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# ON TREE IDEALS

**Abstract.** Let $l^0$ and $m^0$ be the ideals associated with Laver and Miller forcing, respectively. We show that $\mathbf{add}(l^0) < \mathbf{cov}(l^0)$ and $\mathbf{add}(m^0) < \mathbf{cov}(m^0)$ are consistent. We also show that both Laver and Miller forcing collapse the continuum to a cardinal $\leq \mathfrak{h}$.

**Introduction and Notation** In this paper we investigate the ideals connected with the classical tree forcings introduced by Laver [La] and Miller [Mi]. Laver forcing $\mathbb{L}$ is the set of all trees $p$ on ${}^{<\omega}\omega$ such that $p$ has a stem and whenever $s \in p$ extends $stem(p)$ then $Succ_p(s) := \{n : s\hat{\ }n \in p\}$ is infinite. Miller forcing $\mathbb{M}$ is the set of all trees $p$ on ${}^{<\omega}\omega$ such that $p$ has a stem and for every $s \in p$ there is $t \in p$ extending $s$ such that $Succ_p(t)$ is infinite. The set of all these splitting nodes in $p$ we denote by $Split(p)$. For any $t \in Split(p)$, $Split_p(t)$ is the set of all minimal (with respect to extension) members of $Split(p)$ which properly extend $t$. For both $\mathbb{L}$ and $\mathbb{M}$ the order is inclusion.

The Laver ideal $\ell^0$ is the set of all $X \subseteq {}^{\omega}\omega$ with the property that for every $p \in \mathbb{L}$ there is $q \in \mathbb{L}$ extending $p$ such that $X \cap [q] = \emptyset$. Here $[q]$ denotes the set of all branches of $q$. The Miller ideal $m^0$ is defined analogously, using conditions in $\mathbb{M}$ instead of $\mathbb{L}$. By a fusion argument one easily shows that $\ell^0$ and $m^0$ are $\sigma$-ideals.

The additivity (**add**) of any ideal is defined as the minimal cardinality of a family of sets belonging to the ideal whose union does not. The covering number (**cov**) is defined as the least cardinality of a family of sets from the ideal whose union is the whole set on which the ideal is defined – ${}^{\omega}\omega$ in our case. Clearly $\omega_1 \leq \mathbf{add}(\ell^0) \leq \mathbf{cov}(\ell^0) \leq \mathfrak{c}$ and $\omega_1 \leq \mathbf{add}(m^0) \leq \mathbf{cov}(m^0) \leq \mathfrak{c}$ hold.

The main result in this paper says that there is a model of ZFC where $\mathbf{add}(\ell^0) < \mathbf{cov}(\ell^0)$ and $\mathbf{add}(m^0) < \mathbf{cov}(m^0)$ hold. The motivation was that by a result of Plewik [Pl] it was known that the additivity and the covering number of the ideal connected with Mathias forcing are the same and they are equal to the cardinal invariant $\mathfrak{h}$ – the least cardinality of a family of maximal antichains of $\mathcal{P}(\omega)/fin$ without a common refinement. On the other hand, in [JuMiSh] it was shown that $\mathbf{add}(s^0) < \mathbf{cov}(s^0)$ is consistent, where

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$s^0$ is Marczewski's ideal – the ideal connected with Sacks forcing $\mathbb{S}$. Intuitively, $\mathbb{L}$ and $\mathbb{M}$ sit somewhere between Mathias forcing and $\mathbb{S}$. In [GoJoSp] it was shown that under Martin's axiom $\mathbf{add}(\ell^0) = \mathbf{add}(m^0) = \mathfrak{c}$, whereas this is false for $s^0$ (see [JuMiSh]).

The method of proof for $\mathbf{add}(s^0) < \mathbf{cov}(s^0)$ in [JuMiSh] is the following: For a forcing $P$ denote by $\kappa(P)$ the least cardinal to which forcing with $P$ collapses the continuum. In [JuMiSh] it is shown that $\mathbf{add}(s^0) \leq \kappa(\mathbb{S})$. In [BaLa] it was shown that in $V^{\mathbb{S}_{\omega_2}}$ $\kappa(\mathbb{S}) = \omega_1$ holds – where $\mathbb{S}_{\omega_2}$ is the countable support iteration of length $\omega_2$ of $\mathbb{S}$. Hence $V^{\mathbb{S}_{\omega_2}} \models \mathbf{add}(s^0) = \omega_1$. On the other hand, a Löwenheim-Skolem argument shows that $V^{\mathbb{S}_{\omega_2}} \models \mathbf{cov}(s^0) = \omega_2$.

Our method of proof is similar. Denoting by $P_{\omega_2}$ a countable support iteration of length $\omega_2$ of $\mathbb{L}$ and $\mathbb{M}$ (each occuring on a stationary set), in §2 we prove the following:

**Theorem**

$$V^{P_{\omega_2}} \models \omega_1 = \mathbf{add}(\ell^0) = \mathbf{add}(m^0) < \mathbf{cov}(\ell^0) = \mathbf{cov}(m^0) = \omega_2$$

The crucial steps in the proof are to show that $\kappa(\mathbb{L})$, $\kappa(\mathbb{M})$ equal $\omega_1$ and $\mathbf{add}(\ell^0) \leq \kappa(\mathbb{L})$, $\mathbf{add}(m^0) \leq \kappa(\mathbb{M})$ holds.

We will use the standard terminology for set theory and forcing. By $\mathfrak{b}$ we denote the least cardinality of a family of functions in ${}^\omega\omega$ which is unbounded with respect to eventual dominance and $\mathfrak{d}$ will be the least cardinality of a dominating family in ${}^\omega\omega$. Moreover $\mathfrak{p}$ is the least cardinality of a filter base on $([\omega]^\omega, \subseteq^*)$ without any lower bound, and $\mathfrak{t}$ is the least cardinality of a decreasing chain in $([\omega]^\omega, \subseteq^*)$ without any lower bound. It is easy to see that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

## 1. Upper and lower bounds

**1.1 Theorem** *(1)* $\mathfrak{t} \leq \mathbf{add}(\ell^0) \leq \mathbf{cov}(\ell^0) \leq \mathfrak{b}$
*(2)* $\mathfrak{p} \leq \mathbf{add}(m^0) \leq \mathbf{cov}(m^0) \leq \mathfrak{d}$

*Proof of (1):* We have to prove the first and the third inequality. For the third inequality, let $\langle f_\alpha : \alpha < \mathfrak{b}\rangle$ be an unbounded family. Define

$$X_\alpha := \{f \in {}^\omega\omega : (\exists^\infty k)\ f(k) < f_\alpha(k)\}$$

Clearly $\bigcup\{X_\alpha : \alpha < \mathfrak{b}\} = {}^\omega\omega$. We claim $X_\alpha \in \ell^0$. Let $p \in \mathbb{L}$. We define $q \in \mathbb{L}$ as follows: *stem*$(q) :=$ *stem*$(p)$, and for any $s$ extending *stem*$(q)$ we have $s \in q$ if and only if $s \in p$ and $(\forall k)$ if $|stem(q)| \leq k < |s|$ then $s(k) \geq f_\alpha(k)$. Then clearly $q \in \mathbb{L}$, $q$ extends $p$ and $[q] \cap X_\alpha = \emptyset$.

In order to prove the first inequality we use the following notation from [JuMiSh]: Let $Q := \{\bar{A} = \langle A_s : s \in {}^{<\omega}\omega\rangle : (\forall s)\ A_s \in [\omega]^\omega\}$. For $\bar{A} \in Q$ we define a sequence of Laver trees $\langle p_s(\bar{A}) : s \in {}^{<\omega}\omega\rangle$ as follows: $p_s(\bar{A})$ is the unique Laver tree such that $stem(p_s(\bar{A})) = s$ and if $t \in p_s(\bar{A})$ extends $s$ then $Succ_{p_s(\bar{A})}(t) = A_t$.

For $\bar{A}, \bar{B} \in Q$ we define:

$$\bar{A} \subseteq \bar{B} \Leftrightarrow (\forall s)\ A_s \subseteq B_s$$

$$\bar{A} \subseteq^* \bar{B} \Leftrightarrow (\forall s)\ A_s \subseteq^* B_s$$

$$\bar{A} \leq^* \bar{B} \Leftrightarrow (\forall s)\ A_s \subseteq^* B_s \wedge (\forall^\infty s)\ A_s \subseteq B_s$$

Here $\leq^*$ is a slight but important modification of $\subseteq^*$ from [JuMiSh].

**Fact 1.2** *$(Q, \leq^*)$ is $\mathfrak{t}$-closed.*

*Proof of 1.2* Suppose $\langle \bar{A}_\alpha : \alpha < \gamma\rangle$ where $\gamma < \mathfrak{t}$ is a decreasing sequence in $(Q, \leq^*)$. Let $\bar{A}_\alpha := \langle A^\alpha_s : s \in {}^{<\omega}\omega\rangle$. Since $\gamma < \mathfrak{t}$ there is $\bar{B}' = \langle B'_s : s \in {}^{<\omega}\omega\rangle \in Q$ such that $(\forall \alpha < \gamma)\ \bar{B}' \subseteq^* \bar{A}_\alpha$. Define $f_\alpha : {}^{<\omega}\omega \to \omega$ such that $(\forall s)\ B'_s \setminus f_s(\alpha) \subseteq A^\alpha_s$. Since $\mathfrak{t} \leq \mathfrak{b}$ there exists $f : {}^{<\omega}\omega \to \omega$ such that $(\forall \alpha)(\forall^\infty s)\ f_\alpha(s) \leq f(s)$. Now let $B_s := B'_s \setminus f(s)$ and $\bar{B} := \langle B_s : s \in {}^{<\omega}\omega\rangle$. It is easy to check that $(\forall \alpha < \gamma)\ \bar{B} \leq^* \bar{A}_\alpha$.

**Fact 1.3** *Suppose $X \in \ell^0$ and $\bar{A} \in Q$. There exists $\bar{B} \in Q$ such that $\bar{B} \subseteq \bar{A}$ and $(\forall s \in {}^{<\omega}\omega)\ [p_s(\bar{B})] \cap X = \emptyset$.*

*Proof of 1.3:* First note that if $D := \{p \in \mathbb{L} : [p] \cap X = \emptyset\}$ then $D$ is open dense and even 0-dense, i.e. for every $p \in \mathbb{L}$ there exists $q \in D$ extending $p$ such that $stem(q) = stem(p)$. The proof of this is similar to Laver's proof in [La] that the set of Laver trees deciding a sentence in the language of forcing with $\mathbb{L}$ is $0-$dense: Suppose $p \in \mathbb{L}$ has no $0-$extension whose branches are not in $X$. Then inductively we can construct $q \in \mathbb{L}$ extending $p$ such that every extension of $q$ has a branch in $X$, contradicting $X \in \ell^0$.

Using this it is straightforward to construct $\bar{B}$ as desired.

**Fact 1.4:** *Suppose $X \subseteq {}^\omega\omega$, $\bar{A}, \bar{B} \in Q$, $\bar{B} \leq^* \bar{A}$ and $(\forall s)\ [p_s(\bar{A})] \cap X = \emptyset$. Then $(\forall s)\ [p_s(\bar{B})] \cap X = \emptyset$.*

*Proof of 1.4:* Clearly, if $F \subseteq p_s(\bar{B})$ is finite, then

$$[p_s(\bar{B})] = \bigcup\{[p_t(\bar{B})] : t \in p_s(\bar{B}) \setminus F\}$$

But for almost all $t \in p_s(\bar{B})$, $p_t(\bar{B})$ extends $p_t(\bar{A})$. So clearly $[p_s(\bar{B})] \subseteq [p_s(\bar{A})]$ and hence $[p_s(\bar{B})] \cap X = \emptyset$.

*End of the proof of 1.1(1).* Suppose we are given $\langle X_\alpha : \alpha < \gamma \rangle$ and $q \in \mathbb{L}$, where $\gamma < \mathfrak{t}$ and $(\forall \alpha)\ X_\alpha \in \ell^0$. Choose $\bar{A} \in Q$ such that $p_{stem(q)}(\bar{A}) = q$ and let $\bar{B}_0$ be the $\bar{B}$ given by 1.3 for $\bar{A}$ and $X_0$. If $\langle \bar{B}_\alpha : \alpha < \beta \rangle$ has been constructed for $\beta \leq \gamma$ and $\beta$ is a successor, then choose $\bar{B}_\beta$ as given by 1.3 for $\bar{A} = \bar{B}_{\beta-1}$ and $X = X_\beta$. If $\beta$ is a limit, then by 1.2 choose first $\bar{A}$ such that $(\forall \alpha < \beta)\ \bar{A} \leq^* \bar{B}_\alpha$ and then find $\bar{B}_\beta \subseteq \bar{A}$ as given by 1.3 for $\bar{A}$ and $X = X_\beta$. Finally, if we have constructed $\bar{B}_\gamma = \langle B^\gamma_s : s \in {}^{<\omega}\omega \rangle$ define $\bar{B} := \langle B_s : s \in {}^{<\omega}\omega \rangle$ by $B_s := B^\gamma_s \cap Succ_q(s)$ if $s \in q$ extends $stem(q)$ and $B_s := B^\gamma_s$ otherwise. It is easy to check that $\bar{B} \in Q$, $p_{stem(q)}(\bar{B})$ extends $q$ and $(\forall \alpha < \gamma)\ [p_{stem(q)}(\bar{B})] \cap X_\alpha = \emptyset$.

*Proof of 1.1(2)* The proof is similar to (1). For the third inequality, let $\langle f_\alpha : \alpha < \mathfrak{d} \rangle$ be a dominating family. Define

$$X_\alpha := \{f \in {}^\omega\omega : (\forall^\infty k)\ f(k) < f_\alpha(k)\}$$

Then $\bigcup\{X_\alpha : \alpha < \mathfrak{d}\} = {}^\omega\omega$ and in an analogous way as in (1) it can be seen that $X_\alpha \in m^0$.

In order to prove the first inequality we need the following concept from [GoJoSp]. Let $R$ be the set of all $\bar{P} = \langle P_s : s \in {}^{<\omega}\omega \rangle$ where each $P_s \subseteq {}^{<\omega}\omega$ is infinite, $t \in P_s$ implies $s \subset t$ and if $t, t' \in P_s$ are distinct then $t(|s|) \neq t'(|s|)$. Given $\bar{P} \in R$ we can define $\langle p_s(\bar{P}) : s \in {}^{<\omega}\omega \rangle$ as follows: $p_s(\bar{P})$ is the unique Miller tree with stem $s$ such that if $t \in Split(p_s(\bar{P}))$ then $Split_{p_s(\bar{P})}(t) = P_t$.

Define the following relations on $R$:

$$\bar{P} \leq \bar{Q} \Leftrightarrow (\forall s)\ p_s(\bar{P}) \leq p_s(\bar{Q})$$
$$\bar{P} \approx \bar{Q} \Leftrightarrow (\forall s)\ P_s =^* Q_s \wedge (\forall^\infty s)\ P_s = Q_s$$
$$\bar{P} \leq^* \bar{Q} \Leftrightarrow (\exists \bar{P}')\ \bar{P} \approx \bar{P}' \wedge \bar{P}' \leq \bar{Q}$$

**Fact 1.5 [GoJoSp, 4.14]** *Assume $MA_\kappa$($\sigma$-centered). If $\langle \bar{P}_\alpha : \alpha < \kappa \rangle$ is a $\leq^*$-decreasing sequence in $R$, then there exists $\bar{Q} \in R$ such that $(\forall \alpha < \kappa)\ \bar{Q} \leq^* \bar{P}_\alpha$.*

The following two facts have similar proofs as 1.3 and 1.4.

**Fact 1.6** *Suppose $X \in m^0$ and $\bar{P} \in R$. There exists $\bar{Q} \leq \bar{P}$ such that $(\forall s)\ [p_s(\bar{Q})] \cap X = \emptyset$.*

**Fact 1.7** *Suppose $X \in m^0$, $\bar{P}, \bar{Q} \in R$, $\bar{P} \leq^* \bar{Q}$ and $(\forall s)\ [p_s(\bar{Q})] \cap X = \emptyset$. Then $(\forall s)\ [p_s(\bar{P})] \cap X = \emptyset$.*

Now using 1.5, 1.6, 1.7 and the well-known result that for all $\kappa < \mathfrak{p}$ $MA_\kappa$($\sigma$-centered) holds, a similar construction as in 1.1(1) shows that $\mathfrak{p} \leq \mathbf{add}(m^0)$.

## 2 add and cov are distinct

**Definition 2.1** A set $A \subseteq {}^\omega\omega$ is called *strongly dominating* if and only if

$$(\forall f \in {}^\omega\omega)(\exists \eta \in A)(\forall^\infty k)\ f(\eta(k-1)) < \eta(k)$$

**Definition 2.2** For any set $A \subseteq {}^\omega\omega$, we define the domination game $D(A)$ as follows: There are two players, GOOD and BAD. GOOD plays first. The game lasts $\omega$ moves.

| GOOD | BAD |
|---|---|
| $s$ | |
| | $n_0$ |
| $m_0$ | |
| | $n_1$ |
| $m_1$ | |
| $\vdots$ | $\vdots$ |

The rules are: $s$ is a sequence in ${}^{<\omega}\omega$, and the $n_i$ and $m_i$ are natural numbers. (Whoever breaks these rules first, loses immediately).

The GOOD player wins if and only if

(a) For all $i$, $m_i > n_i$.
(b) The sequence $s^\frown m_0{}^\frown m_1{}^\frown \cdots$ is in $A$.

**Lemma 2.3** *Let $A \subseteq {}^\omega\omega$ be a Borel set. Then the following are equivalent:*
*(1) There exists a Laver tree $p$ such that $[p] \subseteq A$.*
*(2) $A$ is strongly dominating.*
*(3) GOOD has a winning strategy in the game $D(A)$.*

*Remark:* Strongly dominating is not the same as dominating. For example, the closed set

$$A := \{\eta \in {}^\omega\omega : (\forall k)\ \eta(2k) = \eta(2k+1)\}$$

is dominating but is not strongly dominating.

*Proof of 2.3* We consider the following condition:

(4) (For all $F : {}^{<\omega}\omega \times \omega \to \omega)(\exists \eta \in A)(\forall^\infty k)(\forall i \leq k)\ \eta(k) > F(\eta\restriction k, i)$.

We will show (1) $\to$ (2) $\to$ (4) $\to$ (3) $\to$ (1).

(1) $\to$ (2) is clear.

(2) → (4): Given $F$, define $f$ by

$$f(m) := \max\{F(s,i) : i \le m, s \in m^{\le m+1}\} + m$$

$f$ is increasing, $f(m) \ge m$ for all $m$.

Find $\eta$ such that $\forall^\infty k\, \eta(k) > f(\eta(k-1))$. Then $\eta$ is increasing. For almost all $k$ we have, letting $m := \eta(k-1)$:
$m \ge k-1$, so $\eta{\restriction}k \in m+1^{m+1}$, so by the definition of $f$ we get $f(m) \ge F(\eta{\restriction}k, i)$ for any $i \le k$. So $\eta(k) > f(\eta(k-1) \ge F(\eta{\restriction}k, i)$.

(4) → (3): Assume that GOOD has no winning strategy. Then BAD has a winning strategy $\sigma$ (since the game $D(A)$ is Borel, hence determined.)

We can find a function $F : {}^{<\omega}\omega \times \omega \to \omega$ such that for all $s, m_0, \ldots, m_k$ we have

$$\sigma(s, m_0, \ldots, m_k) = F(s{}^\frown m_0{}^\frown \cdots {}^\frown m_k, |s|)$$

Find $\eta \in A$ as in (4). So there is $k_0$ such that $\forall k \ge k_0\, \eta(k) \ge F(\eta{\restriction}k, k_0)$. So in the play

| GOOD | BAD |
|---|---|
| $s := \eta{\restriction}k_0$ | |
| | $n_0 := \sigma(s) = F(\eta{\restriction}k_0, k_0)$ |
| $m_0 := \eta(k_0+1)$ | |
| | $n_1 := \sigma(s, m_0) = F(\eta{\restriction}(k_0+1), k_0)$ |
| $m_1 := \eta(k_0+2)$ | |
| $\vdots$ | $\vdots$ |

player BAD followed the strategy $\sigma$, but player GOOD won, a contradiction.

(3) → (1): Let $B$ be the set of all sequences $s{}^\frown m_0{}^\frown m_1{}^\frown \cdots$ that can be played when GOOD follows a specific winning strategy. Clearly $B \subseteq A$, and for some Laver tree $p$, $B = [p]$.

**Lemma 2.4 [Ke]** *Let $A \subseteq {}^\omega\omega$ be an analytic set. Then the following are equivalent:*
*(1) There exists a Miller tree $p$ such that $[p] \subseteq A$.*
*(2) $A$ is unbounded in $({}^\omega\omega, \le^*)$.*

**Lemma 2.5** *(1) Suppose $\mathfrak{b} = \mathfrak{c}$. For every dense open $D \subseteq \mathbb{L}$ there exists a maximal antichain $A \subseteq D$ such that*

$$\forall q \in \mathbb{L}([q] \subseteq \bigcup\{[p] : p \in A\} \;\Rightarrow\; \exists A' \in [A]^{<\mathfrak{c}} \forall p \in A \setminus A'\, p \perp q) \qquad (*)$$

*(2) The same is true for $\mathbb{M}$.*

*Proof:* Let $\mathbb{L} = \{q_\alpha : \alpha < \mathfrak{c}\}$. Inductively we will define a set $S \subseteq \mathfrak{c}$ and sequences $\langle x_\gamma : \gamma < \mathfrak{c}\rangle$ and $\langle p_\gamma : \gamma \in S\rangle$. Finally we will let $A = \{p_\gamma : \gamma \in S\}$.

Let $0 \in S$ and choose $x_0 \in [q_0]$ arbitrarily.

It can be easily seen that every Laver tree contains $\mathfrak{c}$ extensions such that every two of them do not contain a common branch. So clearly we may find $p_0 \in D$ such that $x_0 \notin [p_0]$.

Now suppose that $\langle x_\gamma : \gamma < \alpha\rangle$ and $\langle p_\gamma : \gamma \in S \cap \alpha\rangle$ have been constructed for $\alpha < \mathfrak{c}$.

First choose $x_\alpha \in [q_\alpha]$ arbitrarily, but such that, if $[q_\alpha] \not\subseteq \bigcup\{[p_\gamma] : \gamma < \alpha\}$ then $x_\alpha \notin \bigcup\{[p_\gamma] : \gamma < \alpha\}$.

In order to decide whether $\alpha \in S$ or not we distinguish the following two cases:

**Case 1:** $q_\alpha$ is compatible with some $p_\gamma$, $\gamma < \alpha$. In this case $\alpha \notin S$.

**Case 2:** $q_\alpha$ is incompatible with all $p_\gamma$, $\gamma < \alpha$. Now we let $\alpha \in S$, and we define $p_\alpha$ as follows:

By Lemma 2.3 for each $\gamma \in \alpha$ we may find $f_\gamma : \omega \to \omega$ such that

$$(\forall \eta \in [p_\gamma] \cap [q_\alpha])(\exists^\infty k)\ \eta(k) \le f_\gamma(\eta(k-1)) \tag{$**$}$$

By our assumption on $\mathfrak{b}$ there exists a strictly increasing $f$ which dominates all the $f_\gamma$'s. Now define $p'_\alpha \in \mathbb{L}$ as follows: $stem(p'_\alpha) = stem(q_\alpha)$, and for $t \in p'_\alpha$, if $t \supseteq stem(p'_\alpha)$ and $|t| =: n$ we require

$$Succ_{p'_\alpha}(t) = Succ_{q_\alpha}(t) \cap [f(t(n-1)), \infty)$$

Clearly $p'_\alpha \in \mathbb{L}$, $p'_\alpha \subseteq q_\alpha$, and by $(**)$ and our assumption on $f$ we conclude $[p_\gamma] \cap [p'_\alpha] = \emptyset$ for every $\gamma < \alpha$.

By the remark above that every Laver tree contains $\mathfrak{c}$ extensions such that every two of them do not contain a common branch, we may find $p_\alpha \in D$ such that $p_\alpha$ extends $p'_\alpha$ and $[p_\alpha]$ and $\{x_\gamma : \gamma \le \alpha\}$ are disjoint.

This finishes the construction. Now let $A := \{p_\gamma : \gamma \in S\}$.

Since every $q_\alpha$ is either compatible with some $p_\gamma$, $\gamma < \alpha$ (in case 1) or contains the condition $p_\alpha$ (in case 2) and for $\alpha \ne \gamma$ with $\alpha, \gamma \in S$ we have $[p_\alpha] \cap [p_\gamma] = \emptyset$ we conclude that $A$ is a maximal antichain.

$A$ also satisfies condition $(*)$: Let $q = q_\alpha$. By construction, if $[q_\alpha] \not\subseteq \bigcup\{[p_\gamma] : \gamma \in S \cap \alpha\}$ then $[q_\alpha] \not\subseteq \bigcup\{[p_\gamma] : \gamma \in S\}$.

The proof of (2) is analogous, but instead of Lemma 2.3 we use 2.4.

**Lemma 2.6** *Suppose* $\mathfrak{b} = \mathfrak{c}$. *Then* $\mathbf{add}(\ell^0) \le \kappa(\mathbb{L})$ *and* $\mathbf{add}(m^0) \le \kappa(\mathbb{M})$.

*Proof:* We may assume $\kappa(\mathbb{L}) < \mathfrak{c}$. Let $\dot f$ be a $\mathbb{L}$-name such that $\Vdash_{\mathbb{L}}$ "$\dot f : \kappa(\mathbb{L}) \to \mathfrak{c}$ is onto". For $\alpha < \kappa(\mathbb{L})$ let

$$D_\alpha := \{p \in \mathbb{L} : (\exists \beta)\ p \Vdash_{\mathbb{L}} \dot f(\alpha) = \beta\}$$

For $p \in D_\alpha$ will write $\beta_p = \beta_p(\alpha)$ for the unique $\beta$ satisfying $p \Vdash_{\mathbb{L}} \dot{f}(\alpha) = \beta$.

Clearly $D_\alpha$ is dense and open. So we may choose a maximal antichain $A_\alpha \subseteq D_\alpha$ as in Lemma 2.5. Let

$$X_\alpha := {}^\omega\omega \setminus \bigcup\{[p] : p \in A_\alpha\}$$

Then $X_\alpha \in \ell^0$. We claim that $X = \bigcup_{\alpha<\kappa(\mathbb{L})} X_\alpha \notin \ell^0$. Suppose on the contrary $X \in \ell^0$. So we may find $q \in \mathbb{L}$ such that $[q] \cap X = \emptyset$ and hence $[q] \subseteq \bigcup\{[p] : p \in A_\alpha\}$ for each $\alpha$. By the choice of $A_\alpha$ each of the sets

$$B_\alpha := \{\beta_p(\alpha) : p \in A_\alpha,\ p \text{ compatible with } q\}$$

is bounded in $\mathfrak{c}$. Since $\mathfrak{c}$ is regular by our assumption $\mathfrak{b} = \mathfrak{c}$ we can find $\nu < \mathfrak{c}$ such that for all $\alpha < \kappa(\mathbb{L})$, $B_\alpha \subseteq \nu$. So easily conclude that

$$q \Vdash_{\mathbb{L}} \text{“range}(\dot{f}) \subseteq \nu < \mathfrak{c}\text{”}$$

This is a contradiction.

The proof for $\mathbb{M}$ is similar.

**Theorem 2.7** *$\kappa(\mathbb{L}) \leq \mathfrak{h}$ and $\kappa(\mathbb{M}) \leq \mathfrak{h}$.*

*Proof:* We prove it only for $\mathbb{L}$. The proof for $\mathbb{M}$ is very similar. We work in $V$. Let $\langle \mathcal{A}_\alpha : \alpha < \mathbf{h}\rangle$ be a family of maximal almost disjoint families such that,

(1) if $\alpha < \beta < \mathfrak{c}$ then $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$
(2) there exists no maximal almost disjoint family refining all the $\mathcal{A}_\alpha$
(3) $\bigcup\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ is dense in $([\omega]^\omega, \subseteq^*)$

That such a sequence exists was shown in [BaPeSi].

Since $\mathfrak{h}$ is regular, for every $p \in \mathbb{L}$ there exists $\alpha < \mathfrak{h}$ such that for each $s \in \mathit{Split}(p)$ there is $A \in \mathcal{A}_\alpha$ with $A \subseteq^* \mathit{Succ}_p(s)$. Hence, writing $\mathbb{L}_\alpha$ for the set of those $p \in \mathbb{L}$ for which $\alpha$ has the property just stated, we conclude $\mathbb{L} = \bigcup\{\mathbb{L}_\alpha : \alpha < \mathfrak{h}\}$.

For each $A \in \mathcal{A}_\alpha$ choose $\mathcal{B}_A = \{B^A(p) : p \in \mathbb{L}\}$, an almost disjoint family on $A$.

Now we will define $\mathbb{L}'_\alpha := \{q^\alpha(p) : p \in \mathbb{L}_\alpha\}$ such that $q^\alpha(p)$ extends $p$ for every $p \in \mathbb{L}_\alpha$ and $p_1 \neq p_2$ implies $q^\alpha(p_1) \perp q^\alpha(p_2)$. For $p \in \mathbb{L}_\alpha$, $q^\alpha(p)$ will be defined as follows:

> For each $s \in \mathit{Split}(p)$ let $C^\alpha_s(p) := \mathit{Succ}_p(s) \cap B^A(p)$ where $A \in \mathcal{A}_\alpha$ is such that $A \subseteq^* \mathit{Succ}_p(s)$. So clearly $C^\alpha_s(p)$ is infinite. Now $q^\alpha(p)$ is the unique Laver tree $\subseteq p$ satisfying $\mathit{stem}(q^\alpha(p)) = \mathit{stem}(p)$ and for each $s \in \mathit{Split}(q^\alpha(p))$ we have $\mathit{Succ}_{q^\alpha(p)}(s) = C^\alpha_s(p)$.

It is not difficult to see that $\mathbb{L}'_\alpha$ has the stated properties.

Now we are ready to define a $\mathbb{L}$-name $\dot{f}$ such that $\Vdash_{\mathbb{L}}$ “$\dot{f} : \mathfrak{h}^V \to \mathfrak{c}^V$ is onto”: For each $p \in \mathbb{L}_\alpha$, let $\{r^\alpha_\xi(p) : \xi < \mathfrak{c}\} \subseteq \mathbb{L}$ be a maximal antichain below $q^\alpha(p)$, and define $\dot{f}$

in such a way that $r^\alpha_\xi(p) \Vdash_{\mathbb{L}}$ "$\dot f(\alpha) = \xi$". As $\bigcup\{\mathbb{L}'_\alpha : \alpha < \mathfrak{h}\}$ is dense in $\mathbb{L}$, it is easy to check that $\dot f$ is as desired.

**Theorem 2.8** *Let $\omega_2 = S_{\mathbb{M}} \dot\cup S_{\mathbb{L}}$, where the sets $S_{\mathbb{M}}$ and $S_{\mathbb{L}}$ are disjoint and stationary. Let $(P_\alpha, Q_\alpha : \alpha < \omega_2)$ be a countable support iteration of length $\omega_2$ such that for all $\alpha$ we have $\Vdash_{P_\alpha} Q_\alpha = \mathbb{M}$ whenever $\alpha \in S_{\mathbb{M}}$, and $\Vdash_{P_\alpha} Q_\alpha = \mathbb{L}$ otherwise. Also suppose that $V$ satisfies CH. Then in $V^P$, $\mathfrak{h} = \omega_1$ holds.*

*Proof:* Both $\mathbb{M}$ and $\mathbb{L}$ have the property $(*)_1$ of [JuSh]. (For $\mathbb{L}$, this was proved in [JuSh] and for $\mathbb{M}$ this was proved in [BaJuSh].) [JuSh] also showed that this property is preserved under countable support iterations, so also $P_{\omega_2}$ has this property. Hence the reals of $V$ do not have measure zero in $V^P$, so from $\mathfrak{h} \le \mathfrak{s} \le \mathbf{unif}(\mathcal{L})$ (where $\mathfrak{s}$ is the splitting number and $\mathbf{unif}(\mathcal{L})$ is the smallest cardinality of a set of reals which is not null) we get the desired conclusion.

**Theorem 2.9** *Let $P_{\omega_2}$ be as in 2.8. Then*

$$V^{P_{\omega_2}} \models \omega_1 = \mathbf{add}(\ell^0) = \mathbf{add}(m^0) < \mathbf{cov}(\ell^0) = \mathbf{cov}(m^0) = \omega_2$$

*Proof:* Since $\mathbb{L}$ adds a dominating real, we have $V^{P_{\omega_2}} \models \mathfrak{b} = \mathfrak{c}$, so by 2.6, 2.7 and 2.8, it suffices to prove that the covering coefficients are $\omega_2$ in the respective models. The proof of this is similar to the proof of [JuMiSh, Thm1.2] that **cov** of the Marczewski ideal is $\omega_2$ in the iterated Sacks' forcing model.

We give the proof only for $\ell^0$. Suppose $\langle X_\alpha : \alpha < \omega_1\rangle \in V^{P_{\omega_2}}$ is a sequence of $\ell^0$-sets. In $V^{P_{\omega_2}}$ let $f_\alpha : \mathbb{L} \to \mathbb{L}$ be such that for every $p \in \mathbb{L}$, $f_\alpha(p)$ extends $p$ and $[f_\alpha(p)] \cap X_\alpha = \emptyset$. Since $P_{\omega_2}$ has the $\omega_2$-chain condition, by a Löwenheim-Skolem argument it is possible to find $\gamma < \omega_2$ such that

$$\langle f_\alpha \restriction \mathbb{L}^{V_\gamma} : \alpha < \omega_1\rangle \in V^{P_\gamma}$$

where $V_\gamma := V^{P_\gamma}$. Moreover, it is possible to find such a $\gamma$ in $S_{\mathbb{L}}$. We claim that the Laver real $x_\gamma$ (which is added by $Q_\gamma = \mathbb{L}^{V_\gamma}$) is not in $\bigcup_{\alpha<\omega_1} X_\alpha$, which will finish the proof. Otherwise, for some $p \in \mathbb{L}_{\gamma\omega_2}$ where $\mathbb{L}_{\gamma\omega_2} := \mathbb{L}_{\omega_2}/G_\gamma$ and some $\alpha < \omega_1$ we would have: $p \Vdash x_\gamma \in X_\alpha$. But letting $q := p(\gamma) \in \mathbb{L}$ and letting $r(\gamma) := f_\alpha(q)$ and $r(\beta) := p(\beta)$ for $\beta > \gamma$ we see that $r \Vdash x_\gamma \notin X_\alpha$, a contradiction.

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