Cardinal invariants of ultraproducts of Boolean algebras

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Abstract

We deal with some of problems posed by Monk [Mo 1], [Mo 3] and related to cardinal invariant of ultraproducts of Boolean algebras. We also introduce and investigate several new cardinal invariants.

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0 Introduction

In the present paper we deal with cardinal invariants of Boolean algebras and ultraproducts. Several questions in this area were posed by Monk ([Mo 1], [Mo 2], [Mo 3]) and we address some of them. General schema of these problems can be presented in the following fashion. Let inv be a cardinal function on Boolean algebras. Suppose that B_i are Boolean algebras (for $i < \kappa$) and that D is an ultrafilter on the cardinal κ. We ask what is the relation between $\mathrm{inv}(\prod\limits_{i<\kappa} B_i/D)$ and $\prod\limits_{i<\kappa} \mathrm{inv}(B_i)/D$? For each invariant inv we may consider two questions:

$(<)_{\mathrm{inv}}$ is $\mathrm{inv}(\prod\limits_{i<\kappa} B_i/D) < \prod\limits_{i<\kappa} \mathrm{inv}(B_i)/D$ possible?

$(>)_{\mathrm{inv}}$ is $\mathrm{inv}(\prod\limits_{i<\kappa} B_i/D) > \prod\limits_{i<\kappa} \mathrm{inv}(B_i)/D$ possible?

We deal with these questions for several cardinal invariants. We find it helpful to introduce "finite" versions inv_n of the invariants. This helps us in some problems as $\mathrm{inv}^+(\prod\limits_{i<\kappa} B_i/D) \geq \prod\limits_{i<\kappa} \mathrm{inv}^+_{f(i)}(B_i)/D$ for each function $f : \kappa \longrightarrow \omega$ such that $\lim\limits_D f = \omega$.

In section 1 we will give a general setting of the subject. These results were known much earlier (at least to the second author). We present them here to establish a uniform approach to the invariants and show how the Łoś theorem applies. In the last part of this section we present a simple method which uses the main result of [MgSh 433] to show the consistency of the inequality $\mathrm{inv}(\prod\limits_{i<\kappa} B_i/D) < \prod\limits_{i<\kappa} \mathrm{inv}(B_i)/D$ for several invariants inv. These problems will be fully studied and presented in [MgSh 433].

Section 2 is devoted to the (topological) density of Boolean algebras. We show here that, in ZFC, the answer to the question $(<)_d$ is "yes". This improves Theorem A of [KoSh 415] (a consistency result) and answers (negatively) Problem J of [Mo 3]. It should be remarked here that the answer to $(>)_d$ is "no" (see [Mo 2]).

In the third section we introduce strong λ-systems which are one of tools for our constructions. Then we apply them to build Boolean algebras which (under some set-theoretical assumptions) show that the inequalities $(>)_{\mathrm{h-cof}}$ and $(>)_{\mathrm{inc}}$ are possible (a consistency). These results seem to be new, the second one can be considered as a partial answer to Problem X of [Mo 3]. We get similar constructions for spread, hereditary Lindelöf degree and hereditary density. However they are not sufficient to give in ZFC positive answers to the corresponding questions $(>)_{\mathrm{inv}}$. These investigations are continued in [Sh 620], where the respective Boolean algebras are built in ZFC. The consistencies of the reverse inequalities will be presented in [MgSh 433].

The fourth section deals with the independence number and the tightness. It has been known that both questions $(>)_{\rm ind}$ and $(>)_t$ have the answer "yes". In coming paper [MgSh 433] it will be shown that $(<)_{\rm ind}$, $(<)_t$ may be answered positively (a consistency result; see section 1 too). Our results here were inspired by other sections of this paper and [Sh 503]. We introduce and study "finite" versions of the independence number getting a surprising asymmetry between odd and even cases. A completely new cardinal invariant appears naturally here. It has some reflection in what we can show for the tightness. Finally we re-present and re-formulate the main result of [Sh 503] (on products of interval Boolean algebras) putting it in our general schema and showing explicitly its heart.

History: A regular study of cardinal invariants of Boolean algebras was initialized in [Mo 1], where several problems were posed. Those problems stimulated and directed the work in the area. Some of the problems were naturally related to the behaviour of the invariants in ultraproducts and that found a reflection in papers coming later. Several bounds, constructions and consistency results were proved in [Pe], [Sh 345], [KoSh 415], [MgSh 433], [Sh 479], [Sh 503]. New techniques of constructions of Boolean algebras were developed in [Sh 462] (though the relevance of the methods for ultraproducts was not stated explicitly there).

This paper is, in a sense, a development of the notes "F99: Notes on cardinal invariants…" which the second author wrote in January 1993. A part of these notes is incorporated here, other results will be presented in [MgSh 433] and [RoSh 599].

The methods and tools for building Boolean algebras which we present here will be applied in a coming paper to deal with the problems of attainment in different representations of cardinal invariants.

Notation: Our notation is rather standard. All cardinals are assumed to be infinite and usually they are denoted by λ, κ, θ, Θ (with possible indexes).

We say that a family $\{\langle s^\alpha_0,\ldots,s^\alpha_{m-1}\rangle:\alpha<\lambda\}$ of finite sequences forms a Δ-system with the root $\{0,\ldots,m^*-1\}$ (for some $m^*\leq m$) if the sets $\{s^\alpha_{m^*},\ldots,s^\alpha_{m-1}\}$ (for $\alpha<\lambda$) are pairwise disjoint and

$$(\forall\alpha<\lambda)(\forall l<m^*)(s^\alpha_l=s^0_l).$$

In Boolean algebras we use $\vee$ (and $\bigvee$), $\wedge$ (and $\bigwedge$) and $-$ for the Boolean operations. If B is a Boolean algebra, $x\in B$ then $x^0=x$, $x^1=-x$.

The sign $\circledast$ stands for the operation of the free product of Boolean algebras (see [Ko], def.11.1) and $\prod^w$ denotes the weak product of Boolean algebras (as defined in [Ko], p.112).

All Boolean algebras we consider are assumed to be infinite (and we will not repeat this assumption). Similarly whenever we consider a cardinal invariant $\mathrm{inv}(B)$ we assume that it is infinite.

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1 Invariants and ultraproducts

1.1 Definable cardinal invariants.

In this section we try to systematize the definition of cardinal invariants and we define what is a def.car.invariant (definable cardinal invariant) of Boolean algebras. Then we get immediate consequences of this approach for ultraproducts. Actually, Boolean algebras are irrelevant in this section and can be replaced by any structures.

Definition 1.1 1. *For a (not necessary first order) theory T in the language of Boolean algebras plus one distinguished predicate $P = P_0$ (unary if not said otherwise) plus, possibly, some others $P_1, P_2, \ldots$ we define cardinal invariants* inv_T, inv^+_T *of Boolean algebras by (for a Boolean algebra B):*

$$\mathrm{inv}_T(B) \stackrel{\rm def}{=} \sup\{\|P\| : (B, P_n)_n \text{ is a model of } T\}$$
$$\mathrm{inv}^+_T(B) \stackrel{\rm def}{=} \sup\{\|P\|^+ : (B, P_n)_n \text{ is a model of } T\}$$
$$\mathrm{Inv}_T(B) \stackrel{\rm def}{=} \{\|P\| : (B, P_n)_n \text{ is a model of } T\}$$

We call $\mathrm{inv}^{(+)}_T$ *a* def.car. invariant *(definable cardinal invariant).*

2. *If in 1., T is first order, we call such cardinal invariant a* def.f.o.car. invariant *(definable first order cardinal invariant).*

3. *A theory T is n*-universal in (P_0, P_1) *if all sentences $\phi \in T$ are of the form*

$$(\forall x_1, \ldots, x_n \in P_0)\psi(\bar{x}),$$

where all occurrences of $x_1, \ldots, x_n$ in ψ are free and P_0 does not appear there and any appearance of P_1 in ψ is in the form $P_1(x_{i_0}, \ldots, x_{i_k})$ with no more complicated terms.

If we allow all n then T is said to be universal in (P_0, P_1).

Note: quantifiers can still occur in $\psi(\bar{x})$ on other variables.

4. *If in 1., T is universal in (P_0, P_1), first order except the demand that P_1 is a well ordering of P_0 we call such cardinal invariant* def.u.w.o.car. invariant *(definable universal well ordered cardinal invariant).*

5. *If in 1., P_1 is a linear order on P (i.e. T says so) and in the definition of* $\mathrm{inv}_T(B)$, $\mathrm{inv}^+(B)$ *we replace "$\|P\|$" by the cofinality of (P, P_1) then we call those cardinal invariants* def.cof. invariant *(definable cofinality invariant,* $\mathrm{cf}-\mathrm{inv}_T$*); we can have the f.o. and the u.w.o. versions. We define similarly* $\mathrm{cf}-\mathrm{Inv}_T(B)$ *as the set of such cofinalities. To use* cf$-$inv *we can put it in $^+$ (we may omit "cf-" if the context allows it). We can use the order type instead of the cofinality and cardinality writing* ot-inv*. For the cardinality we may use* car-inv*.*

6. *For a theory T as in 2., the* minimal definable first order cardinal invariant *of B (determined by T) is* min $\mathrm{Inv}_T(B)$.

To avoid a long sequence of definitions we refer the reader to [Mo 1], [Mo 2] for definitions of the cardinal functions below. Those invariants which are studied in this paper are defined in the respective sections.

Proposition 1.2 *1. The following cardinal invariants of Boolean algebras are def.f.o.car. invariants (of course each has two versions:* inv *and* inv^+*):*

> *c (cellularity), Length, irr (irredundance), cardinality, ind (independence), s (spread), Inc (incomparability).*

2. *The following cardinal invariants of Boolean algebras are def.f.o.cof. invariants:*

> *hL (hereditary Lindelof), hd (hereditary density).*

3. *The following cardinal invariants of Boolean algebras are def.u.w.o.car. invariants*

> *Depth, t (tightness), h-cof (hereditary cofinality), hL, hd.*

4. *π (algebraic density) and d (topological density) are minimal def.f.o.card. invariants.*

PROOF: All unclear cases are presented in next sections. ■

Proposition 1.3 *1. If* $\mathrm{inv}^+_T(B)$ *is a limit cardinal then the* sup *in the definition of* $\mathrm{inv}_T(B)$ *is not obtained and* $\mathrm{inv}_T(B) = \mathrm{inv}^+_T(B)$.

2. If $\mathrm{inv}^+_T(B)$ *is not a limit cardinal then it is* $(\mathrm{inv}_T(B))^+$ *and the* sup *in the definition of* $\mathrm{inv}_T(B)$ *is obtained.* ■

Definition 1.4 *A linear order* $(I,<)$ *is* Θ*-like if*

$$\|I\| = \Theta \quad \text{and} \quad (\forall a \in I)(\|\{b \in I : b < a\}\| < \Theta).$$

Proposition 1.5 *Assume that* $\mathrm{inv}^{(+)}_T$ *is a definable first order cardinal invariant. Assume further that: D is an ultrafilter on a cardinal κ, for $i<\kappa$, B_i is a Boolean algebra and* $B \stackrel{\mathrm{def}}{=} \prod_{i<\kappa} B_i/D$. *Then*

(a) *if* $\lambda_i < \mathrm{inv}^+_T(B_i)$ *for* $i<\kappa$ *then* $\prod_{i<\kappa} \lambda_i/D < \mathrm{inv}^+_T(B)$,

(b) $\prod_{i<\kappa} \mathrm{inv}^+_T(B_i)/D \le \mathrm{inv}^+_T(B)$,

(c) *if* $\mathrm{inv}_T(B) < \prod_{i<\kappa} \mathrm{inv}_T(B_i)/D$ *then for the D-majority of $i<\kappa$ we have:* $\lambda_i \stackrel{\mathrm{def}}{=} \mathrm{inv}_T(B_i)$ *is a limit cardinal and the linear order* $\prod_{i<\kappa}(\lambda_i,<)/D$ *is* $(\mathrm{inv}_T(B))^+$*–like; hence for the D-majority of the $i<\kappa$ we have: λ_i is a regular limit cardinal (i.e. weakly inaccessible),*

(d) $\min \mathrm{Inv}_T(B) \le \prod_{i<\kappa} \min \mathrm{Inv}_T(B_i)/D$.

PROOF: (a) This is an immediate consequence of Łoś theorem.

(b) For $i<\kappa$ let $\lambda_i = \mathrm{inv}^+_T(B_i)$. Suppose $\lambda < \prod_{i<\kappa} \lambda_i/D$. As $\prod_{i<\kappa}(\lambda_i,<)/D$ is a linear order of cardinality $>\lambda$ we find $f \in \prod_{i<\kappa} \lambda_i$ such that

$$\|\{g/D \in \prod_{i<\kappa} \lambda_i/D : g/D < f/D\}\| \ge \lambda.$$

Since $f(i) < \mathrm{inv}^+_T(B_i)$ (for $i<\kappa$) we may apply (a) to conclude that

$$\lambda \le \| \prod_{i<\kappa} f(i)/D \| < \mathrm{inv}^+_T(B).$$

(c) Let $\lambda = \mathrm{inv}_T(B)$, $\lambda_i = \mathrm{inv}_T(B_i)$ and assume that $\lambda < \prod_{i<\kappa} \lambda_i/D$. By part (b) we conclude that then

(∗) $\lambda^+ = \prod\limits_{i<\kappa} \mathrm{inv}^+_T(B_i)/D = \prod\limits_{i<\kappa} \mathrm{inv}_T(B_i)/D = \mathrm{inv}^+_T(B)$.

Let $A = \{i < \kappa : \mathrm{inv}_T(B_i) < \mathrm{inv}^+_T(B_i)\}$. Note that $A \notin D$: if not then we may assume $A = \kappa$ and for each $i < \kappa$ we have $\lambda_i < \mathrm{inv}^+_T(B_i)$. By part (a) and (*) above we get $\lambda^+ = \prod\limits_{i<\kappa} \lambda_i/D < \mathrm{inv}^+_T(B)$, a contradiction. Consequently we may assume that $A = \emptyset$. Thus for each $i < \kappa$ we have $\lambda_i = \mathrm{inv}_T(B_i) = \mathrm{inv}^+_T(B_i)$ and λ_i is a limit cardinal, $\lambda_i = \sup \mathrm{Inv}_T(B_i) \notin \mathrm{Inv}_T(B_i)$ (by 1.3).

The linear order $\prod\limits_{i<\kappa} (\lambda_i, <)/D$ is of the cardinality λ^+ (by (∗)). Suppose that $f \in \prod\limits_{i<\kappa} \lambda_i$ and choose $\mu_i \in \mathrm{Inv}_T(B_i)$ such that $f(i) \le \mu_i$ for $i < \kappa$. Then $\|\prod\limits_{i<\kappa} f(i)/D\| \le \prod\limits_{i<\kappa} \mu_i/D \in \mathrm{Inv}_T(\prod\limits_{i<\kappa} B_i/D) \subseteq \lambda^+$. Hence the order $\prod\limits_{i<\kappa} (\lambda_i, <)/D$ is λ^+-like.

Finally assume that $A = \{i < \kappa : \lambda_i$ is singular $\} \in D$, so w.l.o.g. $A = \kappa$. Choose cofinal subsets Q_i of λ_i such that $Q_i \subseteq \lambda_i = \sup Q_i$, $\|Q_i\| = \mathrm{cf}(\lambda_i)$ (for $i < \kappa$) and let $M_i = (\lambda_i, <, Q_i, \ldots)$. Take the ultrapower $M = \prod\limits_{i<\kappa} M_i/D$ and note that $M \models$"Q^M is unbounded in $<^M$". As earlier, $\|Q^M\| = \prod\limits_{i<\kappa} \|Q_i\|/D \le \lambda$ so $\mathrm{cf}(\prod\limits_{i<\kappa} (\lambda_i, <)/D) \le \lambda$ what contradicts λ^+-likeness of the product order.

(d) It follows from (a). ■

Definition 1.6 *Let $(I, <)$ be a partial order.*

1. *The depth* $\mathrm{Depth}(I)$ *of the order I is the supremum of cardinalities of well ordered (by $<$) subsets of I.*
2. *I is Θ-Depth-like if I is a linear ordering which contains a well ordered cofinal subset of length Θ but* $\mathrm{Depth}^+(\{b \in I : b < a\}, <) \le \Theta$ *for each $a \in I$.*

Lemma 1.7 *Let D be an ultrafilter on a cardinal κ, λ_i (for $i < \kappa$) be cardinals. Then:*

1. *if there is a $<_D$-increasing sequence $\langle f_\alpha/D : \alpha \le \mu_0\rangle \subseteq \prod\limits_{i<\kappa} (\lambda^+_i, <)/D$, μ_0 is a cardinal then $\mu_0 < \mathrm{Depth}^+(\prod\limits_{i<\kappa} (\lambda_i, <)/D)$,*
2. $\mathrm{Depth}(\prod\limits_{i<\kappa} (\lambda^+_i, <)/D) \le \mathrm{Depth}^+(\prod\limits_{i<\kappa} (\lambda_i, <)/D)$ *and hence* $\mathrm{Depth}^+(\prod\limits_{i<\kappa} (\lambda^+_i, <)/D) \le (\mathrm{Depth}^+(\prod\limits_{i<\kappa} (\lambda_i, <)/D))^+$.

PROOF: 1) Let $\mu_1 = \mathrm{cf}(\prod_{i<\kappa}(\lambda_i,<)/D)$, so $\mu_1 < \mathrm{Depth}^+(\prod_{i<\kappa}(\lambda_i,<)/D)$. If $\mu_0 \leq \mu_1$ then we are done, so let us assume that $\mu_0 > \mu_1$ and let us consider two cases.

CASE A: $\mathrm{cf}(\mu_0) \neq \mu_1$.

Let $\langle g_\beta/D : \beta < \mu_1\rangle$ be an increasing sequence cofinal in $\prod_{i<\kappa}(\lambda_i,<)/D$. For each $i<\kappa$ choose an increasing sequence $\langle A^i_\xi : \xi < \lambda_i\rangle$ of subsets of $f_{\mu_0}(i)$ such that $f_{\mu_0}(i) = \bigcup_{\xi<\lambda_i} A^i_\xi$ and $\|A^i_\xi\| < \lambda_i$. Then

$$(\forall \alpha < \mu_0)(\exists \beta < \mu_1)(\{i<\kappa : f_\alpha(i) \in A^i_{g_\beta(i)}\} \in D)$$

and, passing to a subsequence of $\langle f_\alpha/D : \alpha < \mu_0\rangle$ if necessary, we may assume that for some $\beta_0 < \mu_1$ for all $\alpha < \mu_0$

$$\{i<\kappa : f_\alpha(i) \in A^i_{g_{\beta_0}(i)}\} \in D$$

(this is the place we use the additional assumption $\mathrm{cf}(\mu_0) \neq \mu_1$). Each set $A^i_{g_{\beta_0}(i)}$ is order–isomorphic to some ordinal $g(i) < \lambda_i$ (as $\|A^i_{g_{\beta_0}(i)}\| < \lambda_i$). These isomorphisms give us a "copy" of the sequence $\langle f_\alpha/D : \alpha < \mu_0\rangle$ below some $g/D \in \prod_{i<\kappa}\lambda_i/D$, witnessing $\mu_0 < \mathrm{Depth}^+(\prod_{i<\kappa}(\lambda_i,<)/D)$.

CASE B: $\mathrm{cf}(\mu_0) = \mu_1 < \mu_0$.

For each regular cardinal $\mu \in (\mathrm{cf}(\mu_0),\mu_0)$ we may apply Case A to μ and the sequence $\langle f_\alpha/D : \alpha \leq \mu\rangle$ and conclude $\mu < \mathrm{Depth}^+(\prod_{i<\kappa}(\lambda_i,<)/D)$. Hence $\mu_0 \leq \mathrm{Depth}^+(\prod_{i<\kappa}(\lambda_i,<)/D)$. Let $\langle \mu^\xi : \xi < \mathrm{cf}(\mu_0)\rangle \subseteq (\mathrm{cf}(\mu_0),\mu_0)$ be an increasing cofinal in μ_0 sequence of regular cardinals. Note that for each $\xi < \mathrm{cf}(\mu_0)$ and a function $f \in \prod_{i<\kappa}\lambda_i$ we can find a $<_D$-increasing sequence $\langle h^*_\alpha : \alpha < \mu^\xi\rangle \subseteq \prod_{i<\kappa}\lambda_i$ such that $f <_D h^*_0$. Using this fact we construct inductively a $<_D$–increasing sequence $\langle h_\alpha/D : \alpha < \mu_0\rangle \subseteq \prod_{i<\kappa}\lambda_i/D$ (which will show that $\mu_0 < \mathrm{Depth}^+(\prod_{i<\kappa}(\lambda_i,<)/D)$):

Suppose we have defined h_α for $\alpha < \mu^\xi$ (for some $\xi < \mathrm{cf}(\mu_0)$). Since μ^ξ is regular and $\mu^\xi \neq \mu_1$ the sequence $\langle h_\alpha/D : \alpha < \mu^\xi\rangle$ cannot be cofinal in $\prod_{i<\kappa}(\lambda_i,<)/D$. Take $f/D \in \prod_{i<\kappa}\lambda_i/D$ which $<_D$-bounds the sequence. By the previous remark we find a $<_D$-increasing sequence $\langle h_\alpha/D : \mu^\xi \leq \alpha < \mu^{\xi+1}\rangle \subseteq \prod_{i<\kappa}\lambda_i/D$ such that $f <_D h_{\mu^\xi}$. So the sequence $\langle h_\alpha : \alpha < \mu^{\xi+1}\rangle$ is increasing.

Now suppose that we have defined h_α/D for $\alpha < \sup_{\xi<\xi_0} \mu^\xi$ for some limit ordinal $\xi_0 < \mathrm{cf}(\mu_0)$. The cofinality of the sequence $\langle h_\alpha/D : \alpha < \sup_{\xi<\xi_0} \mu^\xi \rangle$ is $\mathrm{cf}(\xi_0) < \mu_1$. Consequently the sequence is bounded in $\prod_{i<\kappa} \lambda_i/D$ and we may proceed as in the successor case and define h_α/D for $\alpha \in [\sup_{\xi<\xi_0} \mu^\xi, \mu^{\xi_0})$.

2) It follows immediately from 1). $\blacksquare$

Proposition 1.8 *Assume that* $\mathrm{inv}_T^{(+)}$ *is a definable universal well ordered cardinal invariant. Assume further that: D is an ultrafilter on a cardinal κ, for $i < \kappa$, B_i is a Boolean algebra and $B \stackrel{\mathrm{def}}{=} \prod_{i<\kappa} B_i/D$. Then*

(a) *if* $\lambda_i < \mathrm{inv}_T^+(B_i)$ *for* $i < \kappa$ *then* $\mathrm{Depth}^+ \prod_{i<\kappa} (\lambda_i, <)/D \le \mathrm{inv}_T^+(B)$,

(b) $\mathrm{Depth}(\prod_{i<\kappa} (\mathrm{inv}_T^+(B_i), <)/D) \le \mathrm{inv}_T^+(B)$,

(c) *if* $\mathrm{inv}_T(B) < \mathrm{Depth} \prod_{i<\kappa} (\mathrm{inv}_T(B_i), <)/D$ *then for the D-majority of $i < \kappa$ we have: $\lambda_i \stackrel{\mathrm{def}}{=} \mathrm{inv}_T(B_i)$ is a limit cardinal and the linear order $\prod_{i<\kappa} (\lambda_i, <)/D$ is $(\mathrm{inv}_T(B))^+$-Depth-like; hence for the D-majority of the $i < \kappa$ we have: λ_i is a regular limit cardinal, i.e. weakly inaccesible.*

PROOF: (a) Suppose that $\mu < \mathrm{Depth}^+ \prod_{i<\kappa} (\lambda_i, <)/D$. As $\lambda_i < \mathrm{inv}_T^+(B_i)$ we find $P_0^i, P_1^i, \ldots$ such that $M_i \stackrel{\mathrm{def}}{=} (B_i, P_0^i, P_1^i, \ldots) \models T$, $\|P_0^i\| \ge \lambda_i$. Look at $M \stackrel{\mathrm{def}}{=} \prod_{i<\kappa} M_i/D$. Note that (P_0^M, P_1^M) is a linear ordering such that

$$\mathrm{Depth}^+(P_0^M, P_1^M) \ge \mathrm{Depth}^+ \prod_{i<\kappa} (\lambda_i, <)/D.$$

Thus we find $P_0^* \subseteq P_0^M$ such that $\|P_0^*\| = \mu$ and (P_0^*, P_1^M) is a well ordering. As formulas of T are universal in (P_0, P_1), first order except the demand that P_1 is a well order on P_0 we conclude $M^* \stackrel{\mathrm{def}}{=} (B, P_0^*, P_1^M, \ldots) \models T$. Hence $\mu = \|P_0^*\| < \mathrm{inv}_T^+(B)$.

(b) We consider two cases here

Case 1: For D-majority of $i < \kappa$ we have $\mathrm{inv}_T(B_i) < \mathrm{inv}_T^+(B_i)$.

Then we may assume that for each $i < \kappa$

$$\lambda_i \stackrel{\mathrm{def}}{=} \mathrm{inv}_T(B_i) < \mathrm{inv}_T^+(B_i) = \lambda_i^+.$$

By lemma 1.7(2) we have

$$\mathrm{Depth}(\prod_{i<\kappa}(\lambda_i^+,<)/D) \leq \mathrm{Depth}^+(\prod_{i<\kappa}(\lambda_i,<)/D).$$

On the other hand, it follows from (a) that

$$\mathrm{Depth}^+(\prod_{i<\kappa}(\lambda_i,<)/D) \leq \mathrm{inv}_T^+(B)$$

and consequently we are done (in this case).

Case 2: For D-majority of $i<\kappa$ we have $\mathrm{inv}_T(B_i)=\mathrm{inv}_T^+(B_i)$.

So suppose that $\mathrm{inv}_T(B_i)=\mathrm{inv}_T^+(B_i)$ for each $i<\kappa$. Suppose that

$$\bar{g}=\langle g_\alpha/D:\alpha<\mu\rangle \subseteq \prod_{i<\kappa}\mathrm{inv}_T^+(B_i)/D$$

is a $<_D$–increasing sequence.

If $\bar{g}$ is bounded then we apply (a) to conclude that $\mu<\mathrm{inv}_T^+(B)$. If $\bar{g}$ is unbounded (so cofinal) then there are two possibilities: either μ is a limit cardinal or it is a successor. In the first case we apply the previous argument to initial segments of $\bar{g}$ and we conclude $\mu\leq\mathrm{inv}_T^+(B)$. In the second case we necessarily have $\mu=\mathrm{cf}(\prod\limits_{i<\kappa}(\mathrm{inv}_T^+(B_i),<)/D)=\mu_0^+$ (for some μ_0) and $\mu_0<\mathrm{inv}_T^+(B)$. Thus $\mu\leq\mathrm{inv}_T^+(B)$.

Consequently, if there is an increasing (well ordered) sequence of the length μ in $\prod\limits_{i<\kappa}(\mathrm{inv}_T^+(B_i),<)/D$ then $\mu\leq\mathrm{inv}_T^+(B)$ and the case 2 is done too.

(c) Assume that $\lambda\stackrel{\mathrm{def}}{=}\mathrm{inv}_T(B)<\mathrm{Depth}\prod\limits_{i<\kappa}(\mathrm{inv}_T(B_i),<)/D$. By (b) we get that then

(**) $\lambda^+=\mathrm{Depth}\prod\limits_{i<\kappa}(\mathrm{inv}_T(B_i),<)/D=\mathrm{Depth}\prod\limits_{i<\kappa}(\mathrm{inv}_T^+(B_i),<)/D=\mathrm{inv}_T^+(B)$.

Suppose that $\{i<\kappa:\mathrm{inv}_T(B_i)<\mathrm{inv}_T^+(B_i)\}\in D$. Then by (a) we have

$$\mathrm{Depth}^+\prod_{i<\kappa}(\mathrm{inv}_T(B_i),<)/D\leq\mathrm{inv}_T^+(B),$$

but (by (**) and 1.3) we know that

$$\mathrm{Depth}^+\prod_{i<\kappa}(\mathrm{inv}_T(B_i),<)/D=\lambda^{++}>\mathrm{inv}_T^+(B),$$

a contradiction. Consequently for the D-majority of $i<\kappa$ we have $\lambda_i = \mathrm{inv}_T(B_i) = \mathrm{inv}^+_T(B_i)$ and λ_i is a limit cardinal.

Note that if $f \in \prod\limits_{i<\kappa} \mathrm{inv}_T(B_i)$ then $\mathrm{Depth}^+ \prod\limits_{i<\kappa} (f(i),<)/D \le \lambda^+$ (this is because of the previous remark, $(**)$ and (a)). Moreover, $(**)$ implies that there is an increasing sequence $\langle f_\alpha/D : \alpha<\lambda^+\rangle \subseteq \prod\limits_{i<\kappa} (\mathrm{inv}_T(B_i),<)/D$. By what we noted earlier the sequence has to be unbounded (so cofinal). Consequently the linear order $\prod\limits_{i<\kappa} (\mathrm{inv}_T(B_i),<)/D$ is λ^+–Depth–like. Now assume that $A = \{i<\kappa : \lambda_i \text{ is singular}\} \in D$. Let $Q_i \subseteq \lambda_i$ be a cofinal subset of λ_i of the size $\mathrm{cf}(\lambda_i)$ (for $i<\kappa$). Then $\mathrm{Depth}^+ \prod\limits_{i<\kappa} (Q_i,<)/D \le \lambda^+$ but $\prod\limits_{i<\kappa} Q_i/D$ is cofinal in $\prod\limits_{i<\kappa} \mathrm{inv}_T(B_i)/D$ - a contradiction, as the last order has the cofinality λ^+. ■

Proposition 1.9 *Assume that* $\mathrm{inv}^{(+)}_T$ *is a definable first order cofinality invariant. Assume further that: D is an ultrafilter on a cardinal κ, for $i<\kappa$, B_i is a Boolean algebra and* $B \overset{\mathrm{df}}{=} \prod\limits_{i<\kappa} B_i/D$. *Then*

(a) *if* $\lambda_i \in \mathrm{Inv}_T(B_i)$ *for $i<\kappa$ and* $\lambda = \mathrm{cf}(\prod\limits_{i<\kappa} (\lambda_i,<)/D)$ *then* $\lambda \in \mathrm{Inv}_T(B)$,

(b) *if* $\mathrm{inv}^+_T(B) \le \mathrm{cf}(\prod\limits_{i<\kappa} \mathrm{inv}_T(B_i)/D)$ *then for the D-majority of $i<\kappa$ we have:* $\mathrm{inv}_T(B_i)$ *is a limit cardinal.*

PROOF: should be clear. ■

Proposition 1.10 *Suppose that T is a finite n-universal in (P_0,P_1) theory in the language of Boolean algebras plus two predicates P_0, P_1 and the theory says that P_1 is a linear ordering on P_0. Let* $\mathrm{inv}^{(+)}_T$ *be the respective cardinality invariant. Assume further that: D is an ultrafilter on a cardinal κ, B_i is a Boolean algebra (for $i<\kappa$) and* $B \overset{\mathrm{def}}{=} \prod\limits_{i<\kappa} B_i/D$. *Lastly assume* $\lambda \longrightarrow (\mu)^n_\kappa$, $n \ge 2$ *and* $\lambda \in \mathrm{Inv}(B)$.
Then for the D-majority of the $i<\kappa$, $\mu < \mathrm{inv}^+_T(B_i)$.

PROOF: We may assume that $T = \{\psi_0, \psi\}$, where the sentence ψ_0 says "P_1 is a linear ordering of P_0" (and we denote this ordering by $<$) and

$$\psi = (\forall x_0 < \ldots < x_{n-1})(\phi(\bar{x}))$$

where ϕ is a formula in the language of Boolean algebras. Note that a formula

$$(\forall x_0,\dots,x_{n-1}\in P_0)(\phi(\bar{x}))$$

as in 1.1(3) is equivalent to the formula

$$\bigwedge_{f\in {}^n n}(\forall x_0,\dots,x_{n-1}\in P_0)([\bigwedge_{f(k)=f(l)} x_k=x_l \ \&\ \bigwedge_{f(k)<f(l)} x_k<x_l] \ \Rightarrow\ \phi_0^f(\bar{x})),$$

where, for any $f:n\longrightarrow n$, the formula ϕ_0^f is obtained from ϕ by respective replacing appearances of $P_1(x_i,x_j)$ by either $x_i=x_i$ or $x_i\neq x_i$. Consequently the above assumption is easily justified.

Let $A=\{i<\kappa:\mu<\mathrm{inv}^+_T(B_i)\}$. Assume that $A\notin D$. As $\lambda\in\mathrm{Inv}_T(B)$ we find P_0,P_1 such that $\|P_0\|=\lambda$ and $P_1 =<$ is a linear ordering of P_0 and $(B,P_0,P_1)\models\psi$. For each element of $\prod\limits_{i<\kappa} B_i/D$ we fix a representative of this equivalence class (so we will freely pass from f/D to f with no additional comments). Now, we define a colouring $F:[P_0]^n\longrightarrow\kappa$ by

$$F(f_0/D,\dots,f_{n-1}/D)= \begin{array}{l}\text{the first } i\in\kappa\setminus A \text{ such that}\\ \text{if } f_0/D<\ldots<f_{n-1}/D\\ \text{then } f_0(i),\dots,f_{n-1}(i) \text{ are pairwise distinct and}\\ B_i\models\phi[f_0(i),\dots,f_{n-1}(i)]\end{array}$$

The respective i exists since $A\notin D$ and

$$B\models \text{“}f_0/D,\dots,f_{n-1}/D \text{ are distinct and } \phi[f_0/D,\dots,f_{n-1}/D]\text{”}.$$

By the assumption $\lambda\longrightarrow(\mu)^n_\kappa$ we find $W\in[P_0]^\mu$ which is homogeneous for F. Let i be the constant value of F on W and put $P^i_0=\{f(i):f/D\in W\}$ (recall that we fixed representatives of the equivalence classes). Now we may introduce P^i_1 as the linear ordering of P^i_0 induced by P_1.
Note that $f(i)\neq f'(i)$ for distinct $f/D, f'/D\in W$ and if $f_0(i),\dots,f_{n-1}(i)\in P^i_0$, $f_0(i)<_{P^i_1} f_1(i)<_{P^i_1}\ldots<_{P^i_1} f_{n-1}(i)$ then $f_0/D<\ldots<f_{n-1}/D$ and hence

$$B_i\models\phi[f_0(i),\dots,f_{n-1}(i)].$$

As $\|P^i_0\|=\mu$, $(B_i,P^i_0,P^i_1)\models\psi\wedge\psi_0$ we conclude that $\mu<\mathrm{inv}^+_T(B_i)$ what contradicts $i\notin A$. ■

One of the tools in study the invariants are “finite” versions of them (for invariants determined by infinite theories). Suppose $T=\{\phi_n:n<\omega\}$ and if T is supposed to describe a def.u.w.o.car. invariant then ϕ_0 already says that P_1 is a well ordering of P_0. Let $T^n=\{\phi_m:m<n\}$ for $n<\omega$.

Conclusion 1.11 *Suppose that D is a uniform ultrafilter on κ, $f:\kappa\longrightarrow\omega$ is such that $\lim_D f=\omega$. Let B_i (for $i<\kappa$) be Boolean algebras, $B=\prod_{i<\kappa}B_i/D$.*

1. *If T is first order then:*

   a) *if $\lambda_i\in\mathrm{Inv}_{T^{f(i)}}(B_i)$ (for $i<\kappa$) then $\prod_{i<\kappa}\lambda_i/D\in\mathrm{Inv}_T(B)$,*

   b) $\prod_{i<\kappa}\mathrm{inv}^+_{T^{f(i)}}(B_i)/D\le\mathrm{inv}^+_T(B)$.

2. *If T is u.w.o. then:*

   a) *if $\lambda_i\in\mathrm{Inv}_{T^{f(i)}}(B_i)$ (for $i<\kappa$) and $\lambda<\mathrm{Depth}^+\prod_{i<\kappa}(\lambda_i,<)/D$ then $\lambda\in\mathrm{Inv}_T(B)$,*

   b) $\mathrm{Depth}\prod_{i<\kappa}(\mathrm{inv}^+_{T^{f(i)}}(B_i),<)/D\le\mathrm{inv}^+_T(B)$.

PROOF: Like 1.5 and 1.8. ■

1.2 An example concerning the question $(<)_{\mathrm{inv}}$.

Now we are going to show how the main result of [MgSh 433] may be used to give affirmative answers to the questions $(<)_{\mathrm{inv}}$ for several cardinal invariants.

Proposition 1.12 *Suppose that D is an $\aleph_1$-complete ultrafilter on κ, $B_{i,\alpha}$ are Boolean algebras (for $\alpha<\lambda_i$, $i<\kappa$). Let $C:\prod_{i<\kappa}\lambda_i/D\longrightarrow\prod_{i<\kappa}\lambda_i$ be a choice function (so $C(x)\in x$ for an equivalence class $x\in\prod_{i<\kappa}\lambda_i/D$).*

1. *If $B_i=\circledast_{\alpha<\lambda_i}B_{i,\alpha}$ then*

$$\prod_{i<\kappa}B_i/D\simeq\circledast\{\prod_{i<\kappa}B_{i,C(x)(i)}/D:x\in\prod_{i<\kappa}\lambda_i/D\}.$$

2. *If $B_i=\prod^w_{\alpha<\lambda_i}B_{i,\alpha}$ then*

$$\prod_{i<\kappa}B_i/D\simeq\prod^w\{\prod_{i<\kappa}B_{i,C(x)(i)}/D:x\in\prod_{i<\kappa}\lambda_i/D\}.\qquad\blacksquare$$

Definition 1.13 *Let $\mathbf{O}$ be an operation on Boolean algebras.*

1. *For a theory T we define the property $\square^T_{\mathbf{O}}$:*

   $\square^T_{\mathbf{O}}$ *if μ is a cardinal, B_i are Boolean algebras for $i < \mu^+$ then*

   $$\sup_{i<\mu} \mathrm{inv}_T(B_i) \le \mathrm{inv}_T(\mathop{\mathbf{O}}_{i<\mu} B_i) \quad \text{and} \quad \mathrm{inv}_T(\mathop{\mathbf{O}}_{i<\mu^+} B_i) \le \mu + \sup_{i<\mu^+} \mathrm{inv}_T(B_i).$$

2. *Of course we may define the respective property for any cardinal invariant (not necessary of the form inv_T). But then we additionally demand that $\tau(B) \le \|B\|$ (where τ is the considered invariant).*

Proposition 1.14 *Suppose that a def.car.invariant inv_T (or just an invariant τ) satisfies either $\square^T_{\circledast}$ or $\square^T_{\prod^w}$ and suppose that for each cardinal χ there is a Boolean algebra B such that $\chi \le \mathrm{inv}_T(B)$ and there is no weakly inaccessible cardinal in the interval $(\chi, \|B\|]$. Assume further that*

($\odot$) *$\langle \lambda_i : i < \kappa \rangle$ is a sequence of weakly inaccessible cardinals, $\lambda_i > \kappa^+$, D is an $\aleph_1$-complete ultrafilter on κ and $\prod\limits_{i<\kappa} (\lambda_i, <)/D$ is μ^+-like (for some cardinal μ).*

Then there exist Boolean algebras B_i for $i < \kappa$ such that $\mathrm{inv}_T(B_i) = \lambda_i$ (for $i < \kappa$) and $\mathrm{inv}_T(\prod\limits_{i<\kappa} B_i/D) \le \mu$. So we have

$$\prod_{i<\kappa} \mathrm{inv}_T(B_i)/D = \mu^+ > \mathrm{inv}_T(\prod_{i<\kappa} B_i/D).$$

PROOF: Assume that inv_T satisfies $\square^T_{\circledast}$. For $i < \kappa$ and $\alpha < \lambda_i$ fix an algebra $B_{i,\alpha}$ such that

$$\|\alpha\| \le \mathrm{inv}_T(B_{i,\alpha}) \le \|B_{i,\alpha}\| < \lambda_i$$

(possible by our assumptions on inv_T) and let $B_i = \mathop{\circledast}\limits_{\alpha<\lambda_i} B_{i,\alpha}$. By 1.12 we have

$$\prod_{i<\kappa} B_i/D = \circledast\{\prod_{i<\kappa} B_{i,C(x)(i)}/D : x \in \prod_{i<\kappa} \lambda_i/D\},$$

where $C : \prod\limits_{i<\kappa} \lambda_i/D \longrightarrow \prod\limits_{i<\kappa} \lambda_i$ is a choice function. So by $\square^T_{\circledast}$ (the second inequality):

$$\mathrm{inv}_T(\prod_{i<\kappa} B_i/D) \le \mu + \sup\{\mathrm{inv}_T(\prod_{i<\kappa} B_{i,C(x)(i)}/D) : x \in \prod_{i<\kappa} \lambda_i/D\}.$$

Since $\|B_{i,\alpha}\| < \lambda_i$ and $\prod\limits_{i<\kappa} (\lambda_i, <)/D$ is μ^+-like for each $x \in \prod\limits_{i<\kappa} \lambda_i/D$ we have

$$\mathrm{inv}_T(\prod_{i<\kappa} B_{i,C(x)(i)}/D) \le \prod_{i<\kappa} \|B_{i,C(x)(i)}\|/D \le \mu.$$

Moreover, by the first inequality of $\Box^T_{\circledast}$, for each $\alpha < \lambda_i$

$$\|\alpha\| \leq \mathrm{inv}_T(B_{i,\alpha}) \leq \mathrm{inv}_T(B_i) \leq \|B_i\| = \lambda_i$$

and thus $\mathrm{inv}_T(B_i) = \lambda_i$. ■

Remark: 1. The consistency of $(\odot)$ is the main result of [MgSh 433], where several variants of it and their applications are presented.
2. If inv_T is either def.f.o.car invariant or def.u.w.o.car invariant then we may apply 1.5.c or 1.8.c respectively to conclude that for D-majority of $i < \kappa$ we have $\mathrm{inv}(B_i) = \mathrm{inv}^+(B_i)$. Consequently in these cases we may slightly modify the construction in 1.14 to get additionally $\mathrm{inv}(B_i) = \mathrm{inv}^+(B_i)$ for each $i < \kappa$.
3. Proposition 1.14 applies to several cardinal invariants. For example the condition $\Box^T_{\prod^w}$ is satisfied by:

> Depth (see §4 of [Mo 2]), Length (§7 of [Mo 2]), Ind (§10 of [Mo 2]), π-character (§11 of [Mo 2]) and the tightness t (§12 of [Mo 2]).

Moreover, 1.14 can be applied to the topological density d, as this cardinal invariant satisfies the corresponding condition $\Box^d_{\circledast}$. [Note that $d(\underset{i<\mu^+}{\circledast} B_i) = \max\{\lambda, \sup_{i<\mu^+} d(B_i)\}$, where λ is the first cardinal such that $\mu^+ \leq 2^\lambda$, so $\lambda \leq \mu$; see §5 of [Mo 2].]

2 Topological density

The topological density of a Boolean algebra B (i.e. the density of its Stone space Ult B) equals to $\min\{\kappa : B \text{ is } \kappa\text{-centered}\}$. To describe it as a minimal definable first order cardinality invariant we use the theory defined below.

Definition 2.1 *1. For $n < \omega$ define the formulas ϕ^d_n by:*

$$\phi^d_0 = (\forall x)(\exists y \in P_0)(x \neq 0 \;\Rightarrow\; P_1(y,x)) \;\&\; (\forall x)(\forall y \in P_0)(P_1(y,x) \;\Rightarrow\; x \neq 0)$$

and for $n > 0$:

$$\phi^d_n = (\forall x_0,\ldots,x_n)(\forall y \in P_0)(P_1(y,x_0) \,\&\ldots\&\, P_1(y,x_n) \;\Rightarrow\; x_0 \wedge \ldots \wedge x_n \neq 0).$$

2. For $n \leq \omega$ let $T^n_d = \{\phi_k : k < n\}$.

3. For a Boolean algebra B, $n \leq \omega$ we put $d_n(B) = \min \mathrm{Inv}_{T^n_d}(B)$.

4. *For* $1 \leq n < \omega$, *a subset* X *of a Boolean algebra* B *has* the n–intersection property *provided that the meet of any* n *elements of* X *is nonzero; if* X *has the* n*–intersection property for all* n*, then* X *is* centered*, or has* the finite intersection property*.*

Note that $d_\omega(B)$ is the topological density $d(B)$ of B. Since $T^0_d = \emptyset$, the invariant $d_0(B)$ is just 0. The theory T^{n+1}_d says that for each $y \in P_0$ the set $X_y \stackrel{\text{def}}{=} \{x : P_1(y,x)\}$ has the $n+1$–intersection property and $\bigcup_{y\in P_0} X_y = B \setminus \{0\}$. Thus, for $1 \leq n < \omega$, $d_n(B)$ is the smallest cardinal κ such that $B \setminus \{0\}$ is the union of κ sets having the n–intersection property.

We easily get (like 1.11):

Fact 2.2 *1. For a Boolean algebra* B*, the sequence* $\langle d_n(B) : 1 \leq n \leq \omega\rangle$ *is increasing and* $d(B) \leq \prod_{1\leq n<\omega} d_n(B)$*.*

2. *If* D *is an ultrafilter on a cardinal* κ*,* $f : \kappa \longrightarrow \omega$ *is a function such that* $\lim_D f = \omega$ *and* B_i *(for* $i < \kappa$*) are Boolean algebras then* $d(\prod_{i<\kappa} B_i/D) \leq \prod_{i<\kappa} d_{f(i)}(B_i)/D$*.* ■

Fact 2.3 *1. If* $1 \leq n < \omega$ *and* X *is a dense subset of* $B \setminus \{0\}$*, then* $d_n(B)$ *is the least cardinal* κ *such that* X *can be written as a union of* κ *sets each with the* n*–intersection property.*

2. *If* X *is a dense subset of* $B \setminus \{0\}$*, then* $d_\omega(B)$ *is the least cardinal* κ *such that* X *can be written as a union of* κ *sets each with the finite intersection property.*

3. *If* B *is an interval Boolean algebra then* $d_2(B) = d(B)$*.*

PROOF: Suppose $X \subseteq B \setminus \{0\}$ is dense, $1 \leq n < \omega$. Obviously X can be written as a union of $d_n(B)$ sets each with the n–intersection property. If $X = \bigcup_{i<\kappa} Y_i$, where Y_i have the n–intersection property, let $Z_i \stackrel{\text{def}}{=} \{b \in B : (\exists y \in Y_i)(y \leq b)\}$. Then each Z_i has the n–intersection property and $B \setminus \{0\} = \bigcup_{i<\kappa}$. This proves condition 1; condition 2 is proved similarly. Condition 3 follows since for an interval algebra B intervals are dense in B and if $a_1, \ldots, a_k$ are intervals such that $a_i \wedge a_j \neq 0$ then $\bigwedge_{i=1}^{k} a_i \neq 0$. ■

A natural question that arises here is if we can distinguish the invariants d_n. The positive answer is given by the examples below.

Example 2.4 *Let κ be an infinite cardinal, $n > 2$. There is a Boolean algebra B such that $d_n(B) > \kappa$, $d_{n-1}(B) \leq 2^{<\kappa}$.*

PROOF: Let B be the Boolean algebra generated freely by $\{x_\eta : \eta \in {}^\kappa n\}$ except:

> if $\nu \in {}^{\kappa>}n$, $\nu\hat{\ }\langle l\rangle \subseteq \eta_l \in {}^\kappa n$ (for $l < n$)
> then $x_{\eta_0} \wedge \ldots \wedge x_{\eta_{n-1}} = 0$.

Suppose that $B^+ = \bigcup\limits_{i<\kappa} D_i$. For $\eta \in {}^\kappa n$ let $i(\eta) < \kappa$ be such that $x_\eta \in D_{i(\eta)}$. Now we inductively try to define $\eta^* \in {}^\kappa n$:

> assume that we have defined $\eta^*{\restriction}i$ $(i < \kappa)$ and we want to choose $\eta^*(i)$. If there is $l < n$ such that $i(\eta) \neq i$ for each $\eta \supseteq \eta^*{\restriction}i\hat{\ }\langle l\rangle$ then we choose one such l and put $\eta^*(i) = l$. If there is no such l then we stop our construction.

If the construction was stopped at stage $i < \kappa$ (i.e. we were not able to choose $\eta^*(i)$) then for each $l < n$ we have a sequence $\eta_l \in {}^\kappa n$ such that $\eta^*{\restriction}i\hat{\ }\langle l\rangle \subseteq \eta_l$ and $i(\eta_l) = i$. Thus $x_{\eta_0}, \ldots, x_{\eta_{n-1}} \in D_i$ and $x_{\eta_0} \wedge \ldots \wedge x_{\eta_{n-1}} = 0$, so that D_i does not satisfy the n–intersection property. If we could carry our construction up to κ then we would get $\eta^* \in {}^\kappa n$ such that $x_{\eta^*} \notin \bigcup\limits_{i<\kappa} D_i$. Consequently the procedure had to stop and we have proved that $d_n(B) > \kappa$.

Now we are going to show that $d_{n-1}(B) \leq 2^{<\kappa}$. Let X be the set of all nonzero elements of B of the form

$$x_{\eta_0} \wedge \ldots \wedge x_{\eta_l} \wedge (-x_{\eta_{l+1}}) \wedge \ldots \wedge (-x_{\eta_k})$$

in which the sequences $\eta_0, \ldots, \eta_k \in {}^\kappa n$ are pairwise distinct, $0 < l < k < \omega$. Clearly X is dense in B. We are going to apply fact 2.3(1). To this end, if $0 < l < k$, $\alpha < \kappa$, and $\langle \nu_0, \ldots, \nu_k\rangle$ is a sequence of distinct members of ${}^\alpha n$, let $D^{l,k,\alpha}_{\langle \nu_0,\ldots,\nu_k\rangle}$ be the set

$$\{x_{\eta_0} \wedge \ldots \wedge x_{\eta_l} \wedge (-x_{\eta_{l+1}}) \wedge \ldots \wedge (-x_{\eta_k}) : \nu_0 \subseteq \eta_0 \in {}^\kappa n, \ldots, \nu_k \subseteq \eta_k \in {}^\kappa n\} \setminus \{0\}.$$

Note that X is the union of all these sets. There are $2^{<\kappa}$ possibilities for the parameters, so it suffices to show that each of the sets $D^{l,k,\alpha}_{\langle \nu_0,\ldots,\nu_k\rangle}$ has the $(n-1)$–intersection property.

Before beginning on this, note that if $\eta_0, \ldots, \eta_k \in {}^\kappa n$ are such that $\eta_i \neq \eta_j$ when $i \leq l < j \leq k$ and

$$B \models x_{\eta_0} \wedge \ldots \wedge x_{\eta_l} \wedge (-x_{\eta_{l+1}}) \wedge \ldots \wedge (-x_{\eta_k}) = 0,$$

then necessarily there is $\nu \in {}^{<\kappa}n$ such that

$$(\forall m < n)(\exists i \leq l)(\nu\hat{\ }\langle m\rangle \subseteq \eta_l).$$

Now we check that $D^{l,k,\alpha}_{\langle\nu_0,\ldots,\nu_k\rangle}$ has the $(n-1)$–intersection property, where $0 < l < k < \omega$, $\alpha < \kappa$, and $\nu_0, \ldots, \nu_k$ are pairwise distinct elements of ${}^{\alpha}n$. Thus suppose that

$$x_{\eta^j_0} \wedge \ldots \wedge x_{\eta^j_l} \wedge (-x_{\eta^j_{l+1}}) \wedge \ldots \wedge (-x_{\eta^j_k})$$

are members of $D^{l,k,\alpha}_{\langle\nu_0,\ldots,\nu_k\rangle}$ for each $j < n-1$; and suppose that

$$B \models \bigwedge_{j<n-1} x_{\eta^j_0} \wedge \ldots \wedge \bigwedge_{j<n-1} x_{\eta^j_l} \wedge \bigwedge_{j<n-1} (-x_{\eta^j_{l+1}}) \wedge \ldots \wedge \bigwedge_{j<n-1} (-x_{\eta^j_k}) = 0.$$

By the above remark, choose $\nu \in {}^{<\kappa}n$ such that for all $m < n$ there exist an $i(m) \leq l$ and a $j(m) < n-1$ such that $\nu\hat{\ }\langle m\rangle \subseteq \eta^{j(m)}_{i(m)}$ (note that if $j_0, j_1 < n-1$, $i_0 \leq l$ and $l+1 \leq i_1 \leq k$ then $\eta^{j_0}_{i_0} \neq \eta^{j_1}_{i_1}$ as $\nu_0, \ldots, \nu_k$ are pairwise distinct).

Case 1: $\nu_i \subseteq \nu$ for some $i \leq k$.
Then for each $m < n$ we have $\nu_i \subseteq \nu \subseteq \nu\hat{\ }\langle m\rangle \subseteq \eta^{j(m)}_{i(m)}$ and consequently $i(m) = i$ (for $m < n$). As $j(m) < n-1$ for $m < n$ we find $m_0 < m_1 < n-1$ such that $j(m_0) = j(m_1) = j$. Then $\nu\hat{\ }\langle m_0\rangle \subseteq \eta^j_i$, $\nu\hat{\ }\langle m_1\rangle \subseteq \eta^j_i$ give a contradiction.

Case 2: $\nu_i \not\subseteq \nu$ for all $i \leq k$.
Note that for all $m < n$ the sequences $\nu\hat{\ }\langle m\rangle$ and $\nu_{i(m)}$ are compatible. By the case we are in, it follows that ν is shorter than $\nu_{i(m)}$. So $\nu\hat{\ }\langle m\rangle \subseteq \nu_{i(m)}$, $i(m) < l$. But then by construction, $D^{l,k,\alpha}_{\langle\nu_0,\ldots\nu_k\rangle}$ is empty, a contradiction. ■

Example 2.5 *Let λ_i be cardinals (for $i < \kappa$) such that $2^\kappa < \prod\limits_{i<\kappa} \lambda_i$, $2 < n < \omega$. Then there is a Boolean algebra B such that*

$$d_{n-1}(B) \leq \sum_{\alpha<\kappa} \prod_{i<\alpha} \lambda_i \quad \textit{and} \quad d_n(B) = \|B\| = \prod_{i<\kappa} \lambda_i.$$

In particular, if λ is a strong limit cardinal, $\mathrm{cf}(\lambda) < \lambda$, *$2 < n < \omega$ then there is a Boolean algebra B such that $d_n(B) = \|B\| = 2^\lambda$, $d_{n-1}(B) \leq \lambda$.*

PROOF: Let B be the Boolean algebra generated freely by $\{x_\eta : \eta \in \prod\limits_{i<\kappa} \lambda_i\}$ except that:

if $\alpha<\kappa$, $v\in\prod\limits_{i<\alpha}\lambda_i$, $v\subseteq\eta_l\in\prod\limits_{i<\kappa}\lambda_i$, $\|\{\eta_l(\alpha):l<n\}\|=n$
then $x_{\eta_0}\wedge\ldots\wedge x_{\eta_{n-1}}=0$

The same arguments as in the previous example show that

$$d_{n-1}(B)\leq\sum_{\alpha<\kappa}\prod_{i<\alpha}\lambda_i.$$

Suppose now that $\prod\limits_{i<\kappa}\lambda_i=\bigcup\{D_j:j<\theta\}$, $\theta<\prod\limits_{i<\kappa}\lambda_i$ and if $\eta_0,\ldots,\eta_{n-1}\in D_j$, $j<\theta$ then $x_{\eta_0}\wedge\ldots\wedge x_{\eta_{n-1}}\neq 0$. Thus the trees $T_j=\{\eta\restriction\alpha:\alpha<\kappa,\eta\in D_j\}$ have no splitting into more than $n-1$ points and hence $\|D_j\|\leq n^\kappa<\prod\limits_{i<\kappa}\lambda_i$ for all $j<\theta$ and we get a contradiction, proving $d_n(B)=\prod\limits_{i<\kappa}\lambda_i$. ∎

Corollary 2.6 *Let λ be a strong limit cardinal, $\kappa<\mathrm{cf}(\lambda)<\lambda$. Suppose that D is an ultrafilter on κ which is not $\aleph_1$-complete. Then there exist Boolean algebras B_i (for $i<\kappa$) such that*

$$d(\prod_{i<\kappa}B_i/D)\leq\lambda<2^\lambda=\prod_{i<\kappa}d(B_i)/D.$$

PROOF: As D is not $\aleph_1$-complete we find a function $f:\kappa\longrightarrow\omega\setminus 2$ such that $\lim\limits_D f=\omega$. Let B_i be such that $\|B_i\|=d_{f(i)+1}(B_i)=2^\lambda$, $d_{f(i)}(B_i)\leq\lambda$ (see 2.5). Then, by 2.2, we have $d(\prod\limits_{i<\kappa}B_i/D)\leq\prod\limits_{i<\kappa}d_{f(i)}(B_i)/D\leq\lambda^\kappa=\lambda$. As $d(B_i)=d_{f(i)+1}(B_i)=2^\lambda$ we have $\prod\limits_{i<\kappa}d(B_i)/D=2^\lambda$. ∎

Remark: 1. Corollary 2.6 applied e.g. to $\lambda=\beth_{\omega_1}$, $\kappa=\omega$ gives a negative answer to Problem J of [Mo 3].
2. The algebras B_i in 2.6 are of a quite large size: $\|B_i\|=2^\lambda$, λ strong limit of the cofinality $>\kappa$. Moreover the cardinal λ had to be singular. The natural question if these are real limitations is answered by the theorem below. This example, though more complicated than the previous ones, has several nice properties. E.g. it produces algebras of the size $2^{(2^{\aleph_0})^+}$ already.

Theorem 2.7 *Assume that $\theta=\mathrm{cf}(\theta)$, $\theta^\kappa=\theta$. Then there are Boolean algebras B_γ for $\gamma<\kappa$ such that $d(B_\gamma)=d_2(B_\gamma)=\theta^+$ and $d(\prod\limits_{\gamma<\kappa}B_\gamma/D)\leq\theta$ for every non-principal ultrafilter D on κ.*

PROOF: Let $\lambda=2^\theta$. Choose $\eta_{\alpha,i}\in{}^\theta\theta$ for $\alpha<\lambda$, $i<\theta^+$ such that

1. if $\eta_{\alpha_1,i_1}=\eta_{\alpha_2,i_2}$ then $(\alpha_1,i_1)=(\alpha_2,i_2)$,

2. for each $f \in {}^\theta\theta$ and $i < \theta^+$ the set $\{\alpha < \lambda : (\forall \varepsilon < \theta)(f(\varepsilon) \leq \eta_{\alpha,i}(\varepsilon))\}$ is of the size λ

(the choice is possible as there is $2^\theta = \lambda$ pairs (f, i) to take care of and for each such pair we have 2^θ candidates for $\eta_{\alpha,i}$).

For two functions $f, g \in {}^\theta\theta$ we write $f <^* g$ if and only if

$$\|\{\varepsilon < \theta : f(\varepsilon) \geq g(\varepsilon)\}\| < \theta.$$

We say that a set $A \subseteq \lambda \times \theta^+$ is i-large (for $i < \theta^+$) if for every $f \in {}^\theta\theta$ we have $\|\{\alpha < \lambda : (\alpha, i) \in A \ \&\ f <^* \eta_{\alpha,i}\}\| = \lambda$ and we say that A is large if $\sup\{i < \theta^+ : A \text{ is } i\text{-large}\} = \theta^+$.

Claim 2.7.1 *The union of at most θ sets which are not large is not large.*

Proof of the claim: Should be clear as $\mathrm{cf}(\theta) = \theta < \mathrm{cf}(\lambda)$.

Now we are going to describe the construction of the Boolean algebras we need. First suppose that $S \subseteq \{j < \theta^+ : \mathrm{cf}(j) = \theta\}$ is a stationary set and let $S^+ = \{(\alpha, j) \in \lambda \times \theta^+ : j \in S\}$. Now choose a sequence $\bar{F} = \langle F_\varepsilon : \varepsilon < \theta\rangle$ such that:

3. F_ε is a function with the domain $\mathrm{dom}(F_\varepsilon) = S^+$,

4. if $i \in S$, $\alpha < \lambda$ then $F_\varepsilon(\alpha, i) = (F_{\varepsilon,1}(\alpha, i), F_{\varepsilon,2}(\alpha, i)) \in \lambda \times i$,

5. if $i \in S$, $\alpha < \lambda$ then the sequence $\langle F_{\varepsilon,2}(\alpha, i) : \varepsilon < \theta\rangle$ is strictly increasing with the limit i,

6. if $\langle A_\varepsilon : \varepsilon < \theta\rangle$ is a sequence of large subsets of $\lambda \times \theta^+$ then for some stationary set $S' \subseteq S$ for each $i \in S'$, $f \in {}^\theta\theta$ we have

$$\|\{\alpha < \lambda : f <^* \eta_{\alpha,i} \ \&\ (\forall \varepsilon < \theta)(F_\varepsilon(\alpha, i) \in A_\varepsilon)\}\| = \lambda,$$

7. if $\varepsilon < \zeta < \theta$ then $\eta_{F_\varepsilon(\alpha,i)} <^* \eta_{F_\zeta(\alpha,i)}$.

To construct the sequence $\bar{F}$ fix $i \in S$. Let $\{(f_\alpha, g_\alpha, \bar{j}_\alpha) : \alpha < \lambda\}$ enumerate with λ-repetitions all triples $(f, g, \bar{j})$ such that $f \in {}^\theta\theta$, $\bar{j} = \langle j_\varepsilon : \varepsilon < \theta\rangle$ is an increasing cofinal sequence in i and $g \in {}^\theta\lambda$ is such that

$$(*) \qquad \varepsilon < \zeta < \theta \quad \Rightarrow \quad \eta_{g(\varepsilon), j_\varepsilon} <^* \eta_{g(\zeta), j_\zeta}$$

(recall that $\mathrm{cf}(i) = \theta$, $\lambda = 2^\theta$). Now we inductively choose $\langle \beta_\alpha : \alpha < \lambda\rangle \subseteq \lambda$ such that $\beta_\alpha \notin \{\beta_\delta : \delta < \alpha\}$, $f_\alpha <^* \eta_{\beta_\alpha,i}$ (this is possible by (2)). Finally for $\alpha < \lambda$ and $\varepsilon < \theta$ define $F_\varepsilon(\alpha, i)$ by:

if $\alpha = \beta_\delta$ for some $\delta < \lambda$ then $F_\varepsilon(\alpha, i) = (g_\delta(\varepsilon), j^\delta_\varepsilon)$,
if $\alpha \notin \{\beta_\delta : \delta < \lambda\}$ then $F_\varepsilon(\alpha, i) = (g_\alpha(\varepsilon), j^\alpha_\varepsilon)$

(where $\bar{j}_\delta = \langle j^\delta_\varepsilon : \varepsilon < \theta\rangle$). Easily conditions (3)–(5) and (7) are satisfied. To check clause (6) suppose that $\langle A_\varepsilon : \varepsilon < \theta\rangle$ is a sequence of large sets and let S' be the set of all $i \in S$ such that there exists an increasing cofinal sequence $\langle j_\varepsilon : \varepsilon < \theta\rangle \subseteq i$ such that A_ε is j_ε-large (for each $\varepsilon < \theta$). The set S' is stationary. [Why? For $\varepsilon < \theta$ let C_ε be the set of all points in θ^+ which are limits of increasing sequences from $\{j < \theta^+ : A_\varepsilon$ is j–large$\}$. Clearly each C_ε is a club of θ^+ and thus $\bigcap\limits_{\varepsilon<\theta} C_\varepsilon$ is a club of θ^+. Now one easily checks that $S \cap \bigcap\limits_{\varepsilon<\theta} C_\varepsilon \subseteq S'$.]
We are going to show that S' works for $\langle A_\varepsilon : \varepsilon < \theta\rangle$. Take $i \in S'$ and suppose that $f \in {}^\theta\theta$. Let $\bar{j} = \langle j_\varepsilon : \varepsilon < \theta\rangle \subseteq i$ be an increasing cofinal sequence witnessing $i \in S'$. Take $g \in {}^\theta\lambda$ such that

$$\varepsilon < \zeta < \theta \ \Rightarrow\ [\eta_{g(\varepsilon),j_\varepsilon} <^* \eta_{g(\zeta),j_\zeta} \ \&\ (g(\varepsilon), j_\varepsilon) \in A_\varepsilon]$$

(possible by the j_ε-largeness of A_ε and the regularity of θ). When we defined $F_\varepsilon(\alpha, i)$ (for $\varepsilon < \theta$, $\alpha < \lambda$), the triple $(f, g, \bar{j})$ appeared λ times in the enumeration $\{(f_\alpha, g_\alpha, \bar{j}_\alpha) : \alpha < \lambda\}$. Whenever $(f, g, \bar{j}) = (f_\alpha, g_\alpha, \bar{j}_\alpha)$ we had $F_\varepsilon(\beta_\alpha, i) = (g_\alpha(\varepsilon), j^\alpha_\varepsilon) = (g(\varepsilon), j_\varepsilon) \in A_\varepsilon$ and $f = f_\alpha <^* \eta_{\beta_\alpha,i}$. Consequently if $i \in S'$, $f \in {}^\theta\theta$ then

$$\|\{\alpha < \lambda : f <^* \eta_{\alpha,i} \ \&\ (\forall \varepsilon < \theta)(F_\varepsilon(\alpha, i) \in A_\varepsilon)\}\| = \lambda$$

and condition (6) holds.

For the sequence $\bar{F}$ we define a Boolean algebra $B_{\bar{F}}$: it is freely generated by $\{x_{\alpha,i} : \alpha < \lambda, i < \theta^+\}$ except that

if $F_\varepsilon(\alpha_1, i_1) = (\alpha_2, i_2)$ for some $\varepsilon < \theta$
then $x_{\alpha_1,i_1} \wedge x_{\alpha_2,i_2} = 0$.

Now fix a sequence $\langle S_\gamma : \gamma < \kappa\rangle$ of pairwise disjoint stationary subsets of $\{j \in \theta^+ : \mathrm{cf}(j) = \theta\}$ and for each $\gamma < \kappa$ fix a sequence $\bar{F}_\gamma = \langle F^\gamma_\varepsilon : \varepsilon < \theta\rangle$ satisfying conditions (3)–(7) above (for S_γ).

Claim 2.7.2 *For each $\gamma < \kappa$, $d_2(B_{\bar{F}_\gamma}) > \theta$.*

Proof of the claim: Let $\bar{F} = \bar{F}_\gamma$ and suppose that $B^+_{\bar{F}} = \bigcup\limits_{\varepsilon<\theta} D_\varepsilon$. Let $A_\varepsilon = \{(\alpha, i) : x_{\alpha,i} \in D_\varepsilon\}$ and let $A'_\varepsilon = A_\varepsilon$ if A_ε is large and $A'_\varepsilon = \lambda \times \theta^+$ otherwise. So the sets A'_ε are large (for $\varepsilon < \theta$) and by condition (6) the set

$$A \stackrel{\text{def}}{=} \{(\alpha, i) \in \lambda \times \theta^+ : (\forall \varepsilon < \theta)(F_\varepsilon(\alpha, i) \in A'_\varepsilon)\}$$

is large too. Since $A_\varepsilon \neq A'_\varepsilon$ implies that A_ε is not large we get (by 2.7.1) that

$$A \setminus \bigcup\{A_\varepsilon : A_\varepsilon \neq A'_\varepsilon \ \&\ \varepsilon < \theta\} \neq \emptyset.$$

So take $(\alpha, i) \in A \setminus \bigcup\{A_\varepsilon : A_\varepsilon \neq A'_\varepsilon \ \&\ \varepsilon < \theta\}$. We find $\varepsilon < \theta$ such that $x_{\alpha,i} \in D_\varepsilon$ (so $(\alpha, i) \in A_\varepsilon$). Then $A_\varepsilon = A'_\varepsilon$ and we get $F_\varepsilon(\alpha, i) \in A_\varepsilon$. Hence $x_{\alpha,i}, x_{F_\varepsilon(\alpha,i)} \in D_\varepsilon$ and $x_{\alpha,i} \wedge x_{F_\varepsilon(\alpha,i)} = 0$.

Claim 2.7.3 *Let D be a non-principal ultrafilter on κ.*
Then $d(\prod\limits_{\gamma<\kappa} B_{\bar{F}_\gamma}/D) \leq \theta$.

Proof of the claim: Fix functions $h : \theta^+ \times \theta^+ \longrightarrow \theta$ and $h^* : \theta^+ \times \theta \longrightarrow \theta^+$ such that for $i \in (\theta, \theta^+)$, $\zeta \in \theta$:

$$j_1 < j_2 < i \ \Rightarrow\ h(i, j_1) \neq h(i, j_2), \qquad h^*(i, \zeta) < i \quad \text{and}$$
$$j < i \ \Rightarrow\ h^*(i, h(i, j)) = j.$$

For $\gamma < \kappa$ let $Z_\gamma \subseteq B_{\bar{F}_\gamma}$ be the set of all meets $x_{a_0} \wedge \ldots \wedge x_{a_{n-1}} \wedge (-x_{b_0}) \wedge \ldots \wedge (-x_{b_{m-1}})$ such that:

$a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1} \in \lambda \times \theta^+$ are with no repetition,
for all $k, l < n$, $r < m$ and all $\varepsilon < \theta$

$$F^\gamma_\varepsilon(a_k) \neq a_l \quad \& \quad F^\gamma_\varepsilon(b_r) \neq a_l \quad \& \quad F^\gamma_\varepsilon(a_l) \neq b_r$$

Clearly Z_γ is dense in $B_{\bar{F}_\gamma}$ and $Z \stackrel{\rm def}{=} \prod\limits_{\gamma<\kappa} Z_\gamma/D$ is dense in $\prod\limits_{\gamma<\kappa} B_{\bar{F}_\gamma}/D$. For $e \in \prod\limits_{\gamma<\kappa} Z_\gamma$ and $\gamma < \kappa$ let:

- $e(\gamma) = \bigwedge\limits_{l<n(e,\gamma)} x_{a(e,l,\gamma)} \wedge \bigwedge\limits_{l<m(e,\gamma)} -x_{b(e,l,\gamma)}$,
- $a(e, l, \gamma) = (\alpha(e, l, \gamma), i(e, l, \gamma))$,
- $b(e, l, \gamma) = (\beta(e, l, \gamma), j(e, l, \gamma))$,
- $\text{base}^\gamma(e) = \{a(e, l, \gamma) : l < n(e, \gamma)\} \cup \{b(e, l, \gamma) : l < m(e, \gamma)\}$,
- $\text{base}(e) = \bigcup\limits_{\gamma<\kappa} \text{base}^\gamma(e)$,
- $u_0(e) = \{i < \theta^+ : (\exists \alpha < \lambda)((\alpha, i) \in \text{base}(e))\}$,
- $u_1(e)$ be the (topological) closure of $u_0(e)$,
- $u_2(e)$ be the closure of $u_1(e)$ under the functions h, h^*,

- $\zeta(e)$ be the first $\varepsilon < \theta$ such that

$$(\forall \gamma < \kappa)(\forall (\alpha, i) \in \mathrm{base}(e) \cap \mathrm{dom}(F^\gamma_\varepsilon))(\sup(u_1(e) \cap i) < F^\gamma_{\varepsilon,2}(\alpha, i)).$$

[Note that $\|\mathrm{base}(e)\| \le \kappa < \mathrm{cf}(\theta) = \theta$, so $\|u_0(e)\|, \|u_1(e)\|, \|u_2(e)\| \le \kappa$; looking at the definition of $\zeta(e)$ remember that $(\alpha, i) \in \mathrm{dom}(F^\gamma_\varepsilon)$ implies $\mathrm{cf}(i) = \theta$.]

Next for each $\gamma < \kappa$, $(\alpha, i) \in S^+_\gamma$, $\zeta_0 < \theta$ choose $\varepsilon^\gamma_{\zeta_0}(\alpha, i) < \theta$ such that the sequence $\langle \eta_{F^\gamma_\zeta(\alpha,i)}(\varepsilon^\gamma_{\zeta_0}(\alpha, i)) : \zeta \le \zeta_0 \rangle$ is strictly increasing (it is enough to take $\varepsilon^\gamma_{\zeta_0}(\alpha, i)$ sufficiently large – apply condition (7) for $\bar{F}_\gamma$ remembering $\zeta_0 < \theta$). Further, for $(\alpha, i), (\beta, j) \in \lambda \times \theta^+$, $\gamma < \kappa$ and $\zeta_0 < \theta$ such that $(\beta, j) \notin \{F^\gamma_\varepsilon(\alpha, i) : \varepsilon \le \zeta_0\}$ choose $\varepsilon^\gamma_{\zeta_0}((\alpha, i), (\beta, j)) < \theta$ such that for every $\zeta \le \zeta_0$:

either $\eta_{F^\gamma_\zeta(\alpha,i)}(\varepsilon^\gamma_{\zeta_0}((\alpha, i), (\beta, j))) \ne \eta_{\beta,j}(\varepsilon^\gamma_{\zeta_0}((\alpha, i), (\beta, j)))$
or $\eta_{F^\gamma_\zeta(\alpha,i)}(\varepsilon^\gamma_{\zeta_0}(\alpha, i)) \ne \eta_{\beta,j}(\varepsilon^\gamma_{\zeta_0}(\alpha, i))$

(this is possible as the second condition may fail for at most one $\zeta \le \zeta_0$: the sequence $\langle \eta_{F^\gamma_\zeta(\alpha,i)}(\varepsilon^\gamma_\zeta(\alpha, i)) : \zeta \le \zeta_0 \rangle$ is strictly increasing). Next, for each $e \in \prod_{\gamma<\kappa} Z_\gamma$ and $\gamma < \kappa$ choose a finite set $\mathcal{X}_\gamma(e) \subseteq \theta$ such that:

8. if $a, b \in \mathrm{base}^\gamma(e)$ are distinct then $\eta_a \restriction \mathcal{X}_\gamma(e) \ne \eta_b \restriction \mathcal{X}_\gamma(e)$,

9. if $a \in \mathrm{base}^\gamma(e) \cap S^+_\gamma$ then $\varepsilon^\gamma_{\zeta(e)}(a) \in \mathcal{X}_\gamma(e)$,

10. if $a, b \in \mathrm{base}^\gamma(e)$ then $\varepsilon^\gamma_{\zeta(e)}(a, b) \in \mathcal{X}_\gamma(e)$ (if defined),

(remember that $\mathrm{base}^\gamma(e)$ is finite). Finally we define a function H on $\prod_{\gamma<\kappa} Z_\gamma$ such that for $e \in \prod_{\gamma<\kappa} Z_\gamma$ the value $H(e)$ is the sequence consisting of the following objects:

11. $\langle n(e, \gamma) : \gamma < \kappa \rangle$,

12. $\langle m(e, \gamma) : \gamma < \kappa \rangle$,

13. $\zeta(e)$,

14. $\langle \mathcal{X}_\gamma(e) : \gamma < \kappa \rangle$,

15. $\langle (\gamma, l, \eta_{a(e,l,\gamma)} \restriction \mathcal{X}_\gamma(e)) : \gamma < \kappa, l < n(e, \gamma) \rangle$,

16. $\langle (\gamma, l, \eta_{b(e,l,\gamma)} \restriction \mathcal{X}_\gamma(e)) : \gamma < \kappa, l < m(e, \gamma) \rangle$,

17. $u_2(e) \cap \theta$,

18. $\{(\mathrm{otp}(i \cap u_2(e)), \mathrm{otp}(i \cap u_1(e))) : i \in u_2(e)\}$.

Since $\theta^\kappa = \theta$ we easily check that $\|\mathrm{rng}(H)\| \leq \theta$. For $\Upsilon \in \mathrm{rng}(H)$ let

$$Z_\Upsilon = \{e \in \prod_{\gamma<\kappa} Z_\gamma : H(e) = \Upsilon\} \quad \text{and} \quad Z^*_\Upsilon = \{e/D : e \in Z_\Upsilon\} \subseteq Z.$$

The claim will be proved if we show that

for each $\Upsilon \in \mathrm{rng}(H)$ the set Z^*_Υ is centered.

First note that if $e, e' \in Z_\Upsilon$ then $u_2(e) \cap u_2(e')$ is an initial segment of both $u_2(e)$ and $u_2(e')$. Why? Suppose that $j < i \in u_2(e) \cap u_2(e')$, $j \in u_2(e)$. If $j < \theta$ then $j \in u_2(e')$ since $u_2(e) \cap \theta = u_2(e') \cap \theta$. Suppose that $\theta \leq j < \theta^+$. Then $h(i,j) \in u_2(e) \cap \theta = u_2(e') \cap \theta$ and so $j = h^*(i, h(i,j)) \in u_2(e')$. This shows that $u_2(e) \cap u_2(e')$ is an initial segment of $u_2(e)$. Similarly for $u_2(e')$. Applying to this fact condition (18) we may conclude that $u_1(e) \cap u_1(e')$ is an initial segment of both $u_1(e)$ and $u_1(e')$ for $e, e' \in Z_\Upsilon$. [Why? Assume not. Let $i < \theta^+$ be the first such that there is $j \in u_1(e) \cap u_1(e')$ above i but

$$i \in (u_1(e) \setminus u_1(e')) \cup (u_1(e') \setminus u_1(e)).$$

By symmetry we may assume that $i \in u_1(e) \setminus u_1(e')$. Let i^* be the first element of $u_2(e)$ above i. Then necessarily $i^* \leq j$ (as $j \in u_1(e) \cap u_1(e') \subseteq u_2(e) \cap u_2(e')$) and hence $i^* \in u_2(e')$ (and i^* is the first element of $u_2(e')$ above i). By the choice of i, i^* we have

$$i, i^* \in u_2(e) \cap u_2(e'), \quad i \cap u_1(e) = i \cap u_1(e'), \quad \text{and } i^* \cap u_2(e) = i^* \cap u_2(e').$$

But now we may apply condition (18) to conclude that

$$(\mathrm{otp}(i^* \cap u_2(e)), \mathrm{otp}(i^* \cap u_1(e))) = (\mathrm{otp}(i^* \cap u_2(e')), \mathrm{otp}(i^* \cap u_1(e')))$$

and therefore

$$\mathrm{otp}(i^* \cap u_1(e')) = \mathrm{otp}(i^* \cap u_1(e)) = \mathrm{otp}(i \cap u_1(e)) + 1 = \mathrm{otp}(i \cap u_1(e')) + 1.$$

As there is no point of $u_1(e')$ in the interval $[i, i^*)$ (remember $u_1(e') \subseteq u_2(e')$) we get a contradiction.]

For $e \in Z_\Upsilon$ we have: $n(e,\gamma) = n(\gamma)$, $m(e,\gamma) = m(\gamma)$, $\zeta(e) = \zeta^*$, $\mathcal{X}_\gamma(e) = \mathcal{X}_\gamma$. Let $e_0, \ldots, e_{k-1} \in Z_\Upsilon$. We are going to show that

$$\prod_{\gamma<\kappa} B_{\bar{F}_\gamma}/D \models e_0/D \wedge \ldots \wedge e_{k-1}/D \neq 0$$

and for this we have to prove that

$$I_{e_0,\ldots,e_{k-1}} \stackrel{\rm def}{=} \{\gamma<\kappa : B_{\bar F_\gamma} \models \bigwedge_{j<k}[\bigwedge_{l<n(\gamma)} x_{a(e_j,l,\gamma)} \wedge \bigwedge_{l<m(\gamma)} -x_{b(e_j,l,\gamma)}] \neq 0\} \in D.$$

First let us ask what can be the reasons for

$$\bigwedge_{j<k}[\bigwedge_{l<n(\gamma)} x_{a(e_j,l,\gamma)} \wedge \bigwedge_{l<m(\gamma)} -x_{b(e_j,l,\gamma)}] = 0.$$

There are essentially two cases here: either $x_a\wedge(-x_a)$ appears on the left-hand side of the above equality or $x_a\wedge x_{F^\gamma_\zeta(a)}$ (for some $\zeta<\theta$) appears there. Suppose that the first case happens. Then we have distinct $j_1,j_2<k$ such that $a(e_{j_1},l_1,\gamma)=b(e_{j_2},l_2,\gamma)$ for some l_1,l_2. By (15) (and the definition of Z_Υ) we have $\eta_{a(e_{j_1},l_1,\gamma)}\restriction\mathcal{X}_\gamma=\eta_{a(e_{j_2},l_1,\gamma)}\restriction\mathcal{X}_\gamma$ and by (8) we have $\eta_{a(e_{j_2},l_1,\gamma)}\restriction\mathcal{X}_\gamma\neq\eta_{b(e_{j_2},l_2,\gamma)}\restriction\mathcal{X}_\gamma$. Consequently $\eta_{a(e_{j_1},l_1,\gamma)}\restriction\mathcal{X}_\gamma\neq\eta_{b(e_{j_2},l_2,\gamma)}\restriction\mathcal{X}_\gamma$, a contradiction. Consider now the second case and suppose additionally that $\zeta\leq\zeta^*$. Thus we assume that for some $\zeta\leq\zeta^*$, for some distinct $j_1,j_2<k$ and some $l_1,l_2<n(\gamma)$ we have $F^\gamma_\zeta(a(e_{j_1},l_1,\gamma))=a(e_{j_2},l_2,\gamma)$. Then by (15) we get $\eta_{a(e_{j_2},l_2,\gamma)}\restriction\mathcal{X}_\gamma=\eta_{a(e_{j_1},l_2,\gamma)}\restriction\mathcal{X}_\gamma$. As $\zeta\leq\zeta^*$ we have that [by the choice of $\varepsilon^\gamma_{\zeta^*}(a(e_{j_1},l_1,\gamma),a(e_{j_1},l_2,\gamma))$, $\varepsilon^\gamma_{\zeta^*}(a(e_{j_1},l_1,\gamma))$ – note that $F^\gamma_\varepsilon(a(e_{j_1},l_1,\gamma))\neq a(e_{j_1},l_2,\gamma)$ for all $\varepsilon\leq\zeta^*$]:

either $\eta_{F^\gamma_\zeta(a(e_{j_1},l_1,\gamma))}(\varepsilon^\gamma_{\zeta^*}(a(e_{j_1},l_1,\gamma),a(e_{j_1},l_2,\gamma)))\neq$
$\eta_{a(e_{j_1},l_2,\gamma)}(\varepsilon^\gamma_{\zeta^*}(a(e_{j_1},l_1,\gamma),a(e_{j_1},l_2,\gamma)))$
or $\eta_{F^\gamma_\zeta(a(e_{j_1},l_1,\gamma))}(\varepsilon^\gamma_{\zeta^*}(a(e_{j_1},l_1,\gamma)))\neq\eta_{a(e_{j_1},l_2,\gamma)}(\varepsilon^\gamma_{\zeta^*}(a(e_{j_1},l_1,\gamma)))$

and $\varepsilon^\gamma_{\zeta^*}(a(e_{j_1},l_1,\gamma),a(e_{j_1},l_2,\gamma)),\varepsilon^\gamma_{\zeta^*}(a(e_{j_1},l_1,\gamma))\in\mathcal{X}_\gamma$ (by (9), (10); note that in the definition of $\varepsilon^\gamma_\zeta(a,b)$ we allowed $a=b$ so no problem appears if $l_1=l_2$). Hence $\eta_{F^\gamma_\zeta(a(e_{j_1},l_1,\gamma))}\restriction\mathcal{X}_\gamma\neq\eta_{a(e_{j_1},l_2,\gamma)}\restriction\mathcal{X}_\gamma$ and thus $\eta_{F^\gamma_\zeta(a(e_{j_1},l_1,\gamma))}\restriction\mathcal{X}_\gamma\neq\eta_{a(e_{j_2},l_2,\gamma)}\restriction\mathcal{X}_\gamma$, a contradiction. Consequently, the considered equality may hold only if $x_a\wedge x_{F^\gamma_\zeta(a)}$ appears there for some $\zeta>\zeta^*$.

Asume now that $I_{e_0,\ldots,e_{k-1}}\notin D$. From the above considerations we know that for each $\gamma\in\kappa\setminus I_{e_0,\ldots,e_{k-1}}$ we find distinct $j_1(\gamma),j_2(\gamma)<k$ and $l_1(\gamma),l_2(\gamma)<n(\gamma)$ and $\zeta_\gamma\in(\zeta^*,\theta)$ such that

$$(**)\qquad F^\gamma_{\zeta_\gamma}(a(e_{j_1(\gamma)},l_1(\gamma),\gamma))=a(e_{j_2(\gamma)},l_2(\gamma),\gamma)$$

(note that $(**)$ implies $a(e_{j_1(\gamma)},l_1(\gamma),\gamma)\in\mathrm{dom}(F^\gamma_{\zeta_\gamma})$, $i(e_{j_1(\gamma)},l_1(\gamma),\gamma)\in S_\gamma$). We have assumed that $\kappa\setminus I_{e_0,\ldots,e_{k-1}}\in D$ so we find $j_1,j_2<k$ such that

$$J\stackrel{\rm def}{=}\{\gamma\in\kappa\setminus I_{e_0,\ldots,e_{k-1}} : j_1(\gamma)=j_1, j_2(\gamma)=j_2\}\in D.$$

As we have remarked after (∗∗), $i(e_{j_1}, l_1(\gamma), \gamma) \in S_\gamma$ (for $\gamma \in J$) and consequently there are no repetitions in the sequence $\langle i(e_{j_1}, l_1(\gamma), \gamma) : \gamma \in J\rangle$ (and J is infinite). Choose $\gamma_n \in J$ (for $n \in \omega$) such that the sequence $\langle i(e_{j_1}, l_1(\gamma_n), \gamma_n) : n \in \omega\rangle$ is strictly increasing (so $i(e_{j_1}, l_1(\gamma_n), \gamma_n) \in u_1(e_{j_1}) \cap i(e_{j_1}, l_1(\gamma_{n+1}), \gamma_{n+1}))$ and let $i = \lim\limits_n i(e_{j_1}, l_1(\gamma_n), \gamma_n)$. By the definition of $\zeta(e), \zeta^*$ and the fact that $\zeta_\gamma > \zeta^*$ for all $\gamma \in J$ (and by (5)) we have that for $\gamma \in J$

$$\begin{aligned} i(e_{j_2}, l_2(\gamma), \gamma) = \; & F^{\gamma}_{\zeta_\gamma,2}(a(e_{j_1}, l_1(\gamma), \gamma)) \in \\ & i(e_{j_1}, l_1(\gamma), \gamma) \setminus \sup(u_1(e_{j_1}) \cap i(e_{j_1}, l_1(\gamma), \gamma)). \end{aligned}$$

Applying this for γ_{n+1} we conclude

$$i(e_{j_1}, l_1(\gamma_n), \gamma_n) < i(e_{j_2}, l_2(\gamma_{n+1}), \gamma_{n+1}) < i(e_{j_1}, l_1(\gamma_{n+1}), \gamma_{n+1})$$

and $i = \lim\limits_n i(e_{j_2}, l_2(\gamma_n), \gamma_n)$. Since $u_1(e_{j_1}), u_1(e_{j_2})$ are closed we conclude that $i \in u_1(e_{j_1}) \cap u_1(e_{j_2})$. From the remark we did after the definition of Z_Υ we know that the last set is an initial segment of both $u_1(e_{j_1})$ and $u_1(e_{j_2})$. But this gives a contradiction: $i(e_{j_2}, l_2(\gamma_{n+1}), \gamma_{n+1}) \in u_1(e_{j_2}) \setminus u_1(e_{j_1})$ and it is below $i \in u_1(e_{j_1}) \cap u_1(e_{j_2})$. The claim is proved.

Similarly as in claim 2.7.3 (but much easier) one can prove that really $d(B_{\bar{F}_\gamma}) = \theta^+$. ■

We want to finish this section with posing two questions motivated by 2.5 and 2.7:

Problem 2.8 *Are the following theories consistent?*

1. *ZFC + there is a cardinal κ such that for each Boolean algebra B,*

$$d_n(B) \le \kappa \quad \Rightarrow \quad d_{n+1}(B) < 2^\kappa.$$

2. *ZFC + there is a cardinal θ such that $\theta^{\aleph_0} = \theta$ and for each Boolean algebra B and a non-principal ultrafilter D on ω*

$$d(B) \le \theta \quad \Rightarrow \quad d(B^\omega/D) < 2^\theta.$$

3 Hereditary cofinality and spread

3.1 The invariants

The hereditary cofinality of a Boolean algebra B is the cardinal

$$\text{h}-\text{cof}(B) = \min\{\kappa : (\forall X \subseteq B)(\exists C \subseteq X)(\|C\| \le \kappa \ \&\ C \text{ is cofinal in } X)\}.$$

It can be represented as a def.u.w.o.car. invariant if we use the following description of it (see [Mo 1]):

$(\otimes_{\mathrm{h-cof}})$ $\mathrm{h{-}cof}(B) = \sup\{\|X\| : X \subseteq B \ \&\ (X, <_B) \text{ is well-founded }\}$.

Let the theory $T_{\mathrm{h-cof}}$ introduce predicates P_0, P_1 on which it says that:

- P_1 is a well ordering of P_0,
- $(\forall x_0, x_1 \in P_0)(x_0 < x_1 \ \Rightarrow\ P_1(x_0, x_1))$

(in the above $<$ stands for the respective relation of the Boolean algebra). Clearly $T_{\mathrm{h-cof}}$ determines a def.u.w.o.car. invariant and

$$\mathrm{Inv}_{T_{\mathrm{h-cof}}}(B) = \{\|X\| : X \subseteq B \ \&\ (X, <) \text{ is well-founded}\}.$$

The spread $s(B)$ of a Boolean algebra B is

$$s(B) = \sup\{\|S\| : S \subseteq \mathrm{Ult}\ B \ \&\ S \text{ is discrete in the relative topology}\}.$$

It can be easily described as a def.f.o.car. invariant: the suitable theory T_s introduces predicates P_0, P_1 and it says that for each $x \in P_0$ the set $\{y : P_1(x, y)\}$ is an ultrafilter and the ultrafilters form a discrete set (in the relative topology). Sometimes it is useful to remember the following characterization of $s(B)$ (see [Mo 1]):

$(\otimes_s)$ $s(B) = \sup\{\|X\| : X \subseteq B \text{ is ideal-independent}\}$.

Using this characterization we can write $s(B) = s_\omega(B)$, where

Definition 3.1 *1. ϕ^s_n is the formula saying that no member of P_0 can be covered by union of $n+1$ other elements of P_0.*

2. For $0 < n \leq \omega$ let $T^n_s = \{\phi^s_k : k < n\}$.

3. For a Boolean algebra B and $0 < n \leq \omega$: $s^{(+)}_n(B) = \mathrm{inv}^{(+)}_{T^n_s}(B)$ (so s_n are def.f.o.car. invariants).

The hereditary density of a Boolean algebra B is the cardinal

$$\mathrm{hd}(B) = \sup\{dS : S \subseteq \mathrm{Ult}\ B\}$$

where dS is the (topological) density of the space S. The following characterization of $\mathrm{hd}(B)$ is important for our purposes (see [Mo 1]):

$(\otimes_{\mathrm{hd}})$ $\mathrm{hd}(B) = \sup\{\|\kappa\| : \text{there is a strictly decreasing sequence of ideals (in } B) \text{ of the length } \kappa\ \}$.

We should remark here that on both sides of the equality we have sup but the attainment does not have to be the same. If the sup of the left hand side ($\mathrm{hd}(B)$) is obtained then so is the sup of the other side. If the right hand side sup is obtained AND $\mathrm{hd}(B)$ is regular then the sup of $\mathrm{hd}(B)$ is realized. An open problem is what can happen if $\mathrm{hd}(B)$ is singular.

The hereditary Lindelöf degree of a Boolean algebra B is

$$\mathrm{hL}(B) = \sup\{LS : S \subseteq \mathrm{Ult}\ B\},$$

where for a topological space S, LS is the minimal κ such that every open cover of S has a subcover of size $\leq \kappa$. The following characterization of $\mathrm{hL}(B)$ is crucial for us (see [Mo 1]):

($\otimes_{\mathrm{hL}}$) $\mathrm{hL}(B) = \sup\{\|\kappa\|$: there is a strictly increasing sequence of ideals (in B) of the length κ $\}$.

Note: we may have here differences in the attainment, like in the case of hd.

Definition 3.2 **1.** *Let the formula ψ say that P_1 is a well ordering of P_0 (denoted by $<_1$).*

2. *For $n < \omega$ let ϕ^{hd}_n, ϕ^{hL}_n be the following formulas:*

$\phi^{\mathrm{hd}}_n \equiv \psi \ \&\ (\forall x_0, \ldots, x_{n+1} \in P_0)(x_0 <_1 \ldots <_1 x_{n+1} \ \Rightarrow\ x_0 \not\leq x_1 \vee \ldots \vee x_{n+1})$

$\phi^{\mathrm{hL}}_n \equiv \psi \ \&\ (\forall x_0, \ldots, x_{n+1} \in P_0)(x_{n+1} <_1 \ldots <_1 x_0 \ \Rightarrow\ x_0 \not\leq x_1 \vee \ldots \vee x_{n+1})$.

3. *For $0 < n \leq \omega$ we let $T^n_{\mathrm{hd}} = \{\phi^{\mathrm{hd}}_k : k < n\}$, $T^n_{\mathrm{hL}} = \{\phi^{\mathrm{hL}}_k : k < n\}$.*

4. *For a Boolean algebra B and $0 < n \leq \omega$:*

$$\mathrm{hd}^{(+)}_n(B) = \mathrm{inv}^{(+)}_{T^n_{\mathrm{hd}}}(B), \qquad \mathrm{hL}^{(+)}_n(B) = \mathrm{inv}^{(+)}_{T^n_{\mathrm{hL}}}(B).$$

So hd_n, hL_n are def.u.w.o.car. invariants and $\mathrm{hd}_\omega = \mathrm{hd}$, $\mathrm{hL}_\omega = \mathrm{hL}$ (the sets $\mathrm{Inv}_{T^\omega_{\mathrm{hd}}}(B)$, $\mathrm{Inv}_{T^\omega_{\mathrm{hL}}}(B)$ agree with the sets on the right-hand sides of ($\otimes_{\mathrm{hd}}$), ($\otimes_{\mathrm{hL}}$), respectively).

3.2 Constructions from strong λ-systems.

One of our tools for constructing examples of Boolean algebras is an object taken from the pcf theory.

Definition 3.3 *1.* A weak λ-system *(for a regular cardinal λ) is a sequence $\mathcal{S} = \langle \delta, \bar{\lambda}, \bar{f} \rangle$ such that*

a) *δ is a limit ordinal, $\|\delta\| < \lambda$,*

b) *$\bar{\lambda} = \langle \lambda_i : i < \delta \rangle$ is a strictly increasing sequence of regular cardinals,*

c) *$\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \prod\limits_{i<\delta} \lambda_i$ is a sequence of pairwise distinct functions,*

d) *for every $i < \delta$, $\|\{f_\alpha \restriction i : \alpha < \lambda\}\| \leq \sup\limits_{i<\delta} \lambda_i$.*

2. A λ-system *is a sequence $\mathcal{S} = \langle \delta, \bar{\lambda}, \bar{f}, J \rangle$ such that $\mathcal{S}_0 = \langle \delta, \bar{\lambda}, \bar{f} \rangle$ is a weak λ-system and*

e) *J is an ideal on δ extending the ideal J^{bd}_δ of bounded subsets of δ,*

f) *$\bar{f}$ is a $<_J$-increasing sequence cofinal in $\prod\limits_{i<\delta} (\lambda_i, <)/J$,*

g) *for every $i < \delta$, $\|\{f_\alpha \restriction i : \alpha < \lambda\}\| < \lambda_i$.*

In this situation we say that the system $\mathcal{S}$ extends *the weak system $\mathcal{S}_0$.*

3. $\mathcal{S} = \langle \delta, \bar{\lambda}, \bar{f}, J, (A_\zeta : \zeta < \kappa) \rangle$ *is* a strong λ-system for κ *if $\langle \delta, \bar{\lambda}, \bar{f}, J \rangle$ is a λ-system and*

h) $\mathrm{cf}(\delta) \leq \kappa$, $\sup\limits_{i<\delta} \lambda_i \leq 2^\kappa$,

i) *$A_\zeta \subseteq \delta$, $A_\zeta \notin J$ (for $\zeta < \kappa$) are pairwise disjoint.*

In ZFC, there is a class of cardinals λ for which there are (weak, strong) λ-systems. We can even demand that, for (weak) λ-systems, λ is the successor of a cardinal λ_0 satisfying $\lambda_0^\omega = \lambda_0$ (what is relevant for ultraproducts, see below). More precisely:

Fact 3.4 *1. If $\mu^{<\kappa} < \mu^\kappa = \lambda$ then there is a weak λ-system $\mathcal{S} = \langle \delta, \bar{\lambda}, \bar{f} \rangle$ such that $\sup\limits_{i<\delta} \lambda_i \leq \mu$, $\delta = \kappa$.*

2. If $\kappa = \mathrm{cf}(\kappa)$ and

$(*)$ $\kappa > \aleph_0$, $\mu = \mu^{<\kappa} < \lambda$, $\mathrm{cf}(\lambda) \leq \mu^\kappa$

or even

$(*)^-$ $\mathrm{cf}(\mu) = \kappa$,

$$(\forall \theta)(\exists \mu_\theta < \mu)(\forall \chi)(\mu_\theta < \chi < \mu \ \&\ \mathrm{cf}(\chi) = \theta \ \Rightarrow \mathrm{pp}_\theta(\chi) < \mu)$$

then there is a λ-system $\mathcal{S} = \langle \delta, \bar{\lambda}, \bar{f}, J \rangle$ such that $\mu = \sup_{i<\delta} \lambda_i$, $\delta = \kappa$ (see [Sh 371]).

3. *If $\kappa = \aleph_0$, $\mathrm{cf}(\mu) = \aleph_0 < \mu$ and*

   either $\lambda^* = \mathrm{cov}(\mu, \mu, \aleph_1, 2)$
   or $\lambda^* = \lambda^{\aleph_0}$ & $(\forall \chi < \mu)(\chi^{\aleph_0} < \mu)$

   then for many regular $\lambda \in (\mu, \lambda^*)$ there are λ-systems ($\lambda = \mu_0^+$ really) (see [Sh 430].

4. *There is a class of cardinals λ for which there are strong λ-systems (for some infinite κ), even if we additionally demand that λ is a successor cardinal (see [Sh 400], [Sh 410] or the proof of 4.4 of [Sh 462]).* ■

Theorem 3.5 *Assume that there exists a strong λ-system for κ, λ a regular cardinal. Let θ be an infinite cardinal $\leq \kappa$. Then there are Boolean algebras B_ε (for $\varepsilon < \theta$) such that $\mathrm{inv}^+_{T_{\mathrm{h-cof}}}(B_\varepsilon) \leq \lambda$ and for any ultrafilter D on θ containing all co-bounded sets we have $s^+_\omega(\prod_{\varepsilon<\theta} B_\varepsilon/D) > \lambda$.*

PROOF: The algebras B_ε's are modifications of the algebra constructed in Lemma 4.2 of [Sh 462]. Let $\langle \delta, \bar{\lambda}, \bar{f}, J, (A_\zeta : \zeta < \kappa) \rangle$ be a strong λ-system for κ. For distinct $\alpha, \beta < \lambda$ let $\rho(\alpha, \beta) = \min\{i < \delta : f_\alpha(i) \neq f_\beta(i)\}$.

Take a decreasing sequence $\langle w_\varepsilon : \varepsilon < \theta \rangle$ of subsets of κ such that $\|w_\varepsilon\| = \kappa$ and $\bigcap_{\varepsilon<\theta} w_\varepsilon = \emptyset$.

Fix $\varepsilon < \theta$.

For $i < \delta$ choose a family $\{F_{i,\zeta} : \zeta < \kappa\}$ of subsets of $\{f_\alpha \restriction i : \alpha < \lambda\}$ such that if $X_1, X_2 \in [\{f_\alpha \restriction i : \alpha < \lambda\}]^{<\omega}$ then for some $\zeta < \kappa$ we have $X_1 = F_{i,\zeta} \cap (X_1 \cup X_2)$ (possible as $2^\kappa \geq \lambda_i$). Next take a sequence $\langle (j_i, \zeta_i) : i < \delta \rangle$ such that $j_i \leq i$, $\zeta_i < \kappa$ and the set

$$\{j < \delta : (\forall \zeta < \kappa)(\exists \xi \in w_\varepsilon)(A_\xi \subseteq_J \{i < \delta : j_i = j \ \&\ \zeta_i = \zeta\})\}$$

is unbounded in δ (possible as $\mathrm{cf}(\delta) \leq \kappa$, $\|w_\varepsilon\| = \kappa$).

Now we define a partial order $\prec_\varepsilon$ on λ:

$$\alpha \prec_\varepsilon \beta \quad \text{if and only if} \quad i = \rho(\alpha, \beta) \in \bigcup_{\xi \in w_\varepsilon} A_\xi \text{ and}$$

$$f_\alpha \restriction j_i \in F_{j_i, \zeta_i} \iff f_\alpha(i) < f_\beta(i).$$

The algebra B_ε is the Boolean algebra generated by the partial order $\prec_\varepsilon$. It is the algebra of subsets of λ generated by sets $Z_\alpha = \{\beta < \lambda : \beta \prec_\varepsilon \alpha\} \cup \{\alpha\}$ (for $\alpha < \lambda$).

Claim 3.5.1 **a)** *If $\rho(\alpha,\beta) < \rho(\beta,\gamma)$, $\alpha,\beta,\gamma < \lambda$ then $\beta \prec_\varepsilon \alpha \iff \gamma \prec_\varepsilon \alpha$ and $\alpha \prec_\varepsilon \beta \iff \alpha \prec_\varepsilon \gamma$.*

b) *If $\tau(x_0,\dots,x_{n-1})$ is a Boolean term, $\alpha^l_k < \lambda$ for $k < n$ are pairwise distinct ($l < 2$), $i < \delta$ is such that $\rho(\alpha^0_k, \alpha^1_k) \geq i$ (for $k < n$) but $\rho(\alpha^l_k, \alpha^l_{k'}) < i$ (for $l < 2$, $k < k' < n$) then denoting $X_l = \tau(Z_{\alpha^l_0},\dots,Z_{\alpha^l_{n-1}})$ we have*

1. *$X_0 \cap \{\alpha < \lambda : (\forall k < n)(f_\alpha{\restriction}i \neq f_{\alpha^0_k}{\restriction}i)\} = X_1 \cap \{\alpha < \lambda : (\forall k < n)(f_\alpha{\restriction}i \neq f_{\alpha^0_k}{\restriction}i)\}$*
2. *for each $k < n$,*
   either $X_l \supseteq \{\alpha < \lambda : f_\alpha{\restriction}i = f_{\alpha^0_k}{\restriction}i\}$ for $l < 2$
   or $X_l \cap \{\alpha < \lambda : f_\alpha{\restriction}i = f_{\alpha^0_k}{\restriction}i\} = Z_{\alpha^l_k} \cap \{\alpha < \lambda : f_\alpha{\restriction}i = f_{\alpha^0_k}{\restriction}i\}$ for $l < 2$
   or $X_l \cap \{\alpha < \lambda : f_\alpha{\restriction}i = f_{\alpha^0_k}{\restriction}i\} = \{\alpha < \lambda : f_\alpha{\restriction}i = f_{\alpha^0_k}{\restriction}i\} \setminus Z_{\alpha^l_k}$ for $l < 2$
   or $X_l \cap \{\alpha < \lambda : f_\alpha{\restriction}i = f_{\alpha^0_k}{\restriction}i\} = \emptyset$ for $l < 2$.

Claim 3.5.2 *Suppose that $\langle a_\alpha : \alpha < \lambda\rangle$ are distinct members of B_ε. Then there exist $\alpha < \beta < \lambda$ such that $a_\alpha \geq a_\beta$.*

Proof of the claim: First we may assume that for some integers $n < m < \omega$, a Boolean term $\tau(x_0,\dots,x_{n-1},\dots,x_{m-1})$, ordinals $\alpha_n,\dots,\alpha_{m-1} < \lambda$, an ordinal $i^* < \delta$ and a function $\bar\alpha : \lambda \times n \longrightarrow \lambda \setminus \{\alpha_n,\dots,\alpha_{m-1}\}$ for all $\beta < \lambda$ we have

$a_\beta = \tau(Z_{\bar\alpha(\beta,0)},\dots,Z_{\bar\alpha(\beta,n-1)}, Z_{\alpha_n},\dots,Z_{\alpha_{m-1}})$, and
if $\beta' < \lambda$, $k,k' < n$, $(\beta,k) \neq (\beta',k')$ then $\bar\alpha(\beta,k) \neq \bar\alpha(\beta',k')$, and
$\{f_{\bar\alpha(\beta,k)}{\restriction}i^*, f_{\alpha_{k'}}{\restriction}i^* : k < n, n \leq k' < m\}$ are pairwise distinct.

As we may enlarge i^* we may additionally assume that

$$(\forall \zeta < \kappa)(\exists \xi \in w_\varepsilon)(A_\xi \subseteq_J \{i < \delta : j_i = i^* \ \&\ \zeta_i = \zeta\}).$$

Furthermore, we may assume that $f_{\bar\alpha(\beta,k)}{\restriction}i^* = f_{\bar\alpha(0,k)}{\restriction}i^*$ for all $\beta < \lambda$, $k < n$ (remember that $\|\{f_\alpha{\restriction}i^* : \alpha < \lambda\}\| < \lambda$). Let B be the set of all $i < \delta$ such that

$$(\forall \zeta < \lambda_i)(\exists^\lambda \beta < \lambda)(\forall k < n)(\zeta < f_{\bar\alpha(\beta,k)}(i)).$$

Then the set B is in the dual filter J^c of J (if not clear see Claim 3.1.1 of [Sh 462]). Now apply the choice of $F_{i^*,\zeta}$'s to find $\zeta < \kappa$ such that for $k < n$:

if $a_0 \cap \{\alpha < \lambda : f_{\bar{\alpha}(0,k)} \restriction i^* = f_\alpha \restriction i^*\} = Z_{\bar{\alpha}(0,k)} \cap \{\alpha < \lambda : f_{\bar{\alpha}(0,k)} \restriction i^* = f_\alpha \restriction i^*\}$
then $f_{\bar{\alpha}(0,k)} \restriction i^* \notin F_{i^*,\zeta}$ and
if $a_0 \cap \{\alpha < \lambda : f_{\bar{\alpha}(0,k)} \restriction i^* = f_\alpha \restriction i^*\} = \{\alpha < \lambda : f_{\bar{\alpha}(0,k)} \restriction i^* = f_\alpha \restriction i^*\} \setminus Z_{\bar{\alpha}(0,k)}$
then $f_{\bar{\alpha}(0,k)} \restriction i^* \in F_{i^*,\zeta}$.

Note that by claim 3.5.1 we can replace 0 in the above by any $\beta < \lambda$. Take $\xi \in w_\varepsilon$ such that $A_\xi \subseteq_J \{i < \delta : j_i = i^* \ \& \ \zeta_i = \zeta\}$ and choose $i \in A_\xi \cap B$ such that $j_i = i^*$, $\zeta_i = \zeta$. Since $\|\{f_\alpha \restriction i : \alpha < \lambda\}\| < \lambda_i$ and $i \in B$ we find $\beta_0 < \beta_1 < \lambda$ such that

$$(\forall k < n)(\rho(\bar{\alpha}(\beta_0, k), \bar{\alpha}(\beta_1, k)) = i) \ \& \ \max_{k<n} f_{\bar{\alpha}(\beta_0,k)}(i) < \min_{k<n} f_{\bar{\alpha}(\beta_1,k)}(i).$$

Now by the choice of ζ, claim 3.5.1 and the property of β_0, β_1 we get $a_{\beta_1} \subseteq a_{\beta_0}$.

Claim 3.5.3 $\mathrm{inv}^+_{T_{\mathrm{h-cof}}}(B_\varepsilon) \leq \lambda$

Proof of the claim: Directly from claim 3.5.2 noting that $\lambda \longrightarrow (\lambda, \omega)^2$.

Claim 3.5.4 *Suppose that $\alpha_0, \ldots, \alpha_n < \lambda$ are pairwise $\prec_\varepsilon$-incomparable. Then $Z_{\alpha_0} \not\subseteq Z_{\alpha_1} \cup \ldots \cup Z_{\alpha_n}$.*

Suppose now that D is an ultrafilter on θ containing all co-bounded sets.

Claim 3.5.5 *$s^+_\omega(\prod\limits_{\varepsilon<\theta} B_\varepsilon/D) > \lambda$.*

Proof of the claim: We need to find an ideal–independent subset of $\prod\limits_{\varepsilon<\theta} B_\varepsilon/D$ of size λ. But this is easy: for $\alpha < \lambda$ let $x_\alpha \in \prod\limits_{\varepsilon<\theta} B_\varepsilon/D$ be such that $x_\alpha(\varepsilon) = \{\beta < \lambda : \beta \preceq_\varepsilon \alpha\}$. The set $\{x_\alpha : \alpha < \lambda\}$ is ideal–independent since if $\alpha_0, \ldots, \alpha_n < \lambda$ are distinct and ε is such that $\alpha_0, \ldots, \alpha_n$ are pairwise $\prec_\varepsilon$-incomparable then

$$B_\varepsilon \models x_{\alpha_0}(\varepsilon) \not\leq x_{\alpha_1}(\varepsilon) \vee \ldots \vee x_{\alpha_n}(\varepsilon)$$

(by claim 3.5.4). Now note that if $\varepsilon_0 < \theta$ is such that

$$\bigcup_{\xi \in w_{\varepsilon_0}} A_\xi \cap \{\rho(\alpha_l, \alpha_m) : l < m < n\} = \emptyset$$

then for all $\varepsilon \geq \varepsilon_0$ we have that $\alpha_0, \ldots, \alpha_n$ are pairwise $\prec_\varepsilon$-incomparable. Now by Łoś theorem we conclude

$$\prod_{\varepsilon<\theta} B_\varepsilon/D \models x_{\alpha_0} \not\leq x_{\alpha_1} \vee \ldots \vee x_{\alpha_n}. \quad \blacksquare$$

Remark: 1. For λ such that there exists a strong λ-system and λ is a successor (and for the respective θ, κ's) we have algebras B_ε (for $\varepsilon < \theta$) such that $\mathrm{inv}_T(B_\varepsilon) < \lambda$ and for respective ultrafilters D on κ $\mathrm{inv}_T(\prod_{\varepsilon<\theta} B_\varepsilon/D) \geq \lambda$, where T is one of the following:

$$T_{\mathrm{h-cof}},\ T^\omega_s,\ T^\omega_{\mathrm{hd}},\ T^\omega_{\mathrm{hL}} \text{ or } T_{\mathrm{inc}}.$$

2. We do not know if (in ZFC) we can demand $\lambda = \lambda_0^+$ and $\lambda_0^\omega = \lambda_0$; consistently yes.

Theorem 3.6 *Assume that there exists a strong λ–system for κ, $0 < n < \omega$. Then there is a Boolean algebra B such that $s^+_n(B) = \|B\|^+ = \lambda^+$ (so $\mathrm{hd}^+_n(B) = \mathrm{hL}^+_n(B) = \lambda^+$) but $s^+_\omega(B), \mathrm{hd}^+(B), \mathrm{hL}^+(B) \leq \lambda$.*

PROOF: The construction is slightly similar to the one of 3.5.

Let $\langle \delta, \bar\lambda, \bar f, J, (A_\zeta : \zeta < \kappa)\rangle$ be a strong λ–system for κ, $\rho(\alpha,\beta) = \min\{i < \delta : f_\alpha(i) \neq f_\beta(i)\}$ (for distinct $\alpha, \beta < \lambda$) and let $F_{i,\zeta} \subseteq \{f_\alpha{\restriction} i : \alpha < \lambda\}$ (for $i < \delta$, $\zeta < \kappa$) be such that if $X_1, X_2 \in [\{f_\alpha{\restriction} i : \alpha < \lambda\}]^{<\omega}$ then there is $\zeta < \kappa$ with $X_1 = F_{i,\zeta} \cap (X_1 \cup X_2)$. Like before, fix a sequence $\langle (j_i, \zeta_i) : i < \delta\rangle$ such that $j_i \leq i$, $\zeta_i < \kappa$ and the set

$$\{j < \delta : (\forall \zeta < \kappa)(\exists \xi < \kappa)(A_\xi \subseteq_J \{i < \delta : j_i = j \ \&\ \zeta_i = \zeta\})\}$$

is unbounded in δ.

Let B be the Boolean algebra generated freely by $\{x_\alpha : \alpha < \lambda\}$ except that

(α) if $\alpha_0, \ldots, \alpha_{n+2} < \lambda$, $i < \delta$, $f_0{\restriction} i = \ldots = f_{\alpha_{n+2}}{\restriction} i$, $f_{\alpha_0}(i) < f_{\alpha_1}(i) < \ldots < f_{\alpha_{n+2}}(i)$ and $f_{\alpha_0}{\restriction} j_i \in F_{j_i,\zeta_i}$ then $x_{\alpha_0} \leq x_{\alpha_1} \vee \ldots \vee x_{\alpha_{n+2}}$ and

(β) if $\alpha_0, \ldots, \alpha_{n+2} < \lambda$, $i < \delta$, $f_0{\restriction} i = \ldots = f_{\alpha_{n+2}}{\restriction} i$, $f_{\alpha_0}(i) < f_{\alpha_1}(i) < \ldots < f_{\alpha_{n+2}}(i)$ and $f_{\alpha_0}{\restriction} j_i \notin F_{j_i,\zeta_i}$ then $x_{\alpha_1} \wedge \ldots \wedge x_{\alpha_{n+2}} \leq x_{\alpha_0}$.

Claim 3.6.1 *If $\alpha_0, \ldots, \alpha_n < \lambda$ are pairwise distinct then*

$$B \models x_{\alpha_0} \not\leq x_{\alpha_1} \vee \ldots \vee x_{\alpha_n}.$$

Consequently $s^+_n(B) = \mathrm{hd}^+_n(B) = \mathrm{hL}^+_n(B) = \|B\|^+ = \lambda^+$.

Proof of the claim: Let $h : \lambda \longrightarrow 2$ be such that $h(\alpha_0) = 1$ and for $\alpha \in \lambda \setminus \{\alpha_0\}$

$$h(\alpha) = \begin{cases} 0 & \text{if } f_\alpha \restriction j_i \notin F_{j_i,\zeta_i} \text{ or} \\ & \quad f_\alpha \restriction j_i \in F_{j_i,\zeta_i} \text{ and } f_\alpha \restriction (i+1) \in \{f_{\alpha_l} \restriction (i+1) : l = 1, \ldots, n\}, \\ 1 & \text{otherwise,} \end{cases}$$

where $i = \rho(\alpha_0, \alpha)$. We are going to show that the function h preserves the inequalities imposed on B in (α), (β) above. To deal with (α) suppose that $\beta_0, \ldots, \beta_{n+2} < \lambda$, $f_{\beta_0} \restriction i = \ldots = f_{\beta_{n+2}} \restriction i$, $f_{\beta_0}(i) < \ldots < f_{\beta_{n+2}}(i)$ and $f_{\beta_0} \restriction j_i \in F_{j_i,\zeta_i}$. If $h(\beta_0) = 0$ then there are no problems, so let us assume that $h(\beta_0) = 1$. Since $f_{\beta_k} \restriction (i+1)$ (for $k = 1, \ldots, n+2$) are pairwise distinct we find $k_0 \in \{1, \ldots, n+2\}$ such that

$$f_{\beta_{k_0}} \restriction (i+1) \notin \{f_{\alpha_l} \restriction (i+1) : l \leq n\}.$$

It is easy to check that then $h(\beta_{k_0}) = 1$, so we are done. Suppose now that $\beta_0, \ldots, \beta_{n+2} < \lambda$, $f_{\beta_0} \restriction i = \ldots = f_{\beta_{n+2}} \restriction i$, $f_{\beta_0}(i) < \ldots < f_{\beta_{n+2}}(i)$ but $f_{\beta_0} \restriction j_i \notin F_{j_i,\zeta_i}$ and suppose $h(\beta_0) = 0$ (otherwise trivial). If $\rho(\alpha_0, \beta_0) < i$ then clearly $h(\beta_k) = h(\beta_0) = 0$ for all $k \leq n+2$. If $\rho(\alpha_0, \beta_0) \geq i$ then for some $k_0 \in \{1, \ldots, n+2\}$ we have $\alpha_0 \neq \beta_{k_0}$, $\rho(\alpha_0, \beta_{k_0}) = i$ and easily $h(\beta_{k_0}) = 0$.

Claim 3.6.2 $\mathrm{hd}^+(B), \mathrm{hL}^+(B) \leq \lambda$.

Proof of the claim: Suppose that $\langle a_\beta : \beta < \lambda \rangle \subseteq B$. After the standard cleaning we may assume that for some Boolean term τ, integers $m_0 < m < \omega$, a function $\bar{\alpha} : \lambda \times m \longrightarrow \lambda$, and an ordinal $i_0 < \delta$ for all $\beta < \lambda$ we have:

$(*)_1$ $a_\beta = \tau(x_{\bar{\alpha}(\beta,0)}, \ldots, x_{\bar{\alpha}(\beta,m-1)})$,

$(*)_2$ $\langle f_{\bar{\alpha}(\beta,l)} \restriction i_0 : l < m \rangle$ are pairwise distinct and $f_{\bar{\alpha}(\beta,l)} \restriction i_0 = f_{\bar{\alpha}(0,l)} \restriction i_0$ (for $l < m$) ,

$(*)_3$ $\{\langle \bar{\alpha}(\beta,0), \ldots, \bar{\alpha}(\beta,m-1) \rangle : \beta < \lambda\}$ forms a Δ-system of sequences with the root $\{0, \ldots, m_0 - 1\}$.

Moreover, as we are dealing with hd, hL, we may assume that the term τ is of the form

$$\tau(x_0, \ldots, x_{m-1}) = \bigwedge_{l<m} x_l^{t(l)},$$

where $t : m \longrightarrow 2$. Let $\zeta < \kappa$ be such that for each $l < m$

$$f_{\bar{\alpha}(0,l)} \restriction i_0 \in F_{i_0,\zeta} \iff t(l) = 0.$$

Take $i_1 > i_0$ such that $j_{i_1} = i_0$, $\zeta_{i_1} = \zeta$ and

$$(\forall \zeta < \lambda_{i_1})(\exists^\lambda \beta < \lambda)(\forall l \in [m_0, m))(\zeta < f_{\bar{\alpha}(\beta,l)}(i_1))$$

(like in the proof of claim 3.5.2). Now, as $\|\{f_\alpha \restriction i_1 : \alpha < \lambda\}\| < \lambda_{i_1}$, we may choose distinct $\beta_0, \ldots, \beta_{m\cdot(n+2)} < \lambda$ such that for $k \leq m \cdot (n+2)$, $l < m$

$$f_{\bar{\alpha}(\beta_0,l)} \restriction i_1 = f_{\bar{\alpha}(\beta_k,l)} \restriction i_1 \stackrel{\rm def}{=} \nu_l$$

and for each $l \in [m_0, m)$

$$f_{\bar{\alpha}(\beta_0,l)}(i_1) < f_{\bar{\alpha}(\beta_1,l)}(i_1) < \ldots < f_{\bar{\alpha}(\beta_{m\cdot(n+2)},l)}(i_1).$$

Note that we can demand any order between $\beta_0, \ldots, \beta_{m\cdot(n+2)}$ we wish what allows us to deal with both hd and hL. We are going to show that $a_{\beta_0} \leq \bigvee\limits_{k=1}^{m\cdot(n+2)} a_{\beta_k}$. Suppose that $l \in [m_0, m)$, $1 \leq k_1 < k_2 < \ldots < k_{n+2} \leq m \cdot (n+2)$. If $t(l) = 0$ then, by the choice of ζ and i_1 we may apply clause (α) of the definition of B and conclude that

$$x_{\bar{\alpha}(\beta_0,l)} \leq x_{\bar{\alpha}(\beta_{k_1},l)} \vee \ldots x_{\bar{\alpha}(\beta_{k_{n+2}},l)}.$$

Similarly, if $t(l) = 1$ then

$$x_{\bar{\alpha}(\beta_{k_1},l)} \wedge \ldots x_{\bar{\alpha}(\beta_{k_{n+2}},l)} \leq x_{\bar{\alpha}(\beta_0,l)}.$$

Hence, for any distinct $k_1, \ldots, k_{n+2} \in \{1, \ldots, m\cdot(n+2)\}$ and $l < m$ we have

$$x^{t(l)}_{\bar{\alpha}(\beta_0,l)} \leq x^{t(l)}_{\bar{\alpha}(\beta_{k_1},l)} \vee \ldots x^{t(l)}_{\bar{\alpha}(\beta_{k_{n+2}},l)},$$

and therefore

$$\bigwedge_{l<m} x^{t(l)}_{\bar{\alpha}(\beta_0,l)} \leq \bigvee_{k=1}^{m\cdot(n+2)} \bigwedge_{l<m} x^{t(l)}_{\bar{\alpha}(\beta_k,l)}. \qquad \blacksquare$$

Remark: Theorem 3.6 is applicable to ultraproducts, of course, but we do not know if we can demand (in ZFC) that $\lambda = \lambda_0^+$, $\lambda_0^\omega = \lambda$.
ZFC constructions (using λ–systems) parallel to 3.6 will be presented in a forthcoming paper [Sh 620]. Some related consistency results will be contained in [RoSh 599].

Problem 3.7 *For each $0 < n < \omega$ find (in ZFC) a Boolean algebra B such that $s_n(B) > s_{n+1}(B)$. Similarly for* hL, hd.

3.3 Forcing an example

Theorem 3.8 *Assume that $\aleph_0 \le \kappa < \mu < \lambda = \mu^+ = 2^\mu$. Then there is a forcing notion $\mathbb{P}$ which is $(<\lambda)$–complete of size λ^+ and satisfies the λ^+–cc (so it preserves cardinalities, cofinalities and cardinal arithmetic) and such that in $\mathbf{V}^{\mathbb{P}}$:*

> *there exist Boolean algebras B_ξ (for $\xi < \kappa$) such that $\mathrm{hd}(B_\xi)$, $\mathrm{hL}(B_\xi) \le \lambda$ (so $s(B_\xi) \le \lambda$) but for each ultrafilter D on κ containing co-bounded subsets of κ we have $\mathrm{ind}(\prod\limits_{\xi<\kappa} B_\xi/D) \ge \lambda^+$ (so $\lambda^+ \le \mathrm{hd}(\prod\limits_{\xi<\kappa} B_\xi/D), \mathrm{hL}(\prod\limits_{\xi<\kappa} B_\xi/D), s(\prod\limits_{\xi<\kappa} B_\xi/D)$).*

PROOF: By Theorem 2.5(3) of [Sh 462] there is a suitable forcing notion $\mathbb{P}$ such that in $\mathbf{V}^{\mathbb{P}}$:
there is a sequence $\langle \eta_i : i < \lambda^+ \rangle \subseteq {}^\lambda\lambda$ with no repetition and functions c, d such that:

(a) $c : {}^{\lambda>}\lambda \longrightarrow \lambda$,

(b) the domain $\mathrm{dom}(d)$ of the function d consists of all pairs $(\bar{x}, h)$ such that $h : \zeta \longrightarrow \lambda \times \lambda \times \lambda$ for some $\zeta < \mu$, and $\bar{x} : \mu \longrightarrow {}^\alpha\lambda$ is one-to-one, $\alpha < \lambda$,

(c) for $(\bar{x}, h) \in \mathrm{dom}(d)$, $d(\bar{x}, h)$ is a function from

$$\{\bar{a} \in {}^\mu(\lambda^+) : \bar{a} \text{ is increasing and } (\forall i < \mu)(x_i \lhd \eta_{a_i})\}$$

to λ such that $d(\bar{x}, h)(\bar{a}) = d(\bar{x}, h)(\bar{b})$ implies $\sup \bar{a} \ne \sup \bar{b}$ and denoting $t_i = \eta_{a_i} \wedge \eta_{b_i}$ for some $i^* < \mu$ we have:

(α) $\mathrm{level}(t_i) = \mathrm{level}(t_{i^*})$ for $i > i^*$,

(β) $(\forall \varepsilon < \mu)(\exists^\mu i < \mu)(c(t_i) = \varepsilon)$,

(γ) for μ ordinals $i < \mu$ divisible by ζ we have

(i) either there are $\xi_0 < \xi_1 < \lambda$ such that

$$(\forall \varepsilon < \zeta)(\zeta \cdot \xi_0 \le \eta_{\bar{b}_{i+\varepsilon}}(\mathrm{level}(t_{i+\varepsilon})) < \zeta \cdot \xi_1 \le \eta_{\bar{a}_{i+\varepsilon}}(\mathrm{level}(t_{i+\varepsilon}))),$$

and

$$h = \langle (c(t_{i+\varepsilon}), \eta_{\bar{b}_{i+\varepsilon}}(\mathrm{level}(t_{i+\varepsilon})) - \zeta \cdot \xi_0, \eta_{\bar{a}_{i+\varepsilon}}(\mathrm{level}(t_{i+\varepsilon})) - \zeta \cdot \xi_1) : \varepsilon < \zeta \rangle,$$

(ii) or a symmetrical condition interchanging $\bar{a}$ and $\bar{b}$.

From now on we are working in the universe $\mathbf{V}^{\mathbb{P}}$ using the objects listed above.

For distinct $i, j < \lambda^+$ let $\rho(i,j) = \min\{\xi < \lambda : \eta_i(\xi) \neq \eta_j(\xi)\}$. For $\varepsilon_0, \varepsilon_1 < \kappa$ we put

$$R^\kappa_{\varepsilon_0,\varepsilon_1} = \{(i,j) \in \lambda^+ \times \lambda^+ : \ i \neq j \ \text{ and } \ \eta_i(\rho(i,j)) = \varepsilon_0 \bmod \kappa \ \text{ and } \ \eta_j(\rho(i,j)) = \varepsilon_1 \bmod \kappa\},$$

and now we define Boolean algebras $B_{\kappa,\bar\varepsilon}$ for $\bar\varepsilon = \langle \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3\rangle$, $\varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \kappa$. $B_{\kappa,\bar\varepsilon}$ is the Boolean algebra freely generated by $\{x_i : i < \lambda^+\}$ except:

if $(i,j) \in R^\kappa_{\varepsilon_0,\varepsilon_1}$ then $x_i \leq x_j$,
if $(i,j) \in R^\kappa_{\varepsilon_2,\varepsilon_3}$ then $x_j \leq x_i$.

Claim 3.8.1 *If $i, j < \lambda^+$, $(i,j) \notin R^\kappa_{\varepsilon_0,\varepsilon_1}$, $(j,i) \notin R^\kappa_{\varepsilon_2,\varepsilon_3}$ then $B_{\kappa,\bar\varepsilon} \models x_i \not\leq x_j$. In particular, if $i < j < \lambda^+$ then $B_{\kappa,\bar\varepsilon} \models x_i \neq x_j$ and $\|B_{\kappa,\bar\varepsilon}\| = \lambda^+$.*

Proof of the claim: Fix $i < \lambda^+$. A function $f : \{x_j : j < \lambda^+\} \longrightarrow \mathcal{P}(2)$ (where $\mathcal{P}(2)$ is the Boolean algebra of subsets of $\{0,1\}$) is defined by $f(x_i) = \{0\}$ and for $j \in \lambda^+ \setminus \{i\}$:

if $(i,j) \in R^\kappa_{\varepsilon_0,\varepsilon_1}$ or $(j,i) \in R^\kappa_{\varepsilon_2,\varepsilon_3}$ then $f(x_j) = \{0,1\}$
if $(i,j) \in R^\kappa_{\varepsilon_2,\varepsilon_3}$ or $(j,i) \in R^\kappa_{\varepsilon_0,\varepsilon_1}$ then $f(x_j) = \emptyset$ and
otherwise $f(x_j) = \{1\}$.

We are going to show that f respects all the inequalities we put on x_j's in $B_{\kappa,\bar\varepsilon}$. So suppose that $(j_1, j_2) \in R^\kappa_{\varepsilon_0,\varepsilon_1}$. If $f(x_{j_1}) = \emptyset$ then there are no problems, so assume that both $(i, j_1) \notin R^\kappa_{\varepsilon_2,\varepsilon_3}$ and $(j_1, i) \notin R^\kappa_{\varepsilon_0,\varepsilon_1}$. Similarly, we may assume that $f(x_{j_2}) \neq \{0,1\}$, i.e. that both $(i, j_2) \notin R^\kappa_{\varepsilon_0,\varepsilon_1}$ and $(j_2, i) \notin R^\kappa_{\varepsilon_2,\varepsilon_3}$. Note that these two assumptions imply $j_1 \neq i \neq j_2$. Now we consider three cases:

– if $\rho(j_1, j_2) > \rho(i, j_1) = \rho(i, j_2)$ then $f(x_{j_1}) = f(x_{j_2})$,

– if $\rho(j_1, j_2) = \rho(i, j_1) = \rho(i, j_2)$ then $f(x_{j_1}) = \{1\} = f(x_{j_2})$ (remember $(j_1, j_2) \in R^\kappa_{\varepsilon_0,\varepsilon_1}$, $(i, j_2), (j_1, i) \notin R^\kappa_{\varepsilon_0,\varepsilon_1}$),

– if $\rho(j_1, j_2) < \max\{\rho(j_1, i), \rho(j_2, i)\}$ then either $\rho(j_1, j_2) = \rho(i, j_2) < \rho(i, j_1)$ and $(i, j_2) \in R^\kappa_{\varepsilon_0,\varepsilon_1}$ (what is excluded already) or $\rho(j_1, j_2) = \rho(i, j_1 < \rho(i, j_2$ and $(j_1, i) \in R^\kappa_{\varepsilon_0,\varepsilon_1}$ (what is against our assumption too).

This shows that $f(x_{j_1}) \leq f(x_{j_2})$ whenever $(j_1, j_2) \in R^\kappa_{\varepsilon_0,\varepsilon_1}$. Similarly one shows that $(j_1, j_2) \in R^\kappa_{\varepsilon_2,\varepsilon_3}$ implies $f(x_{j_2}) \leq f(x_{j_1})$. Consequently the function f respects all the inequalities in the definition of $B_{\kappa,\bar\varepsilon}$. Hence it extends to a homomorphism $\bar f : B_{\kappa,\bar\varepsilon} \longrightarrow \mathcal{P}(2)$. But for each $j < \lambda^+$

$$((i,j) \notin R^\kappa_{\varepsilon_0,\varepsilon_1} \ \&\ (j,i) \notin R^\kappa_{\varepsilon_2,\varepsilon_3}) \ \Rightarrow\ (f(x_j) \in \{\emptyset, \{1\}\} \ \&\ f(x_i) = \{0\}).$$

Claim 3.8.2 *Suppose* $\bar i : \lambda^+ \times n \longrightarrow \lambda^+$, $\bar t : n \longrightarrow 2$, $n < \omega$ *are such that* $(\forall \alpha < \lambda^+)(\forall l_1 < l_2 < n)(\bar i(\alpha, l_1) < \bar i(\alpha, l_2))$. *Then*

$$(\oplus_1) \qquad (\exists \alpha < \beta < \lambda^+)(B_{\kappa,\bar\varepsilon} \models \bigwedge_{l<n} (x_{\bar i(\alpha,l)})^{\bar t(l)} \leq \bigwedge_{l<n} (x_{\bar i(\beta,l)})^{\bar t(l)}),$$

$$(\oplus_2) \qquad (\exists \alpha < \beta < \lambda^+)(B_{\kappa,\bar\varepsilon} \models \bigwedge_{l<n} (x_{\bar i(\alpha,l)})^{\bar t(l)} \geq \bigwedge_{l<n} (x_{\bar i(\beta,l)})^{\bar t(l)}).$$

Proof of the claim: To prove $(\oplus_1), (\oplus_2)$ it is enough to show the following:

$$(\oplus_1^*) \qquad (\exists \alpha < \beta < \lambda^+)(\forall l < n)(B_{\kappa,\bar\varepsilon} \models (x_{\bar i(\alpha,l)})^{\bar t(l)} \leq (x_{\bar i(\beta,l)})^{\bar t(l)}),$$

$$(\oplus_2^*) \qquad (\exists \alpha < \beta < \lambda^+)(\forall l < n)(B_{\kappa,\bar\varepsilon} \models (x_{\bar i(\alpha,l)})^{\bar t(l)} \geq (x_{\bar i(\beta,l)})^{\bar t(l)}).$$

By the definition of $B_{\kappa,\bar\varepsilon}$ for $(\oplus_1^*)$ it is enough to have

$(\oplus_1^{**})$ there are $\alpha < \beta < \lambda^+$ such that

$$l < n \ \&\ \bar t(l) = 0 \quad \Rightarrow \quad (\bar i(\alpha,l), \bar i(\beta,l)) \in R^\kappa_{\varepsilon_0,\varepsilon_1} \text{ or } \bar i(\alpha,l) = \bar i(\beta,l),$$

$$l < n \ \&\ \bar t(l) = 1 \quad \Rightarrow \quad (\bar i(\alpha,l), \bar i(\beta,l)) \in R^\kappa_{\varepsilon_2,\varepsilon_3} \text{ or } \bar i(\alpha,l) = \bar i(\beta,l),$$

and similarly for $(\oplus_2^*)$.

We will show how to get $(\oplus_1^{**})$ from the properties of $\langle \eta_i : i < \lambda^+ \rangle$. For this we start with a cleaning procedure in which we pass from the sequence $\langle \langle \bar i(\alpha,l) : l < n \rangle : \alpha < \lambda^+ \rangle$ to its subsequence $\langle \langle \bar i(\alpha,l) : l < n \rangle : \alpha \in A \rangle$ for some $A \subseteq \lambda^+$ of size λ^+ (so we will assume $A = \lambda^+$). First note that if i repeats λ^+ times in $\langle \bar i(\alpha,l) : \alpha < \lambda^+, l < n \rangle$ then for some $l < n$ we have $\|\{\alpha : \bar i(\alpha,l) = i\}\| = \lambda^+$ and we may assume that for all $\alpha < \lambda^+$, $\bar i(\alpha,l) = i$. Consequently $(\oplus_1^{**})$ holds trivially for this l (and every $\alpha < \beta < \lambda^+$). Thus we may assume that each value appears at most λ times in $\langle \bar i(\alpha,l) : \alpha < \lambda^+, l < n \rangle$ and hence we may assume that the sets $\{\bar i(\alpha,l) : l < n\}$ are disjoint for $\alpha < \lambda^+$ (so there are no repetitions in $\langle \bar i(\alpha,l) : \alpha < \lambda^+, l < n \rangle$). Further we may assume that

$$\alpha < \beta < \lambda^+ \ \Rightarrow\ \bar i(\alpha,0) < \ldots < \bar i(\alpha,n-1) < \bar i(\beta,0) < \ldots < \bar i(\beta,n-1).$$

For $l < n$, $\alpha < \lambda^+$ let $a_{n\alpha+l} = \bar{i}(\alpha, l)$. We find $\xi < \lambda$ such that for λ^+ ordinals $\beta < \lambda^+$ divisible by μ the sequence $\langle \eta_{a_{\beta+\varepsilon}} \restriction \xi : \varepsilon < \mu \rangle$ is with no repetitions and does not depend on β (for these β). Since $\lambda = \mu^+ = 2^\mu$ there are $\xi < \lambda$ and a one-to-one sequence $\bar{x} : \mu \longrightarrow {}^\xi\lambda$ such that the set

$$B = \{\beta < \lambda^+ : \ \beta \text{ is divisible by } \mu \text{ and } (\forall \varepsilon < \mu)(\eta_{a_{\beta+\varepsilon}} \restriction \xi = x_\varepsilon)\}$$

is of size λ^+. Let $h : \kappa \longrightarrow \lambda^3$ be such that for $l < 2n$:

$$h(l) = \begin{cases} (0, \varepsilon_0, \varepsilon_1) & \text{if } \ \bar{t}(l) = 0, \ l < n, \\ (0, \varepsilon_1, \varepsilon_0) & \text{if } \ \bar{t}(l-n) = 0, \ n \le l < 2n, \\ (0, \varepsilon_2, \varepsilon_3) & \text{if } \ \bar{t}(l) = 1, \ l < n, \\ (0, \varepsilon_3, \varepsilon_2) & \text{if } \ \bar{t}(l-n) = 1, \ n \le l < 2n. \end{cases}$$

Consider the function $d(\bar{x}, h)$. There are distinct $\beta_0, \beta_1 \in B$ such that $d(\bar{x}, h)(\langle a_{\beta_0+\varepsilon} : \varepsilon < \mu \rangle) = d(\bar{x}, h)(\langle a_{\beta_1+\varepsilon} : \varepsilon < \mu \rangle)$. This implies that we find $\delta < \mu$ divisible by κ such that (possibly interchanging β_0, β_1):
there are $\xi_0 < \xi_1 < \lambda$ such that for some $\gamma < \lambda$ for every $\varepsilon < \kappa$

$\rho(a_{\beta_0+\delta+\varepsilon}, a_{\beta_1+\delta+\varepsilon}) = \gamma$, and

$\kappa \cdot \xi_0 \le \eta_{a_{\beta_0+\delta+\varepsilon}}(\gamma) < \kappa \cdot \xi_1 \le \eta_{a_{\beta_1+\delta+\varepsilon}}(\gamma)$, and

$h = \langle (c(\eta_{a_{\beta_0+\delta+\varepsilon}} \restriction \gamma), \eta_{a_{\beta_0+\delta+\varepsilon}}(\gamma) - \kappa \cdot \xi_0, \eta_{a_{\beta_1+\delta+\varepsilon}} - \kappa \cdot \xi_1) : \varepsilon < \kappa \rangle$.

Suppose that $\beta_0 < \beta_1$ and look at the values $\eta_{a_{\beta_0+\delta+l}}(\gamma)$, $\eta_{a_{\beta_1+\delta+l}}(\gamma)$ for $l < n$. By the definition of h we have that

- if $t(l) = 0$ then $\eta_{a_{\beta_0+\delta+l}}(\gamma) = \varepsilon_0 \mod \kappa$ and $\eta_{a_{\beta_1+\delta+l}}(\gamma) = \varepsilon_1 \mod \kappa$ (so $(a_{\beta_0+\delta+l}, a_{\beta_1+\delta+l}) \in R^\kappa_{\varepsilon_0,\varepsilon_1}$), and
- if $t(l) = 1$ then $\eta_{a_{\beta_0+\delta+l}}(\gamma) = \varepsilon_2 \mod \kappa$ and $\eta_{a_{\beta_1+\delta+l}}(\gamma) = \varepsilon_3 \mod \kappa$ (so $(a_{\beta_0+\delta+l}, a_{\beta_1+\delta+l}) \in R^\kappa_{\varepsilon_2,\varepsilon_3}$).

Consequently $\beta_0 + \delta < \beta_1 + \delta < \lambda^+$ are as required in $(\oplus_1^{**})$. If $\beta_1 < \beta_0$ then we look at the values $\eta_{a_{\beta_0+\delta+n+l}}(\gamma), \eta_{a_{\beta_1+\delta+n+l}}(\gamma)$ (for $l < n$) and similarly we conclude that $\beta_1 + \delta + n < \beta_0 + \delta + n < \lambda^+$ witness $(\oplus_1^{**})$.

Similarly one can get $\oplus_2^*$.

Claim 3.8.3 $\mathrm{hd}(B_{\kappa,\bar{\varepsilon}}) \le \lambda$, $\mathrm{hL}(B_{\kappa,\bar{\varepsilon}}) \le \lambda$.

Proof of the claim: Suppose that $\mathrm{hL}(B_{\kappa,\bar{\varepsilon}}) \ge \lambda^+$ (or $\mathrm{hd}(B_{\kappa,\bar{\varepsilon}}) \ge \lambda^+$). Then there is a sequence $\langle y_\alpha : \alpha < \lambda^+ \rangle \subseteq B_{\kappa,\bar{\varepsilon}}$ such that for each $\alpha < \lambda^+$ the element y_α is not in the ideal generated by $\{y_\beta : \beta < \alpha\}$ ($\{y_\beta : \beta > \alpha\}$, respectively). Moreover we can demand that each y_α is of the form $\bigwedge_{l<n(\alpha)} (x_{\bar{i}(\alpha,i)})^{\bar{t}(\alpha,l)}$ with $\bar{i}(\alpha, l_1) < \bar{i}(\alpha, l_2)$ for $l_1 < l_2 < n(\alpha)$. Next we may assume that $n(\alpha)=n$, $\bar{t}(\alpha,l)=\bar{t}(l)$ for $\alpha<\lambda^+$, $l<n$ and apply $(\oplus_1)$ ($(\oplus_2)$, respectively) of claim 3.8.2 to get a contradiction.

Now, for $\xi<\kappa$ let $B_\xi=B_{\kappa,\langle 4\xi,4\xi+1,4\xi+2,4\xi+3\rangle}$ and $B=\prod\limits_{\xi<\kappa}B_\xi/D$, where D is an ultrafilter on κ such that no its member is bounded in κ.

Claim 3.8.4 $\mathrm{ind}(B)\geq\lambda^+$.

Proof of the claim: Let $f_i\in\prod\limits_{\xi<\kappa}B_\xi$ (for $i<\lambda^+$) be the constant sequence $f_i(\xi)=x_i$. Suppose $i_0<i_1<\ldots<i_{n-1}<\lambda^+$ and look at the set

$$\begin{aligned}\mathcal{X}=\{\xi<\kappa:(\exists j<4)(\exists m<k<n)\quad &(\eta_{i_m}(\rho(i_m,i_k))=4\xi+j \bmod \kappa\\ \text{or}\quad &\eta_{i_k}(\rho(i_m,i_k))=4\xi+j \bmod \kappa)\}.\end{aligned}$$

Obviously, the set $\mathcal{X}$ is bounded in κ. By claim 3.8.1 (or actually by a stronger version of it, but with a similar proof) we have that for $\xi\in\kappa\setminus\mathcal{X}$

$$B_\xi\models \text{“}f_{i_0}(\xi),\ldots,f_{i_{n-1}}(\xi)\text{ are independent elements”}.$$

Therefore, we conclude $B\models$ “$f_{i_0},\ldots,f_{i_{n-1}}$ are independent”. ∎

4 Independence number and tightness

4.1 Independence.

In this section we are interested in the cardinal invariants related to the independence number.

Definition 4.1 *1. ϕ^{ind}_n is the formula which says that any non-trivial Boolean combination of $n+1$ elements of P_0 is non-zero (i.e. ϕ^{ind}_n says that if $x_0,\ldots,x_n\in P_0$ are distinct then $\bigwedge\limits_{l\leq n}x_l^{t(l)}\neq 0$ for each $t\in{}^{n+1}2$).*

2. For $0<n\leq\omega$ let $T^n_{\mathrm{ind}}=\{\phi^{\mathrm{ind}}_k:k<n\}$.

3. For a Boolean algebra B, $0<n\leq\omega$ we define $\mathrm{ind}_n(B)=\mathrm{inv}_{T^n_{\mathrm{ind}}}(B)$ and $\mathrm{ind}^+_n(B)=\mathrm{inv}^+_{T^n_{\mathrm{ind}}}(B)$. We will denote $\mathrm{ind}^{(+)}_\omega$ by $\mathrm{ind}^{(+)}$ too.

*4. A subset X of a Boolean algebra B is n–*independent* if and only if any non-trivial Boolean combination of n elements of X is non-zero.*

Remark: 1. Note that the theory $T^{n+1}_{\rm ind}$ consists of formulas $\phi^{\rm ind}_0,\ldots,\phi^{\rm ind}_n$ and thus it says that the set P_0 is $n+1$–independent. Consequently for each $n<\omega$:

$$\mathrm{ind}^{(+)}_{n+1}(B)=\sup\{\|X\|^{(+)}: X\subseteq B \text{ is } n+1\text{–independent}\}.$$

2. It should be underlined here that the cardinal invariants ind_n (*the n–independence number*) were first introduced and studied by Monk in [Mo 4].

Proposition 4.2 *Suppose that λ is an infinite cardinal, n is an integer greater than 1. Then there is a Boolean algebra B such that*

$$\mathrm{ind}_n(B)=\lambda=\|B\| \quad \& \quad \mathrm{ind}_{n+1}(B)=\aleph_0.$$

PROOF: Surprisingly the example we give depends on the parity of n.

CASE 1: $n=2k$, $k\geq 1$.

Let $\mathcal{X}=\{x\in {}^{\lambda}2: \|x^{-1}[\{1\}]\|\leq k\}$ and for $\alpha<\lambda$ let $Z_\alpha=\{x\in\mathcal{X}: x(\alpha)=1\}$. Let $B^k_0(\lambda)$ be the Boolean algebra of subsets of $\mathcal{X}$ generated by $\{Z_\alpha:\alpha<\lambda\}$.

Claim 4.2.1 $\mathrm{ind}_n(B^k_0(\lambda))=\lambda$.

Proof of the claim: For $\alpha>0$ put $Y_\alpha=Z_0\vartriangle Z_\alpha$ ($\vartriangle$ stands for the symmetric difference). We are going to show that the set $\{Y_\alpha: 0<\alpha<\lambda\}$ is n-independent. For this suppose that $t\in {}^n2$, $0<\alpha_0<\ldots<\alpha_{n-1}<\lambda$. Choose $x\in\mathcal{X}$ such that

if $\|t^{-1}[\{0\}]\|\leq k$ then $x(0)=0$, $x(\alpha_l)=1-t(l)$ for $l<n$,
if $\|t^{-1}[\{0\}]\|>k$ then $x(0)=1$, $x(\alpha_l)=t(l)$ for $l<n$.

Then easily $x\in\bigcap\limits_{l<n} Y^{t(l)}_{\alpha_l}$.

Claim 4.2.2 $\mathrm{ind}_{n+1}(B^k_0(\lambda))=\aleph_0$.

Proof of the claim: It should be clear that $\mathrm{ind}(B^k_0(\lambda))\geq\aleph_0$, so what we have to show is $\mathrm{ind}_{n+1}(B^k_0(\lambda))<\aleph_1$. Suppose that $\langle Y_\alpha:\alpha<\omega_1\rangle\subseteq B^k_0(\lambda)$. We may assume that

- $Y_\alpha=\tau(Z_{\bar{i}(\alpha,0)},\ldots,Z_{\bar{i}(\alpha,m-1)})$, where $m<\omega$, τ is a Boolean term, $\bar{i}:\omega_1\times m\longrightarrow\lambda$ is such that $\bar{i}(\alpha,0),\ldots,\bar{i}(\alpha,m-1)$ are pairwise distinct,
- $\{\langle\bar{i}(\alpha,0),\ldots,\bar{i}(\alpha,m-1)\rangle:\alpha<\omega_1\}$ forms a Δ-system of sequences with the root $\{0,\ldots,m^*-1\}$ (for some $m^*\leq m$).

Further we may assume that $\tau(x_0,\ldots,x_{m-1}) = \bigvee\limits_{t\in A}\bigwedge\limits_{i<m} x_i^{t(i)}$ for some $A \subseteq {}^m2$. If $m = m^*$ (i.e. all the Y_α's are the same) the sequence is not $n+1$-independent. If $m^* = 0$ (i.e. the sets $\{\bar{i}(\alpha,l) : l < m\}$ are disjoint for $\alpha < \omega_1$) then either $Y_0 \wedge \ldots \wedge Y_n = 0$ or $(-Y_0) \wedge \ldots \wedge (-Y_n) = 0$ (e.g. the first holds if $1\hat{}\ldots\hat{}1 = \bar{\mathbf{1}} \notin A$ and otherwise the second equality is true). So we may assume that $0 < m^* < m$.

Suppose that $\bar{\mathbf{1}} \in A$. We claim that then $(-Y_0) \wedge \ldots \wedge (-Y_k) \wedge Y_{k+1} \wedge \ldots \wedge Y_{2k} = 0$. If not then we find $x \in \bigcap\limits_{k<j<2k+1} Y_j \setminus \bigcup\limits_{j<k+1} Y_j$. For $j < 2k+1$ let $t_j \in {}^m2$ be defined by $t_j(l) = 1 - x(\bar{i}(j,l))$. Thus $t_j \in A$ for $k < j < 2k+1$ and $t_j \notin A$ for $j < k+1$. As $\|x^{-1}[\{1\}]\| \leq k$ for some $j_0 \leq k$ we necessarily have $(\forall l \in [m^*,m))(t_{j_0}(l) = 1)$. Since $\mathbf{1} \in A$ and $t_{j_0} \notin A$, necessarily for some $l_0 < m^*$ we have $t_{j_0}(l_0) = 0$. Now look at t_j for $j \in [k+1,2k]$. Since $t_j{\restriction}m^* = t_{j_0}{\restriction}m^*$ and $t_{j_0} \notin A$ (and $t_j \in A$, remember $k+1 \leq j \leq 2k$) we have

$$(\forall j \in [k+1,2k])(\exists l_j \in [m^*,m))(t_j(l_j) = 0).$$

This implies that $x(\bar{i}(j,l_j)) = 1$ for $j \in [k+1,2k]$ and together with $x(\bar{i}(j_0,l_0)) = 1$ we get contradiction to $\|x^{-1}[\{1\}]\| \leq k$.

Suppose now that $\bar{\mathbf{1}} \notin A$. Symmetrically to the previous case we show that then $Y_0 \wedge \ldots \wedge Y_k \wedge (-Y_{k+1}) \wedge \ldots \wedge (-Y_{2k}) = 0$. The claim is proved.

CASE 2: $n = 2k+1$, $k \geq 1$.

In this case we consider

$$\mathcal{X}' = \{x \in {}^\lambda 2 : \|x^{-1}[\{1\}]\| \leq k \text{ or } \|x^{-1}[\{0\}]\| \leq k\}$$

and the Boolean algebra $B_1^k(\lambda)$ of subsets of $\mathcal{X}'$ generated by sets $Z'_\alpha = \{x \in \mathcal{X}' : x(\alpha) = 1\}$. Then the sequence $\langle Z'_\alpha : \alpha < \lambda\rangle$ is n-independent (witnessing $\mathrm{ind}_n(B_1^k(\lambda)) = \lambda$). Similarly as in claim 4.2.2 one can show that $\mathrm{ind}_{n+1}(B_1^k(\lambda)) = \aleph_0$ (after the cleaning consider $(-Y_0) \wedge \ldots \wedge (-Y_k) \wedge Y_{k+1} \ldots \wedge Y_{2k+1}$). ■

Remark: Note that

$$\mathrm{ind}^{(+)}_{2k+1}(B_0^k(\lambda) \times B_0^k(\lambda)) = \lambda^{(+)}$$

as witnessed by the set $\{(Z_\alpha, -Z_\alpha) : \alpha < \lambda\}$.

Corollary 4.3 *Suppose that λ is an infinite cardinal. Then there are Boolean algebras B_n (for $n < \omega$) such that $\mathrm{ind}(B_n) = \aleph_0$ but for every non-principal ultrafilter D on ω, $\mathrm{ind}(\prod\limits_{n<\omega} B_n/D) = \lambda^{\aleph_0}$.* ■

A detailed study of the reasons why we did have to consider two cases in Proposition 4.2 leads to interesting observations concerning the invariant ind_n and products of Boolean algebras. First note that

Fact 4.4 *For any Boolean algebras B_i $(i < \lambda)$ we have*

1. $\mathrm{ind}^+_{2n}(B_0 \times B_0) \leq \mathrm{ind}^+_n(B_0) \leq \mathrm{ind}^+_n(B_0 \times B_0)$,
2. $\mathrm{ind}^+_{\sum\limits_{i<k} n_i}(B_0 \times \ldots \times B_{k-1}) \leq \sum\limits_{i<k} \mathrm{ind}^+_{n_i}(B_i)$,
3. $\mathrm{ind}^+(\prod^w_{i<\lambda} B_i) = \sup\limits_{i<\lambda} \mathrm{ind}^+(B_i)$. ■

However there is no immediate bound on $\mathrm{ind}_{n+1}(B \times B)$ in this context. One can easily show that the algebra $B^k_1(\lambda)$ from the proof of 4.2 (case 2) satisfies

$$\mathrm{ind}_{2k+2}(B^k_1(\lambda) \times B^k_1(\lambda)) = \aleph_0.$$

So we get an example proving:

Corollary 4.5 *If λ is an infinite cardinal, n is an odd integer > 2 then there is a Boolean algebra B such that $\mathrm{ind}_n(B) = \lambda$ and $\mathrm{ind}_{n+1}(B \times B) = \aleph_0$.* ■

The oddity of n in the corollary is crucial. For even n (and λ strong limit) the situation is different. In the lemmas below μ is a cardinal, k is an integer ≥ 1 and B is a Boolean algebra.

Definition 4.6 *For a cardinal μ and an integer $k \in \omega$ we define $\daleth_k(\mu)$ inductively by*[1]

$$\daleth_0(\mu) = \mu, \qquad \daleth_{k+1}(\mu) = (2^{\daleth_k(\mu)})^{++}.$$

Lemma 4.7 *1. Suppose that*

($\oplus$) $\mathrm{ind}_{2k}(B) \geq \beth_{2k}(\mu)^+$

or at least

($\oplus^-$) *there exists a sequence $\langle x_i : i < \beth_{2k}(\mu)^+\rangle \subseteq B$ such that if $i_0 < i_1 < \ldots < i_{2k-2} < i_{2k-1} < \beth_{2k}(\mu)^+$ then $\bigwedge\limits_{l<k} x_{i_{2l}} \wedge (-x_{i_{2l+1}}) \neq 0$.*

[1]Remember that $\daleth$ (daleth) is the second letter after $\beth$ (beth) in the Hebrew alphabet

Then

$\spadesuit^{B,k}_{\mu^+}$ *there is a sequence* $\langle y_j : j < \mu^+\rangle \subseteq B$ *such that for each* $w \in [\mu^+]^k$ *there is an ultrafilter* $D \in \mathrm{Ult}\ B$ *with*

$$(\forall j < \mu^+)(y_j \in D \iff j \in w).$$

2. If $\mathrm{ind}_{2k}(B) \geq \daleth_{k+1}(\mu)$ *then we can conclude* $\spadesuit^{B,k}_{\mu}$.
[In 2) it is enough to assume a suitable variant of $(\oplus^-)$*: see the proof.]*

PROOF: *1.* Assume $(\oplus^-)$. For each $i_0 < \ldots < i_{2k-1} < \beth_{2k}(\mu)^+$ fix an ultrafilter $D^{\{i_0,\ldots,i_{2k-1}\}} \in \mathrm{Ult}\ B$ such that $\bigwedge\limits_{l<k}(x_{i_{2k}} \wedge (-x_{i_{2l+1}})) \in D^{\{i_0,\ldots,i_{2k-1}\}}$. Let $F : [\beth_{2k}(\mu)^+]^{2k+1} \longrightarrow {}^{2k+1}2$ be defined by

$$F(\{i_0,\ldots,i_{2k}\})(l) = 1 \iff x_{i_l} \in D^{\{i_0,\ldots,i_{2k}\}\setminus\{i_l\}}$$

(where $l < 2k+1$, $i_0 < \ldots < i_{2k} < \beth_{2k}(\mu)^+$). By the Erdös—Rado theorem we find a homogeneous for F set I of the size μ^+. We may assume that the sequence $\langle x_i : i < \mu^+\rangle$ behaves uniformly with respect to F.

Put $y_j = x_{\omega\cdot j} \wedge (-x_{\omega\cdot j+5})$ for $j < \mu^+$. We claim that the sequence $\langle y_j : j < \mu^+\rangle$ has the required property. For this suppose that $j_0 < \ldots < j_{k-1} < \mu^+$ and let $i_{2l} = \omega\cdot j_l$, $i_{2l+1} = \omega\cdot j_l + 5$ (for $l < k$). Then $i_0 < \ldots < i_{2k-2} < i_{2k-1} < \mu^+$ so we can take $D = D^{\{i_0,\ldots,i_{2k-1}\}}$. Thus $y_{j_l} = x_{i_{2l}} \wedge (-x_{i_{2l+1}}) \in D$ for $l < k$. On the other hand suppose that $j \notin \{j_0,\ldots,j_{k-1}\}$ and look at $i = \omega\cdot j$, $i' = \omega\cdot j + 5$. Note that for each $l < k$ we have

$$i < i_{2l} \iff i < i_{2l+1} \iff i' < i_{2l+1} \iff i' < i_{2l}.$$

Since $F(\{i, i_0,\ldots,i_{2k-1}\}) = F(\{i', i_0,\ldots,i_{2k-1}\})$ we get that

$$x_i \in D \iff x_{i'} \in D$$

and hence $y_j = x_i \wedge (-x_{i'}) \notin D$.

2. The proof is essentially the same as above but instead of the Erdös–Rado theorem we use 4.26 which is a special case of the canonization theorems of [Sh 95]. We start with a sequence $\langle x_{\alpha,\xi} : \alpha < \daleth_{k+1}(\mu), \xi < \mu\rangle \subseteq B$ such that if $\xi_0,\ldots,\xi_{k-1} < \mu$, $\alpha^0_l < \alpha^1_l < \daleth_{k+1}(\mu)$ (for $l < k$) then $\bigwedge\limits_{l<k}(x_{\alpha^0_l,\xi_l} \wedge (-x_{\alpha^1_l,\xi_l})) \neq 0$. Then we choose the respective ultrafilters $D^{\alpha^0_0\alpha^1_0\ldots\alpha^0_{k-1}\alpha^1_{k-1}}_{\xi_0,\ldots,\xi_{k-1}} \in \mathrm{Ult}\ B$ and we consider a function $F : [\daleth_{k+1}(\mu)\times\mu]^{2k+1} \longrightarrow 2$ such that

$F((\alpha^0_0,\xi_0),(\alpha^1_0,\xi_0),\ldots,(\alpha^0_{k-1},\xi_{k-1}),(\alpha^1_{k-1},\xi_{k-1}),(\alpha,\xi)) = 1$
if and only if $x_{\alpha,\xi} \in D^{\alpha^0_0\alpha^1_0\ldots\alpha^0_{k-1}\alpha^1_{k-1}}_{\xi_0,\ldots,\xi_{k-1}}$.

By 4.26 a) we find $\alpha^0_\xi, \alpha^1_\xi < \beth_{k+1}(\mu)$ (for $\xi < \mu$) such that for each distinct $\xi_0, \ldots, \xi_k \in \mu$

$$F((\alpha^0_{\xi_0}, \xi_0), (\alpha^1_{\xi_0}, \xi_0), \ldots, (\alpha^0_{\xi_{k-1}}, \xi_{k-1}), (\alpha^1_{\xi_{k-1}}, \xi_{k-1}), (\alpha^0_{\xi_k}, \xi_k)) =$$

$$= F((\alpha^0_{\xi_0}, \xi_0), (\alpha^1_{\xi_0}, \xi_0), \ldots, (\alpha^0_{\xi_{k-1}}, \xi_{k-1}), (\alpha^1_{\xi_{k-1}}, \xi_{k-1}), (\alpha^1_{\xi_k}, \xi_k)).$$

Finally put $y_\xi = x_{\alpha^0_\xi, \xi} \wedge (-x_{\alpha^1_\xi, \xi})$. ■

Lemma 4.8 *Suppose that there is a sequence $\langle y_j : j < \mu \rangle \subseteq B$ such that for every $w \in [\mu]^k$ there is an ultrafilter $D \in \mathrm{Ult}\, B$ such that*

$$(\forall j < \mu)(y_j \in D \iff j \in w).$$

Then

$$\mathrm{ind}^+_{2k+1}(B \times B) > \mu.$$

PROOF: Consider the sequence $\langle (y_j, -y_j) : j < \mu \rangle \subseteq B \times B$. To prove that it is $2k+1$-independent suppose that $j_0 < \ldots < j_{2k} < \mu$, $t \in {}^{2k+1}2$. Let $w_0 = \{j_l : t(l) = 0\}$, $w_1 = \{j_l : t(l) = 1\}$. One of these sets has at most k elements so we find an ultrafilter $D \in \mathrm{Ult}\, B$ such that

either $(\forall l < 2k+1)(y_{j_l} \in D \iff t(l) = 0)$
or $(\forall l < 2k+1)(y_{j_l} \in D \iff t(l) = 1)$.

In the first case $\bigwedge_{l<2k+1} y^{t(l)}_{j_l} \in D$, in the second case $\bigwedge_{l<2k+1} (-y_{j_l})^{t(l)} \in D$. Consequently $(\bigwedge_{l<2k+1} y^{t(l)}_{j_l}, \bigwedge_{l<2k+1} (-y_{j_l})^{t(l)}) \neq 0$ and the lemma is proved. ■

Theorem 4.9 *Let k be an integer ≥ 1, B a Boolean algebra, λ a cardinal. Then*

1. $\mathrm{ind}_{2k}(B) \geq \max\{\beth_{2k}(\lambda)^+, \beth_{k+1}(\lambda^+)\}$ *implies* $\mathrm{ind}_{2k+1}(B \times B) \geq \lambda^+$.
2. *If λ is strong limit,* $\mathrm{ind}_{2k}(B) \geq \lambda$ *then* $\mathrm{ind}_{2k+1}(B \times B) \geq \lambda$.
3. $\mathrm{ind}_{2k}(\prod^w_{i<\omega} B) < \beth_\omega(\mathrm{ind}_{2k+1}(\prod^w_{i<\omega} B))$.

PROOF: *1.* It is an immediate consequence of lemmas 4.7 and 4.8.

2. Follows from *1.*

3. It follows from *2* and the following observation.

Claim 4.9.1 *For an integer $n > 1$ and a Boolean algebra B we have*

$$\mathrm{ind}_n(\prod^w_{i<\omega} B) = \mathrm{ind}_n(\prod^w_{i<\omega} B \times \prod^w_{i<\omega} B).$$

Proof of the claim: By 4.4(1) we have

$$\mathrm{ind}_n(\prod^w_{i<\omega} B) \le \mathrm{ind}_n(\prod^w_{i<\omega} B \times \prod^w_{i<\omega} B).$$

For the other inequality assume that

$$\kappa \stackrel{\mathrm{def}}{=} \mathrm{ind}_n(\prod^w_{i<\omega} B) < \mathrm{ind}_n(\prod^w_{i<\omega} B \times \prod^w_{i<\omega} B).$$

Thus we find an n–independent set $X \subseteq \prod^w_{i<\omega} B \times \prod^w_{i<\omega} B$ of size κ^+. For $x \in X$ let $a_x, b_x \in \prod^w_{i<\omega} B$ and $m(x) < \omega$ be such that

$$x = (a_x, b_x) \text{ and } (\forall m \ge m(x))(a_x(m) = a_x(m(x)) \ \& \ b_x(m) = b_x(m(x))).$$

Take $m_0 < \omega$ and $Y \in [X]^{\kappa^+}$ such that $m(x) = m_0$ for $x \in Y$. For $x \in Y$ let

$$c_x \stackrel{\mathrm{def}}{=} (a_x(0), \ldots, a_x(m_0), b_x(0), \ldots, b_x(m_0)) \in B^{2m_0+2}.$$

The set $Z \stackrel{\mathrm{def}}{=} \{c_x : x \in Y\}$ is n-independent as $a_x(m) = a_x(m_0)$, $b_x(m) = b_x(m_0)$ for $m \ge m_0$. As $\|Z\| = \kappa^+$ we conclude that $\kappa^+ \le \mathrm{ind}_n(B^{2m_0+2})$. Now note that the algebras $\prod^w_{i<\omega} B$ and $B^{2m_0+2} \times \prod^w_{i<\omega} B$ are isomorphic, so (by 4.4)

$$\mathrm{ind}_n(B^{2m_0+2}) \le \mathrm{ind}_n(\prod^w_{i<\omega} B),$$

and hence $\kappa^+ \le \mathrm{ind}_n(\prod^w_{i<\omega} B) = \kappa$, a contradiction. ■

Problem 4.10

1. *Can Lemma 4.7 be improved? Can we (consistently?) weaken the variant of the assumption $(\oplus^-)$ for 2) to sequences shorter than $\beth_k(\mu)$ (we are interested in the reduction of the steps in the beth hierarchy)?*
2. *Describe (in ZFC) all dependences between $\mathrm{ind}_k(B^n)$ (for $n, k < \omega$) [note that we may force them distinct].*

4.2 Tightness.

The tightness $t(B)$ of a Boolean algebra B is the minimal cardinal κ such that if F is an ultrafilter on B, $Y \subseteq \mathrm{Ult}B$ and $F \subseteq \bigcup Y$ then there is $Z \in$

$[Y]^{\leq\kappa}$ such that $F \subseteq \bigcup Z$. To represent the tightness as a def.u.w.o.car. invariant we use the following characterization of it (see [Mo 1]):

$$t(B) = \sup\{\|\alpha\| : \text{ there exists a free sequence of the length } \alpha \text{ in } B\}$$

where a sequence $\langle x_\xi : \xi < \alpha\rangle \subseteq B$ is free if

$$(\forall \xi < \alpha)(\forall F \in [\xi]^{<\omega})(\forall G \in [\alpha \setminus \xi]^{<\omega})[\bigwedge_{\eta\in F} x_\eta \wedge \bigwedge_{\eta\in G} -x_\eta \neq 0].$$

Now it is easy to represent $t(B)$ as def.u.w.o.car. invariant. Together with (finite versions of) the tightness we will define a def.f.o.car. invariant ut_k which is inspired by 4.7.

Definition 4.11 *1. Let ψ be the sentence saying that P_1 is a well ordering of P_0 (we denote the respective order by $<_1$). For $k, l < \omega$ let $\phi^t_{k,l}$ be the sentence asserting that*

for each $x_0, \ldots, x_k, y_0, \ldots, y_l \in P_0$
if $x_0 <_1 \ldots <_1 x_k <_1 y_0 <_1 \ldots <_1 y_l$ then $\bigwedge\limits_{i\leq k} x_i \not\leq \bigvee\limits_{i\leq l} y_i$,

and let the sentence $\phi^{\mathrm{ut}}_{k,l}$ say that

for each distinct $x_0, \ldots, x_k, y_0, \ldots, y_l \in P_0$ we have $\bigwedge\limits_{i\leq k} x_i \not\leq \bigvee\limits_{i\leq l} y_i$.

2. For $n, m \leq \omega$ let $T^{n,m}_t = \{\phi^t_{k,l} : k < n, l < m\} \cup \{\psi\}$ and $T^{n,m}_{\mathrm{ut}} = \{\phi^{\mathrm{ut}}_{k,l} : k < n, l < m\}$ and for a Boolean algebra B:

$$t_{n,m}(B) = \mathrm{inv}_{T^{n,m}_t}(B) \qquad \& \qquad \mathrm{ut}_{n,m}(B) = \mathrm{inv}_{T^{n,m}_{\mathrm{ut}}}(B).$$

3. The unordered k-tightness ut_k *is the def.f.o.car. invariant* $\mathrm{ut}_{k,\omega}$.

Remark: Note that $T^{n,m}_t = \{\psi\}$ if either $n = 0$ or $m = 0$ (and thus $t_{n,m}(B) = \|B\|$ whenever $n \cdot m = 0$). The theory $T^{n+1,m+1}_t$ says that P_1 is a well ordering of P_0 and if $x_0 <_1 \ldots <_1 x_n <_1 y_0 <_1 \ldots <_1 y_m$ then the meet $\bigwedge\limits_{i\leq n} x_i$ is not covered by the union $\bigvee\limits_{i\leq m} y_i$. The invariant $t_{\omega,\omega}(B)$ is just the tightness of B. Similarly for $T^{n,m}_{\mathrm{ut}}$.

Corollary 4.12 *For a Boolean algebra B and $n, m \leq \omega$, $0 < k < \omega$:*

1. $\mathrm{ind}^{(+)}_{n+m}(B) \leq \mathrm{ut}^{(+)}_{n,m}(B) = \mathrm{ut}^{(+)}_{m,n}(B) \leq t^{(+)}_{n,m}(B)$,

2. $\mathrm{ut}^{(+)}_k(B) = \sup\{\kappa^{(+)} : \spadesuit^{B,k}_\kappa$ *holds true*$\}$, *where the condition* $\spadesuit^{B,k}_\kappa$ *is as defined in lemma 4.7,*

3. *the condition* $\spadesuit^{B,k}_\kappa$ *is equivalent to:*

   the algebra $B^k_0(\kappa)$ of 4.2 can be embedded into a homomorphic image of B,

4. $\mathrm{ut}^{(+)}_k(B) \leq \mathrm{ind}^{(+)}_{2k}(B), \quad \mathrm{ut}^{(+)}_k(B) \leq \mathrm{ind}^{(+)}_{2k+1}(B \times B)$.

PROOF: *1.* and *2.* should be clear.

3. Assume $\spadesuit^{B,k}_\kappa$ and let $\langle y_j : j < \kappa\rangle \subseteq B$ be a sequence witnessing it. Let I be the ideal of B generated by the set

$$\{y_{j_0} \wedge \ldots \wedge y_{j_k} : j_0 < \ldots < j_k < \kappa\}.$$

Then the algebra $B^k_0(\kappa)$ naturally embeds into the quotient algebra B/I. Moreover, if B' is a homomorphic image of B and $\spadesuit^{B',k}_\kappa$ then clearly $\spadesuit^{B,k}_\kappa$ so the converse implication holds true too.

4. It follows from *3.* and (the proof of) Proposition 4.2 and the remark after the proof of 4.2. ■

Remark: Corollary 4.12(3) is specially interesting if you remember that $s^+(B) > \lambda$ if and only if the finite—cofinite algebra on λ can be embedded into a homomorphic image of B.

From Lemma 4.7 we can conclude the following:

Corollary 4.13 *For* $k > 0$ *and an algebra* B*:*

1. *if either* $\mathrm{ut}_{k,k}(B) > \beth_{2k}(\mu)$ *or* $\mathrm{ut}_{k,k}(B) \geq \gimel_{k+1}(\mu^+)$ *then* $\mathrm{ut}_k(B) \geq \mu^+$,

2. *if* λ *is strong limit,* $\mathrm{ut}_{k,k}(B) \geq \lambda$ *then* $\mathrm{ut}_k(B) \geq \lambda$,

3. $\mathrm{ut}_{k,k}(B) < \beth_\omega(\mathrm{ut}_k(B))$. ■

Proposition 4.14 *Suppose* $n, m < \omega$, $k = \min\{n, m\}$, B *is a Boolean algebra. Then*

$$t_{n,m}(B) \leq \beth_{n+m}(\mathrm{ut}_k(B) + t(B)).$$

PROOF: Let $\mu = \mathrm{ut}_k(B)+t(B)$ and assume that $t_{n,m}(B) > \beth_{n+m}(\mu)$. Then we have a sequence $\langle a_\alpha : \alpha < \beth_{n+m}(\mu)^+\rangle \subseteq B$ such that

$$(\forall\alpha_0 < \ldots < \alpha_{n+m-1} < \beth_{n+m}(\mu)^+)[(\bigwedge_{l<n} a_{\alpha_l} \wedge \bigwedge_{n\le l<n+m} -a_{\alpha_l}) \neq 0].$$

For each $\alpha_0, \ldots, \alpha_{n+m-1}$ as above fix an ultrafilter $D^{\{\alpha_0,\ldots,\alpha_{n+m-1}\}} \in \mathrm{Ult}\, B$ containing the element $\bigwedge_{l<n} a_{\alpha_l} \wedge \bigwedge_{n\le l<n+m} -a_{\alpha_l}$. Look at the function

$$F : [\beth_{n+m}(\mu)^+]^{n+m+1} \longrightarrow {}^{n+m+1}2$$

defined by

$$F(\alpha_0, \ldots, \alpha_{n+m})(l) = 1 \iff a_{\alpha_l} \in D^{\{\alpha_0,\ldots,\alpha_{n+m}\}\setminus\{\alpha_l\}}.$$

By the Erdös-Rado theorem we may assume that μ^+ is homogeneous for F with the constant value $c \in {}^{n+m+1}2$.

If $c(l) = 0$ for each $l \le n+m$ then the sequence $\langle a_\alpha : \alpha < \mu^+\rangle$ witnesses $\mu^+ \le \mathrm{ut}_n(B)$ giving a contradiction to the definition of μ (remember $\mathrm{ut}_k(B) \ge \mathrm{ut}_n(B)$). In fact, given n elements $\alpha_0 < \ldots < \alpha_{n-1}$, choose m additional elements $\alpha_{n-1} < \alpha_n < \ldots < \alpha_{n+m-1}$. Suppose that $\beta \in \mu^+ \setminus \{\alpha_0, \ldots, \alpha_{n+m-1}\}$. Then by homogeneity $-a_\beta \in D^{\{\alpha_0,\ldots,\alpha_{n+m-1}\}}$, proving the result.

If $c(l) = 1$ for each l then the sequence $\langle -a_\alpha : n \le \alpha < \mu^+\rangle$ exemplifies $\mu^+ \le \mathrm{ut}_m(B)$, once again a contradiction. In fact, take any m elements $n-1 < \alpha_n < \ldots < \alpha_{n+m-1}$ and suppose that $\beta \in \mu^+ \setminus \{0, \ldots, n-1, \alpha_n, \ldots, \alpha_{n+m-1}\}$. Then by homogeneity $a_\beta \in D^{\{0,\ldots,n-1,\alpha_n,\ldots,\alpha_{n+m-1}\}}$, as desired.

Finally, suppose that there are $l_0, l_1 \le n+m$ such that $c(l_0) = 0$ and $c(l_1) = 1$.

Case 1: $l_1 < l_0$

Let $\Gamma = \{\beta+\omega : \beta < \mu^+\}$. We claim that $\langle a_\alpha : \alpha \in \Gamma\rangle$ witnesses $\mu^+ \le t(B)$, contradicting $\mu \ge t(B)$. In fact, let $\alpha_0 < \ldots < \alpha_p < \ldots < \alpha_{q-1}$ be elements of Γ; we want to show that

$$\bigwedge_{l<p} a_{\alpha_l} \wedge \bigwedge_{p\le l<q} -a_{\alpha_l} \neq 0.$$

Say $\alpha_p = \beta+\omega$. Define $\gamma_l = l$ for all $l < l_1$, $\gamma_{l_1}, \ldots, \gamma_{l_0-1}$ are consecutive values starting with $\beta+1$, and $\gamma_{l_0}, \ldots, \gamma_{m+n-1}$ are consecutive values starting with $\alpha_{q-1}+1$ (none of the latter if $l_0 = n+m$). Then $a_{\alpha_l} \in D^{\{\gamma_0,\ldots,\gamma_{n+m-1}\}}$ for all $l < p$ and $-a_{\alpha_l} \in D^{\{\gamma_0,\ldots,\gamma_{n+m-1}\}}$ for all $l \ge p$, as desired.

Case 2: $l_1 \geq l_0$
This is similar, using $\langle -a_\alpha : \alpha \in \Gamma \rangle$. ■

Our next proposition is motivated by Theorem 4.9 and the above corollaries.

Proposition 4.15 *Let B be a Boolean algebra, k a positive integer. Then*

1. $\mathrm{ind}_{2k}(\prod^w_{i<\omega} B) \leq \min\{\beth_{2k-1}(\mathrm{ind}_k(B)), \beth_{2k-1}(\mathrm{ut}_k(B))\}$,
2. $\mathrm{ut}_{k+1}(\prod^w_{i<\omega} B) \leq \beth_k(\mathrm{ut}^+_{k+1}(B))$.

PROOF: *1.* Suppose that $\lambda_0 = \beth_{2k-1}(\mathrm{ind}_k(B)) < \mathrm{ind}_{2k}(\prod^w_{i<\omega} B)$. Thus we find a sequence $\langle a_\alpha : \alpha < \lambda^+_0 \rangle \subseteq \prod^w_{i<\omega} B$ which is $2k$-independent. Let $a_\alpha = \langle a_\alpha(i) : i < \omega \rangle$ (for $\alpha < \lambda^+_0$). Consider the function $F : [\lambda^+_0]^{2k} \longrightarrow \omega$ given by $F(\alpha_0, \ldots, \alpha_{2k-1}) =$

$$\min\{i \in \omega : B \models a_{\alpha_0}(i) \wedge (-a_{\alpha_1}(i)) \wedge \ldots \wedge a_{\alpha_{2k-2}}(i) \wedge (-a_{\alpha_{2k-1}}(i)) \neq 0\},$$

where $\alpha_0 < \ldots < \alpha_{2k-1} < \lambda^+_0$. By the Erdös–Rado theorem we find a set I of the size $(\mathrm{ind}_k(B))^+$ homogeneous for F; we may assume that $I = (\mathrm{ind}_k(B))^+$. Let i_0 be the constant value of F (on $[(\mathrm{ind}_k(B))^+]^{2k}$). Look at the sequence $\langle a_\alpha(i_0) : \alpha < (\mathrm{ind}_k(B))^+ \ \&\ \alpha \text{ limit} \rangle$. Any combination of k members of this sequence can be "extended" to a combination of $2k$ elements of $\langle a_\alpha(i_0) : \alpha < (\mathrm{ind}_k(B))^+ \rangle$ of the type used in the definition of F. A contradiction.

Now suppose that $\lambda_1 \stackrel{\mathrm{def}}{=} \beth_{2k-1}(\mathrm{ut}_k(B)) < \mathrm{ind}_{2k}(\prod^w_{i<\omega} B)$. Like in 4.7, we take a sequence $\langle a_\alpha : \alpha < \lambda^+_1 \rangle \subseteq \prod^w_{i<\omega} B$ such that for some $n < \omega$, for each $\alpha < \lambda^+_1$, $a_\alpha \in B^n$ (i.e. the support of a_α is contained in n) and

$$(\forall \alpha_0 < \ldots < \alpha_{2k-1} < \lambda^+_1)(\bigwedge_{l<k} a_{\alpha_{2l}} \wedge (-a_{\alpha_{2l+1}}) \neq 0),$$

and for each $\alpha_0 < \ldots < \alpha_{2k-1} < \lambda^+_1$ we choose an ultrafilter $D^{\{\alpha_0,\ldots,\alpha_{2k-1}\}} \in \mathrm{Ult} \prod^w_{i<\omega} B$ such that

$$\bigwedge_{l<k} a_{\alpha_{2l}} \wedge (-a_{\alpha_{2l+1}}) \in D^{\{\alpha_0,\ldots,\alpha_{2k-1}\}}.$$

Now we consider a colouring $F : [\lambda^+_1]^{2k+1} \longrightarrow {}^{2k+1}(2 \times n)$ given by

$$F(\{\alpha_0, \ldots, \alpha_{2k}\})(l) = (1, m) \iff \begin{array}{l} a_{\alpha_l} \in D^{\{\alpha_0,\ldots,\alpha_{2k}\}\setminus\{\alpha_l\}} \quad \text{and} \\ D^{\{\alpha_0,\ldots,\alpha_{2k}\}\setminus\{\alpha_l\}} \text{ is concentrated on} \\ \text{the } m^{\text{th}} \text{ coordinate.} \end{array}$$

By Erdős–Rado theorem we may assume that the set of the first $(\mathrm{ut}_k(B))^+$ elements of λ_1^+ is homogeneous for F. Now we finish as in 4.7 notifying that for some $m < n$, for all $\alpha_0 < \ldots < \alpha_{2k-1} < (\mathrm{ut}_k(B))^+$ the ultrafilter $D^{\{\alpha_0,\ldots,\alpha_{2k-1}\}}$ is concentrated on the m^{th} coordinate. So we may use elements of the form $a_{\alpha\cdot\omega}(m) \wedge (-a_{\alpha\cdot\omega+5}(m))$ (for $\alpha < (\mathrm{ut}_k(B))^+$) to get a contradiction.

2. Assume that $\mathrm{ut}_{k+1}(\prod^w_{i<\omega} B) > \beth_k(\mu)$, $\mu = \mathrm{ut}^+_{k+1}(B)$. Then we find a sequence $\langle a_\alpha : \alpha < (\beth_k(\mu))^+\rangle \subseteq \prod^w_{i<\omega} B$ such that for any $k+1$ distinct members of this sequence there is an ultrafilter containing all of them and no other member of the sequence. We may assume that for some $n < \omega$ we have $\langle a_\alpha : \alpha < (\beth_k(\mu))^+\rangle \subseteq B^n$. For $\alpha_0, \ldots, \alpha_k < (\beth_k(\mu))^+$ let $D^{\alpha_0,\ldots,\alpha_k}$ be the respective ultrafilter of B^n (i.e. it contains all a_{α_l} (for $l \leq k$) and nothing else from the sequence) and let $F(\alpha_0, \ldots, \alpha_k) < n$ be such that the ultrafilter $D^{\alpha_0,\ldots,\alpha_k}$ is concentrated on that coordinate. By the Erdős–Rado theorem we find a set $A \in [(\beth_k(\mu))^+]^{\mu^+}$ homogeneous for F. Let m be the constant value of F on A. Look at the sequence $\langle a_\alpha(m) : \alpha \in A\rangle$ – it witnesses $\spadesuit^{B,k+1}_{\mu^+}$ contradicting $\mu = \mathrm{ut}^+_{k+1}(B)$. ■

Finally note that for the algebra $B^k_0(\lambda)$ of 4.2 we have:

$$\mathrm{ut}_k(B^k_0(\lambda)) = t_{k,\omega}(B^k_0(\lambda)) = \lambda \qquad \text{and}$$

$$\mathrm{ut}_{k+1}(B^k_0(\lambda)) = \mathrm{ut}_{k+1,k+1}(B^k_0(\lambda)) = t_{k+1,k+1}(B^k_0(\lambda)) = \aleph_0.$$

This gives us an example distinguishing $t_{k,\omega}$ and $t_{k+1,\omega}$ (and in corollary 4.3 we may replace ind by t). But the following problem remains open:

Problem 4.16 *Are the following inequalities possible?:*

$$t_{k,\omega}(B) > \mathrm{ut}_k(B), \quad t_{\omega,k}(B) > \mathrm{ut}_k(B), \quad t_{k,k}(B) > t_{k,k+1}(B).$$

4.3 Independence and interval Boolean algebras.

Now we are going to reformulate (in a stronger form) and put in our general setting the results of [Sh 503].

Definition 4.17 *Let B be a Boolean algebra.*

1. *For a filter D on $[\lambda]^k$ we say that B has the D-*dependence property *if for every sequence $\langle a_i : i < \lambda\rangle \subseteq B$ there is $A \in D$ such that for every $\{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\} \in A$ the set $\{a_{\alpha_0}, a_{\alpha_1}, \ldots, a_{\alpha_{k-1}}\}$ is not independent.*

2. *For a filter D on $[\lambda]^k$ and a Boolean term $\tau(x_0, x_1, \ldots, x_{k-1})$ we say that B has the (D,τ)-*dependence property *if for every $\langle a_i : i < \lambda\rangle \subseteq B$, for some $A \in D$, for every $\{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\} \in A$ with $\alpha_0 < \alpha_1 < \ldots < \alpha_{k-1}$ we have $B \models \tau(a_{\alpha_0}, a_{\alpha_1}, \ldots, a_{\alpha_{k-1}}) = 0$.*

It should be clear that if D is a proper filter on $[\lambda]^k$ and a Boolean algebra B has the D–dependence property then $\lambda \geq \mathrm{ind}^+_k(B)$ (and so $\lambda \geq \mathrm{ind}^+(B)$).

Proposition 4.18 *Let $\tau = \tau(x_0, x_1, \ldots, x_{k-1})$ be a Boolean term and let D be a κ-complete filter on $[\lambda]^k$. Then any reduced product of $< \kappa$ Boolean algebras having the (D,τ)-dependence property has the (D,τ)-dependence property (this includes products and ultraproducts).* ■

Proposition 4.19 *Assume D is a proper filter on $[\lambda]^k$. Then there exists a sequence $\langle \alpha_0, \alpha_1, \ldots, \alpha_{k-1}\rangle$ of ordinals $\leq \lambda$ such that:*

(a) *$\{w \in [\lambda]^k$: for each $\ell < k$ the ℓ-th member of w is $< \alpha_\ell\} \neq \emptyset \bmod D$,*

(b) *if $\alpha'_\ell \leq \alpha_\ell$ for all $\ell < k$, $n < k$ and $\alpha'_n < \alpha_n$ then*

$$\{w \in [\lambda]^k : \text{ for each } \ell < k, \text{ the } \ell\text{-th member of } w \text{ is } < \alpha'_\ell\} = \emptyset \bmod D.$$

[Note that necessarily $\langle \alpha_\ell : \ell < k\rangle$ is non-decreasing.]

PROOF: Let F be the set of all non-decreasing sequences $\langle \alpha_\ell : \ell < k\rangle \subseteq \lambda + 1$ such that the condition **(a)** holds. Then F is upward closed (and $\langle \lambda, \ldots, \lambda\rangle \in F$). Choose by induction $\alpha_0, \ldots, \alpha_{k-1}$ such that for each $\ell < k$

$$\alpha_\ell = \min\{\beta : (\exists \bar{\alpha} \in F)(\bar{\alpha}\restriction\ell = \langle \alpha_0, \ldots, \alpha_{\ell-1}\rangle \ \&\ \alpha_\ell = \beta)\}. \quad ■$$

Definition 4.20 *We call a filter D on $[\lambda]^k$* normal for *$\langle \alpha_0, \alpha_1, \ldots, \alpha_{k-1}\rangle$ if condition* **(b)** *of 4.19 holds and*

(a)$^+$ *$\{w \in [\lambda]^k$: for each $\ell < k$ the $\ell-$th member of w is $< \alpha_\ell\} \in D$.*

Proposition 4.21 *Assume that*

1. *D is a κ-complete filter on $[\lambda]^k$ which is normal for $\langle \alpha_0, \alpha_1, \ldots, \alpha_{k-1}\rangle$, and $\alpha_0, \ldots, \alpha_{k-1}$ are limit ordinals,*
2. *$k(*) = k \cdot 2^k$, $i \mapsto (m_i, l_i) : k(*) \longrightarrow 2^k \times k$ is a one-to-one mapping such that $i_1 < i_2$ implies that, lexicographically, $(\alpha_{\ell_{i_1}}, m_{i_1}, \ell_{i_1}) < (\alpha_{\ell_{i_2}}, m_{i_2}, \ell_{i_2})$; for $(m,\ell) \in 2^k \times k$ the unique $i < k(*)$ such that $(m_i, \ell_i) = (m, \ell)$ is denoted by $i(m,\ell)$,*

3. *κ^* is a regular cardinal such that $(\forall \mu < \kappa^*)(2^\mu < \kappa)$ (e.g. $\kappa^* = \aleph_0$),*

4. *for $X \in D$, $h : X \longrightarrow \mu$, $\mu < \kappa^*$:*

$$A_{X,h} \stackrel{\rm def}{=} \{w \in [\lambda]^{k(*)} : (\forall m, m' < 2^k)(w_m \in X \ \&\ h(w_m) = h(w_{m'}))\}$$

where for $m < 2^k$, $w = \{\beta_0, \ldots, \beta_{k(*)-1}\} \in [\lambda]^{k(*)}$ (the increasing enumeration) the set w_m is $\{\beta_{i(m,\ell)} : \ell < k\}$,

5. *D^* is the κ^*-complete filter on $[\lambda]^{k(*)}$ generated by the family*

$$\{A_{X,h} : X \in D, h : X \longrightarrow \mu, \mu < \kappa^*\},$$

6. *$\tau^* = \tau^*(x_0, x_1, \ldots, x_{k(*)-1}) = \bigwedge\limits_{m<2^k} \bigwedge\limits_{\ell<k} x^{f_m(\ell)}_{i(m,\ell)}$, where $\langle f_m : m < 2^k\rangle$ lists all the functions in ${}^k 2$.*

Then

(a) *D^* is a proper κ^*-complete filter on $[\lambda]^{k(*)}$ which is normal for the sequence $\langle \alpha_{l_i} : i < k(*)\rangle$,*

(b) *if a Boolean algebra B has the D-dependence property then it has the (D^*, τ^*)-dependence property.*

PROOF: Assume that $X_j \in D$, $\mu_j < \kappa^*$, $h_j : X_j \longrightarrow \mu_j$ for $j < \mu < \kappa^*$ and look at the intersection $\bigcap\limits_{j<\mu} A_{X_j,h_j}$. Let $X^* = \bigcap\limits_{j<\mu} X_j$. Then $X^* \in D$ as $\mu < \kappa$ and D is κ–complete. Moreover for some $\langle \xi_j : j < \mu\rangle \in \prod\limits_{j<\mu} \mu_j$ we have

$$X^+ \stackrel{\rm def}{=} \{w \in X^* : (\forall j < \mu)(h_j(w) = \xi_j)\} \neq \emptyset \bmod D,$$

as $\prod\limits_{j<\mu} \mu_j < \kappa$ (remember κ^* is regular and $(\forall \mu < \mu^*)(2^\mu < \kappa)$). Let $r_0 < r_1 < \ldots < r_{\ell^*-1} < r^*_\ell = k-1$ be such that

$$\begin{aligned} &\alpha_0 = \ldots = \alpha_{r_0} < \alpha_{r_0+1} = \ldots = \alpha_{r_1} < \alpha_{r_1+1} = \ldots \\ &\ldots = \alpha_{r_{\ell^*-1}} < \alpha_{r_{\ell^*-1}+1} = \ldots = \alpha_{k-1}. \end{aligned}$$

Now we choose inductively $\{\beta^m_0, \ldots, \beta^m_{k-1}\} \in X^+$ (for $m < 2^k$) such that

$$\begin{aligned} &\beta^m_n < \alpha_n \text{ for } n < k,\ m < 2^k;\ \ \alpha_{r_u} < \beta^0_{r_u+1} \text{ for } u < \ell^*; \\ &\beta^m_n < \beta^m_{n+1} \text{ for } n < k-1,\ m < 2^k, \text{ and} \\ &\beta^m_{r_0} < \beta^{m+1}_0,\ \beta^m_{r_{u+1}} < \beta^{m+1}_{r_u+1} \text{ for } u < \ell^*,\ m+1 < 2^k. \end{aligned}$$

How? Since D is normal for $\langle\alpha_0,\ldots,\alpha_{k-1}\rangle$ and the α_i's are limit, the set

$$Y_0 \stackrel{\rm def}{=} \{w\in[\lambda]^k: \begin{array}{l}\text{for each } n<k \text{ the } n\text{-th member of } w \text{ is } <\alpha_n \text{ and}\\ \text{for each } u<\ell^* \text{ the } r_u+1\text{-th element of } w \text{ is } >\alpha_{r_u}\}\end{array}$$

is in D. Thus we may choose $w_0=\{\beta^0_0,\ldots,\beta^0_{k-1}\}$ in $X^+\cap Y_0$. Now suppose that we defined $\{\beta^m_0,\ldots,\beta^m_{k-1}\}$. The set

$$Y_{m+1} \stackrel{\rm def}{=} \{w\in[\lambda]^k: \begin{array}{l}\text{for each } n<k \text{ the } n\text{-th member of } w \text{ is } <\alpha_n \text{ and}\\ \text{for each } u<\ell^* \text{ the } r_u+1\text{-th element of } w \text{ is } >\beta^m_{r_{u+1}}\\ \text{and the minimal element of } w \text{ is } >\beta^m_{r_0}\}\end{array}$$

is in D and we choose $w_{m+1}=\{\beta^{m+1}_0,\ldots,\beta^{m+1}_{k-1}\}$ in $X^+\cap Y_{m+1}$. Note that then $i_0<i_1 \quad\Rightarrow\quad \beta^{m_{i_0}}_{\ell_{i_0}}<\beta^{m_{i_1}}_{\ell_{i_1}}$ (for $i_0,i_1<k\cdot 2^k$) and hence easily $w\stackrel{\rm def}{=}\{\beta^m_\ell:\ell<k,m<2^k\}\in\bigcap\limits_{j<\mu}A_{X_j,h_j}$. Consequently the κ^*–complete filter D^* generated on $[\lambda]^{k(*)}$ by the sets $A_{X,h}$ is proper. The filter D^* is normal for $\langle\alpha_{\ell_i}:i<k(*)\rangle$ since:
if $X=\{\{\beta_0,\ldots,\beta_{k-1}\}\in[\lambda]^k:(\forall n<k)(\beta_n<\alpha_n)\}$, h is a constant function on X then

$$A_{X,h}=\{\{\beta_0,\ldots,\beta_{k(*)-1}\}\in[\lambda]^{k(*)}:(\forall i<k(*))(\beta_i<\alpha_{l_i})\}\in D^*;$$

if $i<k(*)$, $\alpha'<\alpha_{\ell_i}$ then the complement X of the set

$$\{w\in[\lambda]^k:\ \text{the } \ell_i\text{-th member of } w \text{ is less then } \alpha'\}$$

is in D, and if h is a constant function on X then the set $A_{X,h}$ witnesses that

$$\{w\in[\lambda]^{k(*)}:\ \text{the } i\text{-th member of } w \text{ is less then } \alpha'\}=\emptyset \bmod D^*.$$

It should be clear that the D-dependence property for B implies (D^*,τ^*)-dependence property. ■

This is relevant to the product of linear orders. It was proved in [Sh 503] that if κ is an infinite cardinal, B_ζ (for $\zeta<\kappa$) are interval Boolean algebras then ${\rm ind}(\prod\limits_{\zeta<\kappa}B_\zeta)=2^\kappa$. The next result was actually hidden in the proof of Theorem 1.1 of [Sh 503].

Theorem 4.22 *Let κ be an infinite cardinal and let μ be a regular cardinal such that for every $\chi<\mu$ we have $\chi^\kappa<\mu$ (e.g. $\mu=(2^\kappa)^+$ in* **(1)** *below or $\mu=(2^{2^\kappa})^+$ in* **(2)***).*

(1) *For a regressive function $f:\mu\longrightarrow\mu$ (i.e. $f(\alpha)<1+\alpha$), a two-place function $g:\mu^2\longrightarrow\chi$ for some $\chi<\mu$ and for a closed unbounded subset C of μ we put:*

$$\begin{aligned} A_{C,f,g}=\{\{\alpha_0,\ldots,\alpha_5\}\in[\mu]^6: &\ \alpha_0<\alpha_1<\ldots<\alpha_5 \text{ are from } C,\\ &\ \text{each has cofinality} >\kappa,\\ &\ f(\alpha_0)=f(\alpha_1)=\ldots=f(\alpha_5)\ \text{ and}\\ g(\alpha_0,\alpha_1)= &\ g(\alpha_0,\alpha_2)=g(\alpha_3,\alpha_4)=g(\alpha_3,\alpha_5)\}.\end{aligned}$$

Let $D^6_{\mu,\kappa}$ be the filter on $[\mu]^6$ generated by all the sets $A_{C,f,g}$. Finally, let τ_6 be the following Boolean term:

$$\tau_6(x_0,x_1,\ldots,x_5)\stackrel{\rm def}{=}x_0\wedge(-x_1)\wedge x_2\wedge(-x_3)\wedge x_4\wedge(-x_5).$$

Then $D^6_{\mu,\kappa}$ is a proper κ^+-complete filter normal for $\langle\mu,\mu,\mu,\mu,\mu,\mu\rangle$ and every interval Boolean algebra has the $(D^6_{\mu,\kappa},\tau_6)$-dependence property.

(2) *Let μ_0 be a cardinal such that $(\mu_0)^\kappa=\mu_0$ and $(2^{\mu_0})^+\leq\mu$. For a closed unbounded set $C\subseteq\mu$, a regressive function $f:\mu\longrightarrow\mu$ and a two-place function $g:\mu^2\longrightarrow\mu_0$ we let:*

$$\begin{aligned} A^*_{C,f,g}=\{\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}\in[\mu]^4: &\ \alpha_0<\alpha_1<\alpha_2<\alpha_3 \text{ are from } C,\\ &\ \text{each has cofinality} >\kappa,\\ f(\alpha_0)= &\ f(\alpha_1)=f(\alpha_2)=f(\alpha_3)\ \text{ and}\\ g(\alpha_0,\alpha_2)= &\ g(\alpha_0,\alpha_3)=g(\alpha_1,\alpha_2)=g(\alpha_1,\alpha_3)\}.\end{aligned}$$

Let $D^4_{\mu,\kappa}$ be the κ-complete filter on $[\mu]^4$ generated by all the sets $A^*_{C,f,g}$. Finally, let

$$\tau_4=\tau_4(x_0,x_1,x_2,x_3)\stackrel{\rm def}{=}x_0\wedge(-x_1)\wedge x_2\wedge(-x_3).$$

Then the filter $D^4_{\mu,\kappa}$ is proper, κ^+-complete and normal for $\langle\mu,\mu,\mu,\mu\rangle$ and every interval Boolean algebra has the $(D^4_{\mu,\kappa},\tau_4)$-dependence property.

PROOF: **(1)** Let μ be a regular cardinal such that $(\forall\chi<\mu)(\chi^\kappa<\mu)$ (so $\mu^\kappa=\mu$). First note that all the sets $A_{C,f,g}$ are nonempty. [Why? Let $f:\mu\longrightarrow\mu$ be regressive, $g:\mu^2\longrightarrow\chi$, $\chi<\mu$ and let $C\subseteq\mu$ be a club. Then for some ρ the set

$$S=\{\alpha\in C:\mathrm{cf}(\alpha)>\kappa\ \&\ f(\alpha)=\rho\}$$

is stationary (by Fodor lemma). Next for each $\alpha\in S$ take $h(\alpha)<\chi$ such that the set $\{\alpha'\in S:\alpha<\alpha'\ \&\ g(\alpha,\alpha')=h(\alpha)\}$ is stationary, and note that for some $\delta<\chi$ the set $Z=\{\alpha\in S: h(\alpha)=\delta\}$ is stationary. Take any $\alpha_0\in Z$ and then choose $\alpha_1<\alpha_2$ from $(\alpha_0,\mu)\cap S$ such that

$$g(\alpha_0,\alpha_1)=g(\alpha_0,\alpha_2)=\delta.$$

Next choose $\alpha_3>\alpha_2$ from Z and $\alpha_4,\alpha_5\in(\alpha_3,\mu)\cap S$ such that

$$g(\alpha_3,\alpha_4)=g(\alpha_3,\alpha_5)=\delta.$$

Clearly $\{\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\}\in A_{C,f,g}$.]

Now suppose that $C_\zeta\subseteq\mu$, $f_\zeta:\mu\longrightarrow\mu$, $g_\zeta:\mu^2\longrightarrow\chi_\zeta$, $\chi_\zeta<\mu$ (for $\zeta<\kappa$) are as required in the definition of sets $A_{C_\zeta,f_\zeta,g_\zeta}$. Let $\pi:{}^\kappa\mu\longrightarrow\mu$ be a bijection (remember $\mu=\mu^\kappa$). Choose a club $C\subseteq\mu$ such that $C\subseteq\bigcap\limits_{\zeta<\kappa}C_\zeta$ and

$$\text{if } \alpha\in C,\beta<\alpha, F\in{}^\kappa\beta \ \text{ then } \ \pi(F)<\alpha.$$

Let

$$f:\mu\longrightarrow\mu:\alpha\mapsto\pi(\langle f_\zeta(\alpha):\zeta<\kappa\rangle),$$

$$g:\mu^2\longrightarrow\prod_{\zeta<\kappa}\chi_\zeta:(\alpha,\beta)\mapsto\langle g_\zeta(\alpha,\beta):\zeta<\kappa\rangle.$$

The function f is regressive on $\{\alpha\in C:\mathrm{cf}(\alpha)>\kappa\}$, outside this set we change the values of f to 0. Since $\prod\limits_{\zeta<\kappa}\chi_\zeta<\mu$ we have $A_{C,f,g}\in D^6_{\mu,\kappa}$. It should be clear that $A_{C,f,g}\subseteq\bigcap\limits_{\zeta<\kappa}A_{C_\zeta,f_\zeta,g_\zeta}$. Thus we have proved that the filter $D^6_{\mu,\kappa}$ generated by the sets $A_{C,f,g}$ is proper κ^+–complete. To show that $D^6_{\mu,\kappa}$ is normal for $\langle\mu,\mu,\mu,\mu,\mu,\mu\rangle$ note that for $\alpha<\mu$, $\ell<6$, if we take $C=(\alpha,\mu)$, f,g constant functions then

$$A_{C,f,g}\cap\{\{\alpha_0,\dots,\alpha_5\}\in[\mu]^6:\alpha_\ell<\alpha\}=\emptyset.$$

Suppose now that $(I,<_I)$ is a linear ordering. Let $-\infty$ be a new element (declared to be smaller than all members of I) in the case that I has no minimum element; otherwise $-\infty$ is that minimum element. Further, let ∞ be a new element above all members of I. The interval Boolean algebra $B(I)$ determined by the linear ordering I is the algebra of subsets of I generated by intervals $[x,y)_I=\{z\in I:x\leq_I z<_I y\}$ for $x,y\in I\cup\{-\infty,\infty\}$.

We are going to show that the algebra $B(I)$ has the $(D^6_{\kappa,\mu},\tau_6)$–dependence property. Assume that $\langle a_\alpha:\alpha<\mu\rangle\subseteq B(I)$. Since we can find a subset of I of the size $\leq\mu$ which captures all the dependences in the sequence we may assume that the linear order I is of the size μ, so I is a linear ordering on μ.

Fix a bijection $\phi:[\mu\cup\{-\infty,\infty\}]^{<\omega}\times{}^{\omega>}4\longrightarrow\mu$.

For each $\alpha<\mu$ we have a (unique) $<_I$-increasing sequence

$$\langle s^\alpha_i : i<2n(\alpha)\rangle\subseteq\mu\cup\{-\infty,\infty\},\qquad n(\alpha)<\omega$$

such that $a_\alpha=\bigcup\limits_{i<n(\alpha)}[s^\alpha_{2i},s^\alpha_{2i+1})_I$. Take a closed unbounded set $C\subseteq\mu$ such that for each $\alpha\in C$:

(1) if $w\in[\alpha\cup\{-\infty,\infty\}]^{<\omega}$, $c\in{}^{\omega>}4$ then $\phi(w,c)<\alpha$,

(2) if $\phi(w,c)<\alpha$ then $w\subseteq\alpha\cup\{-\infty,\infty\}$,

(3) if $\beta<\alpha$ then $\{s^\beta_i : i<2n(\beta)\}\subseteq\alpha\cup\{-\infty,\infty\}$.

For each $\alpha<\mu$ fix a finite set $w_\alpha\subseteq\alpha\cup\{-\infty,\infty\}$ such that $-\infty,\infty\in w_\alpha$ and

(4) if $s^\alpha_i\in\alpha\cup\{-\infty,\infty\}$ then $s^\alpha_i\in w_\alpha$ and

(5) if $s,t\in\{s^\alpha_i : i<2n(\alpha)\}\cup\{-\infty,\infty\}$, $s<_I t$ and $(s,t)_I\cap\alpha\neq\emptyset$ then $(s,t)_I\cap w_\alpha\neq\emptyset$.

Next let $c_\alpha : w_\alpha\longrightarrow 4$ (for $\alpha<\mu$) be such that for $s\in w_\alpha$:

$$c_\alpha(s)=\begin{cases}0 & \text{if } (\exists x<_I s)([x,s)_I\subseteq a_\alpha),\\ 1 & \text{if } (\exists x: s<_I x)([s,x)_I\subseteq a_\alpha),\\ 2 & \text{if both of the above,}\\ 3 & \text{otherwise.}\end{cases}$$

We can think of c_α as a member of ${}^{\omega>}4$ and we put $f(\alpha)=\phi(w_\alpha,c_\alpha)$ for $\alpha<\mu$. Note that the function f is regressive on C (so we can modify it outside C to get a really regressive function). Now, if $\alpha_0<\alpha_1$, both in C, $f(\alpha_0)=f(\alpha_1)$ then

$$s^{\alpha_0}_i=s^{\alpha_1}_j\ \&\ i<2n(\alpha_0)\ \&\ j<2n(\alpha_1)\quad\Rightarrow\quad s^{\alpha_1}_j\in w_{\alpha_1},\qquad\text{and}$$

$$w_{\alpha_0}\cap\{s^{\alpha_0}_i : i<2n(\alpha_0)\}=w_{\alpha_1}\cap\{s^{\alpha_1}_i : i<2n(\alpha_1)\}.$$

[Why? For the first statement note that, by (3), $s^{\alpha_0}_i<\alpha_1$ (for each $i<2n(\alpha_0)$) so we may use (4). For the second assertion suppose that $s^{\alpha_0}_{2i}\in w_{\alpha_0}=w_{\alpha_1}$. Then necessarily $c_{\alpha_0}(s^{\alpha_0}_{2i})=1=c_{\alpha_1}(s^{\alpha_0}_{2i})$. Checking when the function c_{α_1} takes value 1 and when 2 we get that $s^{\alpha_0}_{2i}=s^{\alpha_1}_{2j}$ for some $j<n(\alpha_1)$. Next, if $s^{\alpha_0}_{2i+1}\in w_{\alpha_0}=w_{\alpha_1}$ then $c(s^{\alpha_0}_{2i+1})=0$ and $s^{\alpha_0}_{2i+1}=s^{\alpha_j}_{2j+1}$ for some j. Similarly if we start with $s^{\alpha_1}_i$.] Moreover, if $s,t\in w_{\alpha_0}$ are two $<_I$-successive points of w_{α_0}, $s\leq_I s^{\alpha_1}_i<_I s^{\alpha_1}_{i+1}\leq_I t$, $i+1<2n(\alpha_1)$, then $(s^{\alpha_1}_i,s^{\alpha_1}_{i+1})_I\cap\{s^{\alpha_0}_j : j<2n(\alpha_0)\}=\emptyset$.

Let a function $g : \mu^2 \longrightarrow {}^{\omega>}\omega$ be such that if $\alpha < \beta$, $\alpha, \beta \in C$, $f(\alpha) = f(\beta)$ then

$$g(\alpha, \beta) = \langle \|w_\alpha\|, t^0, \ldots, t^{\|w_\alpha\|-1}, v^0, \ldots, v^{\|w_\alpha\|-1} \rangle \in {}^{\omega>}\omega,$$

where $\bar{t}, \bar{v}$ are such that:

if $w_\alpha = \{w_\alpha(0), \ldots, w_\alpha(\|w_\alpha\| - 1)\}$ (the $<_I$-increasing enumeration), $\ell < \|w_\alpha\|$ then

$$t^\ell = 0 \iff \{s^\beta_j : j < 2n(\beta)\} \cap (w_\alpha(\ell), w_\alpha(\ell + 1))_I = \emptyset,$$

and if $s^\beta_i \in (w_\alpha(\ell), w_\alpha(\ell + 1))_I$ then

$$v^\ell = 0 \Rightarrow (w_\alpha(\ell), s^\beta_i)_I \cap \{s^\alpha_j : j < 2n(\alpha)\} = \emptyset,$$

$$v^\ell > 0 \Rightarrow s^\alpha_{v^\ell - 1} \in (w_\alpha(\ell), s^\beta_i)_I \ \&\ (s^\alpha_{v^\ell-1}, s^\beta_i)_I \cap \{s^\alpha_j : j < 2n(\alpha)\} = \emptyset.$$

Suppose now that $\alpha_0 < \ldots < \alpha_5$ from C are such that $f(\alpha_0) = \ldots = f(\alpha_5)$, $g(\alpha_0, \alpha_1) = g(\alpha_0, \alpha_2) = g(\alpha_3, \alpha_4) = g(\alpha_3, \alpha_5) = \langle k, t^0, \ldots, t^{k-1}, v^0, \ldots, v^{k-1} \rangle$. Then $w_{\alpha_0} = \ldots w_{\alpha_5} = w = \{w(0), \ldots, w(k-1)\}$ (the $<_I$-increasing enumeration). We are going to show that for each $\ell < k - 1$

$$(\circledcirc) \qquad \tau_6(a_{\alpha_0}, \ldots, a_{\alpha_5}) \wedge [w(\ell), w(\ell + 1))_I = \emptyset.$$

Fix $\ell < k - 1$. If $t^\ell = 0$ then the interval $(w(\ell), w(\ell + 1))_I$ contains no $s^{\alpha_1}_j, s^{\alpha_2}_j$ and therefore

$$a_{\alpha_1} \wedge [w(\ell), w(\ell + 1))_I = a_{\alpha_2} \wedge [w(\ell), w(\ell + 1))_I \in \{0, [w(\ell), w(\ell + 1))_I\}$$

(remember $c_{\alpha_1} = c_{\alpha_2}$). Hence $(-a_{\alpha_1}) \wedge a_{\alpha_2} \wedge [w(\ell), w(\ell + 1))_I = 0$ and $(\circledcirc)$ holds. So suppose that $t^\ell > 0$. Then for each $k = 1, 2, 4, 5$ the interval $(w(\ell), w(\ell + 1)_I$ contains some $s^{\alpha_k}_j$. We know that if $j < j'$, $k = 1, 2$, $s^{\alpha_k}_j, s^{\alpha_k}_{j'} \in (w(\ell), w(\ell + 1))_I$ then there is no $s^{\alpha_0}_i$ in $[s^{\alpha_k}_j, s^{\alpha_k}_{j'}]_I$ (and similarly for α_3 and $k = 4, 5$. Assume that $v^\ell = 0$ and for $k = 1, 2, 4, 5$ let $j_k < 2n(\alpha_k)$ be the last such that $s^{\alpha_k}_{j_k} \in (w(\ell), w(\ell + 1))_I$. By the definition of the functions g and f and the statement before we conclude that

> either $a_0 \wedge [w(\ell), s^{\alpha_k}_{j_k})_I = 0$ (for $k = 1, 2$) and $a_3 \wedge [w(\ell), s^{\alpha_k}_{j_k})_I = 0$ (for $k = 4, 5$)
> or $a_0 \wedge [w(\ell), s^{\alpha_k}_{j_k})_I = [w(\ell), s^{\alpha_k}_{j_k})_I$ (for $k = 1, 2$) and $a_3 \wedge [w(\ell), s^{\alpha_k}_{j_k})_I = [w(\ell), s^{\alpha_k}_{j_k})_I$ (for $k = 4, 5$)

and the parity of j_k's is the same (just look at $c_{\alpha_k}(w(\ell+1))$). Hence we conclude that either $a_{\alpha_0} \wedge (-a_{\alpha_1}) \wedge a_{\alpha_2} \wedge [w(\ell), w(\ell+1))_I = 0$ or $(-a_{\alpha_3}) \wedge a_{\alpha_4} \wedge (-a_{\alpha_5}) \wedge [w(\ell), w(\ell+1))_I = 0$ (and in both cases we get ($\circledcirc$)). Assume now that $v^\ell > 0$. By similar considerations one shows that if $v^\ell - 1$ is even then

$$(-a_{\alpha_3}) \wedge a_{\alpha_4} \wedge (-a_{\alpha_5}) \wedge [w(\ell), w(\ell+1))_I = 0$$

and if $v^\ell - 1$ is odd then

$$a_{\alpha_0} \wedge (-a_{\alpha_1}) \wedge a_{\alpha_2}) \wedge [w(\ell), w(\ell+1))_I = 0.$$

Since g can be thought of as a function from μ^2 to $\omega < \mu$ the set $A_{C,f,g}$ is in $D^6_{\mu,\kappa}$ and we have shown that it witnesses $(D^6_{\mu,\kappa}, \tau_6)$-dependence for the sequence $\langle a_\alpha : \alpha < \mu \rangle$.

2) It is almost exactly like **1)** above. The only difference is that showing that the sets $A^*_{C,f,g}$ are non-empty we use the Erdős–Rado theorem (to choose $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ suitably homogeneous for g), and then in arguments that $B(I)$ has the dependence property we use triples $\alpha_0, \alpha_2, \alpha_3$ and $\alpha_1, \alpha_2, \alpha_3$. ■

4.4 Appendix: How one can use [Sh 95].

For reader's convenience we recall here some of the notions and results of [Sh 95]. We applied them to reduce the number of steps in the beth hierarchy replacing them partially by passing to successors. This reduction is meaningful if the exponentiation function is far from GCH. Generally we think that κ^+ (or even κ^{++}) should be considered as something less than 2^κ.

Definition 4.23 (see Definition 1 of [Sh 95]) *1. For a sequence* $\bar{r} = \langle n_0, \ldots, n_{k-1} \rangle \in {}^k\omega$ *we denote:* $n(\bar{r}) = \sum\limits_{l<k} n_l$, $k(\bar{r}) = k$, $n_l(\bar{r}) = n_l$.

2. Let B_ξ *(for* $\xi < \mu$*) be disjoint well ordered sets,* $\bar{r} = \langle n_0, \ldots, n_{k-1} \rangle \in {}^k\omega$, $f : [\bigcup\limits_{\xi<\mu} B_\xi]^{n(\bar{r})} \longrightarrow \chi$, $l \le n(\bar{r})$. *We say that* f *is* $(\bar{r})^l$*-canonical (on* $\langle B_\xi : \xi < \mu \rangle$*) if*

> *for every* $\xi_0 < \ldots < \xi_{k-1} < \mu$, $a_0 < \ldots < a_{n_0-1}$ *in* B_{ξ_0}, $a_{n_0} < \ldots < a_{n_0+n_1-1}$ *in* B_{ξ_1} *and so on, the value* $f(a_0, \ldots, a_{n(\bar{r})-1})$ *depends on* $a_0, \ldots, a_{n(\bar{r})-1-l}, \xi_0, \ldots, \xi_{k-1}$ *only (i.e. it does not depend on* $a_{n(\bar{r})-l}, \ldots, a_{n(\bar{r})-1}$*).*

3. A sequence $\langle\lambda_\xi : \xi < \mu\rangle$ (of cardinals) has a $\langle\kappa_\xi : \xi < \mu\rangle$–canonical form for $\Gamma = \{(\bar{r}_i)^{l_i}_{\chi_i} : i < \alpha\}$ (where l_i's are integers, $l_i \leq n(\bar{r}_i)$, χ_i's are cardinals and $\bar{r}_i$'s are finite sequences of integers) if

> *for each disjoint (well ordered) sets A_ξ, $\|A_\xi\| = \lambda_\xi$ (for $\xi < \mu$) and functions $f_i : [\bigcup\limits_{\xi<\kappa} A_\xi]^{n(\bar{r}_i)} \longrightarrow \chi_i$ (for $i < \alpha$) there are sets $B_\xi \subseteq A_\xi$, $\|B_\xi\| = \kappa_\xi$ such that each function f_i is $(\bar{r}_i)^{l_i}$-canonical on $\langle B_\xi : \xi < \mu\rangle$ (for $i < \alpha$).*

Several canonization theorems were proved in [Sh 95], we will quote here two (the simplest actually) which we needed for our applications.

Proposition 4.24 (see Composition Claim 5 of [Sh 95]) *Let Γ_1 be*

$$\begin{aligned}\{((\langle n_0,\ldots,n_{k-1},\ldots,n_{m-1}\rangle)^{p+q}_{2^\mu} :\ & (\langle n_0,\ldots,n_{k-1},\ldots,n_{m-1}\rangle)^{p}_{2^\mu} \in \Gamma_3\ \&\\ ((\langle n_0,\ldots,n_{k-2},n_{k-1}-s\rangle)^{q}_{2^\mu} \in \Gamma_2\ & \&\ p = s + n_k + \ldots + n_{m-1}\ \&\\ & 0 \leq s < n_{k-1}\}.\end{aligned}$$

Suppose that the sequence $\langle\lambda^3_\xi : \xi < \mu\rangle$ has a $\langle\lambda^2_\xi : \xi < \mu\rangle$–canonical form for Γ_3 and the sequence $\langle\lambda^2_\xi : \xi < \mu\rangle$ has a $\langle\lambda^1_\xi : \xi < \mu\rangle$–canonical form for Γ_2.
Then the sequence $\langle\lambda^3_\xi : \xi < \mu\rangle$ has a $\langle\lambda^1_\xi : \xi < \mu\rangle$–canonical form for Γ_1.
■

Proposition 4.25 (see Conclusion 8(1) of [Sh 95]) *The sequence*

$$\langle(2^\mu)^{++} : \xi < \mu\rangle$$

has a $\langle\mu : \xi < \mu\rangle$–canonical form for $\{(\bar{r}\hat{\ }\langle 1\rangle)^2_{2^\mu} : \bar{r} \in {}^k\omega, k < \omega\}$. ■

Recall that for a cardinal μ and an integer k we have defined $\daleth_k(\mu)$ by: $\daleth_0(\mu) = \mu$, $\daleth_{k+1}(\mu) = (2^{\daleth_k(\mu)})^{++}$.

Proposition 4.26 *Suppose that $\langle A_\xi : \xi < \mu\rangle$ is a sequence of disjoint sets, $\|A_\xi\| = \daleth_{k+1}(\mu)$. Let $F : [\bigcup\limits_{\xi<\mu} A_\xi]^{2k+1} \longrightarrow 2^\mu$. Then*

a) *there are $\alpha^0_\xi, \alpha^1_\xi \in A_\xi$ (for $\xi < \mu$), $\alpha^0_\xi \neq \alpha^1_\xi$ such that for each pairwise distinct $\xi_0,\ldots,\xi_k < \mu$*

($\oplus$) $F(\alpha^0_{\xi_0},\alpha^1_{\xi_0},\ldots,\alpha^0_{\xi_{k-1}},\alpha^1_{\xi_{k-1}},\alpha^0_{\xi_k}) = F(\alpha^0_{\xi_0},\alpha^1_{\xi_0},\ldots,\alpha^0_{\xi_{k-1}},\alpha^1_{\xi_{k-1}},\alpha^1_{\xi_k})$

and even more:

b) *there are sets* $B_\xi \in [A_\xi]^\mu$ *(for* $\xi < \mu$*) such that if* $\xi_0, \ldots, \xi_k < \mu$ *are distinct, and* $a^0_{\xi_i}, a^1_{\xi_i} \in B_{\xi_i}$ *are distinct then* $(\oplus)$ *of* **a)** *holds true.*

PROOF: It follows from 4.24 and 4.25 (e.g. inductively) that $\langle \beth_{k+1}(\mu) : \xi < \mu\rangle$ has a $\langle \mu : \xi < \mu\rangle$–canonical form for Γ, where Γ consists of the following elements:

$$(\langle\underbrace{2\ldots2}_{k}1\rangle)^2_{2^\mu}, (\langle\underbrace{2\ldots2}_{k}11\rangle)^2_{2^\mu}, (\langle\underbrace{2\ldots2}_{k-1}121\rangle)^4_{2^\mu},$$
$$(\langle\underbrace{2\ldots2}_{k-2}1221\rangle)^6_{2^\mu}, \ldots, (\langle 1\underbrace{2\ldots2}_{k}1\rangle)^{2k+2}_{2^\mu}.$$

■

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