DECOMPOSING BAIRE CLASS 1 FUNCTIONS INTO CONTINUOUS FUNCTIONS

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1. Introduction

In [1] the authors consider the following question: What is the least cardinal κ such that every function of first Baire class can be decomposed into κ continuous functions? This cardinal κ will be denoted by $\mathfrak{dec}$. The authors of [1] were able to show that $\mathrm{cov}(\mathbb{K}) \leq \mathfrak{dec} \leq \mathfrak{d}$ and asked whether these inequalities could, consistently, be strict. By $\mathrm{cov}(\mathbb{K})$ is meant the least number of closed nowhere dense sets required to cover the real line and by $\mathfrak{d}$ is denoted the least cardinal of a dominating family in ${}^{\omega}\omega$. In [5] it was shown that it is consistent that $\mathrm{cov}(\mathbb{K}) \neq \mathfrak{dec}$. In this paper it will be shown that the second inequality can also be made strict. The model where $\mathfrak{dec}$ is different from $\mathfrak{d}$ is the one obtained by adding ω_2 Miller — sometimes known as super-perfect or rational-perfect — reals to a model of the Continuum Hypothesis. It is somewhat surprising that the model used to establish the consistency of the other inequality, $\mathrm{cov}(\mathbb{K}) \neq \mathfrak{dec}$, is a slight modification of the iteration of super-perfect forcing.

By ${}^{\overset{\omega}{\smile}}\omega$ will be denoted $\cup_{n\in\omega}\{{}^{n}\omega : n \in \omega\}$. As usual, a tree will be defined to mean an initial subset of ${}^{\overset{\omega}{\smile}}\omega$ under $\subseteq$. So if T is a tree and $t \in T$ then $t \restriction k \in T$ for each $k \in \omega$. Also, $T\langle t\rangle$ will be defined to be $\{s \in T : s \subseteq t \text{ or } t \subseteq s\}$. If t and s are both finite sequences then $s \wedge t$ is defined by declaring that $\mathrm{dom}(s \wedge t) = |\mathrm{dom}(t)| + |\mathrm{dom}(s)|$ and

$$s \wedge t(i) = \begin{cases} s(i) & \text{if } i \in \mathrm{dom}(s) \\ t(i - |\mathrm{dom}(s)|) & \text{if } i \notin \mathrm{dom}(s) \end{cases}$$

If $t \in T \subseteq {}^{\overset{\omega}{\smile}}\omega$ then $i \in \omega$ then $t \wedge i$ is defined to be $t \wedge \{(0,i)\}$ and $i \wedge t$ is defined to be $\{(0,i)\} \wedge t$. Finally, $\overline{T} = \{f \in {}^{\omega}\omega : (\forall n \in \omega)(f \restriction n \in T)\}$ and closure in other spaces is denoted similarly.

Definition 1.1. *If $T \subseteq {}^{\overset{\omega}{\smile}}\omega$ is a tree then $\beta(T)$ will be defined to be the set of all $t \in T$ such that $|\{n \in \omega : t \wedge n \in T\}| = \aleph_0$. A tree $T \subseteq {}^{\overset{\omega}{\smile}}\omega$ is said to be* super-perfect *if for each $t \in T$ there is some $s \in \beta(T)$ such that $t \subseteq s$ and if $|\{n \in \omega : t \wedge n \in T\}| \in \{1, \aleph_0\}$ for each $t \in T$. The set of all super-perfect trees will be denoted by $\mathbb{S}$.*

For each $T \in \mathbb{S}$ there is a natural way to assign a mapping $\theta_T : {}^{\overset{\omega}{\smile}}\omega \to \beta(T)$ such that:

- θ_T is one-to-one and onto $\beta(T)$

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- $s \subseteq t$ if and only if $\theta(s) \subseteq \theta(t)$
- $s \leq_{\text{Lex}} t$ if and only if $\theta(s) \leq_{\text{Lex}} \theta(t)$.

Notice that $\theta_T(\emptyset)$ is the root of T. Using the mapping θ_T, it is possible to define a refinement of the ordering on $\mathbb{S}$.

Definition 1.2. *Define $T \prec_n S$ if both S and T are in $\mathbb{S}$, $T \subseteq S$ and $\theta_T \restriction {}^n\omega = \theta_S \restriction {}^n\omega$.*

It should be clear that the ordering $\prec_n$ satisfies Axiom A. The proof of the main result of this paper will use a fusion based on a sequence of the orderings $\prec_n$. Notice that while $\prec_n$ can be used in the same way as the analogous ordering for Sacks reals in the case of adding a single real, is not as easy to deal with in the context of iterations. The chief difficulty is that $\prec_n$ requires deciding an infinite amount of information because branching is infinite. This conflicts with the usual goal of fusion arguments which decide only a finite amount of information at a time.

2. Iterated Super-Perfect Reals

It will be shown that in a model obtained by iterating ω_2 times the partial orders $\mathbb{S}$ with countable support over a ground model where $2^{\aleph_0} = \aleph_1$ yields a model where $\mathfrak{d} = \aleph_2$ and $\mathfrak{dec} = \aleph_1$. The fact that $\mathfrak{d} = \aleph_2$ is well known [3]. The fact that $\mathfrak{dec} = \aleph_1$ is an immediate consequence of the following result.

Lemma 2.1. *Suppose that $\xi \in \omega_2 + 1$, $\mathbb{S}_\xi$ is the iteration with countable support of the partial orders $\mathbb{S}$ and G is $\mathbb{S}_\xi$-generic over V. Then for any $x \in [0,1]$ in $V[G]$ and any Borel function $H : [0,1] \to [0,1]$ in $V[G]$ there is a Borel set $X \in V$ such that $x \in X$ and $H \restriction X$ is continuous.*

Saying that $X \in V$ means, of course, that the real coding the Borel set X belongs to the model V. In order to prove Lemma 2.1 it will be useful to employ a different interpretation of iterated super-perfect forcing. The next sequence of definitions will be used in doing this. If G is $\mathbb{S}_\xi$-generic over some model $\mathfrak{M}$ then there is a natural way to assign a mapping $\Gamma : \xi \cap \mathfrak{M} \to {}^\omega\omega$ such that $\mathfrak{M}[G] = \mathfrak{M}[\Gamma]$. On the other hand, given $\Gamma : \mathfrak{M} \cap \xi \to {}^\omega\omega$ define $G_\Gamma(\mathfrak{M})$ to be the set

$$\{q \in \mathfrak{M} \cap \mathbb{S}_\xi : \forall k \in \omega \forall A \in [\mathfrak{M} \cap \xi]^{<\aleph_0} \exists p \leq q \forall \alpha \in A (p \restriction \alpha \Vdash_{\mathbb{S}_\alpha} \text{"}\Gamma(\alpha) \restriction k \in p(\alpha)\text{"})\}$$

and say that Γ is $\mathbb{S}_\xi$-generic over $\mathfrak{M}$ if and only if G_Γ is $\mathbb{S}_\xi$-generic over $\mathfrak{M}$. Note that if G is $\mathbb{S}_\xi$-generic over $\mathfrak{M}$ and $\Gamma : \mathfrak{M} \cap \xi \to {}^\omega\omega$ is its associated function then $G_\Gamma(\mathfrak{M}) = G$. This will be used without further comment to identify $\mathbb{S}_\xi$-generic sets over $\mathfrak{M}$ with elements of $({}^\omega\omega)^{\mathfrak{M}\cap\xi}$. Whenever a topology on $({}^\omega\omega)^X$ is mentioned, the product topology is intended.

Definition 2.1. *If $p \in \mathbb{S}_\xi$ and $\Lambda \in [\xi]^{\leq\aleph_0}$ then define $S(\Lambda, p)$ to be the set of all functions $\Gamma : \Lambda \to {}^\omega\omega$ such that for all $k \in \omega$ and for all finite subsets $A \subseteq \Lambda$ there is $q \leq p$ such that $q \Vdash_{\mathbb{S}_\xi}$ "$\Gamma(\alpha) \restriction k \in q(\alpha)$" for all $\alpha \in A$.*

Definition 2.2. *Given a countable elementary submodel $\mathfrak{M} \prec H((2^{\aleph_0})^+)$ and $p \in \mathbb{S}_\xi$ define p to be strongly $\mathbb{S}_\xi$-generic over $\mathfrak{M}$ if and only if*

- *each $\Gamma \in S(\mathfrak{M} \cap \xi, p)$ is $\mathbb{S}_\xi$-generic over $\mathfrak{M}$*
- *if ψ is a statement of the $\mathbb{S}_\xi$-forcing language using only parameters from $\mathfrak{M}$, then $\{\Gamma \in S(\mathfrak{M} \cap \xi, p) : \mathfrak{M}[\Gamma] \models \psi\}$ is a clopen set in $S(\mathfrak{M} \cap \xi, p)$.*

A set $X \subseteq ({}^\omega\omega)^\alpha$ will be defined to be *large* by induction on α.

Definition 2.3. *If $\alpha = 1$ then X is large if X is a superperfect tree. If α is a limit then X is large if the projection of X to $({}^\omega\omega)^\beta$ is large for every $\beta \in \alpha$. If $\alpha = \beta+1$ then X is large if there is a large set $Y \subseteq ({}^\omega\omega)^\beta$ such that $X = \cup_{y\in Y}\{y\} \times X_y$ and each X_y is a large subset of ${}^\omega\omega$.*

¿From large closed sets it is possible to obtain, in a natural way, conditions in $\mathbb{S}_\xi$.

Definition 2.4. *If $X \subseteq ({}^\omega\omega)^\alpha$ is a large closed set then define $p_X \in \mathbb{S}_\alpha$ by defining $p_X(\eta)$ to be the $\mathbb{S}_\eta$ name for that subset $T \subseteq {}^\omega\omega$ such that if $\Gamma : \alpha \to {}^\omega\omega$ is $\mathbb{S}_\alpha$ generic then*

$$T = \{f \in {}^\omega\omega | (\exists h)(\Gamma \restriction \eta \cup \{(\eta, f)\} \cup h \in X)\}$$

Observe that, if $X \subseteq ({}^\omega\omega)^\alpha$ is large and closed, it follows that $p_X \in \mathbb{S}_\alpha$. The following result provides a partial converse to this observation.

Lemma 2.2. *If $p \in \mathbb{S}_\xi$ and $\mathfrak{M} \prec H((2^{\aleph_0})^+)$ is a countable elementary submodel containing p then there is $q \leq p$ such that q is strongly $\mathbb{S}_\xi$-generic over $\mathfrak{M}$.*

Proof: The proof consists of merely repeating the proof that the countable support iteration of proper partial orders is proper and checking the assertions in this special case. Only a sketch will be given and the reader should consult [4] for details.

The proof is by induction on ξ. If $\xi = 1$ then a standard fusion argument applied to an enumeration $\{D_n : n \in \omega\}$, of all dense subsets of $\mathbb{S}$ provides the result. In particular, there is a sequence $\{T_i : i \in \omega\}$ such that $T_{i+1} \prec_i T_i$, $T_0 = T$ and such that $T_i\langle\theta_{T_i}(\sigma)\rangle \in D_{i-1}$ for each $\sigma : i \to \omega$. The condition $T_\omega = \cap_{i\in\omega} T_i$ has the desired property. The fact that if ψ is a statement of the $\mathbb{S}_\xi$-forcing language using only parameters from $\mathfrak{M}$, then $\{\Gamma \in S(\mathfrak{M}, T_\omega) : \mathfrak{M}[\Gamma] \models \psi\}$ is a clopen set is obvious because $S(1, T_\omega) = \overline{T_\omega}$.

If $\xi = \mu + 1$ then use the induction hypothesis to find $q' \leq p \restriction \xi$ such that q' is strongly $\mathbb{S}_\mu$-generic over $\mathfrak{M}$. Then, in particular, q' is $\mathbb{S}_\mu$-generic over $\mathfrak{M}$ and so, if G contains q' and is $\mathbb{S}_\mu$-generic over V it is also generic over $\mathfrak{M}$. Therefore $\mathfrak{M}[G]$ is an elementary submodel in $V[G]$ and it is possible to choose an enumeration $\{D_n : n \in \omega\}$ of all dense subsets of $\mathbb{S}$ which are members of $\mathfrak{M}[G]$. It is therefore possible to choose, in $\mathfrak{M}[G]$, as in the case $\xi = 1$, a sequence $\{T_i : i \in \omega\}$ such that $T_{i+1} \prec_i T_i$ and such that $T_i\langle\theta_{T_i}(\sigma)\rangle \in D_{i-1}$ for each $\sigma : i \to \omega$. The condition $T_\omega = \cap_{i\in\omega} T_i$ is then strongly $\mathbb{S}$-generic over $\mathfrak{M}[G]$. Notice that, while T_ω does not itself have a name in $\mathfrak{M}$, each T_n does have a name and so there are enough objects in $\mathfrak{M}[G]$ to construct T_ω.

In order to see that $q = q' * T_\omega$ is strongly $\mathbb{S}_\xi$-generic over $\mathfrak{M}$ suppose that $\Gamma \in S(\mathfrak{M} \cap \xi, q)$. Obvioulsy $\Gamma \restriction \mu \in S(\mathfrak{M} \cap \mu, q')$ and therefore $\mathfrak{M}[\Gamma]$ is an elementary submodel. Hence, by genericity, $T_{i+1} \prec_i T_i$, $T_0 = T$ and $T_i\langle\theta_{T_i}(\sigma)\rangle \in D_{i-1}$ and so it follows that $\cap\{T_i : i \in \omega\}$ is a strongly $\mathbb{S}$-generic condition over $\mathfrak{M}[G]$. Hence $\Gamma(\xi)$ is $\mathbb{S}$-generic over $\mathfrak{M}[G]$ and so Γ is $\mathbb{S}_\xi$-generic over $\mathfrak{M}$.

Just as in the case $\xi = 1$, it is easy to use the induction hypothesis to see that if ψ is a statement of the $\mathbb{S}_\xi$-forcing language using only parameters from $\mathfrak{M}$, then $\{\Gamma \in S(\mathfrak{M} \cap \xi, q) : \mathfrak{M}[\Gamma] \models \psi\}$ is a clopen set.

Finally, suppose that ξ is a limit ordinal. If it has uncountable cofinality then there is nothing to do because of the countable support of the iteration. So assume that $\{\mu_n : n \in \omega\}$ is an increasing sequence of ordinals cofinal in ξ. Let $\{D_n : n \in \omega\}$

enumerate all dense subsets of $\mathfrak{M}$ and choose a sequence of conditions $\{p_i : i \in \omega\}$ such that

- $p_i \restriction \mu_i$ is strongly $\mathbb{S}_{\mu_i}$-generic over $\mathfrak{M}$
- $p \restriction \mu_i \Vdash_{\mathbb{S}_{\mu_i}}$ "$p_i \restriction (\xi \setminus \mu_i) \in D_i/G$" (this is an abbreviation for the more precise statement:
$$p \restriction \mu_i \Vdash_{\mathbb{S}_{\mu_i}} \text{“}(\exists q \in G \cap \mathbb{S}_{\mu_i})(q * p_i \restriction (\xi \setminus \mu_i) \in D_i)\text{”}$$
and will be used later as well)
- $p_i \restriction (\xi \setminus \mu_i)$ belongs to $\mathfrak{M}$
- $p \restriction \mu_i \Vdash_{\mathbb{S}_{\mu_i}}$ "$p_{i+1} \restriction (\mu_{i+1} \setminus \mu_i)$ is $\mathbb{S}_{\mu_{i+1} \setminus \mu_i}$-generic over $\mathfrak{M}[G]$"
- $p_{i+1} \leq p_i$

Notice that the statement that $p_i \restriction (\xi \setminus \mu_i) \in D_i/G$ can be expressed in $\mathfrak{M}$ and so if $\Gamma \in S(\mathfrak{M} \cap \mathbb{S}_{\mu_i}, p_i \restriction \mu_i)$ then $p_i \restriction (\xi \setminus \mu_i) \in D_i/\Gamma$. From this it easily follows that letting $p_\omega = \lim_{n\in\omega} p_n$ yields a strongly $\mathbb{S}_\xi$-generic condition over $\mathfrak{M}$.

To see that if ψ is a statement of the $\mathbb{S}_\xi$-forcing language using only parameters from $\mathfrak{M}$, then $\{\Gamma \in S(\mathfrak{M} \cap \xi, p_\omega) \ : \ \mathfrak{M}[\Gamma] \models \psi\}$ is a clopen set, observe that for any such ψ there corresponds the dense subset of $\mathbb{S}_\xi$ consisting of all conditions which decide ψ. Any such dense set is therefore D_n for some $n \in \omega$. It follows that if $\Gamma \in S(\mathfrak{M} \cap \xi, p_\omega)$ then the interpretation of $p_n \restriction (\xi \setminus \mu_n)$ in $\mathfrak{M}[\Gamma \restriction \mu_n]$ decides the truth value of ψ because $p_n \restriction \mu_n$ is strongly $\mathbb{S}_{\mu_n}$-generic over $\mathfrak{M}$. ¿From the induction hypothesis it follows that there is a clopen set $U \subseteq S(\mathfrak{M} \cap \mu_n, p_n \restriction \mu_n)$ such that for each $\Gamma' \in U$ the model $\mathfrak{M}[\Gamma']$ satisfies that the interpretation of $p_n \restriction (\xi \setminus \mu_n)$ in $\mathfrak{M}[\Gamma \restriction \mu_n]$ decides the truth value of ψ. Let U^* be the lifting of U to $S(\mathfrak{M} \cap \xi, p_\omega)$ — in other words, $\Gamma \in U^*$ if and only if $\Gamma \restriction \mu_n \in U$. Since the interpretation of $p_\omega \restriction (\xi \setminus \mu_n)$ in $\mathfrak{M}[\Gamma \restriction \mu_n]$ is a stronger condition than the interpretation of $p_n \restriction (\xi \setminus \mu_n)$ in $\mathfrak{M}[\Gamma \restriction \mu_n]$, it follows that $U^* \subseteq S(\mathfrak{M} \cap \xi, p_\omega)$ is the desired clopen set. ■

Definition 2.5. *A subset $X \subseteq {}^n\omega$ is said to be a* full subset *if, $X \neq \emptyset$ and for each $x \in X$ and $i \in n$ there is $A \in [\omega]^{\aleph_0}$ such that for all $m \in A$ there is $x_m \in X$ such that $x_m \restriction i = x \restriction i$ and $x_m(i) = m$.*

Lemma 2.3. *If $F : {}^n\omega \to [0,1]$ is a one-to-one function then there is a full subset $T \subseteq {}^n\omega$ such that the image of T under F is discrete.*

Proof: Proceed by induction on n to prove the following stronger assertion: If $F : {}^n\omega \to [0,1]$ is one-to-one then there is a full subset $T \subseteq {}^n\omega$, there is $f \in {}^\omega\omega$ and there is $x \in [0,1]$ such that

A. for any a descending sequence $\{U_i : i \in \omega\}$ of neighbourhoods of x such that $\operatorname{diam}(U_{n+1}) \cdot f(\lceil 1/\operatorname{diam}(U_n) \rceil) < 1$ and for each $X \in [\omega]^{\aleph_0}$ the set $\{t \in T : F(t) \in \cup_{i\in X}(U_i \setminus \overline{U_{i+1}})\}$ is a full subset.

The case $n = 1$ is easy. Choose $A \in [\omega]^{\aleph_0}$ such that $\{F(\emptyset \wedge i) : i \in A\}$ converges to $x \in [0,1]$. Let $f \in {}^\omega\omega$ be any increasing function such that for each $m \in \omega$ there is some $j \in A$ such that $1/m > |F(\emptyset \wedge j)| > 1/f(m)$. Let $T = \{\emptyset \wedge i : i \in A\}$.

Now let $F : {}^{n+1}\omega \to [0,1]$ be one-to-one. Use the induction hypothesis to find, for each $m \in \omega$, full subsets $T_m \subseteq {}^n\omega$ such that the image of F restricted to
$$\{x \in {}^{n+1}\omega : (\exists t \in T_m)(x = \emptyset \wedge m \wedge t)\}$$

is a discrete family and Condition **A.** is witnessed by $f_m \in {}^{\omega}\omega$ and $x_m \in [0,1]$. There are two cases to consider depending on whether or not there is $Z \in [\omega]^{\aleph_0}$ such that $\{x_m | m \in Z\}$ are all distinct.

Case 1

Assume that there is $Z \in [\omega]^{\aleph_0}$ such that $\{x_m : m \in Z\}$ are all distinct. It is then possible to assume that there is some $x \in [0,1]$ such that $\lim_{n \in Z} x_m = x$ and that, without loss of generality, $x_m > x_{m+1} > x$. As in the case $n = 1$, it is possible to find $f \in {}^{\omega}\omega$ such that for any a descending sequence $\{U_i : i \in \omega\}$ of neighbourhoods of x such that $\text{diam}(U_{n+1}) \cdot f(\lceil 1/\text{diam}(U_n) \rceil) < 1$ and for each $X \in [\omega]^{\aleph_0}$ the set $\{m \in \omega : x_m \in \cup_{i \in X}(U_i \setminus \overline{U_{i+1}})\}$ is infinite. Notice that each $U_i \setminus \overline{U_{i+1}}$ is open, so it follows from Condition **A.** that $\{t \in T_m : F(m \wedge t) \in U_i \setminus \overline{U_{i+1}}\}$ is a full subset provided that $x_m \in U_i \setminus \overline{U_{i+1}}$. Hence,

$$\cup\{\{t \in T_m : F(\langle m \rangle \wedge t) \in U_i \setminus \overline{U_{i+1}}\} : x_m \in U_i \setminus \overline{U_{i+1}}\}$$

is a full subset provided that $\text{diam}(U_{n+1}) \cdot f(\lceil 1/\text{diam}(U_n) \rceil) < 1$ and $X \in [\omega]^{\aleph_0}$. Let $T = \{t \in {}^{n+1}\omega : (\exists t' \in T_{t(0)})(t = t(0) \wedge t')\}$. Then T, f and x satisfy the Condition **A.**

Case 2

In this case there exists $x \in [0,1]$ such that $x_m = x$ for all but finitely many $m \in \omega$. Let $f \in {}^{\omega}\omega$ be such that $f \geq^* f_m$ for all $m \in \omega$. Let

$$T = \{t \in {}^{n+1}\omega : (\exists t' \in T_{t(0)})(t = t(0) \wedge t' \text{ and } x_{t(0)} = x)\}$$

To see that this works, suppose that $\{U_i : i \in \omega\}$ is a descending sequence of neighbourhoods of x such that $\text{diam}(U_{i+1}) \cdot f(\lceil 1/\text{diam}(U_i) \rceil) < 1$ and suppose that $X \in [\omega]^{\aleph_0}$.

Let $X = \cup_{j \in \omega} X_j$ be a partition of X into infinite subsets. It may be assumed that $f(i) \geq f_m(i)$ for all $i \in X_m$. By the induction hypothesis it follows that $\{t \in T_m : F(t) \in \cup_{i \in X_m}(U_i \setminus \overline{U_{i+1}})\}$ is a full subset of ${}^{n}\omega$ for each $m \in \omega$ because $f \geq^* f_m$. Hence $\{t \in T : F(t) \in \cup_{i \in X}(U_i \setminus \overline{U_{i+1}})\}$ is a full subset of ${}^{n+1}\omega$. ■

Although this fact will not be used, it should be noted that Lemma 2.3 can be generalised to arbitrary well founded trees.

If $X \subseteq ({}^{\omega}\omega)^{\alpha}$ is large then for each $e : \beta \to {}^{\omega}\omega$ let X_e represent the set of all $f : \alpha \setminus \beta \to {}^{\omega}\omega$ such that $e \cup f \in X$. Note that if $h \in X$ then for every $\beta \in \alpha$, $X_{h \restriction \beta}$ is a large subset of $({}^{\omega}\omega)^{\alpha \setminus \beta}$. Moreover, the projection $X_{h \restriction \beta}$ to $({}^{\omega}\omega)^{\delta \setminus \beta}$ is large provided that $\beta \in \delta$. This set will be denoted by $\pi_{\delta}(X_{f \restriction \beta})$. Note that $\pi_{\beta+1}(X_{f \restriction \beta})$ is the closure of a super-perfect tree, $T_{X,f,\beta}$ and so $\theta_{T_{X,f,\beta}} : {}^{\overset{\omega}{\smile}}\omega \to T_{X,f,\beta}$ is an isomorphism. This induces a natural isomorphism from ${}^{\overset{\alpha}{\smile}}({}^{\overset{\omega}{\smile}}\omega)$ to the open sets of X which will be denoted by Φ_X.

Lemma 2.4. *If $\alpha \in \omega_1$, $\mathfrak{M}$ is a countable elementary submodel, $q \in \mathbb{S}_{\alpha}$ and $F : S(\mathfrak{M} \cap \alpha, q) \to \mathbb{R}$ is continuous satisfying*

- **B.** *for each $\beta \in \alpha$ and each $e \in ({}^{\omega}\omega)^{\beta}$, if $S(\mathfrak{M} \cap \alpha, q)_e \neq \emptyset$, then the range of F restricted to $S(\mathfrak{M} \cap \alpha, q)_e$ is uncountable*

then there is a large closed set $X \subseteq S(\mathfrak{M} \cap \alpha, q)$ such that $F \restriction X$ is one-to-one and, moreover, $F \restriction X$ is a homeomorphism onto its range.

Proof: For $\tau \in {}^{\overset{\alpha}{\smile}}({}^{\overset{\omega}{\smile}}\omega)$ and $\tau' \in {}^{\overset{\alpha}{\smile}}({}^{\overset{\omega}{\smile}}\omega)$ define $\tau \leq \tau'$ if and only if $\tau(\sigma) \subseteq \tau'(\sigma)$ for each σ in the domain of τ and, define τ_1 and τ_2 to be incompatible if there is no τ' such that $\tau_1 \leq \tau'$ and $\tau_2 \leq \tau'$. To begin, let $\{\tau_i : i \in \omega\}$ enumerate a subset of ${}^{\alpha}({}^{\omega}\omega)$ which forms a tree base for $S(\mathfrak{M} \cap \alpha, q)$ — in other words, if i and j are in ω then either $\tau_i < \tau_j$, $\tau_j < \tau_i$ or τ_i and τ_j are incompatible and, moreover, $\{\Phi_{S(\mathfrak{M}\cap\alpha,q)}(\tau_i) : \imath \in \omega\}$ is a base for $S(\mathfrak{M} \cap \alpha, q)$. It may also be assumed that if $\tau_i < \tau_j$ then $i \leq j$ and that for each $k \in \omega$ there is a unique ρ and some $i \in k$ such that $\tau_k(\mu) = \tau_i(\mu)$ if $\mu \neq \rho$ and $\tau_k(\rho) = \tau_i(\rho) \wedge W$ for some integer W. Let $X_0 = S(\mathfrak{M} \cap \alpha, q)$. Construct by induction a sequence $\{(X_k, \{U_i : i \in k\} : k \in \omega\}$ such that:

- **a.** X_k is a large and closed subset of $({}^{\omega}\omega)^{\alpha}$
- **b.** each U_i is an open subset of $\mathbb{R}$
- **c.** $F(\Phi_{X_k}(\tau_i)) \subseteq U_i$
- **d.** $\Phi_{X_{k+1}}(\tau_i) = \Phi_{X_k}(\tau_i) \cap X_{k+1}$ if $i < k$
- **e.** $\overline{U_i} \cap \overline{U_j} = \emptyset$ if τ_i and τ_j are incompatible
- **f.** $U_i \subseteq U_j$ if $\tau_j < \tau_i$
- **g.** if $\tau_i < \tau_j$ then $\overline{U_j} \cap \overline{F(\Phi_{X_k}(\tau_i) \setminus \Phi_{X_k}(\tau_j))} = \emptyset$
- **h.** X_k satisfies Condition **B.** for each $k \in \omega$

If this can be accomplished then let $X = \cap_{k\in\omega} X_k$. It follows that X is large and closed because, by (d), branching is eventually preserved at each node. Moreover $F \restriction X$ is also one-to-one because of the choice of the U_i satisfying (e) for each $i \in \omega$. To see that F is a homeomorphism onto its range suppose that $V \subseteq X$ is an open set and that z belongs to the image of V under F. This means that there is some $i \in \omega$ and z' such that $z' \in \Phi_X(\tau_i) \subseteq V$ and $F(z') = z$. It follows that $z \in U_i \cap F(X)$ and so it suffices to show that $U_i \cap F(X) = F(\Phi_X(\tau_i))$. Clearly (c) implies that $U_i \cap F(X) \supseteq F(\Phi_X(\tau_i))$. On the other hand, if $w \in U_i \cap F(X)$ then there is some $w' \in X$ such that $F(w') = w$. Since $w \in U_i$ it follows that $w' \in \Phi_{X_k}(\tau_i)$ for each $k \geq i$ because $\{\Phi_{X_k}(\tau_j) : j \in \omega\}$ is a tree base. Hence $w \in F(\Phi_X(\tau_i))$.

To perform the induction, use the hypothesis on $\{\tau_i : i \in k\}$ to choose a maximal τ_i below τ_k. Hence there is a unique ρ such that $\tau_k(\mu) = \tau_i(\mu)$ if $\mu \neq \rho$ and $\tau_k(\rho) = \tau_i(\rho) \wedge W$ for some integer W. The open set U_k will be chosen so that $\overline{U_k} \subseteq U_i$ and this will guarantee that if τ_j is incompatible with τ_i then $\overline{U_k} \cap \overline{U_j} = \emptyset$. The hypothesis on $\{\tau_i : i \in k\}$ also implies that there is no $j \in k$ such that $\tau_k < \tau_j$. Moreover, if $\tau_i < \tau_j$ then $\overline{F(\Phi_{X_k}(\tau_i) \setminus \Phi_{X_k}(\tau_j))} \cap \overline{U_j} = \emptyset$.

To satisfy Condition (g), let $\{\delta_m : m \in a\}$ enumerate, in increasing order, the domain of τ_i together with the unique ordinal ρ and define $H : {}^{a}\omega \to \mathbb{R}$ as follows. Choose $y_s \in {}^{\alpha}({}^{\omega}\omega)$ so that for each $s \in {}^{a}\omega$:

- $y_s \in \Phi_{X_k}(\tau_i \wedge s)$ where, in this context, $\tau_i \wedge s$ is defined by $(\tau_i \wedge s)(\delta_m) = \tau_i(\delta_m) \wedge s(m)$
- if $s \restriction j = s' \restriction j$ then $y_s \restriction \delta_j = y_{s'} \restriction \delta_j$
- if $s \neq s'$ then $F(y_s) \neq F(y_{s'})$

This is easily done using Condition **B.** to satisfy the last two conditions. Finally, define $H(s) = F(y_s)$ and observe that this is one-to-one.

Now use Lemma 2.3 to find a full subset $T \subseteq {}^{a}\omega$ such that $H \restriction T$ has discrete image, and furthermore, this is witnessed by $\{\mathcal{V}_t : t \in T\}$. Shrinking T by a finite amount, if necessary, it may be assumed that $\Phi_{X_k}(\tau_j) \cap \Phi_{X_k}(\tau_i \wedge s) = \emptyset$ for all $s \in T$ and $j \in k$ because $a \geq 1$. Let

$$X_{k+1} = (X_k \setminus \Phi_{X_k}(\tau_i)) \cup (\cup\{\Phi_{X_k}(\tau_i \wedge s) : s \in T\}) \cup (\cup\{\Phi_{X_k}(\tau_j) : \tau_i \leq \tau_j\})$$

and define $U_k = \mathcal{V}_{\bar{t}} \cap U_i$ where $\bar{t} \in T$ is lexicographically the first element of T. It is an easy matter to verify that all of the induction hypotheses are satisfied. ■

To finish the proof of the Lemma 2.1 suppose that $\xi \in \omega_2 + 1$, $\mathbb{S}_\xi$ is the iteration with countable support of the partial orders $\mathbb{S}$. Suppose also that $p \Vdash_{\mathbb{S}_\xi}$ "$x \in [0,1]$" and

$$p \Vdash_{\mathbb{S}_\xi} \text{"}H : [0,1] \to [0,1] \text{ is a Borel function"}$$

Let $\eta \in \omega_2$ be such that x occurs for the first time in the model $V[G \cap \mathbb{S}_\eta]$. Let $\mathfrak{M}$ be a countable elementary submodel of $H((2^{\aleph_0})^+)$ containing p and the names x and H. It follows from Lemma 2.2 that it is possible to find $q \leq p$ which is strongly $\mathbb{P}_\eta$-generic over $\mathfrak{M}$. Let $F : S(\mathfrak{M} \cap \xi, q) \to [0,1]$ be defined by $F(\Gamma) = x_\Gamma$ or, in other words, $F(\Gamma)$ is the interpretation of x in $\mathfrak{M}[\Gamma]$. It follows from the second clause of Definition 2.2 that F is a continuous function. Moreover, because it is assumed that x does not belong to any model $\mathfrak{M}[G \cap \mathbb{S}_\mu]$ where $\mu \in \eta$, it follows that Condition **B.** of Lemma 2.4 is satisfied by F. Using this lemma, and the fact that $\eta \cap \mathfrak{M}$ has countable order type, it is possible to find $q' \leq q$ such that $\mathrm{dom}(q) = \mathrm{dom}(q')$ and $F \restriction S(\mathfrak{M} \cap \eta, q')$ is a homeomorphism onto its range.

Now let X be the image of $S(\mathfrak{M} \cap \eta, q')$ under the mapping F. An inspection of the definition of $S(\mathfrak{M} \cap \eta, q')$ reveals it to be a Borel set. Since $F \restriction S(\mathfrak{M} \cap \eta, q')$ is one-to-one, it follows that X is also Borel. Obviously $q' \Vdash_{\mathbb{S}_{\omega_2}}$ "$x \in X$". Because the name H belongs to $\mathfrak{M}$ and F is one-to-one on X, it is possible to define a mapping $H' : X \to [0,1]$ by defining $H'(z)$ to be the interpretation of $H(x)$ in $\mathfrak{M}[F^{-1}(z)]$. Obviously $q' \Vdash_{\mathbb{S}_{\omega_2}}$ "$H(x) = H'(x)$".

All that remains to be shown is that H' is continuous. To see this, let $z \in X$. Then there is some $\Gamma \in S(\mathfrak{M} \cap \eta, q'')$ such that $z = F(\Gamma) = x_\Gamma$. For any interval with rational end-points, (p,q), the statement $\psi_{p,q}$ which asserts that $H(x) \in (p,q)$ has all of its parameters in $\mathfrak{M}$. Moreover, $\mathfrak{M}[\Gamma] \models H(x) = H(x_\Gamma) = H'(z)$. For each interval with rational end-points containing $H'(z)$, (p,q), there is therefore an open neighbourood $U_{p,q}$ of Γ such that $\mathfrak{M}[\Gamma'] \models \psi_{p,q}$ for each $\Gamma' \in U_{p,q}$. Since $F \restriction S(\mathfrak{M} \cap \eta, q'')$ is a homeomorphism, it follows that the image of any $U_{p,q}$ under F is an open neighbourhood $U^*_{p,q}$ of z. Now, if $\bar{z} \in U^*_{p,q}$ then $\bar{z} = x_{\Gamma'}$ for some $\Gamma' \in U_{p,q}$ and, therefore $\mathfrak{M}[\Gamma'] \models \psi_{p,q}$. This means that the interpretation of $H(x)$ in $\mathfrak{M}[\Gamma']$ belongs to (p,q). Hence the image of $U^*_{p,q}$ under H' is contained in (p,q) and so H' is continuous.

3. Remarks

The proof presented here can also be generalised, without difficulty, to apply to the iteration of ω_2 Laver reals as well super-perfect reals. The notion of a large set has its obvious analogue which can be used to deal with the1 iteration. In the single step case use the proof that a Laver real is minimal [2]. The only difference is that, for a Laver condition T, the "frontiers" of [2] should be used in place of the images of $\theta_T \restriction {}^n\omega$. In fact, the proof of the preceding section can be viewed as a generalisation of the fact that adding super-perfect real adds a minimal real in the sense that the structure of the iterated model is shown to depend very predictably on the generic reals added.

References

1. J. Cichoń, M. Morayne, J. Pawlikowski, and S. Solecki, *Decomposing Baire functions*, J. Symbolic Logic **56** (1991), no. 4, 1273–1283.
2. M. Groszek, *Combinatorics on ideals and forcing with trees*, J. Symbolic Logic **52** (1987), no. 3, 582–593.
3. A. Miller, *Rational perfect set forcing*, Axiomatic Set Theory (Providence) (D. A. Martin J. Baumgartner and S. Shelah, eds.), Contemporary Mathematics, vol. 31, American Mathematical Society, pp. 143–159.
4. S. Shelah, *Proper forcing*, Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin, 1982.
5. J. Steprāns, *A very discontinuous Borel function*, J. Symbolic Logic **to appear** (1993), ?–?

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