# On the existence of atomic models

M. C. Laskowski*
Department of Mathematics
University of Maryland

S. Shelah
Department of Mathematics
Hebrew University of Jerusalem

Department of Mathematics
Rutgers University †

December 14, 1992

**Abstract**

We give an example of a countable theory $T$ such that for every cardinal $\lambda \geq \aleph_2$ there is a fully indiscernible set $A$ of power $\lambda$ such that the principal types are dense over $A$, yet there is no atomic model of $T$ over $A$. In particular, $T(A)$ is a theory of size $\lambda$ where the principal types are dense, yet $T(A)$ has no atomic model.

If a complete theory $T$ has an atomic model then the principal types are dense in the Stone space $S_n(T)$ for each $n \in \omega$. In [2, Theorem 1.3], [3, page 168] and [5, IV 5.5], Knight, Kueker and Shelah independently showed that the converse holds, provided that the cardinality of the underlying language has size at most $\aleph_1$.

*Partially supported by an NSF Postdoctoral Fellowship.

†The authors thank the U.S. Israel Binational Science Foundation for its support of this project. This is item 489 in Shelah's bibliography.

In this paper we build an example that demonstrates that the condition on the cardinality of the language is necessary. Specifically, we construct a complete theory $T$ in a countable language having a distinguished predicate $V$ such that if $A$ is any subset of $V^M$ for any model $M$ of $T$, then the principal types are dense in $S_n(T(A))$ for each $n \in \omega$. ($T(A) = \mathrm{Th}(M, a)_{a \in A}$). However, $T(A)$ has an atomic model if and only if $|A| \leq \aleph_1$.

In fact, by modifying the construction (Theorem 0.5) we may insist that there is a particular non-principal, complete type $p$ such that, for any subset $A$ of $V^M$, $p$ is realized in every model of $T(A)$ if and only if $|A| \geq \aleph_2$. With Theorem 0.6 we show that the constructions can be generalized to larger cardinals.

We first build a countable, atomic model in a countable language having an infinite, definable subset of (total) indiscernibles. Let $L$ be the language with unary predicate symbols $U$ and $V$, a unary function symbol $p$, and countable collections of binary function symbols $f_n$ and binary relation symbols $R_n$ for each $n \in \omega$. By an abuse of notation, $p$ and each of the $f_n$'s will actually be partial functions.

For a subset $X$ of an $L$-structure $M$, define the closure of $X$ in $M$, $cl(X)$, to be the transitive closure of $cl_0(X) = \{f_n(b,c) : b, c \in X, n \in \omega\}$. So $cl(X)$ is a subset of the smallest substructure of $M$ containing $X$.

Let $K$ be the set of all finite $L$-structures $\mathcal{A}$ satisfying the following eight constraints:

i) $U$ and $V$ are disjoint sets whose union is the universe $A$;

ii) $p : U \to V$;

iii) each $f_n : U \times U \to U$;

iv) for each $n$, $R_n(x,y) \to (U(x) \wedge U(y))$;

v) the family $\{R_n : n \in \omega\}$ partitions all of $U^2$ into disjoint pieces;

vi) for each $n$ and $m \geq n$, $R_n(x,y) \to f_m(x,y) = x$;

vii) if $x', y' \in cl(\{x,y\})$ and $R_n(x,y)$, then $\bigvee_{j \leq n} R_j(x', y')$;

viii) there is no $cl$-independent subset of $U$ of size 3 (i.e., for all $x_0, x_1, x_2 \in U$, there is a permutation $\sigma$ of $\{0,1,2\}$ such that $x_{\sigma(0)} \in cl(\{x_{\sigma(1)}, x_{\sigma(2)}\})$.)

It is routine to check that $K$ is closed under substructures and isomorphism and that $K$ contains only countably many isomorphism types. We claim that $K$ satisfies the joint embedding property and the amalgamation property. As the proofs are similar, we only verify amalgamation. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in K$ with $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A} \subseteq \mathcal{C}$ and $A = B \cap C$. It suffices to find an element $\mathcal{D}$ of $K$ with universe $B \cup C$ such that $\mathcal{B} \subseteq \mathcal{D}$ and $\mathcal{C} \subseteq \mathcal{D}$. Let $\{b_0, \ldots, b_{l-1}\}$ enumerate $U^{\mathcal{B}}$, let $\{c_0, \ldots, c_{m-1}\}$ enumerate $U^{\mathcal{C}}$ and let $k > l, m$ be large enough that $\bigcup\{R_j^{\mathcal{B}} : j < k\} = (U^{\mathcal{B}})^2$ and $\bigcup\{R_j^{\mathcal{C}} : j < k\} = (U^{\mathcal{C}})^2$. For each $b \in B \setminus A$ and $c \in C \setminus A$, let $f_j(b, c) = c_j$ for $j < m$, and $f_j(b, c) = b$ for $j \geq m$ and let $R_k(b, c)$. Similarly, for $j < l$ let $f_j(c, b) = b_j$ and $f_j(c, b) = c$ for all $j \geq l$ and $R_k(c, b)$. It is easy to check that $\mathcal{D} \in K$.

It follows (see e.g., [4, Theorem 1.5]) that there is a countable, $K$-generic $L$-structure $\mathcal{B}$. That is, (*) $\mathcal{B}$ is the union of an increasing chain of elements of $K$, (**) every element of $K$ isomorphically embeds into $\mathcal{B}$ and (***) if $j : \mathcal{A} \to \mathcal{A}'$ is an isomorphism between finite substructures of $\mathcal{B}$ then there is an automorphism $\sigma$ of $\mathcal{B}$ extending $j$. Such structures are also referred to as homogeneous-universal structures. Let $T$ be the theory of $\mathcal{B}$.

We record the following facts about $\mathcal{B}$ and $T$: First, $V^{\mathcal{B}}$ is infinite as there are elements $\mathcal{A}$ of $K$ with $V^{\mathcal{A}}$ arbitrarily large; $V^{\mathcal{B}}$ is a set of indiscernibles because of property (***) and the fact that any two $n$-tuples of distinct elements from $V^{\mathcal{B}}$ are universes of isomorphic substructures of $\mathcal{B}$; As every finite subset of $\mathcal{B}$ is contained in an element of $K$, it follows that $cl(X)$ is finite for all finite subsets $X$ of $B$ and there is no $cl$-independent subset of $U^{\mathcal{B}}$ of size 3; Finally, $\mathcal{B}$ is atomic as for any tuple $\bar{b}$ from $B$, property (***) guarantees that the complete type of $\bar{b}$ is isolated by finitely much of the atomic diagram of the smallest substructure $\mathcal{A}$ of $\mathcal{B}$ containing $\bar{b}$. (If $n$ is least such that $\bigvee_{j \leq n} R_j(a, a')$ for all $a, a' \in U^{\mathcal{A}}$ then we need only the reduct of the atomic diagram of $\mathcal{A}$ to $L_n = \{U, V, p, R_j, f_j : j \leq n\}$.)

**Lemma 0.1** *Let $\mathcal{C}$ be a model of $T$ and let $A$ be any subset of $V^{\mathcal{C}}$. Then the principal types are dense over $A$.*

**Proof.** Let $\theta(\bar{x}, \bar{a})$ be any consistent formula, where $\bar{a}$ is a tuple of $k$ distinct elements from $A$. Let $\bar{b}$ be any $k$-tuple of distinct elements from $V^{\mathcal{B}}$. As the elements from $V$ are indiscernible, $\mathcal{B} \models \exists \bar{x} \theta(\bar{x}, \bar{b})$. Let $\bar{c}$ from $\mathcal{B}$ realize $\theta(\bar{x}, \bar{b})$. Since $\mathcal{B}$ is atomic, there is a principal formula $\phi(\bar{x}, \bar{y})$ isolating $\mathrm{tp}(\bar{c}, \bar{b})$. It follows from indiscernibility that $\phi(\bar{x}, \bar{a})$ is a principal formula such that $\mathcal{C} \models \forall \bar{x}(\phi(\bar{x}, \bar{a}) \to \theta(\bar{x}, \bar{a}))$.

**Lemma 0.2** *Let $\mathcal{C}$ be an arbitrary model of $T$ and let $A \subseteq V^{\mathcal{C}}$. If $\mathcal{C}$ is atomic over $A$ then:*

*i)* $|U^{\mathcal{C}}| \geq |A|$;

*ii)* $cl(X)$ *is finite for all finite* $X \subseteq U^{\mathcal{C}}$;

*iii) there is no cl-independent subset of size 3 in* $U^{\mathcal{C}}$.

**Proof.**
(i) $|U^{\mathcal{C}}| \geq |A|$ since for each $a \in A$, $p^{-1}(a)$ is non-empty.

(ii) As $\mathcal{C}$ is atomic, $\mathrm{tp}(X/A)$ is isolated by some formula $\theta(\overline{x}, \overline{c})$, where $\overline{c}$ is a $k$-tuple of distinct elements from $A$. As $V$ is indiscernible, $\theta(\overline{x}, \overline{b})$ is principal for any $k$-tuple $\overline{b}$ of distinct elements from $\mathcal{B}$. Choose $\overline{d}$ from $\mathcal{B}$ realizing $\theta(\overline{x}, \overline{b})$ and suppose that $|cl(\overline{d})| = l < \omega$. Then as $\theta(\overline{x}, \overline{a})$ is principal, $\theta(\overline{x}, \overline{a})$ implies $|cl(\overline{x})| \leq l$.

(iii) Assume $c_0, c_1, c_2 \in U^{\mathcal{C}}$ are $cl$-independent. As $\mathcal{C}$ is atomic over $A$, $\mathrm{tp}(\overline{c})$ is principal, so let $\theta(\overline{x}, \overline{a})$ isolate $\mathrm{tp}(\overline{c})$. Again choose $\overline{b}$ from $V^{\mathcal{B}}$ and $\overline{d}$ from $U^{\mathcal{B}}$ such that $\mathcal{B} \models \theta(\overline{d}, \overline{b})$. But then $\overline{d}$ is a $cl$-independent subset of $U^{\mathcal{B}}$, which is a contradiction.

We next record a well-known combinatorial lemma (see e.g., [1]). An abstract closure relation on a set $X$ is a function $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ such that, for all subsets $A, B$ of $X$ and all $b \in X$, $A \subseteq cl(A)$, $cl(cl(A)) = cl(A)$, $A \subseteq B$ implies $cl(A) \subseteq cl(B)$ and $b \in cl(A)$ implies there is a finite subset $A_0$ of $A$ such that $b \in cl(A_0)$.

**Lemma 0.3** *For all ordinals $\alpha$ and all $n \in \omega$, if $|X| \geq \aleph_{\alpha+n}$ and cl is a closure relation on $X$ such that $|cl(A)| < \aleph_\alpha$ for all finite subsets $A$ of $X$. Then $X$ contains a cl-independent subset of size $n+1$.*

**Proof.** Fix an ordinal $\alpha$. We prove the lemma by induction on $n$. For $n = 0$ this is trivial, so assume the lemma holds for $n$. Suppose $X$ has size at least $\aleph_{\alpha+n+1}$. As $cl$ is finitely based we can find a subset $Y$ of $X$, $|Y| = \aleph_{\alpha+n}$, such that $cl(A) \subseteq Y$ for all $A \subseteq Y$. Choose $b \in X \setminus Y$. Define a closure relation $cl'$ on $Y$ by $cl'(A) = cl(A \cup \{b\}) \cap Y$. By induction there is a $cl'$-independent subset $B$ of $Y$ of size $n$. It follows that $B \cup \{b\}$ is the desired $cl$-independent subset of $X$.

Note that by taking $\alpha = 0$ in the lemma above, if $cl$ is a locally finite closure relation on a set $X$ of size $\aleph_2$, then $X$ contains a $cl$-independent subset of size 3.

**Theorem 0.4** *Let $A$ be a subset of $V^{\mathcal{C}}$ for an arbitrary model $\mathcal{C}$ of $T$. Then $A$ is a set of indiscernibles and the principal types over $A$ are dense, but there is an atomic model over $A$ if and only if $|A| \leq \aleph_1$.*

**Proof.** The principal types are dense over $A$ by Lemma 0.1. If $|A| \leq \aleph_1$ then there is an atomic model over $A$ by e.g., Theorem 1.3 of [2]. However, if $|A| \geq \aleph_2$ then there cannot be an atomic model over $A$ by Lemmas 0.2 and 0.3.

Our next goal is to modify the construction given above so that there is a non-principal complete type $p$ that is realized in any model containing $A$, provided that $|A| \geq \aleph_2$. To do this, note that any atomic model of $T(A)$ is locally finite and omits the type of a pair of elements from $U$ with every $R_n$ failing. We shall enrich the language so as to code each of these by a single 1-type.

Let $L' = L \cup \{W, g, h\} \cup \{c_n : n \in \omega\}$, where $W$ is a unary predicate, $g$ and $h$ are respectively binary and ternary function symbols, and the $c_n$'s are new constant symbols. Let $\mathcal{B}'$ be the $L'$-structure with universe $B \cup D$, where $D = \{d_n : n \in \omega\}$ is disjoint from $B$, $W$ is interpreted as $D$, each $c_n$ is interpreted as $d_n$, $g : U \times U \to W$ is given by $g(a, b) = d_n$, where $R_n(a, b)$ holds and $h(a, b, c) = d_n$ if and only if $n = |cl(\{a, b, c\})|$. Let $T'$ be the theory of $\mathcal{B}'$.

Note that any automorphism $\sigma$ of $\mathcal{B}$ extends to an automorphism $\sigma'$ of $\mathcal{B}'$, where $\sigma' \restriction D = id$. It follows that $\mathcal{B}'$ is atomic and $V^{\mathcal{B}'}$ is an indiscernible set. Let $p(x)$ be the non-principal type $\{W(x)\} \cup \{x \neq c_n : n \in \omega\}$. We claim that $p$ is complete. This follows from the fact that for any $L'$-formula $\theta(x)$, if $n$ is greater than the number of terms occurring in $\theta$ and if $\theta(x)$ is an $L'_n = \{U, V, W, g, h, p, R_l, f_l, c_l : l < n\}$-formula then $\mathcal{B}' \models \theta(c_i) \leftrightarrow \theta(c_j)$ for all $i, j \geq n$. (This fact can be verified by finding a back-and-forth system $\mathcal{S} = \{\langle \overline{a}, \overline{b} \rangle : |\overline{a}| = |\overline{b}| < n\}$ such that, for every $\langle \overline{a}, \overline{b} \rangle \in \mathcal{S}$ and every atomic $L'_n \cup \{R_i, f_i, c_i\}$-formula $\phi(\overline{x})$, $|\overline{x}| = |\overline{a}|$,

$$\mathcal{B}' \models \phi(\overline{a}) \leftrightarrow \phi'(\overline{b}),$$

where $\phi'(\overline{x})$ is the atomic $L'_n \cup \{R_j, f_j, c_j\}$-formula generated from $\phi(\overline{x})$ by replacing each occurrence of $R_i, f_i, c_i$ by $R_j, f_j, c_j$, respectively.)

**Theorem 0.5** *Let $A$ be a subset of $V^{\mathcal{C}'}$ for any model $\mathcal{C}'$ of $T'$. Then $A$ is a set of indiscernibles and the principal types over $A$ are dense. Further, if $|A| \leq \aleph_1$ then there is an atomic model over $A$, while if $|A| > \aleph_1$ then any model of $T'(A)$ realizes the complete type $p$.*

**Proof.** The first two statements follow from the atomicity of $\mathcal{B}'$ and the indiscernibility of $V^{\mathcal{B}'}$. If $|A| \leq \aleph_1$ then the existence of the atomic model over $A$ follows from Knight's theorem. So suppose $|A| \geq \aleph_2$ and let $\mathcal{D}'$ be any model of $T'$ containing $A$. By examining the proof of Lemma 0.3 it follows that either there are $a, b, c \in U^{\mathcal{D}'}$ such that $cl(\{a, b, c\})$ is infinite or that there are $cl$-independent elements $a_0, a_1, a_2 \in U^{\mathcal{D}'}$ (i.e., if $cl$ is a closure relation on $X$ and $|X| \geq \aleph_2$ then either $cl(x, y, z)$ is infinite for some $x, y, z \in X$ or there is an independent subset of $X$ of size 3).

In the first case $h(a, b, c)$ realizes $p$. Now assume that the closure of any triple from $U^{\mathcal{D}'}$ is finite. If, in addition, for every two elements $a, b$ from $U^{\mathcal{D}'}$ there were an integer $n$ such that $\mathcal{D}' \models R_n(a, b)$, then $T'$ would ensure that there would not be any 3-element $cl$-independent subset of $U^{\mathcal{D}'}$. (Under these assumptions there would be only finitely many possibilities for the diagram of a triple under the functions $\{f_i : i \in \omega\}$ and no independent triple exists in $\mathcal{B}'$.) Consequently, there must be a pair of elements $a, b$ from $U^{\mathcal{D}'}$ such that $\mathcal{D}' \models \neg R_n(a, b)$ for all $n \in \omega$, so $g(a, b)$ realizes $p$.

We close with the following theorem demonstrating that the behavior between $\aleph_1$ and $\aleph_2$ holds more generally between $\aleph_k$ and $\aleph_{k+1}$ for all $k \geq 1$. The theorem is stated in its most basic form to aid readability. We leave it to the reader to verify that the strengthenings given in Theorems 0.4 and 0.5 (i.e., no atomic model over a given set or the non-atomicity being witnessed by a specific complete type) can be made to hold as well.

**Theorem 0.6** *For every $k$, $1 \leq k < \omega$ there is a countable theory $T_k$ such that $T_k$ has an atomic model of size $\aleph_\alpha$ if and only if $\alpha \leq k$.*

**Proof.** Fix $k$. Let $L_k = \{f_n, R_n : n \in \omega\}$, where each $f_n$ is a $(k+1)$-ary function and each $R_n$ is a $(k+1)$-ary relation. Define $cl(X)$ to be the transitive closure of $cl_0(X) = \{f_n(a_0, \ldots, a_k) : a_i \in X\}$ and let $K$ be the set of all finite $L_k$-structures satisfying the following constraints:

i) $\{R_n : n \in \omega\}$ partitions the $(k+1)$-tuples into disjoint pieces;

ii) for each $n$ and $m \geq n$, $R_n(x_0, \ldots, x_k) \rightarrow f_m(x_0, \ldots, x_k) = x_0$;

iii) if $x'_0, \ldots, x'_k \in cl(\{x_0, \ldots, x_k\})$ and $R_n(x_0, \ldots, x_k)$, then $\bigvee_{j \leq n} R_j(x_0, \ldots, x_k)$;

iv) there is no $cl$-independent subset of size $k+2$.

As before, there is a countable, $K$-generic $L_k$-structure $\mathcal{B}$. Let $T_k = Th(\mathcal{B})$. Just as before, $\mathcal{B}$ is atomic, $cl$ is locally finite on $\mathcal{B}$, $\bigvee_{n \in \omega} R_n(\bar{b})$ holds for all $(k+1)$-tuples $\bar{b}$ from $\mathcal{B}$ and there is no $(k+2)$-element $cl$-independent subset of $\mathcal{B}$. Thus, the proof that there is no atomic model of $T_k$ of power $\lambda > \aleph_k$ is exactly analogous to the proof of Theorem 0.4. What remains is to prove that there is an atomic model of size $\aleph_k$. To help us, we quote the following combinatorial fact, which is a sort of converse to Lemma 0.3.

**Lemma 0.7** *For every $k$, $1 \leq k < \omega$ and every set $A$, $|A| \leq \aleph_{k-1}$, there is a family of functions $\{g_n : A^k \rightarrow A : n \in \omega\}$ such that, letting $cl$ denote the transitive closure under the $g_n$'s:*

*i) $cl$ is locally finite;*

*ii) for all $\bar{a} \in A^k$ there is an $n$ such that $g_m(\bar{a}) \in \bar{a}$ for all $m \geq n$;*

*iii) there is no $cl$-independent subset of $A$ of size $k+1$.*

**Proof.** We prove this by induction on $k$. If $k = 1$, let $\{a_i : i < \alpha \leq \aleph_0\}$ enumerate $A$. Define $g_n$ by

$$g_n(a_i) = \begin{cases} a_n & \text{if } n < i; \\ a_i & \text{otherwise.} \end{cases}$$

Now assume the lemma holds for $k$. Let $\{a_i : i < \alpha \leq \aleph_k\}$ enumerate $A$. Define $g_n : A^{k+1} \rightarrow A$ as follows: Given $\bar{a} = \langle a_{i_0}, \ldots, a_{i_k} \rangle \in A^{k+1}$, let $i^* = \max\{i_0, \ldots, i_k\}$. If $i^* = i_l$ for a unique $l < k+1$ then we can apply the inductive hypothesis to the set $A_{i^*} = \{a_j : j < i^*\}$ and obtain a family of functions $h_n : A^k_{i^*} \rightarrow A_{i^*}$ satisfying the conditions of the lemma. Now define $g_n(\bar{a}) = h_n(\bar{b})$, where $\bar{b}$ is the subsequence of $\bar{a}$ of length $k$ obtained by deleting $a_{i_l}$ from $\bar{a}$. On the other hand, if there are $j < l < k+1$ such that $i_j = i_l = i^*$, then simply let $g_n(\bar{a}) = a_{i^*}$ for all $n \in \omega$.

To show that there is an atomic model of $T_k$ of size $\aleph_k$, as the principal formulas are dense and are $\Sigma_1$ it suffices to show the following:

(#) If $\mathcal{A}$ is an $L_k$-structure of size at most $\aleph_{k-1}$ such that every finitely generated substructure of $\mathcal{A}$ is an element of $K$ and $\phi(x, \overline{d})$ ($\overline{d}$ from $A$) is a principal formula consistent with $T_k$, then there is an extension $\mathcal{C} \supseteq \mathcal{A}$ containing a witness to $\phi(x, \overline{d})$ such that every finitely generated substructure of $\mathcal{C}$ is in $K$.

So choose $\mathcal{A}$ and $\phi(x, \overline{d})$ as above. We may assume that $cl(\overline{d}) = \overline{d}$. We shall produce an extension $\mathcal{C} \supseteq \mathcal{A}$ such that $C \setminus A$ is finite, $\mathcal{C} \models \exists x \phi(x, \overline{d})$ and every finitely generated substructure of $\mathcal{C}$ is in $K$.

Since $\phi(x, \overline{d})$ is consistent with $T_k$ there is an element $\mathcal{D}$ of K such that $\overline{d}$ embeds isomorphically into $\mathcal{D}$ and $\mathcal{D} \models \exists x \phi(x, \overline{d})$. Let $\{b_0, \ldots, b_{l-1}\}$ enumerate the elements of $D \setminus \overline{d}$. We must extend the definitions of $\{f_n : n \in \omega\}$ and $\{R_n : n \in \omega\}$ given in $\mathcal{A}$ and $\mathcal{D}$ to $(k+1)$-tuples $\overline{a}$ from $A \cup \{b_0, \ldots, b_{l-1}\}$ so that every finitely generated substructure is in $K$.

We perform this extension by induction on $i < l$. So fix $i < l$ and assume that $\{f_n, R_n : n \in \omega\}$ have been extended to all $(k+1)$-tuples from $A \cup \{b_j : j < i\}$ so that every finitely generated substructure is in $K$. Let $\overline{a} = \langle a_0, \ldots, a_k \rangle$ be a $(k+1)$-tuple from $A \cup \{b_j : j \leq i\}$ containing $b_i$ with at least one element not in $D$. If $b_i = a_s$ for some $s > 0$, then let $f_n(\overline{a}) = a_0$ for all $n \in \omega$ and let $R_0(\overline{a})$ hold.

On the other hand, if $b_i \neq a_s$ for all $s > 0$, then $a_0 = b_i$, so let $\overline{a}' = \langle a_1, \ldots, a_k \rangle$, apply Lemma 0.7 to $A$ and $k$, and let

$$f_n(b, \overline{a}') = \begin{cases} g_n(\overline{a}') & \text{if } g_n(\overline{a}') \notin \overline{a}'; \\ b & \text{otherwise.} \end{cases}$$

It is easy to verify that $cl$ is locally finite on $A \cup \{b_j : j \leq i\}$ and that there is no $cl$-independent subset of size $k+2$. It is also routine to extend the partition given by the $R_n$'s so as to preserve ii) and iii) in the definition of $K$.

Thus, we have succeeded in showing (#), which completes the proof of Theorem 0.6.

## References

[1] P. Erdös, A. Hajnal, A. Mate, P. Rado, *Combinatorial Set Theory*, North Holland, Amsterdam, 1984.

[2] J. Knight, Prime and atomic models, *Journal of Symbolic Logic* **43** (1978) 385-393.

[3] D. W. Kueker, Uniform theorems in infinitary logic, *Logic Colloquium '77*, A. Macintyre, L. Pacholski, J. Paris (eds), North Holland, 1978.

[4] D. W. Kueker and M. C. Laskowski, On generic structures, *Notre Dame Journal of Formal Logic* **33** (1992) 175-183.

[5] S. Shelah, *Classification Theory*, North Holland, Amsterdam, 1978.