Forcing Isomorphism II

M. C. Laskowski*
Department of Mathematics
University of Maryland

S. Shelah
Department of Mathematics
Hebrew University of Jerusalem

Department of Mathematics
Rutgers University †

September 15, 2020

Abstract

If T has only countably many complete types, yet has a type of infinite multiplicity then there is a c.c.c. forcing notion $\mathcal{Q}$ such that, in any $\mathcal{Q}$-generic extension of the universe, there are non-isomorphic models M_1 and M_2 of T that can be forced isomorphic by a c.c.c. forcing. We give examples showing that the hypothesis on the number of complete types is necessary and what happens if 'c.c.c.' is replaced other cardinal-preserving adjectives. We also give an example showing that membership in a pseudo-elementary class can be altered by very simple cardinal-preserving forcings.

*Partially supported by an NSF Postdoctoral Fellowship and Research Grant DMS 9403701.
†The authors thank the U.S.-Israel Binational Science Foundation for its support of this project. This is item 518 in Shelah's bibliography.

1 Introduction

The fact that the isomorphism type of models of a theory can be altered by forcing was first noted by Barwise in [3]. He observed that the natural back-and-forth system obtained from a pair of $L_{\infty,\omega}$-equivalent structures gives rise to a partial order that makes the structures isomorphic in any generic extension of the universe. Restricting attention to partial orders with desirable combinatorial properties, e.g., the countable chain condition (c.c.c.) and asking which theories have a pair of non-isomorphic models that can be forced isomorphic by such a forcing provides us with an alternate approach to a fundamental question of model theory. The question, roughly stated, asks which (countable) theories admit a 'structure theorem' for the class of models of the theory? Part of the research on this question has been to discover a definition of the phrase 'structure theorem' that leads to a natural dichotomy between theories. In [9] the second author succeeds in characterizing the theories that have the maximal number of non-isomorphic models in every uncountable cardinality and is near a characterization of the theories which have families of 2^κ pairwise non-embeddable models of size κ. These abstract results imply the impossibility of structure theorems (for virtually every definition of 'structure theorem') by the sheer size and complexity of the class of models of such a theory. On the positive side, he defines a *classifiable* theory (i.e., superstable, without the Dimensional Order Property (DOP) and without the Omitting Types Order Property (OTOP)) and shows that any model of a classifiable theory can be described in terms of an independent tree of countable elementary substructures. That is, the class of models of such a theory has a structure theorem in a certain sense. In [8] he analyzes which structures (of a fixed cardinality) can be determined up to isomorphism by their Scott sentences in various infinitary languages (e.g., $L_{\infty,\kappa}$).

In both this paper and in [2], we concentrate on systems of invariants that are preserved under c.c.c. forcings and ask which theories have their models described up to isomorphism by invariants of this sort. It is well-known that c.c.c. forcings preserve cardinality and cofinality, yet such forcings typically add new subsets of ω (reals) to the universe. We call two structures *potentially isomorphic* if they can be forced isomorphic by a forcing with the countable chain condition (c.c.c.). The relevance of this notion is that the existence of a pair of non-isomorphic, potentially isomorphic structures within a class **K** (either in the ground universe or in a c.c.c. forcing extension) implies that the isomorphism type of elements of **K** cannot be described by a c.c.c.-invariant system of invariants.

In [2], it was shown that for countable theories T, if T is not classifiable then there are non-isomorphic, potentially isomorphic models of T of size 2^ω. In addition, certain classifiable theories were shown to have such a pair of models. The main theorem of this paper, Theorem 4.1, states that if T is superstable, $D(T)$ is countable (i.e., T has at most $\aleph_0$ n-types for each n), but has a type of infinite multiplicity (equivalently, T is not $\aleph_0$-stable) then there is a c.c.c. forcing $\mathcal{Q}$ such that $\Vdash_{\mathcal{Q}}$ "There are two non-isomorphic, potentially isomorphic models of T." Combining this with the results from [2] yields the theorem mentioned in the abstract. We remark that the more natural question of whether any such theory has a pair of non-isomorphic, potentially isomorphic models in the ground universe (as opposed to in a forcing extension) remains open.

A consequence of these results is that the system of invariants for the isomorphism type of a model of a classifiable theory mentioned above cannot be simplified significantly. In particular, we conclude that if T is classifiable but not $\aleph_0$-stable and if $D(T)$ is countable then the models of T cannot be described by independent trees of *finite* subsets, for any such tree would be preserved by a c.c.c. forcing.

The idea of the proof of Theorem 4.1 is to build two models of T, each realizing a suitably generic subset of the strong types extending the given type p of infinite multiplicity. The second c.c.c. forcing adds a new automorphism of the algebraic closure of the empty set that extends to an isomorphism of the models. In building these models, we place a natural measure on the space of strong types extending p and introduce a new method of construction. We require that every element of the construction realizes a type over the preceding elements of positive measure. We expect that this technique can be used to solve other problems within the context of superstable theories with a type of infinite multiplicity.

In the final section we give a number of examples. In the first, we show that $(\mathbb{R}, \leq)$ and $(\mathbb{R} \setminus \{0\}, \leq)$ are forced isomorphic by any forcing that adds reals. In particular, this shows that the phenomenon of non-isomorphic models becoming isomorphic in a forcing extension is prevalent, even among very common structures and forcings as simple as Cohen forcing. This example also indicates that membership in a pseudo-elementary class is not absolute, even for very reasonable forcings. The second example shows that there is a difference between the notions of "potentially isomorphic via c.c.c." and "potentially isomorphic via Cohen forcing." The third example shows that the assumption of $D(T)$ countable in Theorem 4.1 cannot be replaced by the weaker assumption of T countable.

We assume only that the reader has a basic understanding of stability theory and forcing. On the model theory side, all that is required is a knowledge of the basic facts of strong types and forking (see [1], [5], [6] or [9]). We assume that our domain of discourse is a large, saturated model $\mathfrak{C}$ of T. That is, all models can be taken as elementary submodels of $\mathfrak{C}$ and all sets of elements are subsets of the universe of $\mathfrak{C}$. In Section 2 we work in an expansion $\mathfrak{C}^{\mathrm{eq}}$ of $\mathfrak{C}$ so that we may consider strong types to be types over algebraically closed sets. The definition of $S^+(A,B)$ does not depend on the choice of the expansion.

Other than a knowledge of the basic techniques of forcing, we assume the reader be familiar with the notion of a complete embedding and basic facts about c.c.c. forcings. The material in [4] is more than adequate.

2 Strong types and measures

Throughout this section, assume that T is countable and stable. As we will be concerned with the space of strong types extending a given type, it is convenient to fix an expansion $\mathfrak{C}^{\mathrm{eq}}$ of $\mathfrak{C}$, where the signature of $\mathfrak{C}^{\mathrm{eq}}$ contains a sort corresponding to each definable equivalence relation E of $\mathfrak{C}^n$, and a function symbol f_E taking each tuple to its corresponding E-class in its sort. The advantage of this assumption is that all types are stationary over algebraically closed sets in $\mathfrak{C}^{\mathrm{eq}}$ (see [6]).

Our goal in this section is to define a measure on the space of strong types extending a given type. Using this measure, we are interested in the subsets having positive measure. This leads to our definition of $S^+(A,B)$.

Definition 2.1 For $p \in S_1(B)$, B finite, let $S^*_p = \{r \in S_1(\mathrm{acl}(B)) : p \subseteq r\}$.

As we are working in $\mathfrak{C}^{\mathrm{eq}}$, there is a natural correspondence between S^*_p and the set of all strong types extending p. We endow S^*_p with a natural topology τ by taking as a base all sets of the form

$$[a/E] = \{r \in S^*_p : r(x) \vdash E(x,a)\}$$

for some equivalence relation E over B with finitely many classes and some realization a of $p^{\mathfrak{C}}$.

As T is countable and B is finite, there are only countably many equivalence relations over B, so τ is separable. In addition $\mathrm{Aut}_B(\mathfrak{C})$ acts naturally on S_p^*, so for each equivalence class $[a/E]$, let $\mathrm{Stab}([a/E])$ denote the setwise stabilizer of $[a/E]$. As E has only finitely many classes, $\mathrm{Stab}([a/E])$ has finite index in $\mathrm{Aut}_B(\mathfrak{C})$. We construct a regular measure μ_p on the Borel subsets of S_p^* by defining $\mu_p([a/E]) = 1/n$, where n is the index of $\mathrm{Stab}([a/E])$ in $\mathrm{Aut}_B(\mathfrak{C})$ and inductively extending the measure to the Borel subsets. This is nothing more than the usual construction of Haar measure on the range of a group action (see e.g., [7]). It is easy to see that the measure μ_p induces a complete metric on S_p^*, which implies that S_p^* is a Polish space.

For a finite set A and $q \in S_1(A)$, let $\Gamma_p^q = \{r \in S_p^* : q \cup r \text{ is consistent}\}$. By compactness, Γ_p^q is a closed, hence measurable subset of S_p^*. For $B \subseteq A$ and A finite, let

$$S^+(A,B) = \{q \in S_1(A) : q \text{ does not fork over } B \text{ and } \mu_p(\Gamma_p^q) > 0, \text{ where } p = q|B\}.$$

We remark that instead of looking at sets of positive measure, we could have defined $S^+(A,B)$ to be the set of non-forking extensions q of p such that Γ_p^q is non-meagre. These two notions are not the same, but they share many of the same properties. In particular, all of the lemmas of this section have analogs in the non-meagre context.

Lemma 2.2 *Assume $C \subseteq B \subseteq A$, A finite and that $q \in S_1(A)$ does not fork over C. Let $p = q|C$. Then $\mu_p(\Gamma_p^q) = 0$ if and only if either $\mu_p(\Gamma_p^{q|B}) = 0$ or $\mu_{q|B}(\Gamma_{q|B}^q) = 0$.*

Proof. For any equivalence class E over C with finitely many classes, say that $[d/E]$ is consistent with q if there is a realization e of q with $E(d,e)$. As q does not fork over C, there is a homeomorphism between $S_{q|B}^*$ and the subspace $\Gamma_p^{q|B}$ of S_p^*, but $\mu_p([d/E])$ may not equal $\mu_{q|B}([d/E])$. However, it follows directly from the definitions of the measures that $\mu_p([d/E]) \leq \mu_{q|B}([d/E])$ for all $[d/E]$ consistent with $q|B$. Hence $\mu_p(\Gamma_p^q) \leq \mu_{q|B}(\Gamma_{q|B}^q)$. Trivially, $\Gamma_p^q \subseteq \Gamma_p^{q|B}$, so $\mu_p(\Gamma_p^q) \leq \mu_p(\Gamma_p^{q|B})$, which completes the proof of the lemma from right to left.

For the converse, let $\lambda = \mu_p(\Gamma_p^{q|B})$. We will show that $\lambda \cdot \mu_{q|B}(\Gamma^q_{q|B}) \leq \mu_p(\Gamma^q_p)$. For this, it suffices to show that

$$\lambda \cdot \mu_{q|B}([d/E]) \leq \mu_p([d/E])$$

for every $[d/E]$ consistent with q. By definition of the measures, $\mu_p([d/E]) = 1/n$, where n is the number of E-classes consistent with p and $\mu_{q|B}([d/E]) = 1/m$, where m is the number of E-classes consistent with $q|B$. Thus, we must show that $\lambda \leq m/n$. To see this, let $d_0, \ldots, d_{m-1}$ enumerate the E-classes consistent with $q|B$. As $\bigcup_{i<m}[d_i/E]$ is a disjoint open cover of $\Gamma_p^{q|B}$ and $\mu_p([d_i/E]) = 1/n$ for each i, the regularity of μ_p implies that $\lambda \leq m/n$. ∎

Lemma 2.3 *If $C \subseteq B \subseteq A$ and A is finite then for every a,* $\mathrm{tp}(a/A) \in S^+(A,C)$ *if and only if* $\mathrm{tp}(a,A) \in S^+(A,B)$ *and* $\mathrm{tp}(a/B) \in S^+(B,C)$*.*

Proof. Let $q = \mathrm{tp}(a/A)$. As non-forking is transitive, q does not fork over C if and only if q does not fork over B and $q|B$ does not fork over A. Further, by Lemma 2.2, $\mu_{q|C}(\Gamma^q_{q|C}) > 0$ if and only if $\mu_{q|C}(\Gamma^{q|B}_{q|C}) > 0$ and $\mu_{q|B}(\Gamma^q_{q|B}) > 0$. ∎

Suppose that $p_0, \ldots, p_{n-1} \in S_1(B)$. Let $S^*_{p_0,\ldots,p_{n-1}} = \{r \in S_n(\mathrm{acl}(B)) : r \restriction x_i = p_i$ and if $\bar{c}$ realizes r then $\{c_i : i < n\}$ is independent over $B\}$.

We endow $S^*_{p_0,\ldots,p_{n-1}}$ with the analogous topology as τ. As types over algebraically closed sets (in $\mathfrak{C}^{\mathrm{eq}}$) have unique non-forking extensions to any superset of their domain, $S^*_{p_0,\ldots,p_{n-1}}$ is homeomorphic to the topological product $\Pi_{i<n} S^*_{p_i}$. Via this identification, endow $S^*_{p_0,\ldots,p_{n-1}}$ with the product measure $\mu_{p_0,\ldots,p_{n-1}} = \mu_{p_0} \times \ldots \times \mu_{p_{n-1}}$ on the basic open sets and extend the measure to the Borel subsets.

For $q \in S_n(A)$, let $\Gamma^q_p = \{r \in S^*_{p_0,\ldots,p_{n-1}} : r \cup q$ is consistent$\}$. As before, Γ^q_p is a closed, hence measurable subset of $S^*_{p_0,\ldots,p_{n-1}}$. For $B \subseteq A$, A finite, let

$$S^+_n(A,B) = \{q \in S_n(A) : q \text{ does not fork over } B \text{ and } \mu_p(\Gamma^q_p) > 0, \text{ where } p = q|B\}.$$

The proof of the following lemma is basically an application of Fubini's Lemma to our context.

Lemma 2.4 *Assume that $q(x,y) \in S_2(A)$, $B \subseteq A$, A finite and that q does not fork over B. Let $q_0 = q \restriction x$, $q_1 = q \restriction y$, and let p, p_0, p_1 denote the restrictions of q, q_0, q_1 (respectively) to B. Let b be any realization of q_1 and let $\Gamma^{q_b}_{p_0} = \{r \in S^*_{p_0} : r(x) \cup q(x,b)$ is consistent$\}$. Then*

$$\mu_{p_0 p_1}(\Gamma^q_p) = \int_{S^*_{p_1}} \mu_{p_0}(\Gamma^{q_b}_{p_0}) d\mu_{p_1} = \mu_{p_0}(\Gamma^{q_b}_{p_0}) \cdot \mu_{p_1}(\Gamma^{q_1}_{p_1}).$$

Proof. The first equality is literally Fubini's Lemma and the second follows from the fact that $\Gamma^{q_b}_{p_0} = \emptyset$ unless b realizes q_1 and the fact that μ_{p_0} is invariant under translations by elements of $\mathrm{Aut}_B(\mathfrak{C})$. ∎

Lemma 2.5 *If $B \subseteq A$ and A is finite, then for all a, b, $\mathrm{tp}(ab/A) \in S^+_2(A,B)$ if and only if $\mathrm{tp}(a/A \cup \{b\}) \in S^+(A \cup \{b\}, B)$ and $\mathrm{tp}(b/A) \in S^+(A,B)$.*

Proof. This follows from Lemma 2.4 in the same manner as Lemma 2.3 followed from Lemma 2.2. ∎

The following two lemmas are the goals of this section. The first is the key ingredient in the proof of the Generalized Symmetry Lemma (Lemma 3.6). The reader should compare it to Axiom VI in [[9], Section IV.1]. The second, the Extendibility Lemma, makes critical use of the added hypothesis that $|D(T)| = \aleph_0$ that will be assumed throughout the next section.

Lemma 2.6 *Assume that T is stable and countable, $B, C \subseteq A$, A finite, $\mathrm{tp}(a/A) \in S^+(A,B)$ and $\mathrm{tp}(b/A \cup \{a\}) \in S^+(A \cup \{a\}, C)$. Then $\mathrm{tp}(a/A \cup \{b\}) \in S^+(A \cup \{b\}, B)$.*

Proof. Let $D = B \cup C$. By Lemma 2.3, $\mathrm{tp}(a/A) \in S^+(A,D)$ and $\mathrm{tp}(b/A \cup \{a\}) \in S^+(A \cup \{a\}, D)$. By Lemma 2.5 (switching the roles of a and b), $\mathrm{tp}(ab/A) \in S^+_2(A,D)$. Using Lemma 2.5 again, $\mathrm{tp}(a/A \cup \{b\}) \in S^+(A \cup \{b\}, D)$. So $\mathrm{tp}(a/A \cup \{b\}) \in S^+(A \cup \{b\}, B)$ using Lemma 2.3. ∎

Lemma 2.7 (Extendibility Lemma) *Assume that T is countable and stable and that $|D(T)| = \aleph_0$. Let $C \subseteq B \subseteq A$ be finite, let E be an equivalence relation with finitely many classes and let a be arbitrary. If $q \in S^+(B,C)$, $p = q|C$ and $\mu_p([a/E] \cap \Gamma^q_p) > 0$, then there is $q^+ \in S^+(A,C)$ extending $q \cup \{E(x,a)\}$.*

Proof. Let $\{q_i : i \in \omega\}$ enumerate the non-forking extensions of q to $S_1(A)$ that are consistent with $E(x, a)$. We claim that $[a/E] \cap \Gamma^q_p = \bigcup_{i \in \omega} \Gamma^{q_i}_p$. For, if $r \in [a/E] \cap \Gamma^q_p$, then as $r \cup q \cup E(x, a)$ is consistent we can choose a realization b of it with $b \underset{B}{\perp} A$. Then $r \in \Gamma^{q_i}_p$, where $q_i = \mathrm{tp}(b/A)$. As μ_p is countably additive, $\mu_p(\Gamma^{q_i}_p) > 0$ for some $i \in \omega$. As non-forking is transitive this $q_i \in S^+(A, C)$, as desired. ∎

3 Positive measure constructions

In this section we define two partial orders $(\mathcal{P}, \leq_{\mathcal{P}})$ and $(\mathcal{R}, \leq_{\mathcal{R}})$ that will be used in the proof of Theorem 4.1. The forcing $\mathcal{P}$ will force the existence of countable subsets B and C_α $(\alpha \in \omega_1)$ of $\mathfrak{C}$ such that $\mathrm{acl}(B)$ and $\mathrm{acl}(B \cup C_\alpha)$ are $\aleph_0$-saturated models of T, $\mathrm{acl}(B) \preceq \mathrm{acl}(B \cup C_\alpha)$, and $\{C_\alpha : \alpha \in \omega_1\}$ are independent over B. *Throughout this section, assume that T is stable, $|D(T)| = \aleph_0$ (hence $|T| = \aleph_0$) and we have a fixed type $r^* \in S_1(\emptyset)$ of infinite multiplicity.*

Definition 3.1 Let $\mathcal{V} = X \cup \bigcup_{\alpha \in \omega_1} Z_\alpha$, where $X = \{x_m : m \in \omega\}$ and each set $Z_\alpha = \{z^\alpha_m : m \in \omega\}$, $\alpha \in \omega_1$ is a countable set of distinct variable symbols. A *$\mathcal{V}$-type* q is a complete type in finitely many variables of $\mathcal{V}$. Let $\mathrm{var}(q)$ denote this set of variables.

A $\mathcal{V}$-type should be thought of as the type of a finite subset of $A \cup \bigcup_{\alpha \in \omega_1} B_\alpha$. As notation, given a sequence $\langle a_i : i < n \rangle$ and $u \subseteq n$, let $A_u = \{a_j : j \in u\}$. Note that as a special case, $A_i = \{a_j : j < i\}$.

Definition 3.2 A *positive measure construction* (PM-construction) t (of length n) is a sequence of triples $\langle (a_i, u_i, v_i) : i < n \rangle$ satisfying the following conditions for each $i < n$:

1. $\mathrm{tp}(a_i/A_i)$ is not algebraic and $u_i \subseteq i$;
2. $\mathrm{tp}(a_i/A_i) \in S^+(A_i, A_{u_i})$;
3. If $v_i \in X$ and $j \in u_i$ then $v_j \in X$;
4. If $v_i \in Z_\alpha$ for some α and $j \in u_i$ then $v_j \in X \cup Z_\alpha$;

5. If $v_i = z_0^\alpha$ then $u_i = \emptyset$ and $\mathrm{tp}(a_i/\emptyset) = r^*$.

A PM-construction t may be thought of as a way of building the $\mathcal{V}$-type $\mathrm{tp}(a_i : i < n)$ in the variables $\langle v_i : i < n\rangle$. Let $\mathrm{tp}(t)$ denote this type and let $\mathrm{var}(t) = \{v_i : i < n\}$. If $\mathrm{tp}(t) = q$ then we call t a *PM-construction of* q. A $\mathcal{V}$-type q is *PM-constructible* if there is a PM-construction of it.

Intuitively, $(a_i, u_i, v_i) \in t$ ensures that $\mathrm{tp}(a_i/A_i)$ is as generic as possible, given that it extends $\mathrm{tp}(a_i/A_{u_i})$. Clause (5) implies that the set $\{z_0^\alpha : \alpha \in \omega_1\} \cap \mathrm{var}(t)$ realizes a generic subset of the strong types extending r^*. In particular, no two such variables can realize the same strong type.

Definition 3.3 Let $\mathcal{P}$ denote the set of all PM-constructible $\mathcal{V}$-types. For $p, q \in \mathcal{P}$, say $p \leq_{\mathcal{P}} q$ if and only if there is a PM-construction t of q and an $m \in \omega$ such that $t \restriction m$ is a PM-construction of p. That $\leq_{\mathcal{P}}$ induces a partial order on $\mathcal{P}$ follows from the lemma below.

Lemma 3.4 *Assume that $p \leq_{\mathcal{P}} q$. Then any PM-construction of p can be continued to a PM-construction of q.*

Proof. Suppose that $t = \langle (a_i, u_i, v_i) : i < n\rangle$ is a PM-construction of q such that $t \restriction m$ is a PM-construction of p and let $s = \langle (b_j, u'_j, v'_j) : j < m\rangle$ be any PM-construction of p. Since $\{v_i : i \in m\} = \{v'_j : j \in m\}$ setwise there is a unique permutation σ of n such that $v_i = v'_{\sigma(i)}$ for all $i < m$ and $\sigma(i) = i$ for all $m \leq i < n$. As $\mathrm{tp}(a_i : i < m) = \mathrm{tp}(b_{\sigma(i)} : i < m)$, we can choose an automorphism ψ of $\mathfrak{C}$ such that $\psi(a_i) = b_{\sigma(i)}$ for each i. It is now easy to verify that $s^\frown\langle(\psi(a_k), \sigma''(u_k), v_k) : m \leq k < n\rangle$ is a PM-construction of q continuing s. ∎

The following lemma will be used to show that a generic subset of $\mathcal{P}$ generates a family of $\aleph_0$-saturated models of T.

Lemma 3.5 *Let $t = \langle (a_i, u_i, v_i) : i < n\rangle$ be any PM-construction.*

1. *If $x_m \in X \smallsetminus \mathrm{var}(t)$ and $u \subseteq n$ such that $j \in u$ implies $v_j \in X$ and p is a non-algebraic 1-type over A_u then there is a realization a_n of p such that $t^\frown\langle(a_n, u, x_m)\rangle$ is a PM-construction.*

2. *If $z^\alpha_m \in Z_\alpha \smallsetminus \mathrm{var}(t)$, $m \neq 0$, $u \subseteq n$ such that $j \in u$ implies $v_j \in X \cup Z_\alpha$ and p is a non-algebraic 1-type over A_u then there is a realization a_n of p such that $t^\frown\langle(a_n, u, z^\alpha_m)\rangle$ is a PM-construction.*

3. *If $z^\alpha_0 \in Z_\alpha \smallsetminus \mathrm{var}(t)$, then there is an a_n such that $t^\frown\langle(a_n, \emptyset, z^\alpha_0)\rangle$ is a PM-construction.*

Proof. These follow immediately from the Extendibility Lemma and Clauses (3), (4), (5) of Definition 3.2. ∎

In order to establish the independence of the B_α's over A and to analyze the complexity of the partial order $(\mathcal{P}, \leq_\mathcal{P})$, we seek a 'standard form' for a PM-construction. The primary complication is that the restriction of a PM-constructible type to a subset of its free variables need not be PM-constructible. We characterize when a permutation σ of a PM-construction t is again a PM-construction. Call a permutation σ *permissible* if $\sigma''(u_i) \subseteq \sigma(i)$ for all $i < n$. Clearly, if σ is not permissible then σt violates Clause (1) of being a PM-construction. The following lemma, known as the Generalized Symmetry Lemma, establishes the converse. Its proof simply amounts to bookkeeping once we have Lemma 2.6.

Lemma 3.6 (Generalized Symmetry Lemma) *If t is a PM-construction of q and σ is a permissible permutation then σt is a PM-construction of q as well.*

Proof. Suppose that $t = \langle(a_i, u_i, v_i) : i < n\rangle$ is a PM-construction of q. Then Lemma 2.6 insures that $\sigma_k(t)$ is a PM-construction, where σ_k is the (permissible) permutation exchanging k and $k+1$ whenever $k \neq n-1$ and $k \notin u_{k+1}$. The lemma now follows easily by induction on the length of t. The reader is encouraged to compare this with [9, IV, Theorem 3.3]. ∎

As an application of Lemma 3.6, we obtain a 'standard form' for a PM-construction. Given any $p \in \mathcal{P}$, there is a PM-construction $t = \langle(a_i, u_i, v_i) : i < n\rangle$ of p such that, for all $i < j < n$,

1. if $v_j \in X$ then $v_i \in X$;

2. if $v_i \in Z_\alpha$ and $v_j \in Z_{\alpha'}$ then $\alpha \leq \alpha'$;

3. if $v_j = z_0^\alpha$ then $v_i \notin Z_\alpha$.

To see this, let s be any PM-construction of p, find an appropriate permissible permutation σ and let $t = \sigma s$. The following lemma is a consequence of this representation.

Lemma 3.7 *Let $p(\overline{x}, \overline{z}^{\alpha_0}, \dots, \overline{z}^{\alpha_{k-1}}) \in \mathcal{P}$, where $\overline{x} \subseteq X$ and $\overline{z}^{\alpha_i} \subseteq Z_{\alpha_i}$ and let $\overline{b}\overline{c}^{\alpha_0} \dots \overline{c}^{\alpha_{k-1}}$ realize p. Then $\{\overline{c}^{\alpha_i} : i < k\}$ is independent over $\overline{b}$.*

Proof. We argue by induction on var(p). Choose $p \in \mathcal{P}$ with $n+1$ free variables. We can find a PM-construction $t = \langle (a_i, u_i, v_i) : i < n+1 \rangle$ of p with the variables arranged as in the application above. By elementarity, we may assume that $\overline{a} = \overline{b}\overline{c}^{\alpha_0} \dots \overline{c}^{\alpha_{k-1}}$. If $v_n \in X$ there is nothing to prove, so say $v_n \in Z_{\alpha_{k-1}}$ and let $\overline{d} = \overline{c}^{\alpha_{k-1}} \setminus \{a_n\}$. From our inductive hypothesis, $\{\overline{c}^{\alpha_i} : i < k-1\} \cup \{\overline{d}\}$ is independent over $\overline{b}$. In particular, $\overline{d} \underset{\overline{b}}{\downarrow} \{\overline{c}^{\alpha_i} : i < k-1\}$. However, $\text{tp}(a_n/A_n) \in S^+(A_n, A_{u_n})$ and $A_{u_n} \subseteq \overline{b} \cup \overline{d}$, so $\text{tp}(a_n/A_n)$ does not fork over $\overline{b} \cup \overline{d}$. Hence, $\{\overline{c}^{\alpha_i} : i < k\}$ is independent over $\overline{b}$ by the transitivity of non-forking. ∎

Lemma 3.8 *Assume that $p, q_1, q_2 \in \mathcal{P}$, $p \leq_{\mathcal{P}} q_1$, $p \leq_{\mathcal{P}} q_2$ and var$(q_1) \cap$* var$(q_2) =$ var(p). *Then there is an upper bound $p^* \in \mathcal{P}$ of both q_1 and q_2.*

Proof. Say $|\text{var}(p)| = n_0$. Let s be any PM-construction for p and, using Lemma 3.4, let $t_1 = \langle (a_i, u_i, v_i) : i < n_1 \rangle$ and $t_2 = \langle (b_i, u_i', v_i') : i < n_2 \rangle$ be PM-constructions for q_1, q_2 respectively, each continuing s. We form a PM-construction t^* by concatenating a 'copy' of $t_2 \setminus s$ to t_1. More formally, let $\overline{d} = \langle a_i : i < n_0 \rangle$ and for each k, $n_0 \leq k < n_2$, let $u_k'' = (u_k' \cap n_0) \cup \{j + (n_1 - n_0) : j \in u_k' \cap (n_2 \setminus n_0)\}$. Using the Extendibility Lemma, we can successively find a sequence $\langle c_k : n_0 \leq k < n_2 \rangle$ such that $t^* = t_1 {}^\frown \langle (c_k, u_k'', v_k') : n_0 \leq k < n_2 \rangle$ is a PM-construction and $\text{tp}(\overline{d}\overline{c}) = q_2$. Let p^* be the $\mathcal{V}$-type generated by t^*. Visibly, $q_1 \leq_{\mathcal{P}} p^*$. That $q_2 \leq_{\mathcal{P}} p^*$ follows from Lemma 3.6 by taking the permissible permutation of t^* exchanging $t_1 \setminus s$ and the copy of $t_2 \setminus s$. ∎

By using the full strength of the Extendibility Lemma, using the notation in the proof above, if E is an equivalence relation with finitely many classes, we may further require that $E(v_i, v_j') \in p^*$ if and only if $\mathfrak{C} \models E(a_i, b_j)$. This improvement will be crucial in the proof of Claim 3 of Lemma 4.3.

A partially ordered set $\mathcal{P}$ has the *Knaster condition* if, given any uncountable subset X of $\mathcal{P}$, one can find an uncountable $Y \subseteq X$ such that any two elements of Y are compatible. Evidently, if a partially ordered set has the Knaster condition, then it satisfies the countable chain condition (c.c.c.). However, in contrast to the case for c.c.c. posets, it is routine to check that the product of two posets with the Knaster condition must have the Knaster condition.

Lemma 3.9 *$(\mathcal{P}, \leq_{\mathcal{P}})$ satisfies the Knaster condition, hence $\mathcal{P} \times \mathcal{P}$ satisfies the countable chain condition.*

We begin with a combinatorial lemma that is of independent interest. It is not claimed to be new, but the authors know of no published reference.

Lemma 3.10 *There is a partition of $[\omega_1]^{<\omega}$ into $\bigcup\{A_i : i \in \omega\}$ such that $c \cap d$ is an initial segment of c whenever $i \in \omega$ and $c, d \in A_i$.*

Proof. Clearly it suffices to partition each $[\omega_1]^l$, so fix $l \in \omega$. We define three families of functions. First, for each $\beta \in \omega_1$, choose an injective function $g_\beta : \beta \to \omega$. Next, for each $n \in \omega$, define a partial function $f_n : \omega_1 \to \omega_1$ by $f_n(\beta) = g_\beta^{-1}(n)$. Note that if $f_n(\beta)$ is defined, then it is less than β. So define a (total) function $h_n : \omega_1 \to \omega$, where $h_n(\beta)$ is the least m such that the m-fold composition $f_n^{(m)}(\beta)$ is undefined.

Define an equivalence relation $\sim$ on $[\omega_1]^l$ by putting $\{\alpha_0, \ldots, \alpha_{l-1}\} \sim \{\beta_0, \ldots, \beta_{l-1}\}$ if and only if, for all $i < j < l$ there is an $n \in \omega$ such that $n = g_{\alpha_j}(\alpha_i) = g_{\beta_j}(\beta_i)$ and $h_n(\alpha_j) = h_n(\beta_j)$. Clearly, $\sim$ partitions $[\omega_1]^l$ into countably many classes, so let each A_i denote a $\sim$-class.

To see that this works, suppose $c = \{\alpha_0, \ldots, \alpha_{l-1}\} \sim d = \{\beta_0 \ldots, \beta_{l-1}\}$. We first observe that if $\alpha_i = \beta_j$ then $i = j$. For, if not, we could assume by symmetry that $i < j$. Let $n = g_{\alpha_j}(\alpha_i) = g_{\beta_j}(\beta_i)$. Now, $f_n(\alpha_j) = \alpha_i = \beta_j$, hence $h_n(\alpha_j) = h_n(\beta_j) + 1$, contradicting $c \sim d$. Next, suppose $\alpha_j = \beta_j$ for some $j < l$ and fix $i < j$. As $c \sim d$, $g_{\alpha_j}(\alpha_i) = g_{\beta_j}(\beta_i)$, so $\alpha_i = \beta_i$ as g_{α_j} is injective. Hence $c \cap d$ is an initial segment of both c and d. ∎

Proof of Lemma 3.9. As notation, for each $p \in \mathcal{P}$, let u^p denote the (finite) set of all α such that $\mathrm{var}(p) \cap Z_\alpha \neq \emptyset$. Given f a permutation of ω_1,

f induces a permutation of $\mathcal{V}$ (also called f), defined by $f(x_m) = x_m$ and $f(z_m^\alpha) = z_m^{f(\alpha)}$. This f induces a permutation of $\mathcal{P}$, where

$$f(p) = \{\phi(f(v_0), \ldots, f(v_{n-1})) : \phi(v_0, \ldots, v_{n-1}) \in p\}.$$

From the lemma above and the fact that $D(T)$ is countable, it is easy to find a partition $\mathcal{P} = \bigcup\{P_n : n \in \omega\}$ such that, for each $n \in \omega$ and each $p, q \in P_n$,

1. $|u^p| = |u^q|$;
2. $u^p \cap u^q$ is an initial segment of both p and q;
3. if f is any permutation of ω_1 fixing $u^p \cap u^q$ pointwise and $f''(u^p) = u^q$, then $f(p) = q$.

We claim that every pair $p, q \in P_n$ are compatible. For, let $w = u^p \cap u^q$ and let $\mathcal{V}_w = X \cup \bigcup\{Z_\alpha : \alpha \in w\}$. By Clause (3), $\mathrm{var}(p) \cap \mathcal{V}_w = \mathrm{var}(q) \cap \mathcal{V}_w$. Further, letting $p_0 = p \restriction (\mathrm{var}(p) \cap \mathcal{V}_w)$, it follows from Clause (3) and the standard form following Lemma 3.6 that $p_0 \leq_{\mathcal{P}} p$ and $p_0 \leq_{\mathcal{P}} q$. Thus, by Lemma 3.8, p and q are compatible. ∎

We next define our second forcing notion, $\mathcal{R}$. We begin by defining a dense suborder of $\mathcal{R}$. The intuition behind a faithful triple (p, q, h) is that p and q are finite approximations to models of T and h is an elementary map between the approximations.

Definition 3.11 A triple (p, q, h) is *faithful* if $p, q \in \mathcal{P}$ and $h : \mathrm{var}(p) \to \mathrm{var}(q)$ satisfy:

1. h is onto;
2. $\phi(\overline{v}) \in p$ if and only if $\phi(h(\overline{v})) \in q$ for all formulas $\phi(\overline{x})$;
3. for $v \in \mathrm{var}(p)$, $v \in X$ if and only if $h(v) \in X$;
4. for $v \in \mathrm{var}(p)$, $\alpha \in \omega_1$, $v \in Z_\alpha$ if and only if $h(v) \in Z_\alpha$.

Lemma 3.12 *Suppose (p, q, h) is faithful and $p \leq_{\mathcal{P}} p'$. There is $q' \geq_{\mathcal{P}} q$ and $h' \supseteq h$ such that (p', q', h') is faithful. Further, for any finite $F \subseteq \mathcal{V}$, we may assume* $\mathrm{var}(q') \cap F \subseteq \mathrm{var}(q)$.

Proof. Let $m = |\mathrm{var}(p)|$. Arguing by induction on the size of the difference, we may assume that $|\mathrm{var}(p')| = m + 1$. Let $s' = \langle (a_i, u_i, v_i) : i \leq m \rangle$ be a PM-construction of p' such that $s = s' \restriction m$ is a PM-construction of p and let $t = \langle (b_j, u_j, w_j) : j < m \rangle$ be a PM-construction of q. Our h induces a map $h^* : A_m \to B_m$ by putting $h^*(a_i) = b_j$, where $h(v_i) = w_j$. As (p, q, h) is faithful, h^* is elementary. Let $p_m = \mathrm{tp}(a_m/A_m)$ and let $q_m = h^*(p_m)$. By elementarity, q_m is a non-algebraic 1-type over B_m. Pick $w_m \in \mathcal{V} \setminus (\mathrm{var}(q) \cup F)$ such that $w_m \in X$ if $v_m \in X$ and $w_m \in Z_\alpha$, where $v_m \in Z_\alpha$, otherwise. By Lemma 3.5, there are b_m and u_m such that $t' = t^\frown\langle (b_m, u_m, w_m) \rangle$ is a PM-construction and $\mathrm{tp}(b_m/B_m) = q_m$. Thus, (p', q', h') is faithful, where $q' = \mathrm{tp}(t')$ and $h' = h \cup \{(v_m, w_m)\}$. ∎

Definition 3.13 $\mathcal{R} = \{(p, q, h) :$ there are $p_1 \leq_{\mathcal{P}} p$ and $q_1 \leq_{\mathcal{P}} q$ such that (p_1, q_1, h) is faithful$\}$. Define a preorder $\leq_0$ on $\mathcal{R}$ by $(p, q, h) \leq_0 (p', q', h')$ if and only if either $p \leq p'$, $q \leq q'$ and $h = h'$; or $p = p'$, $q = q'$ $h \subseteq h'$ and (p, q, h') is faithful. Let $\leq_{\mathcal{R}}$ be the transitive closure of $\leq_{\mathcal{R}}$. It is clear that $\leq_{\mathcal{R}}$ is a partial order on $\mathcal{R}$.

Lemma 3.14 *The set of faithful triples is a dense suborder of $\mathcal{R}$.*

Proof. Pick $(p, q, h) \in \mathcal{R}$ and assume that $h : \mathrm{var}(p_1) \to \mathrm{var}(q_1)$, where $p_1 \leq_{\mathcal{P}} p$ and $q_1 \leq_{\mathcal{P}} q$. By Lemma 3.12 there is $q_2 \geq_{\mathcal{P}} q_1$ with $\mathrm{var}(q_2) \cap \mathrm{var}(q) = \mathrm{var}(q_1)$ and $h_2 \supseteq h$ such that (p, q_2, h_2) is faithful. By Lemma 3.8 there is an upper bound $q^* \in \mathcal{P}$ of both q_2 and q. Now consider the triple (p, q^*, h_2). By Lemma 3.12 again (with the roles of p and q reversed) there is $p^* \geq_{\mathcal{P}} p$ and $h^* \supseteq h_2$ such that (p^*, q^*, h^*) is faithful. Also, $(p, q, h) \leq_0 (p^*, q^*, h) \leq_0 (p^*, q^*, h^*)$, so $(p, q, h) \leq_{\mathcal{R}} (p^*, q^*, h^*)$. ∎

Lemma 3.15 *The natural embedding $i : \mathcal{P} \times \mathcal{P} \to \mathcal{R}$, defined by $i(p, q) = (p, q, \emptyset)$ is a complete embedding.*

Proof. Fix a maximal antichain $A \subseteq \mathcal{P} \times \mathcal{P}$. We must show that $i''(A)$ is a maximal antichain in $\mathcal{R}$. So, fix $(p, q, h) \in \mathcal{R}$. Choose an element $(p', q') \in \mathcal{P} \times \mathcal{P}$ that is an upper bound of (p, q) and some $(p_0, q_0) \in A$. By Lemma 3.14, there is a faithful triple $(p^*, q^*, h^*) \geq_{\mathcal{R}} (p', q', h)$. It is easy to check that (p^*, q^*, h^*) is an upper bound of both (p, q, h) and $i(p_0, q_0)$. ∎

4 The main theorem

This section is devoted to proving the following theorem.

Theorem 4.1 *Assume that T is superstable, $|D(T)| = \aleph_0$ and there is a type of infinite multiplicity. There is a c.c.c. partial order $\mathcal{Q}$ such that $\Vdash_{\mathcal{Q}}$ "There are two non-isomorphic, potentially isomorphic models of T."*

Remark 4.2 The forcing $\mathcal{Q}$ will be $\mathcal{P} \times \mathcal{P}$ from the last section, which, in addition to having the c.c.c., satisfies the Knaster condition. The second forcing (i.e., $\mathcal{R}/H$) is almost an element of the ground model V. That is, the forcing $\mathcal{R} \in V$ and for any $\mathcal{Q}$-generic filter H, $\mathcal{R}/H$ will be a partial order forcing the two models isomorphic.

Proof. By Theorem 1.13 of [2], we may assume that in addition, T has NDOP and NOTOP. In particular, prime and minimal models exist over independent trees of models of T. Given an n-type of infinite multiplicity, one can find a finite set $\overline{a}$ and a 1-type $r^* \in S_1(\overline{a})$ of infinite multiplicity. Let T' denote the $L(\overline{a})$-theory of $(\mathfrak{C}, \overline{a})$. Thus, working with T' as our basic theory, r^* is a type over the empty set, so our results from Section 3 apply.

Fix $\mathcal{P}$ and $\mathcal{R}$ from Section 3 and let $\mathcal{Q} = \mathcal{P} \times \mathcal{P}$. By Lemma 3.9 $\mathcal{Q}$ satisfies the c.c.c., and by Lemma 3.15 the natural embedding of $\mathcal{Q}$ into $\mathcal{R}$ is a complete embedding. If H is $\mathcal{Q}$-generic, then R/H embeds naturally into the set of finite partial functions $f : \omega_1 \times \omega \to \omega_1 \times \omega$ that fix the first coordinate. Thus R/H satisfies the c.c.c. In the remainder of the section we show that $\mathcal{Q}$ 'constructs' two non-isomorphic models and that $\mathcal{R}/H$ forces them isomorphic.

We first show that the forcing $\mathcal{P}$ constructs a new model of our theory, i.e., one that is not isomorphic to any structure in the original universe V. Fix a $\mathcal{P}$-generic filter G. We associate a model $\mathfrak{B}^*[G]$ of T with G as follows. First, by applying Lemma 3.5, for every $\overline{v} \subseteq \mathcal{V}$, $\{p \in \mathcal{P} : \overline{v} \subseteq \mathrm{var}(p)\}$ is dense, hence there is a $p \in G$ such that $\overline{v} \subseteq \mathrm{var}(p)$. In addition, as any $p, q \in G$ have a common upper bound, $p \restriction \overline{v} = q \restriction \overline{v}$ for any $\overline{v} \subseteq \mathrm{var}(p) \cap \mathrm{var}(q)$. Let

$$\Gamma_G = \{\phi(\overline{v}) : \overline{v} \subseteq \mathcal{V} \text{ and } \phi(\overline{v}) \in p \text{ for some } p \in G\}.$$

Let $A = A_X \cup \bigcup_{\alpha \in \omega_1} A_\alpha$ be a realization of Γ_G in $\mathfrak{C}$ i.e., for all $\overline{b} \subseteq A$, $\mathfrak{C} \models \phi(\overline{b})$ if and only if $\phi(\overline{v}) \in \Gamma_G$, where $\overline{v}$ is the tuple from $\mathcal{V}$ corresponding to $\bar{b}$. Working inside $\mathfrak{C}$, let $\mathfrak{A}_\emptyset[G] = \mathrm{acl}(A_X)$ and for each $\alpha \in \omega_1$, $\mathfrak{A}_\alpha[G] = \mathrm{acl}(A_X \cup A_\alpha)$.

We first claim that $\mathfrak{A}_\emptyset[G]$ and each $\mathfrak{A}_\alpha[G]$ is an $\aleph_0$-saturated model of T. To see that this holds of $\mathfrak{A}_\emptyset[G]$, note that by Lemma 3.5(1), A_X realizes every non-algebraic 1-type over a finite subset of itself. It is a straightforward exercise to show that this fact, together with $\mathfrak{A}_\emptyset[G] = \mathrm{acl}(A_X)$ implies that $\mathfrak{A}_\emptyset[G]$ is an $\aleph_0$-saturated model of T. The proof for each $\mathfrak{A}_\alpha[G]$ is analogous, using Lemma 3.5(2) to show that $A_X \cup A_\alpha$ realizes every non-algebraic 1-type over a finite subset of itself.

Also, it follows from Lemma 3.7 that $\{\mathfrak{A}_\alpha[G] : \alpha \in \omega_1\}$ is independent over $\mathfrak{A}_\emptyset[G]$. As T satisfies NDOP and NOTOP, we can form a continuous, increasing chain of countable models $\langle \mathfrak{B}_\alpha[G] : \alpha \in \omega_1 \rangle$ such that $\mathfrak{B}_\alpha[G]$ is prime and minimal over $\bigcup_{\beta<\alpha} \mathfrak{A}_\beta[G]$. By replacing the chain by an isomorphic copy, we may assume that the universe B_α of each $\mathfrak{B}_\alpha[G]$ is a countable subset of ω_1 and that $\mathfrak{B}^*[G] = \bigcup_{\alpha\in\omega_1} \mathfrak{B}_\alpha[G]$ has universe ω_1. The crucial fact is that this model $\mathfrak{B}^*[G]$ is not L-isomorphic to any structure in the ground universe.

Lemma 4.3 *In $V[G]$ there is no L-elementary embedding of $\mathfrak{B}^*[G]$ into any model $\mathcal{D} \in V$.*

Proof. Fix $\mathcal{D} \in V$. As $\mathfrak{C}$ is sufficiently saturated, we may assume that $\mathcal{D}$ is an elementary substructure of $\mathfrak{C}$. Assume by way of contradiction that such an embedding f exists. From our assumption above, f is an elementary map between two subsets of $\mathfrak{C}$. Fix a $\mathcal{P}$-name $\tilde{f}$ and a condition $g_0 \in G$ such that

$$g_0 \Vdash \tilde{f} : \tilde{\mathfrak{B}^*}[G] \to D.$$

Also, as $\{\mathfrak{B}_\alpha : \alpha < \omega_1\}$ is a continuous, increasing chain we can find $\mathcal{P}$-names $\tilde{B}_\alpha$ such that $\alpha < \beta$ implies $\tilde{B}_\alpha \subseteq \tilde{B}_\beta$ and $\tilde{B}_\delta = \bigcup\{\tilde{B}_\alpha : \alpha < \delta\}$. Further, for any $\alpha < \omega_1$, since $\mathcal{P}$ satisfies c.c.c. and $\Vdash \tilde{B}_\alpha$ is countable, there is $\beta < \omega_1$ such that $\Vdash \tilde{B}_\alpha \subseteq \beta$. Consequently, we may assume that each $\tilde{B}_\alpha$ is a countable $\mathcal{P}$-name.

For each $\delta \in \omega_1$, let $\mathcal{V}_\delta = X \cup \bigcup_{\alpha<\delta} Z_\alpha$, let $\mathcal{P}_\delta = \{p \in \mathcal{P} : \mathrm{var}(p) \subseteq \mathcal{V}_\delta\}$, and let $G_\delta = G \cap P_\delta$.

Claim 2. For all $\delta \in \omega_1$, the identity map $i : \mathcal{P}_\delta \to \mathcal{P}$ is a complete embedding.

Proof. Let A be a maximal antichain in $\mathcal{P}_\delta$ and let $p \in \mathcal{P}$. Let $p_0 = p \restriction \mathcal{V}_\delta$. As A is maximal, there is $q_0 \in A$ and $q \in \mathcal{P}_\delta$, q an upper bound of both p_0 and q_0. By Lemma 3.8, there is an upper bound of both p and q (hence of q_0). Thus, A is a maximal antichain of $\mathcal{P}$ as well. ∎

Let $\mathcal{P}/G_\delta = \{p \in \mathcal{P} : p \text{ is compatible with each } g \in G_\delta\}$ and let G^*_δ be the $\mathcal{P}/G_\delta$-generic filter induced by G. As the identity is a complete embedding, $V[G] = V[G_\delta][G^*_\delta]$ (see e.g., [4]). It is easily verified that a condition $p \in \mathcal{P}$ is an element of $\mathcal{P}/G_\delta$ if and only if $p \restriction \mathcal{V}_\delta \subseteq \Gamma_{G_\delta}$. Let

$$C = \{\delta < \omega_1 : \tilde{B}_\delta \text{ is a } \mathcal{P}_\delta\text{-name}, \tilde{B}_\delta \subseteq \delta, \text{ for all } \alpha < \delta, \tilde{f(\alpha)} \text{ is a } \mathcal{P}_\delta\text{-name}\}.$$

Visibly, $C \in V$. Using the fact that $\mathcal{P}$ satisfies c.c.c. again, C is a club subset of ω_1. Note that $\mathfrak{B}_\delta[G] \in V[G_\delta]$ and $f \restriction \delta \in V[G_\delta]$ for each $\delta \in C$. Fix an element $\delta \in C$.

Claim 3. Working in $V[G_\delta]$, for each $e \in \mathcal{D}$, the set

$$D^*_e = \{p^* \in \mathcal{P}/G_\delta : p^* \Vdash_{\mathcal{P}/G_\delta} \operatorname{tp}(z^\delta_0, \mathfrak{B}_\delta[G]) \neq \operatorname{tp}(e, f(\mathfrak{B}_\delta[G]))\}$$

is dense in $\mathcal{P}/G_\delta$.

Proof. Fix e and choose $p \in \mathcal{P}/G_\delta$. By Lemma 3.5 and our characterization of $\mathcal{P}/G_\delta$ we may assume $z^\delta_0 \in \operatorname{var}(p)$. Let $p_0 = p \restriction \mathcal{P}_\delta$ and let $m = |\operatorname{var}(p_0)|$. Let $t = \langle (a_i, u_i, v_i) : i < n\rangle$ be a PM-construction of p such that $t_1 =_{\text{def}} t \restriction m$ is a PM-construction of p_0 and $v_m = z^\delta_0$. Let ψ be an automorphism of $\mathfrak{C}$ fixing A_m pointwise such that $\operatorname{stp}(a_m) \neq \operatorname{stp}(\psi(a_m))$. (One exists since $r^* = \operatorname{tp}(a_m/\emptyset)$ has infinite multiplicity and $\operatorname{tp}(a_m/A_m) \in S^+(A_m, \emptyset)$.) Fix a definable equivalence relation E with finitely many classes such that $\mathfrak{C} \models \neg E(a_m, \psi(a_m))$ and pick a set of representatives $\{c_i : i < k\}$ of E's classes from $\mathfrak{B}_\delta[G]$. Say $E(e, f(c_i))$ holds in $\mathfrak{C}$. Choose $g \in G_\delta$ such that $g \Vdash_{\mathcal{P}_\delta} E(e, f(c_i))$ and $\operatorname{var}(p_0) \cup \{c_i\} \subseteq \operatorname{var}(g)$. Let s be a PM-construction of g and suppose $(b, u, c_i) \in s$. We may assume that $\mathfrak{C} \models \neg E(a_m, b)$, since otherwise we could replace a_m by $\psi(a_m)$ in the argument below. As in the proof of Lemma 3.8 (and the remark following the proof) it follows from the Extendibility Lemma that there is a sequence s^* and an element d such that $s^\frown s^*$ is a PM-construction, $\operatorname{tp}(s^*) = \operatorname{tp}(t \setminus t_1)$, $(d, \emptyset, z^\delta_0) \in s^*$, and $\mathfrak{C} \models E(a_m, d)$. Let $p^* = \operatorname{tp}(s^\frown s^*)$. As E is an equivalence relation,

$\mathfrak{C} \models \neg E(d, b)$ so $\neg E(z_0^\delta, c_i) \in p^*$. Further, $p^* \restriction \mathcal{V}_\delta = g$, $p^* \in \mathcal{P}/G_\delta$ as required. ∎

Thus, working in $V[G] = V[G_\delta][G_\delta^*]$, Claim 3 implies that $\mathrm{tp}(z_0^\delta, \mathfrak{B}_\delta[G]) \neq \mathrm{tp}(e, f(\mathfrak{B}_\delta[G]))$ for all $e \in D$, contradicting the elementarity of f. ∎

Continuing with the proof of Theorem 4.1, fix $H = G_1 \times G_2$, a $\mathcal{P} \times \mathcal{P}$-generic filter. Following the procedure above, we can build elementary substructures $\mathfrak{B}^*[G_1]$ and $\mathfrak{B}^*[G_2]$ of $\mathfrak{C}$ in $V[H]$. It follows from Lemma 4.3 and the fact that $V[H] = V[G_1][G_2]$ that there is no L-isomorphism $f : \mathfrak{B}^*[G_1] \to \mathfrak{B}^*[G_2]$ in $V[H]$.

To complete the proof of the theorem, it remains to show that $\mathfrak{B}^*[G_1]$ can be forced isomorphic to $\mathfrak{B}^*[G_2]$ by a c.c.c. forcing. Let $\mathcal{R}/H = \{(p, q, h) \in \mathcal{R} : (p, q, h)$ is compatible with $i(p', q')$ for every $(p', q') \in H\}$. As noted above, $\mathcal{R}/H$ satisfies the c.c.c. We claim that $\mathcal{R}/H$ forces an $L(\overline{a})$ isomorphism between $\mathfrak{B}^*[G_1]$ and $\mathfrak{B}^*[G_2]$. Indeed, let

$$h^* = \bigcup\{h : (p, q, h) \in \mathcal{R}/G_1 \times G_2 \text{ for some } p, q \in \mathcal{P}\}.$$

By Lemma 3.12, h^* is an $L(\overline{a})$-elementary map from a set of realizations of Γ_{G_1} to a set of realizations of Γ_{G_2}. Now h^* easily extends to an $L(\overline{a})$-elementary map of the algebraic closures of these sets. That is, h^* maps the independent tree $\bigcup\{\mathfrak{A}_\alpha[G_1] : \alpha \in \omega_1\}$ of models of T' to the independent tree $\bigcup\{\mathfrak{A}_\alpha[G_2] : \alpha \in \omega_1\}$. As the prime and minimal model of such a tree is unique, h^* extends to an $L(\overline{a})$-isomorphism of $\mathfrak{B}^*[G_1]$ and $\mathfrak{B}^*[G_2]$. ∎

5 Some examples

Our first example demonstrates the ubiquity of the phenomenon of non-isomorphic models becoming isomorphic in a forcing extension. It implies that even very weak forcings such as Cohen forcing are able to alter the isomorphism type of some very simple structures.

Example 5.1 Let $M_1 = (\mathbb{R}^V, \leq)$ and $M_2 = (\mathbb{R}^V \setminus \{0\}, \leq)$. Then M_1 is not isomorphic to M_2 in the ground universe V, but M_1 and M_2 become isomorphic in any transitive $V' \supseteq V$ with $\mathbb{R}^{V'} \neq \mathbb{R}^V$.

Proof. It is clear that M_1 and M_2 are not isomorphic in V. Fix V', a transitive extension of V that adds reals. We will construct an isomorphism $f \in V'$ between M_1 and M_2. Towards this end, first note that as V and V' are both transitive, ω, $\mathbb{Z}$ and $\mathbb{Q}$ are all absolute between V and V'. In particular, $\mathbb{Q}^{V'} = \mathbb{Q}^V$. As $\mathbb{R}^{V'}$ is defined as the set of all Dedekind cuts of rationals, $\mathbb{Q}^V$ is dense in $\mathbb{R}^{V'}$.

Next, for any $a, b \in \mathbb{R}^{V'}$ with $a < b$, fix $\{x_n : n \in \mathbb{Z}\}$, a strictly increasing sequence from (a, b) that is both cofinal and coinitial in (a, b). Using the density of $\mathbb{Q}^V$ in $\mathbb{R}^{V'}$, we may successively choose $y_n \in \mathbb{Q}^V \cap (x_n, x_{n+1})$ to obtain a cofinal, coinitial sequence of order-type $\mathbb{Z}$ in (a, b) with each element in V.

Using, this, we claim that if $a < b$ and $c < d$, then there is an isomorphism $g : (a, b) \cap \mathbb{R}^V \to (c, d) \cap \mathbb{R}^V$. To see this, choose strictly increasing sequences $\langle y_n : n \in \mathbb{Z}\rangle$ and $\langle z_n : n \in \mathbb{Z}\rangle$ from $\mathbb{Q}^V$, cofinal and coinitial in (a, b) and (c, d), respectively. Now, as $(y_n, y_{n+1}) \cap \mathbb{R}^V$ and $(z_n, z_{n+1}) \cap \mathbb{R}^V$ are each open intervals in $\mathbb{R}^V$, there is an isomorphism $g_n \in V$ between them. Piecing these isomorphisms together yields an isomorphism between $(a, b) \cap \mathbb{R}^V$ and $(c, d) \cap \mathbb{R}^V$.

We are now ready to build our isomorphism between M_1 and M_2. Fix $a < b < c$ in $\mathbb{R}^{V'} \setminus \mathbb{R}^V$ with $a < 0 < c$. From the paragraph above, let g_1 be an isomorphism between $(a, b) \cap \mathbb{R}^V$ and $(a, 0) \cap \mathbb{R}^V$ and let g_2 be an isomorphism between $(b, c) \cap \mathbb{R}^V$ and $(0, c) \cap \mathbb{R}^V$. Define $f : M_1 \to M_2$ by

$$f(x) = \begin{cases} x & \text{if } x < a \text{ or } c < x \\ g_1(x) & \text{if } a < x < b \\ g_2(x) & \text{if } b < x < c. \end{cases}$$

The (pseudo-elementary) class $\mathbf{K}_{\mathrm{hom}}$ of homogeneous linear orders is the class of all dense linear orders with no endpoints such that any non-empty open interval is isomorphic to the entire linear order. Examples include $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$. It is well known that forcing preserves satisfaction for models. Thus, the relation "$M \in \mathbf{K}$" is absolute between transitive models of set theory for elementary classes $\mathbf{K}$. Similarly, if $\mathbf{K}$ is a pseudo-elementary class (i.e., a class of reducts of an elementary class) and $M \in \mathbf{K}$ in the ground universe, then $M \in \mathbf{K}$ in any forcing extension. However, Example 5.1 indicates that the converse need not hold. That is, $M_2 \notin \mathbf{K}_{\mathrm{hom}}$ in V, while $M_2 \in \mathbf{K}_{\mathrm{hom}}$ in any transitive $V' \supseteq V$ that adds reals.

The class $\mathbf{K}_{\mathrm{hom}}$ can also be used to show that 'potential isomorphism via c.c.c. forcings' is distinct from 'potential isomorphism via Cohen forcings.' As $\mathbf{K}_{\mathrm{hom}}$ is unstable, it follows from Theorem 1.7 of [2] that there is a pair of non-isomorphic structures in $\mathbf{K}_{\mathrm{hom}}$ that can be forced isomorphic by a c.c.c. forcing. This contrasts with the theorem below.

Theorem 5.2 *Let $\mathcal{Q} = ({}^{<\omega}\omega, \lhd)$ be Cohen forcing. For all $M_1, M_2 \in \mathbf{K}_{\mathrm{hom}}$, $M_1 \cong M_2$ if and only if $\Vdash_{\mathcal{Q}} M_1 \cong M_2$.*

Proof. Right to left is clear by absoluteness. Choose homogeneous linear orders $M_1 = (I_1, \leq)$ and $M_2 = (I_2, \leq)$ such that $\Vdash_{\mathcal{Q}} M_1 \cong M_2$. We will construct an isomorphism $g : M_1 \to M_2$ in the ground universe as a countable union of approximations in the sense described below.

Fix a $\mathcal{Q}$-name $\tilde{f}$ such that $\Vdash_{\mathcal{Q}}$ "$\tilde{f}$ is an isomorphism between M_1 and M_2." For each $q \in \mathcal{Q}$, let $f_q = \{(a, b) \in I_1 \times I_2 : q \Vdash \tilde{f}(a) = b\}$. To ease notation, let $I'_i = I_i \cup \{-\infty, \infty\}$ $(i = 1, 2)$, where $-\infty$ is the smallest element of I'_i and ∞ is the largest. For h a partial 1-1 function from I'_1 to I'_2, let $D_1(h) = \mathrm{dom}(h)$ and $D_2(h) = \mathrm{dom}(h^{-1})$.

An *approximation on* $[x_0, x_1]$ is a partial, order-preserving function $g : [x_0, x_1] \to I'_2$ such that, for each $a \in [x_0, x_1] \setminus D_i(g)$ there are $b, c \in D_i(g)$ with $b < a < c$ and $(b, c) \cap \mathcal{D}_i(g) = \emptyset$. If $[x_0, x_1] = I'_1$, g is simply called an approximation.

Trivially, $g_0 = \{(-\infty, -\infty), (\infty, \infty)\}$ is an approximation. As noted above, we will construct an increasing sequence $\langle g_n : n \in \omega\rangle$ of approximations such that for each $q \in \mathcal{Q}$ there is $n \in \omega$ such that $D_i(g_n) \supseteq D_i(f_q)$ $(i = 1, 2)$. Once we build such a sequence, $g = \bigcup g_n$ will be an isomorphism between I'_1 and I'_2 since every $a \in I_1$ is in $D_i(f_q)$ for some $q \in \mathcal{Q}$. Thus, all that remains is to prove the following claim.

Claim. For every approximation g and $q \in \mathcal{Q}$ there is an approximation $g' \supseteq q$ with $D_i(g') \supseteq D_i(f_q)$, $i = 1, 2$.

Proof. Fix an approximation g and $q \in \mathcal{Q}$. By symmetry it suffices to find $g' \supseteq g$ with $D_1(g') \supseteq D_1(f_q)$. As $D_1(g)$ partitions I_1 into convex sets, we may independently find approximations g' on $[x_0, x_1]$ extending $g \restriction [x_0, x_1]$ for each pair $x_0, x_1 \in D_1(g)$ with $x_0 < x_1$ and $(x_0, x_1) \cap D_1(g) = \emptyset$. So fix such a pair (x_0, x_1). Choose $p \geq q$ such that $x_0, x_1 \in D_1(f_p)$. Say $p \Vdash \tilde{f}(x_0) = y_0$ and $\tilde{f}(x_1) = y_1$. As $M_2 \in \mathbf{K}_{\mathrm{hom}}$, it suffices to find an approximation $h : [x_0, x_1] \to [y_0, y_1]$ with $D_1(h) \supseteq D_1(f_q)$, since then $k \circ h$ would be an approximation extending g for any order-preserving isomorphism $k : I_2 \restriction (y_0, y_1) \to I_2 \restriction (g(x_0), g(x_1))$.

For $a \in (x_0, x_1)$, let $PV(a)$ denote the set of possible values of $\tilde{f}(a)$, i.e., the set of all $b \in (y_0, y_1)$ such that $r \Vdash \tilde{f}(a) = b$ for some $r \geq p$. If $a < a'$ then as $\Vdash \tilde{f}(a) < \tilde{f}(a')$, there always exist elements $b \in PV(a)$ and $b' \in PV(a')$ such that $b < b'$. By contrast, we say $PV(a)$ and $PV(a')$ *overlap* if there are $c \in PV(a)$ and $c' \in PV(a')$ such that $c' \leq c$. Let the symmetric relation $R(a, a')$ hold if $PV(a)$ and $PV(a')$ overlap. It is easy to verify that the set of elements R-related to a is a convex subset of (x_0, x_1). Let $\sim$ be the transitive closure of R. For notation, let $[a] = \{a' \in (x_0, x_1) : a \sim a'\}$. Each $[a]$ is convex. Similarly, for $b \in (y_0, y_1)$, let $PV(b) = \{a \in (x_0, x_1) : f_p(a) = b\}$. Define the relation R on (y_0, y_1) and $[b]$ analogously. Note that if $b, c \in PV(a)$, then as $a \in PV(b) \cap PV(c)$, $[b] = [c]$. As each of the equivalence classes are convex, it follows that if $R(a, a')$ holds and $b \in PV(a)$, $b' \in PV(a')$ then $[b] = [b']$. Thus, for all $a \in (x_0, x_1)$ and all $b \in PV(a)$, $p \Vdash \tilde{f} : [a] \to [b]$. It is easy to see that if $a \in \mathrm{dom}(f_p)$ then $[a] = \{a\}$. On the other hand,

Subclaim. If $a \notin \mathrm{dom}(f_p)$ then there is a strictly increasing, cofinal and coinitial sequence $\langle a_n : n \in \mathbb{Z} \rangle$ in $[a]$.

Proof. Suppose $a \notin \mathrm{dom}(f_p)$. We first claim that there is an $a' > a$, $a' \in [a]$. To see this, pick distinct elements $b_1, b_2 \in PV(a)$ with $b_1 < b_2$. Pick $r \geq p$ with $r \Vdash \tilde{f}(a) = b_1$. Pick $s \geq r$ with $b_2 \in \mathrm{dom}(f_2^{-1})$ and let $a' = f_s^{-1}(b_2)$. Then $a < a'$ and $b_2 \in PV(a) \cap PV(a')$, so $a \sim a'$. Similarly, there is $a' < a$ with $a' \in [a]$. By symmetry, to complete the proof of the subclaim we need only show that there is no strictly increasing sequence $\langle a_\alpha : \alpha \in \omega_1 \rangle$ in $[a]$. By way of contradiction, assume that such a sequence exists. For each $\alpha \in \omega_1$, let $A_\alpha = (x_0, a_\alpha)$ and let $B_\alpha = (y_0, b_\alpha)$. As $\mathcal{Q}$ is countable, $PV(a')$ is countable for all a', hence there is a club $C \subseteq \omega_1$ such that, for all $\delta \in C$, $a' \in A_\delta$ implies $PV(a') \subseteq B_\delta$ and $b' \in B_\delta$ implies $PV(b') \subseteq A_\delta$. Thus, $p \Vdash \tilde{f} : A_\delta \to B_\delta$ for $\delta \in C$, contradicting the definition of $[a]$. ∎

Note that by symmetry, if $b \notin \mathrm{dom}(f_p^{-1})$ then there is a strictly increasing, cofinal and coinitial sequence of order type $\mathbb{Z}$ in $[b]$. We build our function $h : (x_0, x_1) \to (y_0, y_1)$ as follows: Let $h(a) = f_p(a)$ for each $a \in \mathrm{dom}(f_p)$. For each non-trivial equivalence class $[a]$, let $b \in PV(a)$ and choose strictly increasing, cofinal and coinitial sequences $\langle a_n : n \in \mathbb{Z}\rangle$ and $\langle b_n : n \in \mathbb{Z}\rangle$ in $[a]$ and $[b]$, respectively. Let $h(a_n) = b_n$ for each $n \in \mathbb{Z}$. It is easy to verify that h is an approximation on $[x_0, x_1]$. ∎

We close with the following example that shows that the assumption of $D(T)$ countable in Theorem 4.1 cannot be weakened.

Example 5.3 There is a countable, superstable theory with a complete type of infinite multiplicity, yet non-isomorphism of models of T is preserved under all cardinal-preserving forcings.

Let T be the theory of countably many binary splitting, cross-cutting equivalence relations. That is $L = \{E_n : n \in \omega\}$ and the axioms of T state that:

1. Each E_n is an equivalence relation with two classes, each infinite and
2. For each $n \in \omega$ and $w \subseteq n$, $\forall x \exists y (\bigwedge_{i\in w} E_i(x,y) \wedge \bigwedge_{i\in n\setminus w} \neg E_i(x,y))$.

T admits elimination of quantifiers, is superstable and the unique 1-type has infinite multiplicity. However, for any model M of T and any $a \in M$, every $p \in S_1(\{a\})$ is stationary.

Further, it is easy to verify that for all models M, N of T and all $a \in M$, $b \in N$, there is an isomorphism $g : M \to N$ with $g(a) = b$ if and only if for all 2-types $p(x,y) \in S_2(\emptyset)$, $|p(M,a)| = |p(N,b)|$.

Now assume that $\Vdash_Q M \cong N$ for some cardinal-preserving forcing Q. Then for some $q \in Q$, some $a \in M$ and some $b \in N$,

$$q \Vdash \text{"for all } p \in S_2(\emptyset),\ |p(M,a)| = |p(N,b)|.\text{"}$$

As Q is cardinal preserving, this implies that $|p(M,a)| = |p(N,b)|$ for all $p \in S_2(\emptyset)$, so $M \cong N$.

References

[1] J.T. Baldwin. *Fundamentals of Stability Theory.* Springer-Verlag, 1988.

[2] J.T. Baldwin, M.C. Laskowski, and S. Shelah. Forcing isomorphism. *Journal of Symbolic Logic*, 58, 1993.

[3] J. Barwise. Back and forth through infinitary logic. In M. Morley, editor, *Studies in Model Theory*, pages 5–34. Mathematical Association of America, 1973.

[4] K. Kunen. *Set Theory*. North-Holland, 1980.

[5] D. Lascar. *Stability in Model Theory*. Longman, 1987. originally published in French as Stabilité en Théorie des Modèles (1986).

[6] M. Makkai. Survey of basic stability with particular emphasis on orthogonality and regular types. *Israel Journal of Mathematics*, 1984.

[7] L. Nachbin. *The Haar Integral*. van Nostrand, 1965.

[8] S. Shelah. Existence of many $L_{\infty,\lambda}$-equivalent non-isomorphic models of T of power λ. *Annals of Pure and Applied Logic*, 34, 1987.

[9] S. Shelah. *Classification Theory*. North-Holland, 1991.