CONSTRUCTING STRONGLY EQUIVALENT NONISOMORPHIC MODELS FOR UNSUPERSTABLE THEORIES, PART B

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Abstract

We study how equivalent nonisomorphic models of unsuperstable theories can be. We measure the equivalence by Ehrenfeucht-Fraisse games. This paper continues [HS].

1. Introduction

In [HT] we started the studies of so called strong nonstructure theorems. By strong nonstructure theorem we mean a theorem which says that if a theory belongs to some class of theories then it has very equivalent nonisomorphic models. Usually the equivalence is measured by the length of the Ehrenfeucht-Fraisse games (see Definition 2.2) in which $\exists$ has a winning strategy. These theorems are called nonstructure theorems because intuitively the models must be complicated if they are very equivalent but still nonisomorphic. Also structure theorems usually imply that a certain degree of equivalence gives isomorphism (see f.ex. [Sh1] (Chapter XIII)).

In [HT] we studied mainly unstable theories. We also looked unsuperstable theories but we were not able to say much if the equivalence is measured by the length of the Ehrenfeucht-Fraisse games in which $\exists$ has a winning strategy. In this paper we make a new attempt to study the unsuperstable case.

The main result of this paper is the following: if $\lambda = \mu^+$, $cf(\mu) = \mu$, $\kappa = cf(\kappa) < \mu$, $\lambda^{<\kappa} = \lambda$, $\mu^\kappa = \mu$ and T is an unsuperstable theory, $|T| \le \lambda$ and $\kappa(T) > \kappa$, then there are models $\mathcal{A}$, $\mathcal{B} \models T$ of cardinality λ such that

$$\mathcal{A} \equiv^{\lambda}_{\mu\times\kappa} \mathcal{B} \ \ \textit{and} \ \ \mathcal{A} \not\cong \mathcal{B}.$$

In [HS] we proved this theorem in a special case.

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FromTheorem 4.4 in [HS] we get the following theorem easily: Let T_c be the canonical example of unsuperstable theories i.e. $T_c = Th((^{\omega}\omega, E_i)_{i<\omega})$ where $\eta E_i \xi$ iff for all $j \leq i$, $\eta(j) = \xi(j)$.

1.1 Theorem. *([HS]) Let $\lambda = \mu^+$ and I_0 and I_1 be models of T_c of cardinality λ. Assume $\lambda \in I[\lambda]$. Then*

$$I_0 \equiv^{\lambda}_{\mu\times\omega+2} I_1 \quad\Leftrightarrow\quad I_0 \cong I_1.$$

So the main result of Chapter 3 is essentially the best possible.

In the introduction of [HT] there is more background for strong nonstructure theorems.

2. Basic definitions

In this chapter we define the basic concepts we shall use and construct two linear orders needed in Chapter 3.

2.1 Definition. *Let λ be a cardinal and α an ordinal. Let t be a tree (i.e. for all $x \in t$, the set $\{y \in t |\ y < x\}$ is well-ordered by the ordering of t). If $x, y \in t$ and $\{z \in t \mid z < x\} = \{z \in t \mid z < y\}$, then we denote $x \sim y$, and the equivalence class of x for $\sim$ we denote $[x]$. By a λ,α-tree t we mean a tree which satisfies:*

(i) $|[x]| < \lambda$ for every $x \in t$;

(ii) there are no branches of length $\geq \alpha$ in t;

(iii) t has a unique root;

(iv) if $x, y \in t$, x and y have no immediate predecessors and $x \sim y$, then $x = y$.

Note that in a λ,α-tree each ascending sequence of a limit length has at most one supremum.

2.2 Definition. *Let t be a tree and κ a cardinal. The Ehrenfeucht-Fraisse game of length t between models $\mathcal{A}$ and $\mathcal{B}$, $G^{\kappa}_t(\mathcal{A},\mathcal{B})$, is the following. At each move α:*

(i) player $\forall$ chooses $x_\alpha \in t$, $\kappa_\alpha < \kappa$ and either $a^\beta_\alpha \in \mathcal{A}$, $\beta < \kappa_\alpha$ or $b^\beta_\alpha \in \mathcal{B}$, $\beta < \kappa_\alpha$, we will denote this sequence by X_α;

(ii) if $\forall$ chose from $\mathcal{A}$ then $\exists$ chooses $b^\beta_\alpha \in \mathcal{B}$, $\beta < \kappa_\alpha$, else $\exists$ chooses $a^\beta_\alpha \in \mathcal{A}$, $\beta < \kappa_\alpha$, we will denote this sequence by Y_α.

$\forall$ must move so that $(x_\beta)_{\beta\leq\alpha}$ form a strictly increasing sequence in t. $\exists$ must move so that $\{(a^\beta_\gamma, b^\beta_\gamma) | \gamma \leq \alpha, \beta < \kappa_\gamma\}$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$. The player who first has to break the rules loses.

We write $\mathcal{A} \equiv^{\kappa}_t \mathcal{B}$ if $\exists$ has a winning strategy for $G^{\kappa}_t(\mathcal{A},\mathcal{B})$.

2.3 Definition. *Let t and t' be trees.*

(i) If $x \in t$, then $pred(x)$ denotes the sequence $(x_\alpha)_{\alpha<\beta}$ of the predecessors of x, excluding x itself, ordered by $<$. Alternatively, we consider $pred(x)$ as a set. The notation $succ(x)$ denotes the set of immediate successors of x. If $x, y \in t$ and there is z, such that $x, y \in succ(z)$, then we say that x and y are brothers.

(ii) By $t^{<\alpha}$ we mean the set

$$\{x \in t| \text{ the order type of } pred(x) \text{ is } < \alpha\}.$$

Similarly we define $t^{\leq\alpha}$.

(iii) The sum $t \oplus t'$ is defined as the disjoint union of t and t', except that the roots are identified.

2.4 Definition. *Let ρ_i, $i < \alpha$, ρ and θ be linear orders.*

(i) We define the ordering $\rho \times \theta$ as follows: the domain of $\rho \times \theta$ is $\{(x, y)| \ x \in \rho, \ y \in \theta\}$, and the ordering in $\rho \times \theta$ is defined by last differences, i.e., each point in θ is replaced by a copy of ρ;

(ii) We define the ordering $\rho + \theta$ as follows: The domain of $\rho + \theta$ is $(\{0\} \times \rho) \cup (\{1\} \times \theta)$ and the ordering in $\rho + \theta$ is defined by the first difference i.e. $(i, x) < (j, y)$ iff $i < j$ or $i = j$ and $x < y$.

(iii) We define the ordering $\sum_{i<\alpha} \rho_i$ as follows: The domain of $\sum_{i<\alpha} \rho_i$ is $\{(i, x)| \ i \in \alpha, x \in \rho_i\}$ and the ordering in $\sum_{i<\alpha} \rho_i$ is defined by the first difference i.e. $(i, x) < (j, y)$ iff $i < j$ or $i = j$ and $x < y$.

2.5 Definition. *We define generalized Ehrenfeucht-Mostowski models (E-M-models for short). Let K be a class of models we call index models. In this definition the notation $tp_{at}(\overline{x}, A, \mathcal{A})$ means the atomic type of $\overline{x}$ over A in the model $\mathcal{A}$.*

Let Φ be a function. We say that Φ is proper for K, if there is a vocabulary τ_1 and for each $I \in K$ a model $\mathbf{M}_1$ and tuples $\overline{a}_s$, $s \in I$, of elements of $\mathbf{M}_1$, such that:

(i) each element in $\mathbf{M}_1$ is an interpretation of some $\mu(\overline{a}_{\overline{s}})$, where μ is a τ_1-term;

(ii) $tp_{at}(\overline{a}_{\overline{s}}, \emptyset, \mathbf{M}_1) = \Phi(tp_{at}(\overline{s}, \emptyset, I))$.

Here $\overline{s} = (s_0, ..., s_n)$ denotes a tuple of elements of I and $\overline{a}_{\overline{s}}$ denotes $\overline{a}_{s_0} \frown \cdots \frown \overline{a}_{s_n}$.

Note that if $\mathbf{M}_1$, $\overline{a}_s$, $s \in I$, and $\mathbf{M}_1'$, $\overline{a}_s'$, $s \in I$, satisfy the conditions above, then there is a canonical isomorphism $\mathbf{M}_1 \cong \mathbf{M}_1'$ which takes $\mu(\overline{a}_{\overline{s}})$ in $\mathbf{M}_1$ to $\mu(\overline{a}_{\overline{s}}')$ in $\mathbf{M}_1'$. Therefore we may assume below that $\mathbf{M}_1$ and $\overline{a}_s$, $s \in I$, are unique for each I. We denote this unique $\mathbf{M}_1$ by $EM^1(I, \Phi)$ and call it an Ehrenfeucht-Mostowski model. The tuples $\overline{a}_s$, $s \in I$, are the generating elements of $EM^1(I, \Phi)$, and the indexed set $(\overline{a}_s)_{s \in I}$ is the skeleton of $EM^1(I, \Phi)$.

Note that if

$$tp_{at}(\overline{s}_1, \emptyset, I) = tp_{at}(\overline{s}_2, \emptyset, J),$$

then

$$tp_{at}(\overline{a}_{\overline{s}_1}, \emptyset, EM^1(I, \Phi)) = tp_{at}(\overline{a}_{\overline{s}_2}, \emptyset, EM^1(J, \Phi)).$$

2.6 Definition. *Let θ be a linear order and κ infinite regular cardinal. Let $K^\kappa_{\rm tr}(\theta)$ be the class of models of the form*

$$I = (M, <, \ll, H, P_\alpha)_{\alpha \leq \kappa},$$

where $M \subseteq \theta^{\leq\kappa}$ and:

(i) M is closed under initial segments;

(ii) $<$ *denotes the initial segment relation;*
(iii) $H(\eta, \nu)$ *is the maximal common initial segment of* η *and* ν*;*
(iv) $P_\alpha = \{\eta \in M \mid length(\eta) = \alpha\}$*;*
(v) $\eta \ll \nu$ *iff either* $\eta < \nu$ *or there is* $n < \kappa$ *such that* $\eta(n) < \nu(n)$ *and* $\eta \restriction n = \nu \restriction n$*.*

Let $K^\kappa_{\rm tr} = \bigcup\{K^\kappa_{\rm tr}(\theta) \mid \theta$ *a linear order* $\}$*.*

If $I \in K^\kappa_{\rm tr}(\theta)$ *and* $\eta, \nu \in I$*, we define* $\eta <_s \nu$ *iff* η *and* ν *are brothers and* $\eta < \nu$*. But we do not put* $<_s$ *to the vocabulary of* I*.*

Thus the models in $K^\kappa_{\rm tr}$ are lexically ordered trees of height $\kappa+1$ from which we have removed the relation $<_s$ and where we have added relations indicating the levels and a function giving the maximal common predecessor.

The following theorem gives us means to construct for T E-M-models such that the models of $K^\kappa_{\rm tr}$ act as index models. Furthermore the properties of the models of $K^\kappa_{\rm tr}$ are reflected to these E-M-models.

2.7 Theorem. *([Sh1]). Suppose* $\tau \subseteq \tau_1$*,* T *is a complete* τ*-theory,* T_1 *is a complete* τ_1*-theory with Skolem functions and* $T \subseteq T_1$*. Suppose further that* T *is unsuperstable,* $\kappa(T) > \kappa$ *and* $\phi_n(\overline{x}, \overline{y}_n)$*,* $n < \kappa$*, witness this. (The definition of witnessing is not needed in this paper. See [Sh1].)*

Then there is a function Φ*, which is proper for* $K^\kappa_{\rm tr}$*, such that for every* $I \in K^\kappa_{\rm tr}$*,* $EM^1(I, \Phi)$ *is a* τ_1*-model of* T_1*, for all* $\eta \in I$*,* $\overline{a}_\eta$ *is finite and for* $\eta, \xi \in P^I_n, \nu \in P^I_\kappa$*,*
(i) if $I \models \eta < \nu$*, then* $EM^1(I, \Phi) \models \phi_n(\overline{a}_\nu, \overline{a}_\eta)$*;*
(ii) if η *and* ξ *are brothers and* $\eta < \nu$ *then* $\xi = \eta$ *iff* $EM^1(I, \Phi) \models \phi_n(\overline{a}_\xi, \overline{a}_\nu)$*.*
□

Above $\phi_n(\overline{x}, \overline{y}_n)$ is a first-order τ-formula. We denote the reduct

$$EM^1(I, \Phi) \restriction \tau$$

by $EM(I, \Phi)$. In order to simplify the notation, instead of $\overline{a}_\eta$, we just write η. It will be clear from the context, whether η means $\overline{a}_\eta$ or η.

Next we construct two linear orders needed in the next chapter. The first of these constructions is a modification of a linear order construction in [Hu] (Chapter 9).

2.8 Definition. *Let* γ *be an ordinal closed under ordinal addition and let* $\theta_\gamma = ({}^{<\omega}\gamma, <)$*, where* $<$ *is defined by* $x < y$ *iff*
(i) y *is an initial segment of* x
or
(ii) there is $n < min\{length(x), length(y)\}$ *such that* $x \restriction n = y \restriction n$ *and* $x(n) < y(n)$*.*

2.9 Lemma. *Assume* γ *in an ordinal closed under ordinal addition. Let* $x \in \theta_\gamma$*,* $length(x) = n < \omega$ *and* $\alpha < \gamma$*. Let* A^α_x *be the set of all elements* y *of* θ_γ *which satisfy:*
(i) x *is an initial segment of* y *(not necessarily proper);*
(ii) if $length(y) > n$ *then* $y(n) \geq \alpha$*.*
Then $(A^\alpha_x, < \restriction A^\alpha_x) \cong \theta_\gamma$*.*

Proof. Follows immediately from the definition of θ_γ. □

If $\alpha \leq \beta$ are ordinals then by $(\alpha, \beta]$ we mean the unique ordinal order isomorphic to

$$\{\delta | \ \alpha < \delta \leq \beta\} \cup \{\delta | \ \delta = \alpha \text{ and limit}\}$$

together with the natural ordering. Notice that if $(\alpha_i)_{i<\delta}$ is strictly increasing continuous sequence of ordinals, $\alpha_0 = 0$, $\beta = sup_{i<\delta}\alpha_i$ and for all successor $i < \delta$, α_i is successor, then $\sum_{i<\delta}(\theta \times (\alpha_i, \alpha_{i+1}]) \cong \theta \times \beta$, for all linear-orderings θ.

2.10 Lemma. *Let γ be an ordinal closed under ordinal addition and not a cardinal.*

(i) Let $\alpha < \gamma$ be an ordinal. Then

$$\theta_\gamma \cong \theta_\gamma \times (\alpha + 1).$$

(ii) Let $\alpha < \beta < |\gamma|^+$. Then

$$\theta_\gamma \cong \theta_\gamma \times (\alpha, \beta].$$

Proof. (i) For all $i < \alpha$ we let $x_i = (i)$. Then by the definition of θ_γ,

$$\theta_\gamma \cong (\sum_{i<\alpha} A^0_{x_i}) + A^\alpha_{()},$$

where by $()$ we mean the empty sequence. By Lemma 2.9

$$(\sum_{i<\alpha} A^0_{x_i}) + A^\alpha_{()} \cong \theta_\gamma \times (\alpha + 1).$$

(ii) We prove this by induction on β. For $\beta = 1$ the claim follows from (i). Assume we have proved the claim for $\beta < \beta'$ and we prove it for β'. If $\beta' = \delta + 1$, then by induction assumption

$$\theta_\gamma \cong \theta_\gamma \times (\alpha, \delta]$$

and so

$$\theta_\gamma \times (\alpha, \delta + 1] \cong \theta_\gamma + \theta_\gamma \cong \theta_\gamma$$

by (i).

If β' is limit, then we choose a strictly increasing continuous sequence of ordinals $(\beta_i)_{i<cf(\beta')}$, so that $\beta_0 = \alpha$, $sup_{i<cf(\beta')}\beta_i = \beta'$ and for all successor $i < cf(\beta')$, β_i is successor. Then

$$\theta_\gamma \times (\alpha, \beta'] \cong \sum_{i<cf(\beta')} (\theta_\gamma \times (\beta_i, \beta_{i+1}]) + \theta_\gamma.$$

By induction assumption

$$\sum_{i<cf(\beta')} (\theta_\gamma \times (\beta_i, \beta_{i+1}]) + \theta_\gamma \cong \theta_\gamma \times (cf(\beta') + 1).$$

Because γ is not a cardinal, $cf(\beta') < \gamma$ and so by (i)

$$\theta_\gamma \times (cf(\beta') + 1) \cong \theta_\gamma.$$

□

2.11 Corollary. *Let γ be an ordinal closed under ordinal addition and not a cardinal. If $\alpha < |\gamma|^+$ is a successor ordinal then $\theta_\gamma \cong \theta_\gamma \times \alpha$.*

Proof. Follows immediately from Lemma 2.10 (ii). □

2.12 Lemma. *Assume μ is a regular cardinal and $\lambda = \mu^+$. Then there are linear order θ of power λ, one-one and onto function $h : \theta \to \lambda \times \theta$ and order isomorphisms $g_\alpha : \theta \to \theta$ for $\alpha < \lambda$ such that the following holds:*
(i) if $g_\alpha(x) = y$ then $x \neq y$ and either
(a) $h(x) = (\alpha, y)$
or
(b) $h(y) = (\alpha, x)$
but not both,
(ii) if for some $x \in \theta$, $g_\alpha(x) = g_{\alpha'}(x)$ then $\alpha = \alpha'$,
(iii) if $h(x) = (\alpha, y)$ then $g_\alpha(x) = y$ or $g_\alpha(y) = x$.

Proof. Let the universe of θ be $\mu \times \lambda$. The ordering will be defined by induction. Let

$$f : \lambda \to \lambda \times \lambda$$

be one-one, onto and if $\alpha < \alpha'$, $f(\alpha) = (\beta, \gamma)$ and $f(\alpha') = (\beta', \gamma')$ then $\gamma < \gamma'$. This f is used only to guarantee that in the induction we pay attention to every $\beta < \lambda$ cofinally often.

By induction on $\alpha < \lambda$ we do the following: Let $f(\alpha) = (\beta, \gamma)$. We define $\theta^\alpha = (\mu \times (\alpha + 1), <^\alpha)$, $h^\alpha : \theta^\alpha \to \lambda \times \theta^\alpha$ and order isomorphisms (in the ordering $<^\alpha$)

$$g^\alpha_\beta : \theta^\alpha \to \theta^\alpha$$

so that
(i) if $\alpha < \alpha'$ then $h^\alpha \subseteq h^{\alpha'}$ and $<^\alpha \subseteq <^{\alpha'}$,
(ii) if $\alpha < \alpha'$, $f(\alpha) = (\beta, \gamma)$ and $f(\alpha') = (\beta, \gamma')$ then $g^\alpha_\beta \subseteq g^{\alpha'}_\beta$,
(iii) if $g^\alpha_\beta(x) = y$ then $x \neq y$ and either
(a) $h^\alpha(x) = (\beta, y)$
or
(b) $h^\alpha(y) = (\beta, x)$
but not both.

The induction is easy since at each stage we have μ "new" elements to use: Let $B \subseteq \mu \times \alpha$ be the set of those element from $\mu \times \alpha$ which are not in the domain of any $g^{\alpha'}_\beta$ such that $\alpha' < \alpha$ and $f(\alpha') = (\beta, \gamma')$ for some γ'. (Notice that B is also the set of those element from $\mu \times \alpha$ which are not in the range of any $g^{\alpha'}_\beta$ such that $\alpha' < \alpha$ and $f(\alpha') = (\beta, \gamma')$ for some γ'.) Clearly if $B \neq \emptyset$ then $|B| = \mu$.

Let A_i, $i \in \mathbf{Z}$, be a partition of $\mu \times \{\alpha\}$ into sets of power μ. We first define g^α_β so that the following is true:
(a) g^α_β is one-one,
(b) if $B \neq \emptyset$ then $g^\alpha_\beta \restriction A_0$ is onto B otherwise $g^\alpha_\beta \restriction A_0$ is onto A_{-1},
(c) if $B \neq \emptyset$ then $g^\alpha_\beta \restriction B$ is onto A_{-1},
(d) for all $i \neq 0$, $g^\alpha_\beta \restriction A_i$ is onto A_{i-1}.

By an easy induction on $|i| < \omega$ we can define $<^\alpha$ so that $<^{\alpha'} \subseteq <^\alpha$ for all $\alpha' < \alpha$ and g^α_β is an order isomorphism. We define the function $h^\alpha \restriction (\mu \times \{\alpha\})$ as follows:

(a) if $B = \emptyset$ then $h^\alpha(x) = (\beta, g^\alpha_\beta(x))$,

(b) if $B \neq \emptyset$ and $i \geq 0$ and $x \in A_i$ then $h^\alpha(x) = (\beta, g^\alpha_\beta(x))$,

(c) if $B \neq \emptyset$ and $i < 0$ and $x \in A_i$ then $h^\alpha(x) = (\beta, y)$ where $y \in A_{i+1}$ or B is the unique element such that $g^\alpha_\beta(y) = x$.

It is easy to see that (iii) above is satisfied.

We define $\theta = (\mu \times \lambda, <)$, where $< = \bigcup_{\alpha<\lambda} <^\alpha$, $h = \bigcup_{\alpha<\lambda} h^\alpha$ and for all $\beta < \lambda$ we let $g_\beta = \bigcup\{g^\alpha_\beta |\ \alpha < \lambda,\ f(\alpha) = (\beta, \gamma)$ for some $\gamma\}$. Clearly these satisfy (i). (ii) follows from the fact that if $g^\alpha_\beta(x) = y$ then either $x \in \mu \times \{\alpha\}$ and $y \in \mu \times (\alpha+1)$ or $y \in \mu \times \{\alpha\}$ and $x \in \mu \times (\alpha+1)$. (iii) follows immediately from the definition of h. □

3. On nonstructure of unsuperstable theories

In this chapter we will prove the main theorem of this paper i.e. Conclusion 3.19. The idea of the proof continues III Claim 7.8 in [Sh2]. Throughout this chapter we assume that T is an unsuperstable theory, $|T| < \lambda$ and $\kappa(T) > \kappa$. The cardinal assumptions are: $\lambda = \mu^+$, $cf(\mu) = \mu$, $\kappa = cf(\kappa) < \mu$, $\lambda^{<\kappa} = \lambda$, $\mu^\kappa = \mu$.

If $i < \kappa$ we say that i is of type n, $n = 0, 1, 2$, if there are a limit ordinal $\alpha < \kappa$ and $k < \omega$ such that $i = \alpha + 3k + n$.

We define linear orderings θ_n, $n < 3$, as follows. Let $\theta_0 = \lambda$ and θ_1, h' and g_α, $\alpha < \lambda$, as θ, h and g_α in Lemma 2.12. Let $\theta_2 = \theta_{\mu\times\omega} \times \lambda$, where $\theta_{\mu\times\omega}$ is as in Definition 2.8.

For $n < 2$, let J^-_n be the set of sequences η of length $< \kappa$ such that

(i) $\eta \neq ()$;

(ii) $\eta(0) = n$;

(iii) if $0 < i < length(\eta)$ is of type $m < 3$ then $\eta(i) \in \theta_m$.

Let

$$f : (\lambda - \{0\}) \to \{(\eta, \xi) \in J^-_0 \times J^-_1 \mid length(\eta) = length(\xi) \text{ is of type } 1\}$$

be one-one and onto. Then we define

$$h : \theta_1 \to J^-_0 \cup J^-_1$$

and order isomorphisms

$$g_{\eta,\xi} : succ(\eta) \to succ(\xi),$$

for $(\eta, \xi) \in rng(f)$, as follows:

(i) $g_{\eta,\xi}(\eta \frown (x)) = \xi \frown (g_\alpha(x))$, where α is the unique ordinal such that $f(\alpha) = (\eta, \xi)$;

(ii) Assume $h'(x) = (\alpha, y)$, $\alpha \neq 0$, and $f(\alpha) = (\eta, \xi)$. Then $h(x) = \xi \frown (y)$ if $g_\alpha(x) = y$ otherwise $h(x) = \eta \frown (y)$. If $h'(x) = (0, y)$ then $h(x) = (0)$ (here the idea is to define $h(x)$ so that $length(h(x))$ is not of type 2).

3.1 Lemma. *Assume $\eta \in J_0^-$ and $\xi \in J_1^-$ are such that $m = length(\eta) = length(\xi)$ is of type 2. Let $m = n+1$. If $g_{\eta,\xi}(\eta') = \xi'$ then either*
(a) $h(\eta'(n)) = \xi'$
or
(b) $h(\xi'(n)) = \eta'$
but not both.

Proof. We show first that either (a) or (b) holds. So we assume that (a) is not true and prove that (b) holds. Let $\eta'(n) = x$, $\xi'(n) = y$ and $f(\alpha) = (\eta, \xi)$. Now $g_\alpha(x) = y$, $x \neq y$ and either $h'(x) = (\alpha, y)$ or $h'(y) = (\alpha, x)$. Because (a) is not true $h'(x) \neq (\alpha, y)$ and so $h'(y) = (\alpha, x)$. We have two cases:

(i) Case $y > x$: Because g_α is order-precerving, $g_\alpha(y) > y > x$. So $g_\alpha(y) \neq x$ and by the definition of h, $h(y) = \eta \frown (x) = \eta'$.

(ii) Case $y < x$: As the case $y > x$.

Next we show that it is impossible that both (a) and (b) holds. For a contradiction assume that this is not the case. Then (a) implies that there is β such that $h'(x) = (\beta, y)$ and $g_\beta(x) = y$. On the other hand (b) implies that there is γ such that $h'(y) = (\gamma, x)$ and $g_\gamma(y) \neq x$. By Lemma 2.12 (iii), $g_\gamma(x) = y$. By Lemma 2.12 (ii) $\beta = \gamma$. So $h'(y) = (\beta, x)$ and $h'(x) = (\beta, y)$, which contradicts Lemma 2.12 (i). □

For $n < 2$, let J_n^+ be the set of sequences η of length $\leq \kappa$ such that
(i) $\eta \neq ()$;
(ii) $\eta(0) = n$;
(iii) if $0 < i < length(\eta)$ is of type $m < 3$ then $\eta(i) \in \theta_m$.

Let $e : \theta_1 \to \lambda$ be one-one and onto. We define functions s and d as follows: if $i < length(\eta)$ is of type 0 then $d(\eta, i) = \eta(i)$ and $s(\eta, i) = \eta(i)$, if $i < length(\eta)$ is of type 1 then $d(\eta, i) = \eta(i)$ and $s(\eta, i) = e(\eta(i))$ and if $i < length(\eta)$ is of type 2 and $\eta(i) = (d, s)$ then $d(\eta, i) = d$ and $s(\eta, i) = s$.

For $n < 2$ and $\gamma < \lambda$, we define

$$J_n^+(\gamma) = \{\eta \in J_n^+ |\ \text{for all } i < length(\eta),\ s(\eta, i) < \gamma\},$$

$J_n^-(\gamma) = J_n^+(\gamma) \cap J_n^-$.

Let us fix $d \in \theta_1$ so that $h(d) = (0)$.

3.2 Definition. *For all $\eta \in J_0^-$ and $\xi \in J_1^-$ such that $n = length(\eta) = length(\xi)$ is of type 1, let $\alpha(\eta, \xi)$ be the set of ordinals $\alpha < \lambda$ such that for all $\eta' \in succ(\eta)$, $s(\eta', n) < \alpha$ iff $s(g_{\eta,\xi}(\eta'), n) < \alpha$ and $e(d) < \alpha$. Notice that $\alpha(\eta, \xi)$ is a closed and unbounded subset of λ. By $\alpha(\beta)$, $\beta < \lambda$, we mean*

$$Min \bigcap \{\alpha(\eta, \xi) |\ \eta \in J_0^-(\beta),\ \xi \in J_1^-(\beta),\ length(\eta) = length(\xi) \text{ is of type } 1\}.$$

3.3 Definition. *For all $\eta \in J_0^+$ and $\xi \in J_1^+$, we write $\eta R^- \xi$ and $\xi R^- \eta$ iff*
(i) $\eta(j) = \xi(j)$ for all $0 < j < min\{length(\eta), length(\xi)\}$ of type 0;
(ii) for all $j < min\{length(\eta), length(\xi)\}$ of type 1 $\xi \restriction (j+1) = g_{\eta \restriction j, \xi \restriction j}(\eta \restriction (j+1))$.

Let $length(\eta) = length(\xi) = j+1$, j of type 1, and $\eta R^- \xi$. We write $\eta \to \xi$ if $h(\eta(j)) = \xi$. We write $\xi \to \eta$ if $h(\xi(j)) = \eta$.

3.4 Remark. *If $\xi \to \eta$ and $\xi \to \eta'$ then $\eta = \eta'$ and if $\eta R^- \xi$ then $\eta \to \xi$ or $\xi \to \eta$ but not both.*

3.5 Definition. *Let $\eta \in J_0^+ - J_0^-$ and $\xi \in J_1^+ - J_1^-$. We write $\eta R \xi$ and $\xi R \eta$ iff*

(i) $\eta R^- \xi$;

(ii) for every $j < \kappa$ of type 2, η and ξ satisfy the following: if $\eta \restriction j \to \xi \restriction j$ then $s(\eta, j) \le s(\xi, j)$ and if $\xi \restriction j \to \eta \restriction j$ then $s(\xi, j) \le s(\eta, j)$;

(iii) the set $W^\kappa_{\eta,\xi}$ is bounded in κ, where $W^\kappa_{\eta,\xi}$ is defined in the following way: Let $\eta \in J_0^+ - J_0^{<\delta}$ (see Definition 2.3 (ii)) and $\xi \in J_1^+ - J_1^{<\delta}$ then

$$W^\delta_{\eta,\xi} = W^\delta_{\xi,\eta} = V^\delta_{\eta,\xi} \cup U^\delta_{\eta,\xi},$$

where

$$V^\delta_{\eta,\xi} = \{j < \delta |\ j \text{ is of type 2 and } \xi \restriction j \to \eta \restriction j \text{ and } cf(s(\eta, j)) = \mu \text{ and } s(\xi, j) = s(\eta, j)\}$$

and

$$U^\delta_{\eta,\xi} = \{j < \delta |\ j \text{ is of type 2 and } \eta \restriction j \to \xi \restriction j \text{ and } cf(s(\xi, j)) = \mu \text{ and } s(\eta, j) = s(\xi, j)\}.$$

Our next goal is to prove that if J_0 and J_1 are such that

(i) $J_n^- \subseteq J_n \subseteq J_n^+$, $n = 0, 1$ and

(ii) if $\eta \in J_0^+$, $\xi \in J_1^+$ and $\eta R \xi$ then $\eta \in J_0$ iff $\xi \in J_1$,

then $(J_0, <, <_s) \equiv^\lambda_{\mu\times\kappa} (J_1, <, <_s)$, where $<$ is the initial segment relation and $<_s$ is the union of natural orderings of $succ(\eta)$ for all elements η of the model. Fromnow on in this chapter we assume that J_0 and J_1 satisfy (i) and (ii) above.

The relation R designed not only to guarantee the equivalence but also to make it possible to prove that the final models are not isomorphic. Here (iii) in the definition of R plays a vital role. The pressing down elements η such that $cf(s(\eta, i)) = \mu$, i of type 2, in (iii) prevents us from adding too many elements to $J_n - J_n^-$, $n < 2$.

For $n < 2$, we write $J_n(\gamma) = J_n^+(\gamma) \cap J_n$.

3.6 Definition. *Let $\alpha < \kappa$. G_α is the family of all partial functions f satisfying:*

(a) f is a partial isomorphism from J_0 to J_1;

(b) $dom(f)$ and $rng(f)$ are closed under initial segments and for some $\beta < \lambda$ they are included in $J_0(\beta)$ and $J_1(\beta)$, respectively;

(c) if $f(\eta) = \xi$ then $\eta R^- \xi$;

(d) if $\eta \in J_0^+$, $\xi \in J_1^+$, $f(\eta) = \xi$ and $j < length(\eta)$ of type 2, then η and ξ satisfy the following: if $\eta \restriction j \to \xi \restriction j$ then $s(\eta, j) \le s(\xi, j)$ and if $\xi \restriction j \to \eta \restriction j$ then $s(\xi, j) \le s(\eta, j)$;

(e) assume $\eta \in J_0^+ - J_0^{<\delta}$ and $\{\eta \restriction \gamma |\ \gamma < \delta\} \subseteq dom(f)$ and let

$$\xi = \bigcup_{\gamma<\delta} f(\eta \restriction \gamma),$$

then $W^{\delta}_{\eta,\xi}$ has order type $\le \alpha$;

(f) if $\eta \in dom(f)$ and $length(\eta)$ is of type 2 then

$$\{i < \lambda | \text{ for all } d \in \theta_2,\ \eta \frown ((d,i)) \in dom(f)\} =$$

$$\{i < \lambda | \text{ for some } d \in \theta_2,\ \eta \frown ((d,i)) \in dom(f)\} =$$

$$\{i < \lambda | \text{ for all } d \in \theta_2,\ f(\eta) \frown ((d,i)) \in rng(f)\} =$$

$$\{i < \lambda | \text{ for some } d \in \theta_2,\ f(\eta) \frown ((d,i)) \in rng(f)\}$$

is an ordinal.

We define $F_\alpha \subseteq G_\alpha$ by replacing (f) above by

(f') if $\eta \in dom(f)$ and $length(\eta)$ is of type 2 then

$$\{i < \lambda | \text{ for all } d \in \theta_2,\ \eta \frown ((d,i)) \in dom(f)\} =$$

$$\{i < \lambda | \text{ for some } d \in \theta_2,\ \eta \frown ((d,i)) \in dom(f)\} =$$

$$\{i < \lambda | \text{ for all } d \in \theta_2,\ f(\eta) \frown ((d,i)) \in rng(f)\} =$$

$$\{i < \lambda | \text{ for some } d \in \theta_2,\ f(\eta) \frown ((d,i)) \in rng(f)\}$$

is an ordinal and of cofinality $< \mu$.

The idea in the definition above is roughly the following: If $f \in G_\alpha$ and $f(\eta) = \xi$ then $\eta R \xi$ and the order type of $W^{\delta}_{\eta,\xi}$ is $\le \alpha$. If $f \in F_\alpha$ then not only $f \in G_\alpha$ but f is such that for all small $A \subset J_0 \cup J_1$ we can find $g \supset f$ such that $A \subset dom(g) \cup rng(g)$ and $g \in F_\alpha$.

3.7 Definition. *For $f, g \in G_\alpha$ we write $f \le g$ if $f \subseteq g$ and if $\gamma < \delta \le \kappa$, $\eta \in J_0^+ - J_0^{<\delta}$, $\eta \restriction \gamma \in dom(f)$, $\eta \restriction (\gamma+1) \notin dom(f)$, $\eta \restriction j \in dom(g)$ for all $j < \delta$ and $\xi = \bigcup_{j<\delta} g(\eta \restriction j)$, then $W^{\gamma}_{\eta,\xi} = W^{\delta}_{\eta,\xi}$.*

Notice that $f \le g$ is a transitive relation.

3.8 Remark. *Let $f \in G_\alpha$. We define $\overline{f} \supseteq f$ by*

$$dom(\overline{f}) = dom(f) \cup \{\eta \in J_0 | \ \eta \restriction \gamma \in dom(f) \text{ for all } \gamma < length(\eta)$$

$$\text{and } length(\eta) \text{ is limit}\}$$

and if $\eta \in dom(\overline{f}) - dom(f)$ then

$$\overline{f}(\eta) = \bigcup_{\gamma < length(\eta)} f(\eta \restriction \gamma).$$

If $f \in F_\alpha$ then $\overline{f} \in F_\alpha$ and if $f \in G_\alpha$ then $\overline{f} \in G_\alpha$.

3.9 Lemma. *Assume $\alpha < \kappa$, $\delta \le \mu$, $f_i \in F_\alpha$ for all $i < \delta$ and $f_i \le f_j$ for all $i < j < \delta$.*

(i) $\bigcup_{i<\delta} f_i \in G_\alpha$.

(ii) If $\delta < \mu$ then $\bigcup_{i<\delta} f_i \in F_\alpha$ and $f_j \le \bigcup_{i<\delta} f_i$ for all $j \le \delta$.

Proof. (i) We have to check that $f=\bigcup_{i<\delta} f_i$ satisfies (a)-(f) in Definition 3.6. Excluding purhapse (e), all of these are trivial.

Without loss of generality we may assume δ is a limit ordinal. So assume $\eta \in J_0^+ - J_0^{<\beta}$ and $\{\eta \restriction \gamma |\ \gamma<\beta\} \subseteq dom(f)$ and let

$$\xi = \bigcup_{\gamma<\beta} f(\eta \restriction \gamma).$$

We need to show that $W^\beta_{\eta,\xi} \leq \alpha$.

If there is $i<\delta$ such that $\eta \restriction \gamma \in dom(f_i)$ for all $\gamma<\beta$ then the claim follows immediately from the assumption $f_i \in F_\alpha$. Otherwise for all $\gamma<\beta$ we let $i_\gamma<\delta$ be the least ordinal such that $\eta\restriction\gamma \in dom(f_{i_\gamma})$. Let $\gamma^*<\beta$ be the least ordinal such that $i_{\gamma^*+1} > i_{\gamma^*}$. Because for all $\gamma<\beta$, $f_{i_\gamma} \in F_\alpha$, we get $W^\gamma_{\eta\restriction\gamma,\xi\restriction\gamma}$ has order type $\leq\alpha$. If $\gamma^*<\gamma'<\beta$ then $f_{i_{\gamma^*}} \leq f_{i_{\gamma'}}$ and so $W^{\gamma^*}_{\eta\restriction\gamma^*,\xi\restriction\gamma^*} = W^{\gamma'}_{\eta\restriction\gamma',\xi\restriction\gamma'}$. Because $W^\beta_{\eta,\xi} = \bigcup_{\gamma<\beta} W^\gamma_{\eta\restriction\gamma,\xi\restriction\gamma}$, we get $W^\beta_{\eta,\xi}\leq\alpha$.

(ii) As (i), just check the definitions. □

3.10 Lemma. *If $\delta<\kappa$, $f_i \in G_i$ for all $i<\delta$ and $f_i \subseteq f_j$ for all $i<j<\delta$ then*

$$\bigcup_{i<\delta} f_i \in G_\delta.$$

Proof. Follows immediately from the definitions. □

3.11 Lemma. *If $f\in F_\alpha$ and $A\subseteq J_0\cup J_1$, $|A|<\lambda$, then there is $g\in F_\alpha$ such that $f\leq g$ and $A\subseteq dom(g)\cup rng(g)$.*

Proof. We may assume that A is closed under initial segments. Let $A' = A\cap(J_0^-\cup J_1^-)$. We enumerate $A'=\{a_i |\ 0<i<\mu\}$ so that if a_i is an initial segment of a_j then $i<j$. Let $\gamma<\lambda$ be such that $A\cup dom(f)\cup rng(f)\subseteq J_0(\gamma)\cup J_1(\gamma)$. By induction on $i<\mu$ we define functions g_i.

If $i=0$ we define $g_i = f\cup\{((0),(1))\}$.

If $i<\mu$ is limit then we define

$$g_i = \overline{\bigcup_{j<i} g_j}.$$

If $i=j+1$ then there are two different cases. For simplicity we assume $a_i\in J_0$.

(i) $n = length(a_i)$ is of type 0 or 1: Then we choose g_i to be such that

(a) $g_j \leq g_i$;

(b) $g_i\in F_\alpha$;

(c) if $\xi\in dom(g_i)-dom(g_j)$ then $\xi\in succ(a_i)$;

(d) if $\xi\in succ(a_i)$ and $s(\xi,n)<\gamma$ then $\xi\in dom(g_i)$;

(e) if $\xi\in succ(g_j(a_i))$ and $s(\xi,n)<\gamma$ then $\xi\in rng(g_i)$.

Trivially such g_i exists.

(ii) $n = length(a_j)$ is of type 2: Then we choose g_i to be such that (a)-(c) above and (d')-(f') below are satisfied.

Let

$$\beta = sup\{i+1<\lambda \mid for\ all\ d \in \theta_2,\ a_i \frown ((d,i)) \in dom(g_j)\}.$$

(d') if $\xi \in succ(a_i)$ then $s(\xi,n) < \gamma+2$ iff $\xi \in dom(g_i)$;
(e') if $\xi \in succ(g_j(a_i))$ then $s(\xi,n) < \gamma+2$ iff $\xi \in rng(g_i)$;
(f') $g_i \restriction \{\eta \in succ(a_i) \mid \beta \leq s(\eta,n) < \gamma+1\}$ is an order isomorphism to $\{\eta \in succ(g_j(a_i)) \mid \beta \leq s(\eta,n) < \beta+1\}$ and $g_i \restriction \{\eta \in succ(a_i) \mid \gamma+1 \leq s(\eta,n) < \gamma+2\}$ is an order isomorphism to $\{\eta \in succ(g_j(a_i)) \mid \beta+1 \leq s(\eta,n) < \gamma+2\}$.
By Corollary 2.11 it is easy to satisfy (d')-(f'). Because $g_j \in F_\alpha$, $cf(\beta) < \mu$ and we do not have problems with (a) and (b). So there is g_i satisfying (a)-(c) and (d')-(f').

Finally we define

$$g = \overline{\bigcup_{i<\mu} g_i}.$$

It is easy to see that g is as wanted (notice that $f \leq g$ follows from the construction, not from Lemma 3.9). □

3.12 Lemma. *If $f \in G_\alpha$ and $A \subseteq J_0 \cup J_1$, $|A| < \lambda$, then there is $g \in F_{\alpha+1}$ such that $f \subseteq g$ and $A \subseteq dom(g) \cup rng(g)$.*

Proof. Essentially as the proof of Lemma 3.11. □

3.13 Theorem. *If J_0 and J_1 are such that*
(i) $J_n^- \subseteq J_n \subseteq J_n^+$, $n = 0,1$ and
(ii) if $\eta R \xi$, $\eta \in J_0^+$ and $\xi \in J_1^+$ then $\eta \in J_0$ iff $\xi \in J_1$,
then $(J_0, <, <_s) \equiv^\lambda_{\mu\times\kappa} (J_1, <, <_s)$.

Proof. Because $\emptyset \in F_0$, the theorem follows from the previous lemmas. □

3.14 Corollary. *If J_0 and J_1 are as above and Φ is proper for T, then*

$$EM(J_0,\Phi) \equiv^\lambda_{\mu\times\kappa} EM(J_1,\Phi).$$

Proof. Follows immediately from the definition of E-M-models and Theorem 3.13. □

In the rest of this chapter we show that there are trees J_0 and J_1 which satisfy the assumptions of Corollary 3.14 and

$$EM(J_0,\Phi) \not\cong EM(J_1,\Phi).$$

3.15 Lemma. *(Claim 7.8B [Sh2]) There are closed increasing cofinal sequences $(\alpha_i)_{i<\kappa}$ in α, $\alpha < \lambda$ and $cf(\alpha) = \kappa$, such that if i is successor then $cf(\alpha_i) = \mu$ and for all cub $A \subseteq \lambda$ the set*

$$\{\alpha < \lambda \mid cf(\alpha) = \kappa \text{ and } \{\alpha_i \mid i < \kappa\} \subseteq A \cap \alpha\ \}$$

is stationary.

We define $J_0 - J_0^-$ and $J_1 - J_1^-$ by using Lemma 3.15. For all $\alpha < \lambda$ we define I_0^α and I_1^α. Let $I_0^0 = J_0^-$ and $I_1^0 = J_1^-$. If $0 < \alpha < \lambda$, $cf(\alpha) = \kappa$, and there are sequence $(\beta_i)_{i<\kappa}$ and $\eta \in J_0^+ - J_0^-$ such that

(i) $(\beta_i)_{i<\kappa}$ is properly increasing and cofinal in α;

(ii) for all $i < \kappa$, $cf(\beta_{i+1}) = \mu$, $\beta_{i+1} > \alpha(\beta_i)$ and $\beta_i \in \{\alpha_i | \ i < \kappa\}$;

(iii) for all $0 < i < \kappa$ of type 0 or 2, $s(\eta, i) = \beta_i$;

(iv) for all $i < \kappa$ of type 1, $\eta(i) = d$;

then we choose some such η, let it be η_α, and define I_0^α and I_1^α to be the least sets such that

(i) $\{\eta_\alpha\} \cup \bigcup_{\beta<\alpha} I_0^\beta \subseteq I_0^\alpha$ and $\bigcup_{\beta<\alpha} I_1^\beta \subseteq I_1^\alpha$

(ii) $I_0^\alpha \cup I_1^\alpha$ is closed under R.

Otherwise we let $I_0^\alpha = \bigcup_{\beta<\alpha} I_0^\beta$ and $I_1^\alpha = \bigcup_{\beta<\alpha} I_1^\beta$. Finally we define $J_0 = \bigcup_{\alpha<\lambda} I_0^\alpha$ and $J_1 = \bigcup_{\alpha<\lambda} I_1^\alpha$.

3.16 Lemma. *For all $\alpha < \lambda$ and $\eta \in (J_0 \cup J_1) - (J_0^- \cup J_1^-)$, the following are equivalent:*

(i) $\eta \in (I_0^\alpha \cup I_1^\alpha) - (\bigcup_{\beta<\alpha} I_0^\beta \cup \bigcup_{\beta<\alpha} I_1^\beta)$.

(ii) $sup\{s(\eta, i) | \ i < \kappa\} = \alpha$.

Proof. By the construction it is enough to show that (i) implies (ii). So assume (i). Because of levels of type 0, it is enough to show that for all $i < \kappa$, $s(\eta, i) < \beta_{i+1}$. We prove this by induction on $i < \kappa$. If i is of type 0, the claim is clear. If i is of type 1 this follows from $\beta_{i+1} > \alpha(\beta_i)$ and $e(d) < \alpha(\beta_i)$ together with the induction assumption. For i is of type 2, $i = j+1$, it is enough to show that $s(\eta_\alpha, i) \geq s(\eta, i)$. This follows easily from the fact that $\eta_\alpha(j) = d$ and $length(h(d)) \neq i$. □

3.17 Definition. *Let $g : EM(J_0, \Phi) \to EM(J_1, \Phi)$ be an isomorphism. We say that $\alpha < \lambda$ is g-saturated iff for all $\eta \in J_0$ and $\xi_0, ..., \xi_n \in J_1$ the following holds: if*

(i) $length(\eta) = l + 1$ and for all $i < l$, $s(\eta, i) < \alpha$;

(ii) for all $k \leq n$ and $i < length(\xi_k)$, $s(\xi_k, i) < \alpha$;

(iii) $g(\eta) = t(\delta_0, ..., \delta_m)$, for some term t and $\delta_0, ..., \delta_m \in J_1$;

then there are $\eta' \in J_0$ and $\delta'_0, ..., \delta'_n \in J_1$ such that

(a) $g(\eta') = t(\delta'_0, ..., \delta'_m)$;

(b) $length(\eta') = l + 1$ and $\eta' \restriction l = \eta \restriction l$;

(c) $s(\eta', l) < \alpha$;

(d) the basic type of $(\xi_0, ..., \xi_n, \delta_0, ..., \delta_m)$ in $(J_1, <, \ll, H, P_j)$ is the same as the basic type of $(\xi_0, ..., \xi_n, \delta'_0, ..., \delta'_m)$.

Notice that for all isomorphisms $g : EM(J_0, \Phi) \to EM(J_1, \Phi)$ the set of g-saturated ordinals is unbounded in λ and closed under increasing sequences of length $\alpha < \lambda$ if $cf(\alpha) > \kappa$.

3.18 Lemma. *Let Φ be proper for T. Then*

$$EM(J_0, \Phi) \not\cong EM(J_1, \Phi).$$

Proof. We write $\mathcal{A}_\gamma$ for the submodel of $EM(J_0, \Phi)$ generated (in the extended language) by $J_0(\gamma)$. Similarly, we write $\mathcal{B}_\gamma$ for the submodel of $EM(J_1, \Phi)$ generated by $J_1(\gamma)$. Let g be an one-one function from $EM(J_0, \Phi)$ onto $EM(J_1, \Phi)$. We say that g is closed in γ, if $\mathcal{A}_\gamma \cup \mathcal{B}_\gamma$ is closed under g and g^{-1}.

For a contradiction we assume that g is an isomorphism from $EM(J_0, \Phi)$ to $EM(J_1, \Phi)$. By Lemma 3.15 we choose $\alpha < \lambda$ to be such that

(i) $cf(\alpha) = \kappa$, for all $i < \kappa$, g is closed in α_i and for all $i < \kappa$, $cf(\alpha_{i+1}) = \mu$ and α_{i+1} is g-saturated;

(ii) there are sequence $(\beta_i)_{i<\kappa}$ and $\eta = \eta_\alpha \in J_0 - J_0^-$ satisfying (i)-(iv) in the definition of $(J_0 - J_0^-) \cup (J_1 - J_1^-)$.

Let $g(\eta) = t(\xi_0, ..., \xi_n)$, $\xi_0, ..., \xi_n \in J_1$. Now for all $k \leq n$, either $\xi_k \in J_1(\beta_i)$ for some $i < \kappa$ or there is $j < \kappa$ such that $s(\xi_k, j) \geq \alpha$ or $length(\xi_k) = \kappa$, $sup\{s(\xi_k, j) | \ j < \kappa\} = \alpha$ and for all $j < \kappa$, $s(\xi_k, j) < \alpha$. By Lemma 3.16, in the last case ξ_k has been put to J_1 at stage α.

We choose $i < \kappa$ so that

(a) i is of type 2 and > 2;

(b) for all $k < l \leq n$, $\xi_k \restriction i \neq \xi_l \restriction i$;

(c) for all $k \leq n$, if $length(\xi_k) = \kappa$, $sup\{s(\xi_k, j) | \ j < \kappa\} = \alpha$ and for all $j < \kappa$, $s(\xi_k, j) < \alpha$ then there are $\rho_0, ..., \rho_r \in J_0 \cup J_1$ such that

(i) $\rho_o = \eta$ and $\rho_r = \xi_k$;

(ii) if $p < r$ then $\rho_p R \rho_{p+1}$;

(iii) if $p < r$ then $W^\kappa_{\rho_p, \rho_{p+1}} \subseteq i$;

(iv) for all $p < q \leq r$, $\rho_p \restriction i \neq \rho_q \restriction i$;

(d) for all $k \leq n$, if $\xi_k \in J_1(\beta_j)$ for some $j < \kappa$ then $\xi_k \in J_1(\beta_i)$;

(e) for all $k \leq n$, if $s(\xi_k, j) \geq \alpha$ for some $j < \kappa$ then $\xi_k \restriction j_k \in J_1(\beta_i)$ and $j_k < i$, where $j_k = min\{j < i | \ s(\xi_k, j) \geq \alpha\}$.

Let $l \leq l' \leq n+1$ be such that $\xi_k \in J_1(\beta_i)$ iff $k < l$, $length(\xi_k) = \kappa$, $sup\{s(\xi_k, j) | \ j < \kappa\} = \alpha$ and for all $j < \kappa$, $s(\xi_k, j) < \alpha$ iff $l \leq k < l'$ and $\xi_k \restriction i \notin J_1(\alpha)$ iff $l' \leq k \leq n$. (Of course we may assume that we have ordered $\xi_0, ..., \xi_m$ so that l and l' exist.) If $l \leq k < l'$ then there are $\rho_0, ..., \rho_r \in J_1 \cup J_0$ satisfying (c)(i)-(c)(iv) above. By the choice of $\eta(i-1)$, $\rho_p \restriction i \leftarrow \rho_{p+1} \restriction i$, for all $p < r$, and so $\xi_k \restriction (i+1) \in J_1(\beta_i)$. For all $k \leq n$ we define ξ'_k as follows:

(α) if $k < l$ then $\xi'_k = \xi_k$;

(β) if $l \leq k < l'$ then $\xi'_k = \xi_k \restriction (i+1)$;

(γ) if $l' \leq k \leq n$ then $\xi'_k = \xi_k \restriction j_k$.

Let $g(\eta \restriction (i+1)) = u(\delta_0, ..., \delta_m)$, u a term and $\delta_0, ..., \delta_m \in J_1(\beta_{i+1})$. Because β_i is g-saturated there is $\eta' \in J_0(\beta_i)$ and $\delta'_0, ..., \delta'_m \in J_1(\beta_i)$ such that

(a) $g(\eta') = u(\delta'_0, ..., \delta'_m)$;

(b) $length(\eta') = i+1$ and $\eta' \restriction i = \eta \restriction i$;

(c) the basic type of $(\xi'_0, ..., \xi'_n, \delta_0, ..., \delta_m)$ in $(J_1, <, \ll, H, P_j)$ is the same as the basic type of $(\xi'_0, ..., \xi'_n, \delta'_0, ..., \delta'_m)$.

Because for all $l \leq k < l'$, $s(\xi_k, i+1) \geq \beta_{i+1}$ and for all $l' \leq k \leq n$, $s(\xi_k, j_k) > \beta_{i+1}$, it is easy to see that the basic type of $(\xi_0, ..., \xi_n, \delta_0, ..., \delta_m)$ in $(J_1, <, \ll, H, P_j)$ is the same as the basic type of $(\xi_0, ..., \xi_n, \delta'_0, ..., \delta'_m)$.

Let ϕ_n, $n < \kappa$, be as in Theorem 2.7. Then

$$EM^1(J_1, \Phi) \models \phi_{i+1}(u(\delta'_0, ..., \delta'_m), t(\xi_0, ..., \xi_n)).$$

So $\eta' \neq \eta \restriction (i+1)$, $\eta' \restriction i = \eta \restriction i$ and

$$EM^1(J_0, \Phi) \models \phi_{i+1}(\eta', \eta).$$

This is impossible by Theorem 2.7 (ii). □

3.19 Conclusion. *Let $\lambda = \mu^+$, $cf(\mu) = \mu$, $\kappa = cf(\kappa) < \mu$, $\lambda^{<\kappa} = \lambda$ and $\mu^\kappa = \mu$. Assume T is an unsuperstable theory, $|T| \leq \lambda$ and $\kappa(T) > \kappa$. Then there are models $\mathcal{A}$, $\mathcal{B} \models T$ of cardinality λ such that*

$$\mathcal{A} \equiv^{\lambda}_{\mu\times\kappa} \mathcal{B} \text{ and } \mathcal{A} \not\cong \mathcal{B}.$$

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