

Finally, let's prove clause (4) again by contradiction. Suppose clause (4) is not true. Then there is an unbounded set  $b \subseteq A_i \setminus n_i$  such that  $cf(\sup\{f_\xi^i(k) : \xi \in C_{\delta_i}^i\}) < \kappa_i$  for some  $k \in b$ . So as o.t.  $(C_{\delta_i}^i) = \kappa_i$  for each  $k \in b$ , there is  $\xi_k \in C_{\delta_i}^i$  such that for all  $\xi \in C_{\delta_i}^i$  with  $\xi \geq \xi_k$  we have:  $\sup\{f_\zeta^i(k) : \zeta \in C_{\delta_i}^i\} = \sup\{f_\zeta^i(k) : \zeta \in C_{\delta_i}^i \cap \xi\}$ . Let  $\beta \in C_{\delta_i}^i$  be such that  $\beta > \sup(\{\xi_k : k \in b\})$ . For each  $k \in A_i$ , let  $\mu_k = \sup(\{f_\zeta^i(k) : \zeta \in C_\beta^i\})$ . Then  $\langle \mu_k : k \in A_i \rangle \in N_{\bar{\delta}} \cap \prod_{k \in A_i} \aleph_k$  since  $C_\beta^i \in N_{\bar{\delta}}$ . So as above there is  $\beta' \in N_{\bar{\delta}} \cap \aleph_{\omega+i}$  such that  $\langle \mu_k : k \in A_i \rangle \leq^* f_{\beta'}^i$ . So we have  $\langle \sup(N_{\bar{\delta}} \cap \aleph_k) : k \in b \rangle = \langle \mu_k : k \in B \rangle \leq^* f_{\beta'}^i \upharpoonright b$  which contradicts to  $f_{\beta'}^i \in N_{\bar{\delta}} \cap \prod_{k \in A_i} \aleph_k$  (the initiated could have used “wlog  $\vec{f}^i$  obeys  $\vec{C}^i$ ”). This completes the proof of the claim.

Now, Theorem 3.1 follows from the claim, Proposition 2.1 and Lemma 2.1.  $\square$

By the remarks following Theorem 3.1, the theorem is not trivial only when  $pp\aleph_\omega > \aleph_{\omega+1}$ . In particular GCH does not hold at  $\aleph_\omega$ . But using the following observation, we can make GCH hold at  $\aleph_\omega$  by collapsing  $2^{\aleph_\omega}$  and still have the desired conclusion in the forcing extension, i.e., the set  $S$  in Theorem 3.1 is still stationary in the generic extension. (So by well-known consistency results we can even have GCH.)

**Proposition 3.1.** *Let  $P$  be an  $< \aleph_{\omega+1}$ -closed forcing notion. Suppose  $S$  is stationary in  $[\aleph_\omega]^{< \aleph_{n+1}}$  in  $V$ . Then  $V^P \Vdash$  “ $S$  is stationary in  $[\aleph_\omega]^{< \aleph_{n+1}}$ ”.*

**Proof.** It suffices to show that in  $V^P$ , for any given structure  $A = \langle \aleph_\omega, \dots \rangle$  of a countable language, there is  $B \prec A$  such that  $|B| = \aleph_n$  and  $B \in S$ .

Let  $p \in P$  for that  $\dot{A} = \langle \aleph_\omega, (f_i)_{i \in \omega} \rangle$  is a structure on  $\aleph_\omega$  with skolem functions  $f_i$  closed under compositions.

Since  $P$  is  $(< \aleph_{\omega+1})$ -closed, we can find  $\langle p_\alpha : \alpha < \aleph_\omega \rangle$  (such that  $p_\alpha$  is stronger than  $p$  and  $p_\beta$  for  $\beta < \alpha$  and  $f'_i$  in  $V$  such that for each  $i$ , for any  $\vec{\alpha} \in [\aleph_\omega]^{< \omega}$  there is a  $\beta$  such that  $p_\beta \Vdash f_i(\vec{\alpha}) = f'_i(\vec{\alpha})$  whenever  $\vec{\alpha} \in \text{dom}(f_i)$ ).

Consider  $A' = \langle \aleph_\omega, (f'_i)_{i \in \omega} \rangle$  in  $V$ . Let  $p$  be such that  $p$  is stronger than  $p_\alpha$  for all  $\alpha < \aleph_\omega$ . Let  $B \prec A'$  be such that  $|B| = \aleph_n$  and  $B \in S$ . Then  $p \Vdash B \prec \dot{A}$  since  $p \Vdash f'_i = f_i$ . This is as required.  $\square$

#### 4. APPLICATIONS OF LARGE IDEALS.

In this section, we prove two results under the existence of large ideals (on the  $\aleph_n$ 's). Before we state our results, we need some terminology.

**Definition 4.1.** *Let  $\kappa > \lambda$  be cardinals. Let  $D$  be a filter on  $\kappa$ .*

(1) *We define the game  $\overline{GM}_\lambda(D)$  as follows: the game lasts  $\lambda$  moves. At  $\xi^{\text{th}}$  move, the first player chooses a subset  $A_\xi$  of  $\kappa$  such that  $A_\xi \subseteq \bigcap_{\eta < \xi} B_\eta$ , and if  $\bigcap_{\eta < \xi} B_\eta \neq \emptyset \text{ mod } (D)$  then  $A_\xi \neq \emptyset \text{ mod } (D)$ . The second player chooses a subset  $B_\xi$  of  $A_\xi$  with  $B_\xi \neq \emptyset \text{ mod } (D)$ .*

*A player without a legal move loses the game immediately. (Note this can only happen to the second player.) If the game lasts for  $\lambda$  moves, the second player wins if  $\bigcap_{\xi < \lambda} B_\xi$  is unbounded.*

(2) *Let's also assume that  $D$  is  $\kappa$ -complete. We now define the “cut-and-choose” game  $\overline{GM}'_{\lambda, \kappa}(D)$  of length  $\lambda$ : at the  $0^{\text{th}}$  move, the first player chooses a set  $A_0 \neq \emptyset \text{ mod } (D)$  and then partitions  $A_0$  into less than  $\kappa$  parts; the second player chooses one of the parts, say*

$B_0 \in D^+$ . At the  $\xi^{th}$  move for  $\xi > 0$ , the first player partitions the set  $\bigcap_{\eta < \xi} B_\eta$  into less than  $\kappa$  parts, and the second player chooses one of the parts, call it  $B_\xi$  such that  $B_\xi \in D^+$ .

The winning conditions for each player is exactly as in the game defined in part (1) above.

Let's first prove the following theorem:

**Theorem 4.1.** *Assume  $A \subseteq \omega, 0 < n^* < \min(A)$  and for each  $n \in A$  there is an  $\aleph_n$  such that  $F_n$  contains the cobounded subsets of  $\alpha_n$  and the second player has a winning strategy in the game  $GM_{\alpha_{n^*}}(F_n)$ .*

*Then the set*

$$S = \{x \subset \aleph_\omega : |x| \leq \alpha_{n^*}, (\forall n \in A) cf(x \cap \aleph_n) = \aleph_0 \text{ and } (\forall n)[\forall n^* < n \notin A \rightarrow cf(x \cap \aleph_n) = \aleph_{n^*}]\}$$

*is stationary in  $[\aleph_\omega]^{<\aleph_{n^*+1}}$ .*

### Remarks.

(1) Before we prove Theorem 4.1, we would like to see how we can get the filters as required in the hypothesis of the theorem. Magidor (see [Mg1]) has shown the consistency of the existence of the filters. Also, Laver has proved that if we collapse a measurable cardinal  $\kappa$  to some  $\aleph_n$ , then in the generic extension, there is a normal ideal on  $F_n$  on  $\aleph_n$  such that  $\mathcal{P}(\aleph_n)/F_n$  has a  $< \aleph_{n-1}$ -closed dense subset. Therefore, if there are infinitely many measurable cardinals, say  $\langle \kappa_n : n < \omega \rangle$ ,  $A = \{m_m | n < \omega\}$ ,  $n^* < m_n < n, m_n + 1 < m_{n+1}$  we can Levy collapse each  $\kappa_n$  to  $\aleph_{n^*+m_n}$  to get the normal filters as required in the hypothesis of Theorem 4.1.

(2) We can also use (in the assumption of Theorem 4.1) the games  $GM'_{\aleph_{n^*}, \aleph_n}(F_n)$  in place of  $GM_{\aleph_{n^*}}(F_n)$ . We can weaken it further using for  $n \in A$ , the following game for  $F_n$  (see [Sh 250]) (see better [Sh:f, Ch XIV]: in the  $\xi^{th}$  move, player one choose  $m_\xi \in \omega \setminus A, n^* < m_\xi < n$  and  $F_\xi : \aleph_n \rightarrow \aleph_{m_\xi}$  and player two has to choose  $B_\xi \subseteq \bigcap_{\zeta < \xi} B_\zeta \cap A_0$  such that the range of  $f_\xi \upharpoonright B_\xi$  is bounded in  $\aleph_{m_\xi}$  and  $B_\xi \neq \emptyset \text{ mod } F_n$ ; in the  $0^{th}$  move player one also choose  $A_0 \subseteq \aleph_{m_\xi}, A_0 \neq \emptyset \text{ mod } F_n$ .

If  $F$  is a filter on  $\lambda$ , and the cardinal player two choose is from  $S$ , and a play last  $\theta$  moves we call the game  $GM'_\theta(F, S)$ .

In order to prove Theorem 4.1, let's consider tagged trees of the form  $\langle T, \mathcal{I} \rangle$ , which by definition means that

1.  $T \subset [ON]^{<\omega}$  is a tree, i.e.  $T$  consists of finite sequences of ordinals closed under initial segments.
2.  $\mathcal{I} = \langle I_\sigma : \sigma \in T \rangle$  is such that for each  $\sigma \in T, I_\sigma$  is an ideal on  $Suc_T(\sigma)$  which is the set of immediate successors of  $\sigma$  in  $T$ . Also,  $I_\sigma$  can be thought of as an ideal on  $\{\alpha : \sigma^{(\alpha)} \in T\}$ .

If  $T_1$  is a subtree of  $T$ , we can view  $\langle T_1, \mathcal{I}' \rangle$  as a tagged tree with  $\mathcal{I}' = \langle I_\sigma | Suc_{T_1}(\sigma) : \sigma \in T_2 \rangle$ . By abuse of notation, we still denote it by  $\langle T_1, \mathcal{I} \rangle$ . If the family  $\mathcal{I}$  of ideals is clear from context, we will simply say  $T$  is a tagged tree without mentioning  $\mathcal{I}$  explicitly.

For  $X \subseteq T$ , let  $T[X] = \{\sigma \in T : \exists \eta \in X (\eta \leq_T \sigma \vee \sigma \leq_T \eta)\}$ . Clearly  $T[X]$  is a subtree of  $T$ . The following lemma is from [RuSh 117] or [Sh:b, Ch X] or [Sh:f] and we will not give the proof here.

**Lemma 4.1.** *Let  $\langle T, \mathcal{I} \rangle$  be a tagged tree such that for each  $\sigma \in T, I_\sigma$  is a proper ideal such that  $\text{Suc}_T(\sigma) \notin I_\sigma$ . Let  $\lambda$  be a regular, uncountable cardinal and for every  $\sigma \in T, I_\sigma$  is  $\lambda$ -indecomposable, i.e. if  $A \subseteq \text{Suc}_T(\sigma) \setminus A \neq \emptyset \pmod{I_\sigma}$  and  $f : A \rightarrow \lambda$  then for some  $\zeta < \lambda$  we have  $\{x \in A : f(x) < \zeta\} \neq \emptyset \pmod{I_\sigma}$ . (This holds if for every  $\sigma \in T, I_\sigma$  is a  $\lambda^+$  complete ideal or  $|\text{Suc}_T(\sigma)| < \lambda$ ).*

(\*) *for every function  $F : T \rightarrow \lambda$ , there is a subtree  $T_1$  of  $T$  such that for all  $\sigma \in T_1$ ,  $\text{Suc}_{T_1}(\sigma) \notin I_\sigma$  and  $\text{Sup}(F''T_1) < \lambda$ .*

We now proceed to prove Theorem 4.1.

**Proof of Theorem 4.1.** Let  $\langle m_i : i < \omega \rangle$  be such that each  $m_i \in A$  and for each  $m \in A$  there are infinitely many  $i$  with  $m = m_i$ . Let  $\mathcal{T} = \{T : T \text{ is a subtree of } \bigcup_{l < \omega} \prod_{i < l} \aleph_{m_i} \text{ and } \forall \eta \in T \text{ with } lh(\eta) = i \text{ we have } \{\alpha < \aleph_{m_i} : \hat{\eta} \langle \alpha \rangle \in T\} \in F_{m_i}^+\}$ , where  $lh(\eta)$  means the length of the finite sequence  $\eta$ .

Note that each  $T \in \mathcal{T}$  can be considered as a tagged tree where for each  $\sigma \in T$ , the associated ideal  $I_\sigma$  is just the dual ideal to the filter  $F_{m_{lh(\sigma)}}$ .

Suppose  $A = \langle \aleph_{\omega, \dots} \rangle$  is an arbitrary structure on  $\aleph_\omega$ . We are going to find a  $B \prec A$  such that  $|B| = \aleph_{n^*}$  and for each  $n^* < m < \omega$  if  $m \in A$  then  $cf(B \cap \aleph_m) = \aleph_0$ ; if  $m \notin A$  then  $cf(B \cap \aleph_m) = \aleph_{n^*}$ . This is enough to prove the theorem by Proposition 2.1.

By induction on  $\xi < \aleph_{n^*}$ , we are going to build  $T_\xi, \langle \alpha_{\xi, m} : n^* < m \notin A \rangle, \langle A_{\xi, \eta}, B_{\xi, \eta} : \eta \in T_{\xi+1} \rangle$  and  $\langle N_{\xi, \eta} : \eta \in T_\xi \rangle$  such that

1.  $T_\xi \in \mathcal{T}$  and for any  $\xi < \xi', T_{\xi'} \subseteq T_\xi$ .
2. For  $\eta \in T_{\xi+1}, B_{\xi, \eta} = \{\alpha < \aleph_{m_{lh(\eta)}} : \hat{\eta} \langle \alpha \rangle \in T_{\xi+1}\}$ . Furthermore,  $\langle A_{\xi', \eta}, B_{\xi', \eta} : \xi' \leq \xi \rangle$  is an initial segment of a play of the game  $GM_{\aleph_{n^*}}(F_{m_{lh(\eta)}})$  with the second player following his winning strategy.
3. If  $\xi$  is a limit ordinal,  $T_\xi = \bigcap_{\xi' < \xi} T_{\xi'}$ .
4. For  $\eta \in T_\xi, N_{\xi, \eta} = sk^A(\text{ran}(\eta) \cap \{\alpha_{\xi', m} : \xi' < \xi \text{ and } n^* < m \notin A\})$ .
5. For  $\eta \in T_{\xi+1}$  and  $n^* < m \notin A$ , we have  $\text{Sup}(N_{\xi+1, \eta} \cap \aleph_m) < \alpha_{\xi+1, m}$ .
6. For each  $n^* < m \notin A, \langle \alpha_{\xi, m} : \xi < \aleph_{n^*} \rangle$  is an increasing sequence of ordinals in  $\aleph_m$ .

Take any  $T_0 \in \mathcal{T}$  to start with. For  $\xi$  limit, let  $T_\xi = \bigcap_{\xi' < \xi} T_{\xi'}$ . Since the second player has a winning strategy in the game  $GM_{\aleph_{n^*}}(F_m)$  for each  $m \in A, T_\xi \in \mathcal{T}$  for  $\xi$  limit.

If  $\xi = 0$ , we let  $\alpha_{\xi, m} = 0$  for  $n^* < m \notin A$ . If  $\xi$  is limit, we let  $\alpha_{\xi, m} = \text{Sup}(\{\alpha_{\xi', m} : \xi' < \xi\})$  for  $n^* < m \notin A$ .

Suppose  $T_\xi$  and  $\langle \alpha_{\xi, m} : n^* < m \notin A \rangle$  have been constructed. We now construct  $T_{\xi+1}, \langle \alpha_{\xi+1, m} : n^* < m \notin A \rangle$  and  $\langle A_{\xi, \eta}, B_{\xi, \eta} : \eta \in T_{\xi+1} \rangle$ .

Let  $\langle k_i : i < \omega \rangle$  be an enumeration of  $\{m \notin A : n^* < m < \omega\}$ . We will define  $\langle T'_i : i < \omega \rangle, \langle A_{i, \eta}, B_{i, \eta} : i < \omega \rangle$  for  $\eta \in T'_{i+1}$  and  $\langle \alpha_i : i < \omega \rangle$  by induction on  $i$  such that

1. for each  $i, T'_{i+1} \subseteq T'_i \in \mathcal{T}$  and  $\text{Sup}(N_{\xi+1, \eta} \cap \aleph_{k_i}) < \alpha_i$  for  $\eta \in T'_{i+1}$ ;
2.  $\langle A_{i, \eta}, B_{i, \eta} : i < \omega \rangle$  is an initial segment of the play of the game  $GM_{\aleph_{n^*}}(F_{lh(\eta)})$  with player two following his winning strategy.

Let  $T'_0 = T_\xi$ . Suppose we have defined  $T'_i, \alpha_{i-1}$  and  $A_{i-1, \eta}, B_{i-1, \eta}$  for  $\eta \in T'_i$ . Consider the function  $F : T'_i \rightarrow \aleph_{k_i}$  defined by  $F(\eta) = \text{Sup}(N_{\xi+1, \eta} \cap \aleph_{k_i})$ . Then  $F$  has a value  $< \aleph_{k_i}$ . Since  $k_i \notin A$ , we have  $m_i \neq k_i$ , so  $F_{m_i}$  is  $\aleph_{m_i}$ -complete on a set of cardinality  $\aleph_{m_i}$  so the assumptions of Lemma 4.1 holds. Hence there is  $T''_{i+1} \subseteq T'_i$  such that  $T''_{i+1} \in \mathcal{T}$  and  $\text{Sup}(F''T''_{i+1}) < \aleph_{k_i}$  by Lemma 4.1. Let  $A_{i, \eta} = \text{Suc}_{T''_{i+1}}(\eta)$  for  $\eta \in T''_{i+1}$ . Let  $B_{i, \eta}$  be the



move of the second player following his winning strategy in the game  $GM_{\aleph_{n^*}}(F_{lh(\eta)})$ . Let  $T'_{i+1} \in \mathcal{T}$  be such that  $B_{i,\eta} = \text{Suc}_T(\eta)$  for each  $\eta \in T$ . Let  $\alpha_i$  be such that  $\text{Sup}(F''T'_{i+1}) < \alpha_i < \aleph_{k_i}$ .

Now, let  $T'_{\xi+1} = \bigcap_{i < \omega} T'_i$  and  $\alpha_{\xi+1, k_i} = \alpha_i$ . Since  $\langle A_{i,\eta}, B_{i,\eta} : i \in \omega \rangle$  is an initial segment of the play of the game  $GM_{\aleph_{n^*}}(F_{lh(\eta)})$  with player two following his winning strategy, we have that  $T'_{\xi+1} \in \mathcal{T}$ . For each  $\eta \in T'_{\xi+1}$ , let  $A_{\xi,\eta} = \text{Suc}_{R'_{\xi+1}}(\eta)$ . Note this is a legal move for the first player. Now player two chooses  $B_{\xi,\eta}$  according to his winning strategy. Let  $T_{\xi+1} \in \mathcal{T}$  be such that  $B_{i,\eta} = \text{Suc}_{T_{\xi+1}}(\eta)$  for each  $\eta \in T_{\xi+1}$ . This completes the construction as required. (Alternatively demand that in  $\langle k_i : i < \omega \rangle$ , each  $m \in \{m < \omega : m \notin A, n^* < m < \omega\}$  appear  $\omega$  many times and if  $\xi = i \bmod \omega$ , take care only of  $\aleph_{k_i}$ .)

Finally, let  $T_{\aleph_{n^*}} = \bigcap_{\xi < \aleph_{n^*}} T_{\xi}$ . Since  $\langle A_{\xi,\eta}, B_{\xi,\eta} : \xi < \aleph_{n^*} \rangle$  is a play of the game  $GM_{\aleph_{n^*}}(F_{lh(\eta)})$  with the second player following his winning strategy, it's easy to see that for each  $\eta \in T_{\aleph_{n^*}}$  we have  $|\text{Suc}_{T_{\aleph_{n^*}}}(\eta)| = \aleph_{lh(\eta)}$ . Now let  $b$  be an infinite branch of  $T_{\aleph_{n^*}}$  such that  $b(i) > \text{Sup}(N_{\aleph_{n^*}, b \upharpoonright i} \cap \aleph_{m_i})$ , where  $N_{\aleph_{n^*}, b \upharpoonright i}$  is defined in the same way as  $N_{\xi,\eta}$  was defined above. Such a branch  $b$  clearly exists.

Now, let  $B = \text{sk}^A(\{b(i) : i \in \omega\} \cap \{\alpha_{\xi,m} : \xi < \aleph_{n^*} \wedge n^* < m \notin A\})$ . Then for each  $m \in A$ , the set  $\langle b(i) : i < \omega \wedge m_i = m \rangle$  is cofinal in  $B \cap \aleph_m$ . Furthermore, for  $n^* < m \notin A$ ,  $\langle \alpha_{\xi,m} : \xi < \aleph_{n^*} \rangle$  is cofinal in  $B \cap \aleph_m$ . Hence  $B$  is as required.  $\square$

**Theorem 4.2.** *Assume GCH. Let  $0 \leq m < n^* < \omega$  and  $E \subseteq \omega \setminus (n^* + 1)$  be such that for all  $i \in E, i+1 \notin E$  and let  $j(i) = \text{Max}(E \cap i)$ . Let  $\langle n_i : i \in E \rangle$  with each  $n_i < n^*$ . Suppose that for each  $i \in E$ , there is an  $\aleph_i$ -complete filter  $F_i$  on  $\aleph_i$  containing all clubs of  $\aleph_i$  such that  $W_i = \{\alpha < \aleph_i : \text{cf}(\alpha) = \aleph_{n_i}\} \in F_i^+$  and the second player has a winning strategy in the same  $GM'_{\aleph_m, \aleph_i}(F_i)$ .*

Let  $f : \omega \rightarrow n^*$  be the function defined by

$$f(i) = \begin{cases} n_i & \text{if } i \in E, \\ m & \text{if } n^* < i \notin E \end{cases}$$

Then the set  $S' = \{x \subset \aleph_\omega : |x| \leq \aleph_{n^*} \text{ and } (\forall i > n^*) \text{cf}(x \cap \aleph_i) = \aleph_{f(i)}\}$  is stationary in  $[\aleph_\omega]^{< \aleph_{n^*+1}}$ .

**Remarks.** 1. Instead of GCH, it's enough to assume for  $i < j$  in  $E$ , we have  $2^{\aleph_i} < \aleph_j$ .

2. The assumption is consistent, but not so if we strengthen it using  $GM_{\aleph_m}(F_i)$ . (By [Sh 542])

**Proof.** Let  $\lambda \gg \aleph_\omega$  be a regular cardinal and  $A = \langle \mathcal{H}(\lambda), \in, \triangleleft, \langle n_i : i \in E \rangle, (\tau_i)_{i < \omega}, \dots \rangle$  be a fully skolemized structure with skolem functions closed under compositions, where  $\triangleleft$  is a well-ordering on  $\mathcal{H}(\lambda)$ . In order to prove the theorem, it suffices to show that there exists  $B \triangleleft A$  such that  $|B| = \aleph_{n^*}$ ,  $\text{cf}(B \cap \aleph_i) = \aleph_{n_i}$  for each  $i \in E$  and  $\text{cf}(B \cap \aleph_i) = \aleph_m$  for  $n^* < i \notin E$ .

For each  $i > n^*$ , let  $h_i : W_i \rightarrow [\aleph_i]^{< \aleph_{n^*}}$  be defined by  $h_i(\delta) = X_{i,\delta}$ , where  $X_{i,\delta}$  is the  $\triangleleft$ -least cofinal subset of  $\delta$  of cardinality  $\aleph_{n_i}$ . Note that each  $h_i$  is definable in  $A$ .

We now define  $\langle A_{i,0}, B_{i,\xi} : i \in E, \xi < \aleph_m \rangle$  and  $\langle A_\xi : \xi \in \aleph_m \rangle$  by induction on  $\xi < \aleph_m$  such that

- 1) For each  $i \in E$ ,  $\langle A_{i,0}, P_{i,\xi}, B_{i,\xi} : \xi < \aleph_m \rangle$  is a play of the game  $GM'_{\aleph_m}(F_i)$  with the second player following his winning strategy;
- 2)  $A_{i,0} = W_i$ ,  $A_0 = sk^A(\{\emptyset\})$  and  $A_\xi = \cup_{\xi' < \xi} A_{\xi'}$  if  $\xi$  is limit;
- 3)  $A_\xi \prec A$ ,  $|A_\xi| < \aleph_{n^*}$  and  $A_\xi \subseteq A_{\xi+1}$ ;
- 4) for each  $n^* < j < \omega$ ,  $j \notin E$  we have  $\sup(A_\xi \cap \aleph_j) < \sup(A_{\xi+1} \cap \aleph_j)$ ;
- 5) for each  $i \in E$ , for all  $\delta \in B_{i,\xi+1}$ , we have  $sk^A(A_\xi \cup X_{i,\delta}) \cap \aleph_{i-1} \subseteq A_{\xi+1}$  and  $sk^A(A_\xi \cup \delta) \cap \aleph_i = \delta$ .

We simulate the games  $GM'_{\aleph_m, \aleph_i}(F_i)$  for  $i \in E$  simultaneously. The first player chooses,  $A_{i,0} = W_i$  and then divides it into less than  $\aleph_i$  parts for his (or her)  $0^{th}$ -move in the game  $GM'_{\aleph_m, \aleph_i}(F_i)$ . The second player always follows the winning strategy. For successor stage, suppose we have constructed  $\langle B_{i,\xi} : i \in E \rangle$  and  $A_\xi$ . For  $i \in E$ , let  $C_i = \{\alpha < \aleph_i : sk^A(\alpha \cup A_\xi) \cap \aleph_i = \alpha\}$ . Then  $C_i$  is a club in  $\aleph_i$ . So  $C_i \in F_i$ . For each  $i \in E$ , consider the function  $f_i : B_{i,\xi} \cap C_i \rightarrow [\aleph_{i-1}]^{<\aleph_{n^*}}$  defined by  $f_i(\delta) = sk^A(A_\xi \cup X_{i,\delta}) \cap \aleph_{i-1}$ . The first player divides  $B_{i,\xi}$  into  $\aleph_{i-1}$  parts as follows:  $P_{i,\xi} = \{f_i^{-1}\{x\} : x \in [\aleph_{i-1}]^{<\aleph_{n^*}}\} \cup \{B_{i,\xi} \setminus C_i\}$ . (Note that  $||[\aleph_{i-1}]^{<\aleph_{n^*}}| = \aleph_{i-1}$  by GCH.) The second player chooses one of the parts, say  $B_{i,\xi+1}$  according to his winning strategy. Note that the second player will not choose  $B_{i,\xi} \setminus C_i$  as his move since  $B_{i,\xi} \setminus C_i = \emptyset \text{ mod } (F_i)$ . (Otherwise he will lose right away). So there must be some  $X_i \in [\aleph_{i-1}]^{<\aleph_{n^*}}$  such that  $f_i'' B_{i,\xi+1} = \{X_i\}$ . Now let  $X = \cup_{i \in E} X_i$  and  $\alpha_j = \sup(A_\xi \cap \aleph_j)$  for  $j \notin E$  and  $A_{\xi+1} = sk^A(A_\xi \cup X \cup \{\alpha_j : j \notin E\})$ .

For limit stage, having defined  $\langle B_{i,\xi'} : \xi' < \xi \rangle$ , the first player just divides  $\cap_{\xi' < \xi} B_{i,\xi'}$  into  $\aleph_{i-1}$  parts anyway he wants. We let  $B_{i,\xi}$  be the move of the second player following his winning strategy. This completes the construction and the sequences  $\langle A_{i,0}, B_{i,\xi} : i \in E, \xi \in \aleph_m \rangle$  and  $\langle A_\xi : \xi \in \aleph_m \rangle$  clearly satisfy clauses (1)-(5) above.

Now, let  $A^* = \cup_{\xi < \aleph_m} A_\xi$  and  $W'_i = \cap_{\xi < \alpha_m} B_{i,\xi}$ . Then  $cf(A^* \cap \aleph_j) = \aleph_m$  for  $n^* < j \notin E$  by clause (4) above, and each  $W'_i$  is unbound in  $\aleph_i$ .

Let's enumerate  $E$  as  $\langle i_n : n \in \omega \rangle$  in increasing order. We choose  $\delta_{i_n} \in W'_{i_n}$ . Let  $B_n = sk^A(A^* \cup \cup_{k \leq n} X_{i_k, \delta_{i_k}})$  and  $B'_n = sk^A(A^* \cup X_{i_n, \delta_{i_n}})$ . Then we have that  $\sup(B'_n \cap \aleph_{i_n}) = \delta_{i_n}$  by clause (5) above. Also, we have that  $B'_n \cap \aleph_{i_n-1} \subseteq A^*$  since if  $\alpha \in B'_n \cap \aleph_{i_n-1}$  then  $\alpha \in sk^A(A_\xi \cup X_{i_n, \delta_{i_n}}) \cap \aleph_{i_n-1} \subseteq A_{\xi+1} \subseteq A^*$  for some  $\xi < \aleph_m$ .

**Claim.** For all  $n < \omega$ , we have

- a)  $B_n \cap \aleph_{i_n-1} = B_{n-1} \cap \aleph_{i_n-1}$  for  $n > 0$
- b)  $\sup(B_n \cap \aleph_{i_n}) = \delta_{i_n}$
- c)  $(\forall i_0 < j \notin E)[\sup(B_n \cap \aleph_j) = \sup(A^* \cap \aleph_j)]$

**Proof.** To prove a), it suffices to show that for any  $\alpha \in B_n \cap \aleph_{i_n-1}$ ,  $\alpha \in B_{n-1}$ . Let  $\alpha \in B_n \cap \aleph_{i_n-1}$ . For simplicity, we may assume  $\alpha = \tau(a^*, x_0, \dots, x_{n-1}, x_n)$  for  $a^* \in A^*$ ,  $x_k \in X_{i_k, \delta_{i_k}}$  for  $k \leq n$  and for some skolem function  $\tau$ .

Let  $f : \prod_{k < n} \aleph_{i_k} \rightarrow \aleph_{i_n-1}$  be the function defined by letting  $f(\vec{\beta}) = \tau(a^*, \vec{\beta}, x_n)$  if  $\tau(a^*, \vec{\beta}, x_n) < \aleph_{i_n-1}$  and  $f(\vec{\beta}) = 0$  otherwise. Then  $f$  is definable from  $a^*$  and  $x_n$ . So  $f \in B'_n$ .

Now, let  $\vec{f} = \langle f_\xi : \xi < \aleph_{i_n-1} \rangle$  be a list of all the functions from  $\prod_{k < n} \aleph_{i_k}$  to  $\aleph_{i_n-1}$ . (Note this is possible by GCH and  $i_{n-1} < i_n - 1$ .) By definability, we can choose  $\vec{f} \in A^*$ . But then  $B'_n \models (\exists \xi < \aleph_{i_n-1}) f_\xi = f$ . Let  $\xi \in B'_n \cap \aleph_{i_n-1}$  be such that  $f_\xi = f$ . Then  $\xi \in$

$A^* \subseteq B_{n-1}$  since  $B'_n \cap \aleph_{i_n-1} \subseteq A^*$ . So  $f = f_\xi \in B_{n-1}$ . Therefore,  $\alpha = f(x_0, \dots, x_{n-1}) \in B_{n-1}$  since  $x_k \in B_{n-1}$  for all  $k < n$ . We have thus proved part a) of the claim.

Clause b) follows from Lemma 2.1 and  $\sup(B'_n \cap \aleph_{i_n}) = \delta_{i_n}$ . By Lemma 2.1,  $\sup(B_n \cap \aleph_{i_n}) = \sup(B'_n \cap \aleph_{i_n}) = \delta_{i_n}$ .

We prove c) by induction on  $n$ . If  $n = 0$ , clause c) follows from Lemma 2.1. Now suppose c) holds for  $n-1$ . We want to show c) holds for  $n$ . By a) and induction hypothesis,  $\sup(B_n \cap \aleph_j) = \sup(B_{n-1} \cap \aleph_j) = \sup(A^* \cap \aleph_j)$  if  $i_0 < j < i_n$  and  $j \notin E$ . For  $i_n < j \notin E$ ,  $\sup(B_n \cap \aleph_j) = \sup(B'_n \cap \aleph_j) = \sup(A^* \cap \aleph_j)$  by Lemma 2.1. This finishes the proof of the claim.  $\square$

We now can complete the proof of Theorem 4.2. Let  $B^* = \cup_{n < \omega} B_n$ . Then  $B^* \prec A$  and  $(\forall i \geq \min(E))[cf(B^* \cap \aleph_i) = \aleph_{f(i)}]$  by the above claim. (Note that if  $i > n^*$  and  $i \leq i_n$ , then  $B^* \cap \aleph_i = B_n \cap \aleph_i$  by the claim.)

Finally, let  $B = sk^A(B^* \cup \aleph_{n^*})$ .  $B$  is as required again by Lemma 2.1. So we have finished the proof of Theorem 4.2.  $\square$

**Concluding Remarks.** The most natural context (at least for the second author) is having a constant cardinal  $\kappa$ , set  $\mathfrak{a}$  of regular cardinals. Let  $\lambda = \sup(\mathfrak{a})$  and we look for stationary subsets of  $[\lambda]^{<\kappa}$ . Let  $\mathcal{F}_\alpha^\kappa = \{f : f \text{ is a function with domain } \mathfrak{a} \text{ and } f(\theta) \text{ is a regular cardinal } < \theta \text{ and } \sup \text{Rang}(f) < \kappa\}$ . For an ideal  $J$  on  $\mathfrak{a}$  and  $f \in \mathcal{F}_\alpha^\kappa$  we define  $S_f^J = \{A \subseteq \lambda : |A| < \kappa \text{ and for some } \mathfrak{b} \in J \text{ for each } \theta \in \mathfrak{a} \setminus \mathfrak{b} \text{ the set } A \cap \theta \text{ is a bounded subset of } \theta \text{ with order type of cofinality } f(\theta)\}$ . Note that  $\mathfrak{a}, \lambda$  and  $\kappa$  can be reconstructed from  $f$  so we can just say “ $S_f = S_f^J$  is stationary”. We call the framework simple if  $\kappa \leq \text{Min}(\mathfrak{a})$ , and we concentrate on it. If  $J = \{\emptyset\}$  we may omit it.

(\*)<sub>1</sub> If  $\mathfrak{a}$  has a maximal element  $\theta$  and  $f \in \mathcal{F}_\alpha^\kappa$  then  $(f \upharpoonright (\mathfrak{a} \setminus \{\theta\})) \in \mathcal{F}_{\mathfrak{a} \setminus \{\theta\}}^\kappa$  and  $S_f^J$  is stationary iff  $S_{f \upharpoonright (\mathfrak{a} \setminus \{\theta\})}$  is stationary.

(\*)<sub>2</sub> for  $f \in \mathcal{F}_\alpha^\kappa$  and  $\theta$  we have  $S_f^J$  is stationary iff  $S_{f \upharpoonright (\mathfrak{a} \cap \theta)}^J$  is stationary and  $S_{f \upharpoonright (\mathfrak{a} \setminus \theta)}^J$  is stationary.

(\*)<sub>3</sub> assume  $\mathfrak{a}$  has no last element and  $f \in \mathcal{F}_\alpha^\kappa$ , then  $S_f$  is stationary iff (a) + (b) where:  
 (a) for every algebra  $M$  with universe  $\sup(\mathfrak{a})$  for some  $N \prec M$  we have: for every  $\theta \in \mathfrak{a}$  large enough,

$$cf(\sup(N \cap \theta)) = f(\theta).$$

(b) for every  $\theta \in \mathfrak{a}$  the set  $S_{f \upharpoonright (\mathfrak{a} \cap \theta)}^J$  is stationary.

(\*)<sub>4</sub> Assume  $n^* < \omega$  and  $\lambda_1 < \lambda_2 < \dots < \lambda_{n^*}$  are member of  $pcf(\mathfrak{a})$  which are  $> \sup(\mathfrak{a})$ . Assume further  $f \in \mathcal{F}_\alpha^\kappa$ ,  $\langle \mathfrak{b}_1, \dots, \mathfrak{b}_{n^*} \rangle$  is a partition of  $\mathfrak{a}$ , and  $f \upharpoonright \mathfrak{b}_e$  is constant and  $\lambda_e = tcf(\prod \mathfrak{b}_e, <_{J \upharpoonright \mathfrak{b}_e})$  then  $S_f^J$  is stationary.

[Why? by the proof of Theorem 3.1. I.e. by (\*)<sub>2</sub> used several times wlog  $\text{Min}(\mathfrak{a}) > |\mathfrak{a}|^{n^*+2}$ , then by the proof of 3.1 if  $1 \leq i \leq n^* \Rightarrow f(i) > |\mathfrak{a}|$  we succeed. Lastly use (\*)<sub>3</sub> possibly several times.]

(\*)<sub>5</sub> Assume (a)  $\mathfrak{b} \subseteq \mathfrak{a}$  is countable,  $f \in \mathcal{F}_\alpha^\kappa$ ,  $f \upharpoonright \mathfrak{b}$  is constantly  $\aleph_0$  and  $f \upharpoonright (\mathfrak{a} \setminus \mathfrak{b})$  is constantly  $\sigma$ ,  $\delta$  is a unit ordinal of cofinally  $\sigma$  but  $< \kappa$  and  $\delta$  is divisible by  $|\mathfrak{a}|$ .  
 (b) for  $\theta \in \mathfrak{a}$ , the second player has a winner strategy in the game  $GM_{\sigma_0}(F_\theta, \theta \cap (\mathfrak{a} \setminus \mathfrak{b}))$  (see second remark to Theorem 4.1).

Then  $S_f$  is stationary.

[Why? repeat the proof of Theorem 4.1, but we let  $\langle \theta_i : i < \omega \rangle$  list  $\mathfrak{b}$ , each appearing infinitely often, and  $\mathcal{T} = \{T : T \text{ subtree of } \cup_{e < \omega} \prod_{i < e} \theta_i \text{ such that for every } \eta \in T \text{ of length } i \text{ we have } \{\alpha < \theta_i : \hat{\eta}(i) \in T\} \neq \emptyset \text{ mod } F_{\theta_i}\}$  let  $\langle (\theta_\xi, m_\xi) : \xi < \delta \rangle$  be such that:  $\theta_\alpha \in \mathfrak{a} \setminus \mathfrak{b}$ ,  $m_\alpha < \omega$ , and each such pair occurs boundedly often. Then define the  $T_\xi \in \mathcal{T}$  as in the proof of Theorem 4.1, in  $T_{\xi+1}$  we take care of every  $\eta \in T_{\xi+1}$  of length  $\leq m_\xi$ .]

(\*)<sub>6</sub> Assume (a)  $\mathfrak{b} \subseteq \mathfrak{a}$ ,  $\sigma = cf(\sigma) < \kappa$ ,  $f \in \mathcal{F}_\mathfrak{a}^\kappa$ ,  $f \upharpoonright (\mathfrak{a} \setminus \mathfrak{b})$  is constantly  $\sigma$ ,  $\delta$  is an ordinal  $< \kappa$  of cofinality  $< \sigma$  and let  $\sigma_\theta = [(\sup(\mathfrak{a} \cap \theta))^{\sigma + \sup(\mathfrak{b} \cap \theta)}]^+$ .

(b) for  $\theta \in \mathfrak{b}$ ,  $F_\theta$  is a  $\sigma_\theta$ -complete filter on  $\theta$  extending the club filter such that player two has a winning strategy in the game  $GM'_{\delta, \sigma_\theta}(F_\theta)$ .

Then  $S_f$  is stationary.

[Why? by the proof of Theorem 4.2]

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