Math. Japonica **39,** No. 1(1994), 1–5

THE NUMBER OF INDEPENDENT ELEMENTS IN THE PRODUCT OF INTERVAL BOOLEAN ALGEBRAS

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Received August 19, 1993

Abstract. We prove that in the product of κ many Boolean algebras we cannot find an independent set of more than 2^κ elements solving a problem of Monk (earlier it was known that we cannot find more than 2^{2^κ} but can find 2^κ).

§0 Introduction

In his systematic investigation of cardinal invariants of Boolean Algebras, Monk ([M], problem 26, p. 71 or p. 146) has raised the following question, where we define:

Definition: For a Boolean algebra B, $\mathrm{Ind}(B)$ is the supremum of $|X|$, for $X \subseteq B$ independent, which means that for distinct $a_1, \ldots, a_n \in X$ and non trivial Boolean term $\tau(x_1, \cdots, x_n)$ we have $B \models \tau(a_1 \cdots, a_n) \neq 0$.

Interval Boolean algebras are defined in 1.2 below (non trivial is defined in 1.5 below).

Problem: If A_i is a non-trivial interval algebra for each $i \in I$, where I is infinite, is $\mathrm{Ind}(\Pi_{i \in I} A_i) = 2^{|I|}$? Equivalently, is it true that for every infinite cardinal κ there is no linear order L and sequence $\langle x_\alpha : \alpha < (2^\kappa)^+\rangle$ with the following properties?

(1) For all $\alpha < (2^\kappa)^+$, x_α is a sequence $\langle x_{\alpha,\beta} : \beta < \kappa\rangle$ such that for $\beta < \kappa$, $x_{\alpha,\beta}$ is a finite collection of half-open intervals of L.

(2) For all finite disjoint $\Gamma, \nabla \subseteq (2^\kappa)^+$ there is a $\beta < \kappa$ such that

$$\bigcap_{\alpha \in \Gamma} (\bigcup x_{\alpha,\beta}) \cap \bigcap_{\alpha \in \nabla} (L \setminus \bigcup x_{\alpha,\beta}) \neq 0.$$

It was known that $2^{2^\kappa} \geq \mathrm{Ind}(\prod_{\zeta < \kappa} B_\zeta) \geq 2^\kappa$, and it was felt that the answer to the question is independent of ZFC. Monk phrased some weaker related questions of interest to set theorists on this see Shelah and Soukup [ShSo376].

But not all problems in set theory are independent of set theory and as we can see below, the main point is to get those with an answer and here we even get a reasonable one, so we discuss to some length how simple. Those results are essentially best possible as will be shown elsewhere. In his lecture in Jerusalem, Monk raised the question again, for which we thank him, and Mati Rubin had enough interest in the solution to rewrite it beautifully for which I thank him and the reader should too. Lately I have learned that: Just and Weese

Partially supported by the Deutsche Forschungsgemeinschaft, grant Ko 490/7-1. Publication no. 503

had gotten a restricted positive answer (assuming $B_i = B(I_i)$, I_i a well order), and Rubin [R, lemma 5.4] proved: if λ is a regular cardinal and $\{a_\alpha | \alpha < \lambda\}$ is a sequence of elements of an interval algebra B, then $\{a_\alpha | \alpha < \lambda\}$ contains a semihomogeneous subsequence of length λ.

Notation: The operations in a Boolean algebra are denoted by $\cdot$ (product, intersection) $+$ (addition, union), $-$ (complement), and $\triangle$ is the symmetric difference.

§1 The main result

Theorem 1.1. *Let κ be an infinite cardinal, and for every $\zeta < \kappa$ let B_ζ be an interval Boolean algebra. Then $Ind(\prod_{\zeta<\kappa} B_\zeta) = 2^\kappa$.*

Moreover, there is $n \in \omega$ and a nontrivial Boolean term $\tau(x_{0,\cdots,}x_{n-1})$ such that for every $\{a_\alpha | \alpha < (2^\kappa)^+\} \subseteq \prod_{\zeta<\kappa} B_\zeta$, there are $\alpha_0 < \cdots < \alpha_{n-1}$ such that $\tau(a_{\alpha_{0,\cdots,}}a_{\alpha_{n-1}}) = 0$. In fact τ can be taken to be $\tau(x_0, \cdots, x_5) \stackrel{\text{def}}{=} x_0 \cdot x_1 \cdot (-x_2) \cdot (-x_3) \cdot x_4 \cdot (-x_5)$.

We need some notations and definitions.

Definition 1.2: If $\langle I, < \rangle$ is a linear ordering let $I^+ \stackrel{\text{def}}{=} I \cup \{-\infty, \infty\}$. We assume that $-\infty, \infty \notin I$, and define the linear ordering $<^+$ on I^+ in the obvious way. For $s, t \in I^+$, let $(s,t) \stackrel{\text{def}}{=} \{x \in I | s <^+ x <^+ t\}$, $[s,t) \stackrel{\text{def}}{=} \{x \in I | s \leq^+ x <^+ t\}$ etc. Let $B(\langle I, < \rangle)$ be the subalgebra of $\mathcal{P}(I)$ generated by $\{[s,t) | s, t \in I^+\}$. We abbreviate $<^+$ by $<$ and $B(\langle I, < \rangle)$ by $B(I)$. A Boolean algebra of this form is called an interval Boolean algebra. Let $I^* = \{x \in I |$ there is $y \in I$ such that $y < x\} \cup \{-\infty, \infty\}$. Every $a \in B(I)$ has a unique representation of the form $a = \bigcup_{i<n} [s_{2i}, s_{2i+1})$, where $n \geq 0$, $s_{0,\cdots,}s_{2n-1} \in I^*$ and $s_0 < s_1 < \ldots < s_{2n-1}$. We denote $\sigma_a^- = \{s_{0,\cdots,}s_{2n-1}\}$, $\sigma_a = \sigma_a^- \cup \{-\infty, \infty\}$, $n_a = |\sigma_a|$ and $\vec{\sigma}_a = \langle s'_{0,\cdots,}s'_{n_a-1}\rangle$, where $\{s'_i | i < n_a\} = \sigma_a$ and $s'_0 < \cdots < s'_{n_a-1}$.

Definition 1.3: Let $\langle I, < \rangle$ be a linear ordering, S be a set of ordinals, and $\vec{a} = \{a_\alpha | \alpha \in S\} \subseteq B(I)$.

(a) $\vec{a}$ is homogeneous, if:

(1) there is $k \in \omega$ such that for every $\alpha \in S$, $|\sigma_{a_\alpha}| = k$;

(2) for every $\alpha, \beta \in S$, $\sigma_{a_\alpha}^- \cap \{-\infty, \infty\} = \sigma_{a_\beta}^- \cap \{-\infty, \infty\}$, and

(3) for every $\alpha < \beta$ in S there is $\ell_{\alpha,\beta} < n_{a_\alpha} - 1$ such that for every $s \in \sigma_{a_\beta} \setminus \{-\infty, \infty\}$, $\vec{\sigma}_{a_\alpha}(\ell_{\alpha,\beta}) < s < \vec{\sigma}_{a_\alpha}(\ell_{\alpha,\beta} + 1)$.

(b) $\vec{a}$ is semi-homogeneous, if there are $-\infty = t_0 < \cdots < t_k = \infty$ in I^+ such that for every $\ell < k$ we have: $\{a_\alpha \cap [t_\ell, t_{\ell+1}) | \alpha \in S\}$ is homogeneous in $B(I) \restriction [t_\ell, t_{\ell+1})$. (Note that $B(I) \restriction [t_\ell, t_{\ell+1})$ is the interval algebra of the linear subordering of I whose universe is $[t_\ell, t_{\ell+1})$). Now $\{t_{0,\cdots,}t_k\}$ is called a partitioning set for $\vec{a}$.

If $\{B_\zeta | \zeta < \kappa\}$ is a family of $BA's$, then the members of $\prod_{\zeta<\kappa} B_\zeta$ are denoted by $\langle a_\zeta | \zeta < \kappa \rangle$. So for every $\zeta < \kappa$, $a_\zeta \in B_\zeta$.

The following claim is our main lemma.

Lemma 1.4. *Let κ be an infinite cardinal. For every $\zeta < \kappa$ let B_ζ be an interval Boolean algebra. Let $\vec{a} = \{a_\alpha | \alpha < (2^\kappa)^+\} \subseteq \prod_{\zeta<\kappa} B_\zeta$, and denote $a_\alpha = \langle a_{\alpha,\zeta} | \zeta < \kappa \rangle$. Then there is $S \subseteq (2^\kappa)^+$ of cardinality $(2^\kappa)^+$ such that for every $\zeta < \kappa$ we have: $\{a_{a,\zeta} | \alpha \in S\}$ is semi-homogeneous. In fact, S can be taken to be a stationary subset of $(2^\kappa)^+$.*

Proof: Let $\langle I,<\rangle$ be a linear ordering, $a \in B(I)$ and $A \subseteq I$. A set $C \subseteq A \cup \{-\infty,\infty\}$ is called an A-partition of a, if: $\{-\infty,\infty\} \subseteq C$, $\sigma_a \cap A \subseteq C$, and for every $\ell < n_a - 1$: if $(\vec{\sigma}_a(\ell),\vec{\sigma}_a(\ell+1)) \cap A \neq \emptyset$, then $(\vec{\sigma}_a(\ell),\vec{\sigma}_a(\ell+1)) \cap C \neq \emptyset$.

Let $\{\langle I_\zeta,<_\zeta\rangle | \zeta<\kappa\}$ be a family of linear orderings, such that $B_\zeta = B(I_\zeta)$, let $\lambda = (2^\kappa)^+$ and $\{a_\alpha|\alpha<\lambda\} \subseteq \prod_{\zeta<\kappa} B_\zeta$. We may assume that $|I_\zeta| = \lambda$ for every $\zeta<\kappa$, and hence we may further assume that $I_\zeta = \lambda$. For $s,t \in \lambda$ and $\zeta<\kappa$, we use $[s,t)_\zeta, (s,t)_\zeta$ etc. to denote intervals of $<_\zeta$ whose endpoints are s and t. For every $\alpha<\lambda$ let $a_\alpha = \langle a_{\alpha,\zeta}|\zeta<\kappa\rangle$. For every $\alpha<\lambda$, $\zeta<\kappa$ and $\ell \le n_{a_{\alpha,\zeta}}-1$, let $\sigma_{\alpha,\zeta} = \sigma_{a_{\alpha,\zeta}}, \vec{\sigma}_{\alpha,\zeta} = \vec{\sigma}_{a_{\alpha,\zeta}}$ and $s_{\alpha,\zeta,\ell} = \vec{\sigma}_{a_{\alpha,\zeta}}(\ell)$.

We next perform on the sequence $\{a_\alpha|\alpha<\lambda\}$ several steps of cleaning. Let $S_0 = \{\alpha < \lambda|\mathrm{cf}(\alpha) = \kappa^+\}$.

<u>Step 1</u>: By partitioning S_0 into $4^\kappa < \lambda$ sets, we obtain a stationary set $S_1 \subseteq S_0$ such that for every $\alpha,\beta \in S_1$ and $\zeta<\kappa$: $\sigma^-_{\alpha,\zeta} \cap \{-\infty,\infty\} = \sigma^-_{\beta,\zeta} \cap \{-\infty,\infty\}$.

<u>Step 2</u>: For every $\alpha<\lambda$ let $\vec{n}_\alpha = \langle n_{a_{\alpha,\zeta}}|\zeta<\kappa\rangle$. For every $\vec{n} \in {}^\kappa\omega$ let $S^1_{\vec{n}} = \{\alpha \in S_1|\vec{n}_\alpha = \bar{n}\}$. So $\{S^1_{\vec{n}}|\ \vec{n} \in {}^\kappa\omega\}$ is a partition of S_1 into $\le 2^\kappa < \lambda$ sets. Hence for some $\vec{n} \in {}^\kappa\omega$, $S_2 \stackrel{\text{def}}{=} S^1_{\vec{n}}$ is stationary. Let $\vec{n} = \langle n_\zeta|\zeta<\kappa\rangle$.

<u>Step 3</u>: For every $\alpha \in S_2$ let $C_\alpha \subseteq \alpha \cup \{-\infty,\infty\}$ have the following properties:
(1) $|C_\alpha| \le \kappa$; and
(2) for every $\zeta<\kappa, C_\alpha$ is an α-partition of $a_{\alpha,\zeta}$ with respect to $<_\zeta$.

Since $S_2 \subseteq \{\alpha<\lambda|\mathrm{cf}(\alpha)=\kappa^+\}$, clearly $\sup(C_\alpha \setminus \{-\infty,\infty\}) < \alpha$. By Fodor's theorem and the fact that $(2^\kappa)^\kappa = 2^\kappa$, there is a stationary set $S_3 \subseteq S_2$, $\delta<\lambda$ and $C \subseteq \delta \cup \{-\infty,\infty\}$ such that for every $\alpha \in S_3, C_\alpha = C$.

<u>Step 4</u>: By partitioning S_3 into $\le 2^\kappa$ sets we obtain a stationary set $S_4 \subseteq S_3$ and a system $T_\zeta = \{t_{\zeta,0},\ldots,t_{\zeta,m_\zeta}\} \subseteq C$ for $\zeta<\kappa$, such that for every $\alpha,\beta \in S_4$ and $\zeta<\kappa$:
(1) $-\infty = t_{\zeta,0} <_\zeta t_{\zeta,1} <_\zeta \cdots <_\zeta t_{\zeta,m_\zeta} = \infty$;
(2) T_ζ is a $(C \setminus \{-\infty,\infty\})$-partition of $a_{\alpha,\zeta}$;
(3) for every $\ell \le n_\zeta - 1$: if $s_{\alpha,\zeta,\ell} \in T_\zeta$, then $s_{\beta,\zeta,\ell} = s_{\alpha,\zeta,\ell}$, and
(4) for every $\ell < n_\zeta - 1$, $T_\zeta \cap (s_{\alpha,\zeta,\ell}, s_{\alpha,\zeta,\ell+1})_\zeta = T_\zeta \cap (s_{\beta,\zeta,\ell}, s_{\beta,\zeta,\ell+1})_\zeta$.

<u>Step 5</u>: Let $F \subseteq \lambda$ be a closed and unbounded set such that for every $\gamma \in F$ and $\alpha<\gamma$ and $\zeta<\kappa$ we have $\sigma_{a_{\alpha,\zeta}} \setminus \{-\infty,\infty\} \subseteq \gamma$. Let $S = S_4 \cap F$.

We shall show that for every $\zeta<\kappa$:
$(*)_\zeta$ $\quad \vec{a}^\zeta \stackrel{\text{def}}{=} \{a_{\alpha,\zeta}|\alpha \in S\}$ is semi-homogeneous, and T_ζ is a partitioning set for $\vec{a}^\zeta$.

Let $m < m_\zeta$, and we show that $\{a_{\alpha,\zeta} \cap [t_{\zeta,m}, t_{\zeta,m+1})_\zeta|\alpha \in S\}$ is homogeneous in $B_\zeta \restriction [t_{\zeta,m}, t_{\zeta,m+1})_\zeta$. So for the rest of the proof of 1.4, we fix ζ and m. Let $a'_\alpha = a_{\alpha,\zeta} \cap [t_{\zeta,m}, t_{\zeta,m+1})_\zeta$. For every $\alpha \in S$:

T_ζ is a $(C \setminus \{-\infty,\infty\})$-partition of $a_{\alpha,\zeta}$ and C is an α-partition of $a_{\alpha,\zeta}$.
Hence
(i) for every $\alpha \in S$: T_ζ is an α-partition of $a_{\alpha,\zeta}$.

Let $I' = [t_{\zeta,m}, t_{\zeta,m+1})_\zeta$ and $<' = <_\zeta \restriction I'$ and $B' = B(I')$. For every $\alpha \in S$ let $n'_\alpha = n_{a'_\alpha}, \sigma'_\alpha = \sigma_{a'_\alpha}$ and $\langle s'_{\alpha,0,\cdots}, s'_{\alpha,n'_\alpha-1}\rangle = \vec{\sigma}_{a'_\alpha}$, where the representation is taken with respect to B'. For $\alpha \in S$: if $t_{\zeta,m} \neq -\infty$ let ℓ^α_0 be such that $t_{\zeta,m} \in [s_{\alpha,\zeta,\ell^\alpha_0}, s_{\alpha,\zeta,\ell^\alpha_0+1})_\zeta$ and $\ell^\alpha_0 = -\infty$ otherwise, and if $t_{\zeta,m+1} \neq \infty$, let ℓ^α_1 be such that $t_{\zeta,m+1} \in [s_{\alpha,\zeta,\ell^\alpha_1}, s_{\alpha,\zeta,\ell^\alpha_1+1})_\zeta$ and $\ell^\alpha_1 = \infty$ otherwise. By conditions (3) and (4) in step 4, for every $\alpha,\beta \in S$: $\ell^\alpha_0 = \ell^\beta_0$, and $\ell^\alpha_1 = \ell^\beta_1$. It follows that for every $\alpha,\beta \in S$: $n'_\alpha = n'_\beta$ and $\{-\infty,\infty\} \cap \sigma^-_{a'_\alpha} = \{-\infty,\infty\} \cap \sigma^-_{a'_\beta}$. Let $n' = n'_\alpha$. This means that $\{a'_\alpha|\alpha \in S\}$ satisfies conditions (1) and (2) in 1.3(a).

Let $\alpha < \beta$ be in S. By the choice of F and S, (ii) holds:

(ii) $\sigma'_\alpha \setminus \{-\infty,\infty\} \subseteq \sigma_{a_{\alpha,\zeta}} \setminus \{-\infty,\infty\} \subseteq \beta$.

By (i), T_ζ is a β-partition of $a_{\beta,\zeta}$, and hence

$$\sigma'_\beta \cap \beta \setminus \{-\infty,\infty\} \subseteq \sigma_{a_{\beta,\zeta}} \cap \beta \cap (t_{\zeta,m}, t_{\zeta,m+1})_\zeta = \emptyset.$$

It follows that for every distinct $\alpha,\beta \in S$, $(\sigma'_\alpha \setminus \{-\infty,\infty\}) \cap (\sigma'_\beta \setminus \{-\infty,\infty\}) = \emptyset$.

Suppose by contradiction that for some $\alpha < \beta \in S, k \le n'-1$ and $\ell < n'-1$ we have: $s'_{\beta,\ell}, s'_{\beta,\ell+1} \notin \{-\infty,\infty\}$ and $s'_{\beta,\ell} <_\zeta s'_{\alpha,k} <_\zeta s'_{\beta,\ell+1}$, $s'_{\alpha,k} \notin \{-\infty,\infty\}$, and hence by (ii), $s'_{\alpha,k} < \beta$. So $(s'_{\beta,\ell}, s'_{\beta,\ell+1})_\zeta \cap \beta \neq \emptyset$. Since $s'_{\beta,\ell}, s'_{\beta_{\ell+1}} \notin \{-\infty,\infty\}$, there is $\ell_1 < n_\zeta - 1$, such that $s'_{\beta,\ell} = s_{\beta,\zeta,\ell_1}$ and $s'_{\beta,\ell+1} = s_{\beta,\zeta,\ell_1+1}$. So $(s_{\beta,\zeta,\ell_1}, s_{\beta,\zeta,\ell_1+1})_\zeta \cap \beta \neq \emptyset$. Since T_ζ is a β-partition for $a_{\beta,\zeta}$, we have $T_\zeta \cap (s_{\beta,\zeta,\ell_1}, s_{\beta,\zeta,\ell_1+1})_\zeta \neq \emptyset$. A contradiction. This shows that $\{a'_\alpha | \alpha \in S\}$ satisfies condition (3) in 1.3(a). So $\{a'_\alpha|\alpha \in S\}$ is homogeneous, and hence $\{a_{\alpha,\zeta}|\alpha \in S\}$ is semi-homogeneous. □

Definition 1.5: Let $\tau(x_1,\cdots,x_n)$ be a Boolean term. τ is nontrivial, if for some Boolean algebra B and $a_1,\cdots,a_n \in B$ we have $\tau(a_1,\cdots,a_n) \neq 0$.

Lemma 1.6. *There are nontrivial Boolean terms* $\tau_1(x_0,x_1,x_2)$, $\tau_2(x_0,x_1,x_2)$, $\tau_3(x_1,x_2)$ *and* $\tau_4(x_1,x_2)$ *such that for every interval Boolean algebra B and a homogeneous sequence* $\{a_i|i<3\} \subseteq B$ *we have:* $B \models \bigvee_{i=1}^{4} (\tau_i = 0)[v]$, *where v is the assignment that takes each* x_i *to* a_i.

Proof: Let $\tau_1 = x_0 \cdot x_1 \cdot (-x_2), \tau_2 = (-x_0)\cdot(-x_1)\cdot x_2, \tau_3 = x_1 \cdot x_2$ and $\tau_4 = (-x_1)\cdot(-x_2)$. Let $B = B(\langle I,<\rangle)$ be an interval algebra and $\{a_i|i<3\}$ be homogeneous.

Let $\vec\sigma_{a_i} = \langle s^i_0,\cdots,s^i_{n-1}\rangle$. For every $i<j<3$ let $\ell_{i,j} < n-1$ be such that for every $s \in \sigma_{a_j} \setminus \{-\infty,\infty\}$, $s^i_{\ell_{i,j}} < s < s^i_{\ell_{i,j}+1}$ (it exists - see Def 1.3(a)(3)).

Case 1 $\ell_{1,2} \neq 0$ and $\ell_{1,2}+1 \neq n-1$. It follows that $\ell_{0,2} = \ell_{0,1}$, and so $a_1 \triangle a_2 \subseteq [s^0_{\ell_{0,1}}, s^0_{\ell_{0,1}+1}) \stackrel{\text{def}}{=} J$. Either (i) $J \subseteq a_0$, or (ii) $J \cap a_0 = \emptyset$. Now (i) implies that $B \models (\tau_2 = 0)[v]$, and (ii) implies that $B \models (\tau_1 = 0)[v]$.

Case 2 $\ell_{1,2} = 0$ or $\ell_{1,2}+1 = n-1$. W.l.o.g. $\ell_{1,2} = 0$. Let $J_1 = [-\infty, s^1_1)$ and $J_2 = [s^2_{n-2},\infty)$. So:

(1) $J_1 \cup J_2 = I$;

(2) $J_1 \subseteq a_1$ or $J_1 \subseteq -a_1$; and

(3) $J_2 \subseteq a_2$ or $J_2 \subseteq -a_2$.

Hence there are four possibilities: $a_1 + a_2 = 1, a_1 + (-a_2) = 1, (-a_1) + a_2 = 1$ or $(-a_1) + (-a_2) = 1$. It is now trivial to check that $B \models \bigvee_{i=1}^{4} (\tau_i = 0)[v]$ in all the above subcases. (Note, if $\ell_{1,2}+1 = n-1$. Use $J'_1 = [s^1_{n-2},\infty), J'_2 = [-\infty, s^2_1)$). □

1.7 Proof of theorem 1.1: Let

$$\tau(x_0,\cdots,x_{11}) = \tau_1(x_0,x_1,x_2)\cdot\tau_2(x_3,x_4,x_5)\cdot\tau_3(x_7,x_8)\cdot\tau_4(x_{10},x_{11}).$$

For every $\zeta < \kappa$ let $B_\zeta = B(\langle I_\zeta,<_\zeta\rangle)$ be an interval algebra. Let $\lambda = (2^\kappa)^+$, and $\{a_\alpha|\alpha < \lambda\} \subseteq \prod_{\zeta<\kappa} B_\zeta$, where $a_\alpha = \langle a_{\alpha,\zeta}|\zeta<\kappa\rangle$. By lemma 1.4 we may assume that for every $\zeta < \kappa$, $\{a_{\alpha,\zeta}|\alpha < \lambda\}$ is semi-homogeneous with the partitioning sequence $\langle t_{\zeta,0},\cdots t_{\zeta,m_\zeta}\rangle$. For every $\zeta < \kappa$ and $m < m_\zeta$ let $B_{\zeta,m} = B_\zeta \restriction [t_{\zeta,m}, t_{\zeta,m+1})$ and $a_{\alpha,\zeta,m} = a_{\alpha,\zeta} \cap [t_{\zeta,m}, t_{\zeta,m+1})$. Hence

$B = \prod\{B_{\zeta,m}|\zeta < \kappa, m < m_\zeta\}$ and $a_\alpha = \langle a_{\alpha,\zeta,m}|\zeta < \kappa, m < m_\zeta\rangle$, and for every $\zeta < \kappa$ and $m < m_\zeta$, $\{a_{\alpha,\zeta,m}|\alpha < \lambda\}$ is homogeneous in $B_{\zeta,m}$. So by renaming $\{B_{\zeta,m}|\zeta < \kappa, m < m_\zeta\}$ as $\{B'_\zeta|\zeta < \kappa\}$ and $\{a_{\alpha,\zeta,m}|\zeta < \kappa, m < m_\zeta\}$ as $\{a'_{\alpha,\zeta}|\zeta < \kappa\}$, we may assume that for every $\zeta < \kappa$, $\{a_{\alpha,\zeta}|\zeta < \kappa\}$ is homogeneous in B_ζ. For every $\alpha < \beta < \lambda$ and $\zeta < \kappa$ let $\vec{\sigma}_{a_{\alpha,\zeta}} = \langle s^{\alpha,\zeta}_0, \cdots, s^{\alpha,\zeta}_{n_{\zeta-1}}\rangle$, and let $\ell = \ell^\zeta_{\alpha,\beta} < n_\zeta$ be such that for every $s \in \sigma_{a_{\beta,\zeta}} \setminus \{-\infty, \infty\}$ we have $s^{\alpha,\zeta}_\ell < s < s^{\alpha,\zeta}_{\ell+1}$. Let $\vec{\ell}_{\alpha,\beta} = \langle \ell^\zeta_{\alpha,\beta}|\zeta < \kappa\rangle$.

There are four triples $\alpha_0 < \cdots < \alpha_{11} < \lambda$, such that for every $i = 1,2,3$ $\vec{\ell}_{\alpha_0\alpha_1} = \vec{\ell}_{\alpha_{3i},\alpha_{3i+1}}$ and $\vec{\ell}_{\alpha_1,\alpha_2} = \vec{\ell}_{\alpha_{3i+1},\alpha_{3i+2}}$. So for every Boolean term $\tau(x_0,x_1,x_2), \zeta < \kappa$ and $i = 1,2,3$ we have: $B_\zeta \models \tau(a_{\alpha_0,\zeta}, a_{\alpha_1\zeta}, a_{\alpha_2,\zeta}) = 0$ iff $B_\zeta \models \tau(a_{\alpha_{3i,\zeta}}, a_{\alpha_{3i+1,\zeta}}, a_{\alpha_{3i+2},\zeta}) = 0$. Since $\bigvee_{i=1}^{4} (\tau_i(a_{\alpha_0,\zeta}, a_{\alpha_1,\zeta}, a_{\alpha_2,\zeta}) = 0)$ holds in B_ζ clearly,

$$\tau(a_{\alpha_0,\zeta}, \cdots, a_{\alpha_{11},\zeta}) = \prod_{i=1}^{4} \tau_i(a_{\alpha_{3i,\zeta}}, a_{\alpha_{3i+1,\zeta}}, a_{\alpha_{3i+2,\zeta}}) = 0.$$

That is, every coordinate of $\tau\,(a_{\alpha_0,\ldots}, a_{\alpha_{11}})$ is equal to 0. So $\tau(a_{\alpha_0,\ldots}, a_{\alpha_{11}}) = 0$. Note that τ really has only eight variables.

1.8 Proof with the shorter term: In order to show that τ can be taken to be $x_0 \cdot x_1 \cdot (-x_2) \cdot (-x_3) \cdot x_4 \cdot (-x_5)$, we first notice that for every $\alpha < \beta < \gamma$: if $\vec{\ell}_{\alpha,\beta} = \vec{\ell}_{\alpha,\gamma}$, then for every $\zeta < \kappa$: $(a_{\beta,\zeta} \triangle a_{\gamma,\zeta}) \cdot a_{\alpha,\zeta} = 0$, or $(a_{\beta,\zeta} \triangle a_{\gamma,\zeta}) \cdot -a_{\alpha,\zeta} = 0$, moreover from the value of $\vec{\ell}_{\alpha,\beta} = \vec{\ell}_{\alpha,\gamma}$: (but not $\vec{\ell}_{\beta,\gamma}$) we can compute an equation, which is one of those two and holds (possibly both holds).

For every $\alpha < \lambda$ let $L_\alpha = \{\vec{\ell}| \quad |\{\beta > \alpha | \vec{\ell}_{\alpha,\beta} = \vec{\ell}\}| = \lambda\}$. So since the number of possible $\vec{\ell}_{\alpha,\beta}$'s is 2^κ, $L_\alpha \neq \emptyset$. For every $\vec{\ell}$ let $\Lambda_{\vec{\ell}} = \{\alpha | \vec{\ell} \in L_\alpha\}$. So for some $\vec{\ell}^0$, $|\Lambda_{\vec{\ell}^0}| = \lambda$. Let $\alpha_0 < \cdots < \alpha_5$ be such that $\vec{\ell}_{\alpha_0,\alpha_1} = \vec{\ell}_{\alpha_0,\alpha_2} = \vec{\ell}_{\alpha_3,\alpha_4} = \vec{\ell}_{\alpha_3,\alpha_5} = \vec{\ell}^0$. (We can demand that $\vec{\ell}_{\alpha_1,\alpha_2} = \vec{\ell}_{\alpha_4,\alpha_5}$, but it is not needed). Let $\zeta < \kappa$. Then either

$$\bigwedge_{i=0}^{1} (a_{\alpha_{3i+1},\zeta} \triangle a_{\alpha_{3i+2},\zeta}) \cdot a_{\alpha_{3i},\zeta} = 0$$

or

$$\bigwedge_{i=0}^{1} (a_{\alpha_{3i+1},\zeta} \triangle a_{\alpha_{3i+2},\zeta}) \cdot (-a_{3i,\zeta}) = 0.$$

It follows that $(a_{\alpha_1,\zeta} \triangle a_{\alpha_2,\zeta}) \cdot a_{\alpha_0,\zeta} \cdot (a_{\alpha_4,\zeta} \triangle a_{\alpha_5,\zeta}) \cdot (-a_{\alpha_3,\zeta}) = 0$. So $\tau(a_{\alpha_0,\zeta}, \cdots, a_{\alpha_5,\zeta}) = 0$. Hence τ is as required. □

Claim 1.9. *In 1.1 we can use* $(x_0 \triangle x_1) \cdot x_2 \cdot (x_3 \triangle x_4) \cdot (-x_5)$

Proof: As above but in 1.8 demand also $\vec{\ell}_{\alpha_1,\alpha_2} = \vec{\ell}_{\alpha_4,\alpha_5}$.

Claim 1.10. *Let κ be an infinite cardinal, and for $\zeta < \kappa$ let B_ζ be an interval Boolean algebra. If $a_\alpha \in \prod_{\zeta<\kappa} B_\zeta$ for $\alpha < (2^\kappa)^{++}$, then for some $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < (2^\kappa)^{++}$ we have*

$$(a_{\alpha_0} \triangle a_{\alpha_1}) \cdot (a_{\alpha_2} \triangle a_{\alpha_3}) = 0.$$

Proof: Similar only in 1.7, 1.8 we use the easy fact (which follows from Erdös–Rado)

(∗) if c is a two place function from $(2^\kappa)^{++}$ to κ then for some $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$ we have $c(\alpha_0,\alpha_2) = c(\alpha_0,\alpha_3) = c(\alpha_1,\alpha_2) = c(\alpha_1,\alpha_3)$.

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