On a theorem of Shapiro

Sakaé Fuchino, Saharon Shelah and Lajos Soukup

May 26, 1994

Abstract

We show that a theorem of Leonid B. Shapiro which was proved under MA, is actually independent from ZFC. We also give a direct proof of the Boolean algebra version of the theorem under MA(*Cohen*).

1 Introduction

L.B. Shapiro [8] recently proved the following theorem:

Theorem 1.1 (L.B. Shapiro) (MA(*Cohen*)) *For any compact Hausdorff space X of weight $< 2^{\aleph_0}$ and $\aleph_0 \leq \tau < 2^{\aleph_0}$ the following assertions are equivalent:*

i) *There exists a continuous surjection from X onto ${}^{\tau}\mathbb{I}$;*

ii) *There exists a continuous injection from ${}^{\tau}2$ into X;*

iii) *There exists a closed subset $Y \subseteq X$ such that $\chi(y, Y) \geq \tau$ for every $y \in Y$.*

The original proof of Theorem 1.1 by L.B. Shapiro in [8] was formulated under MA. However practically the same proof still works when merely MA(Cohen) is assumed where MA(Cohen) stands for Martin's Axiom restricted to the partial orderings of the form $\mathrm{Fn}(\kappa, 2)$.

A part of the theorem above can be translated into the language of Boolean algebras:

The first author would like to thank Professor L.B. Shapiro for calling his attention to [8], and Professor K. Eda for informing him of an example in [3]. The second author is partially supported by the Deutsche Forschungsgemeinschaft(DFG) grant Ko 490/7–1. He also gratefully acknowledges partial support by the Edmund Landau Center for research in Mathematical Analysis, supported by the Minerva Foundation (Germany). The present paper is the second author's Publication No. 543. The third author is partially supported by the Hungarian National Foundation for Scientific Research grant No. 1908 and the Deutsche Forschungsgemeinschaft(DFG) grant Ko 490/7–1.

Corollary 1.2 (Boolean algebra version of Shapiro's theorem) (MA(*Cohen*)) *For any infinite Boolean algebra B of cardinality $< 2^{\aleph_0}$ and any infinite τ, the following are equivalent:*

i′) There exists an injective Boolean mapping from $\mathrm{Fr}\,\tau$ into B;
ii′) There exists a surjective Boolean mapping from B onto $\mathrm{Fr}\,\tau$.

The implication from *ii′*) to *i′*) as well as the implication from *ii*) to *i*) can be proved already in ZFC. For the proof of *ii*) from *i*), let $g : {}^{\tau}2 \to X$ be a continuous injection. Note that $g[{}^{\tau}2]$ is a closed subset of X. For any fixed $y_0 \in {}^{\tau}2$ let $f' : X \to {}^{\tau}2$ be defined by

$$f'(x) = \begin{cases} g^{-1}(x) & ; \text{if } x \in g[{}^{\tau}2], \\ y_0 & ; \text{otherwise.} \end{cases}$$

Then f' is a continuous surjection from X onto ${}^{\tau}2$. Let f'' be a continuous surjection from ${}^{\tau}2$ to ${}^{\tau}\mathbb{I}$. E.g. let $h : {}^{\omega}2 \to \mathbb{I}$ be the continuous surjection defined by $u \mapsto$ the real represented by the binary expression $0.u(0)u(1)u(2)\cdots$. $h^{\kappa} : {}^{\kappa}({}^{\omega}2) \to {}^{\kappa}\mathbb{I}$ is then a continuous surjection. Since ${}^{\kappa}({}^{\omega}2)$ is homeomorphic to ${}^{\kappa}2$ we can find a continuous surjection f'' from ${}^{\tau}2$ onto ${}^{\tau}\mathbb{I}$ corresponding to h^{κ}. The mapping $g = f'' \circ f'$ is then as desired. In the next section we shall give a direct proof of *i′*) $\Rightarrow$ *ii′*). For *iii*) $\Rightarrow$ *i*) we need some deep results by Shapiro on dyadic compactum (see [8]).

The equivalence of the assertions *i′*) and *ii′*) above is not true in general for Boolean algebras of cardinality $\geq 2^{\aleph_0}$: For any σ-complete Boolean algebra B and any infinite κ, there exits no surjective Boolean mapping $f : B \to \mathrm{Fr}\,\kappa$ (see Lemma 1.3 below). Hence e.g. for Boolean algebra $B = \overline{\mathrm{Fr}\,\omega}$ we have that $|\,B\,| = 2^{\aleph_0}$; $\mathrm{Fr}\,2^{\aleph_0}$ is embeddable into B (by Balcar-Fraňek-Theorem, see [1]) but there exists no surjective Boolean mapping from B onto $\mathrm{Fr}\,2^{\aleph_0}$. The non-existence of surjective Boolean mapping from a σ-complete Boolean algebra in the ground model onto $\mathrm{Fr}\,\tau$ is preserved in a generic extension by a partial ordering of cardinality $< \tau$ though B may be no more σ-complete in such a generic extension:

Lemma 1.3 *Let B be a σ-complete Boolean algebra and P a partial ordering. For any $\kappa > |\,P\,|$ we have that*

$$\Vdash_P \text{“}\textit{there exists no surjective Boolean mapping from } B \textit{ onto } \mathrm{Fr}\,\kappa\text{”}.$$

Proof Suppose that there would be a P-name $\dot{f}$ such that

$$\Vdash_P \text{“}\dot{f} : B \to \mathrm{Fr}\,\kappa \text{ is a surjective Boolean mapping”}.$$

For each $p \in P$ let

$$B_p = \{\, b \in B \;:\; p \Vdash_P \text{“}\dot{f}(b) = c \text{ for some } c \in \mathrm{Fr}\,\kappa\text{”}\,\}$$

and

$$C_p = \{ c \in \operatorname{Fr}\kappa : p \Vdash_P \text{“} \dot{f}(b) = c \text{ for some } b \in B \text{”} \}.$$

Then B_p and C_p are subalgebras of B and $\operatorname{Fr}\kappa$ respectively. Since $\bigcup_{p\in P} C_p = \operatorname{Fr}\kappa$ and $\kappa > |P|$ there exists some $p \in P$ such that C_p is infinite. Let c_n, $n < \omega$ be pairwise disjoint positive elements of C_p. By the definition of B_p and C_p, there exits pairwise disjoint positive elements b_n, $n < \omega$ of B_p such that $p \Vdash_P$ “$\dot{f}(b_n) = c_n$” holds for every $n < \omega$. Let $X \subseteq \omega$ be such that there exists no $c \in \operatorname{Fr}\kappa$ such that $c \cdot c_n = c_n$ holds for all $n \in X$ and $c \cdot c_n = 0$ for all $n < \omega \setminus X$. Let $d = \Sigma^B_{n\in X} b_n$. Then for any $q \le p$ there can be no $c \in \operatorname{Fr}\kappa$ such that $q \Vdash_P$ “$\dot{f}(d) = c$”. This is a contradiction. ❑ (Lemma 1.3)

The lemma above together with Corollary 1.2 yields the following:

Proposition 1.4 *Let B be a complete Boolean algebra with $|B| = \tau \ge \aleph_0$. Then*

$$\Vdash_{\operatorname{Fn}(\kappa,2)} \text{“}\textit{there exists no surjective Boolean mapping from } B \textit{ onto } \operatorname{Fr}\tau\text{”}$$

holds if and only if $\kappa < \tau$.

Proof If $\kappa < \tau$ then $|\operatorname{Fn}(\kappa, 2)| = \kappa < \tau$. Hence by Lemma 1.3,

$$\Vdash_{\operatorname{Fn}(\kappa,2)} \text{“there exists no surjective Boolean mapping from } B \text{ onto } \operatorname{Fr}\tau\text{”}$$

holds.

Suppose now that $\kappa \ge \tau$. Then as in the proof of Proposition 2.1, we can show that

$$\Vdash_{\operatorname{Fn}(\kappa,2)} \text{“there exists a surjective Boolean mapping from } B \text{ onto } \operatorname{Fr}\tau\text{”}$$

holds. ❑ (Proposition 1.4)

Now, (⫯) (read “stick”, see [2]) is the following principle:

(⫯): There exists a sequence $(x_\alpha)_{\alpha<\omega_1}$ of countable subsets of ω_1 such that for any $y \in [\omega_1]^{\aleph_1}$ there exists $\alpha < \omega_1$ such that $x_\alpha \subseteq y$.

Clearly (⫯) follows from CH. Another combinatorial principle (♣), a strengthning of (⫯), is introduced in Ostaszewski [7]. Let $Lim(\omega_1) = \{ \gamma < \omega_1 : \gamma \text{ is a limit} \}$.

(♣): There exists a sequence $(x_\gamma)_{\gamma\in Lim(\omega_1)}$ of countable subsets of ω_1 such that for every $\gamma \in Lim(\omega_1)$, x_γ is a cofinal subset of γ, $otp(x_\gamma) = \omega$ and for every $X \in [\omega_1]^{\aleph_1}$ there is $\gamma \in Lim(\omega_1)$ such that $x_\gamma \subseteq X$.

Clearly (⫯) follows from $(\clubsuit)$. Unlike (⫯), $(\clubsuit)$ does not follow from CH, since $(\clubsuit)$ + CH is equivalent with $\diamondsuit$ (K. Devlin, see [7]). For more about the combinatorial principles (⫯) and $(\clubsuit)$, and independence results connected with them, see [4].

MA(*countable*) — Martin's axiom restricted to countable partial orderings — and MA(*Cohen*) both add a lot of Cohen reals over any small model of (a sufficiently large finite subset of) ZFC and in many cases where this property is needed, MA(*countable*) is just enough. Hence it seems to be quite natural to ask if these axioms are perhaps equivalent. However they are not. I. Juhász proved in an unpublished note that $\neg$CH + MA(*countable*) + $(\clubsuit)$ is consistent (two other constructions of models of $\neg$CH + MA(*countable*) + $(\clubsuit)$ are to be found in [5] and [4].). On the other hand, it is easy to see that the negation of $\mathrm{MA}(\mathrm{Fn}(\aleph_1, 2))$ follows from $\neg$CH + $(\clubsuit)$: using (⫯) we can obtain a Boolean algebra B of cardinality $\aleph_1$ such that $\mathrm{Fr}\,\omega_1$ is embeddable into B but there is no surjection from B onto $\mathrm{Fr}\,\omega_1$ (see Theorem 4.4). By Proposition 2.1, this shows that $m_{\mathrm{Fn}(\aleph_1,2)} = \aleph_1 < 2^{\aleph_0}$. It follows also that the assertions of Theorem 1.1 and Corollary 1.2 are independent from ZFC and MA(*countable*) is not enough to prove them.

Corollary 1.2 for other variety than Boolean algebras can be simply false. E.g., this is the case in the variety of abelian groups: in [3], an $\aleph_1$-free abelian group G in $\aleph_1$ is constructed (in ZFC) which contains uncountable free subgroup but $Hom(G, Z) = 0$.

2 A proof of the Boolean algebra version of the theorem

In this section we shall prove Corollary 1.2. More precisely we prove the following Proposition 2.1. For any class $\mathcal{C}$ of partial orderings Let

$$\begin{aligned} m_{\mathcal{C}} = \min\{\, |\,\mathcal{D}\,| \,:\, & \mathcal{D} \text{ is a family of dense subsets of } P \text{ for some } P \in \mathcal{C} \\ & \text{such that there exists no } \mathcal{D}\text{-generic filter over } P\,\} \end{aligned}$$

If $\mathcal{C}$ is a singleton $\{\,P\,\}$, we shall write simply m_P in place of $m_{\{\,P\,\}}$. Let us say that two partial orderings P, Q are coabsolute when their completions are isomorphic. It is easy to see that for any class $\mathcal{C}$ of partial orderings $m_{\mathcal{C}} = m_{\tilde{\mathcal{C}}}$ where $\tilde{\mathcal{C}} = \{\, Q \,:\, Q \text{ is coabsolute with some } P \in \mathcal{C}\,\}$. If the class $\mathcal{C}$ is introduced by a property $\mathcal{P}$ of Boolean algebras, we also write $m_{\mathcal{P}}$ in place of $m_{\mathcal{D}}$. We also write $m_{countable} = m_{\{\,P\,:\,P \text{ is countable}\,\}}$ and $m_{Cohen} = m_{\{\,P\,:\,P=\mathrm{Fn}(\kappa,2) \text{ for some } \kappa\,\}}$. Hence MA(*Cohen*) (MA(*countable*), MA etc. respectively) holds if and only if $m_{Cohen} = 2^{\aleph_0}$ ($m_{countable} = 2^{\aleph_0}$, $m_{ccc} = 2^{\aleph_0}$ etc. respectively) and we have $m_{ccc} \le m_{Cohen} \le m_{countable}$.

Proposition 2.1 *Let B be a Boolean algebra containing* $\mathrm{Fr}\,\kappa$ *as a subalgebra. If* $|\, B \,| < m_{\mathrm{Fn}(\kappa,2)}$, *then there exists a surjective Boolean mapping from B onto* $\mathrm{Fr}\,\kappa$.

Proof By Sikorski's theorem, there is a Boolean mapping from B to $\overline{\mathrm{Fr}\,\kappa}$ — the completion of $\mathrm{Fr}\,\kappa$, extending the inverse of the canonical embedding of $\mathrm{Fr}\,\kappa$ into B. Hence without loss of generality we may assume that B is a subalgebra of $\overline{\mathrm{Fr}\,\kappa}$. Now let $P = \mathrm{Fn}(\kappa, 3)$. Note that P is coabsolute with $\mathrm{Fn}(\kappa, 2)$. We shall define a family $\mathcal{D}$ of dense subsets of P such that $|\, D \,| < m_{\mathrm{Fn}(\kappa,2)}$ so that among other things (see below), for $\mathcal{D}$-generic set G, $g = \bigcup G$ will be a function from κ to 3 and $X = \{\, \alpha < \kappa \,:\, g(\alpha) = 2 \,\}$ will be of cardinality κ. Then we let f be the function on κ defined by:

$$f(\alpha) = \begin{cases} 0_B & ; \text{ if } g(\alpha) = 0, \\ 1_B & ; \text{ if } g(\alpha) = 1, \\ \alpha & ; \text{ otherwise.} \end{cases}$$

Let $\bar{f}$ be the Boolean mapping from $\mathrm{Fr}\,\alpha$ to $\mathrm{Fr}\,X$ generated by f.

Now we are done, if we can show that $\bar{f}$ extends to a Boolean mapping $\tilde{f}$ from B onto $\mathrm{Fr}\,X$. But by the following Lemma 2.2, we can choose $\mathcal{D}$ appropriate for this purpose.

For $p \in P$, let $B_p = \mathrm{Fr}\,\mathrm{dom}(p)$ (hence $B_p \leq B$) and $f_p : B_p \to \mathrm{Fr}\,(p^{-1}[\{2\}])$ be the Boolean mapping generated by the mapping f^0_p on $\mathrm{dom}(p)$ defined by:

$$f^0_p(\alpha) = \begin{cases} 0_B & ; \text{ if } p(\alpha) = 0, \\ 1_B & ; \text{ if } p(\alpha) = 1, \\ \alpha & ; \text{ otherwise.} \end{cases}$$

Lemma 2.2 *For any $b \in B$ and $p \in P$ there exists $q \leq p$ and b_1, $b_2 \in B_q$ such that $b_1 \leq b$, $b_2 \leq -b$ and $f_q(b_1) + f_q(b_2) = 1$ (i.e, q "forces" $\tilde{f}(b) = f_q(b_1)$).*

For the proof of the Lemma 2.2 we use the following Lemma whose proof is left to the reader:

Lemma 2.3 *Let $b \in \overline{\mathrm{Fr}\,\kappa}$ and let $Y \subseteq \kappa$ be a countable set such that $b \in \overline{\mathrm{Fr}\,Y}$ holds. Let $Y = \{\, \alpha_n \,:\, n < \omega \,\}$. Then there exist an increasing sequence $(l_n)_{n<\omega}$ with $l_n < \omega$ for $n < \omega$ and a sequcence $(i_n)_{n<\omega}$ with $i_n \in {}^{l_n}\{\, -1, 1 \,\}$ for $n < \omega$ such that, letting $p_n = \Sigma_{k<l_n} i_n(k) \cdot \alpha_k$ for $n < \omega$,*

i) *either $p_n \leq b$ or $p_n \leq -b$ and*

ii) $\Sigma_{n<\omega} p_n = 1$.

In particular we have $b = \Sigma\{\, p_n \,:\, n < \omega,\, p_n \leq b \,\}$. ❑

Proof of Lemma 2.2 Let $Y = \{\, \alpha_n \,:\, n < \omega \,\}$, $(l_n)_{n<\omega}$, $(i_n)_{n<\omega}$ and p_n, $n < \omega$ be as in Lemma 2.3 for our $b \in B$. Without loss of generality we may assume that $\mathrm{dom}(p) \cap Y = \{\, \alpha_n \,:\, n < k \,\}$ for some $k < \omega$. Let ${}^k\{\,-1, 1\,\} = \{\, \tau_m \,:\, m < 2^k \,\}$. By induction we can take $n_m < \omega$ for $m < 2^k$ such that

a) i_{n_m} is compatible (as an element of $\mathrm{Fn}(Y, \{\,-1, 1\,\})$) with τ_m and

b) $\{\, i_{n_m}\restriction (\mathrm{dom}(i_{n_m}) \setminus k) \,:\, m < 2^k \,\}$ is pairwise compatible.

Let $\tilde{n} = \max\{\, n_m \,:\, m < 2^k \,\}$, $\tilde{l} = l_{\tilde{n}}$ and $\tilde{i} = \bigcup\{\, i_{n_m}\restriction (\mathrm{dom}(i_{n_m}) \setminus k) \,:\, m < 2^k \,\}$. Let $q \le p$ be such that $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{\, \alpha_k, \ldots, \alpha_{\tilde{l}-1} \,\}$, $q\restriction \mathrm{dom}(p) = p$ and

$$q(\alpha_m) = \begin{cases} 1 & ; \text{if } \tilde{i}(\alpha_m) = 1, \\ 0 & ; \text{if } \tilde{i}(\alpha_m) = -1. \end{cases}$$

Then q as above together with $b_1 = \Sigma\{\, p_n \,:\, n < \tilde{n},\, p_n \le b \,\}$ and $b_2 = \Sigma\{\, p_n \,:\, n < \tilde{n},\, p_n \le -b \,\}$ is as desired. ❑ (Lemma 2.2)

Now by the lemma above

$$\begin{aligned} \mathcal{D} = & \{\, \{\, p \in P \,:\, \alpha \in \mathrm{dom}(p) \,\} \,:\, \alpha < \kappa \,\} \\ & \cup \{\, \{\, p \in P \,:\, \exists \beta > \alpha \; p(\beta) = 2 \,\} \,:\, \alpha < \kappa \,\} \\ & \cup \{\, \{\, q \in P \,:\, f_q(b_1) + f_q(b_2) = 1 \text{ for some } b_1 \le b,\, b_2 \le -b \,\} \,:\, b \in B \,\} \end{aligned}$$

is a family of dense subsets of P. Clearly the mapping $\bar{f}$ defined as above with respect to this $\mathcal{D}$ can be extended to a Boolean mapping $\tilde{f}$ from B onto $\mathrm{Fr}\, X$.

❑ (Proposition 2.1)

3 Pcf and the theorem of Shapiro

Proposition 3.1 *Assume that*

$\oplus_{\mu,\kappa,\lambda}$ *for any $\mathcal{F} \subseteq [\lambda]^{\aleph_0}$ with $|\, \mathcal{F} \,| < \mu$, there is $Y \in [\lambda]^{\kappa}$ such that $a \cap Y$ is finite for all $a \in \mathcal{F}$.*

Then, for any Boolean algebra B of cardinality $< \mu$, if $\mathrm{Fr}\, \lambda$ *is embeddable into B then there is a surjective Boolean mapping from B onto* $\mathrm{Fr}\, \kappa$.

Proof As in the proof of Proposition 2.1, we may assume without loss of generality that $\mathrm{Fr}\, \lambda \le B \le \overline{\mathrm{Fr}\, \lambda}$ holds. Let $|\, B \,| = i^*$ $(< \mu)$ and let $(y_i)_{i<i^*}$ be an enumeration of B. Let $y_i = \sum_{n<\omega} \tau_i^n(\alpha(i, n, 0), \ldots, \alpha(i, n, m_{i,n}))$ where τ_i^n is a Boolean term with $m_{i,m} + 1$ variables and $\alpha(i, n, 0), \ldots, \alpha(i, n, m_{i,n}) < \lambda$ for $i < i^*$ and $n < \omega$. For $i < i^*$, let $w_i = \{\, \alpha(i, n, l) \,:\, n < \omega,\, l \le m_{i,n} \,\}$. By the assumption, there exists a $Y \in [\lambda]^\kappa$ such that $w_i \cap Y$ is finite for everly $i < i^*$. Let $g : B \to \mathrm{Fr}\, Y$ be defined by

$$g(y_i) = \sum_{n<\omega} \tau_i^n(\alpha^*(i,n,0), \ldots, \alpha^*(i,n,m_{i,n}))$$

where

$$\alpha^*(i,n,l) = \begin{cases} \alpha(i,n,l) & ;\ \text{if } \alpha(i,n,l) \in Y \\ 0_B & ;\ \text{otherwise.} \end{cases}$$

The function g is well-defined since, for each $i < \omega$, $\tau_i^n(\alpha^*(i,n,0), \ldots, \alpha^*(i,n,m_{i,n}))$ is an element of $\mathrm{Fr}\,(w_i \cap Y)$ and $\mathrm{Fr}\,(w_i \cap Y)$ is finite. Clearly this g is as desired.
❏ (Proposition 3.1)

Corollary 3.2 *For any Boolean algebra of cardinality $< \mathbf{a}$ (where $\mathbf{a}$ is the minimal cardinality of a maximal almost disjoint family in $[\omega]^{\aleph_0}$), if $\mathrm{Fr}\,\omega$ is embeddable into B then there is a surjection from B onto $\mathrm{Fr}\,\omega$.*

Proof By Proposition 3.1 for $\oplus_{\mathbf{a},\aleph_0,\aleph_0}$. ❏ (Corollary 3.2)

Theorem 3.3 *Assume that*

$(*)_{\mu,\lambda,\kappa}$ *there are $\underset{\circ}{a}_i \in [Reg \cap (\lambda^+ \setminus \kappa^+)]^{<\aleph_0}$ for $i < \kappa$ such that for every $a \in [\kappa]^{\aleph_0}$, $\max \mathrm{pcf}(\bigcup_{i\in a} \underset{\circ}{a}_i) \geq \mu$ holds.*

Then for any Boolean algebra B of cardinality $< \mu$, if $\mathrm{Fr}\,\kappa$ is embeddable into B then there is a surjective Boolean mapping g from B onto $\mathrm{Fr}\,\kappa$.

(For more about $(*)_{\mu,\lambda,\kappa}$ see [10]. For pcf theory in general, the reader may consult [11].) The theorem follows from Proposition 3.1 and the following:

Lemma 3.4 *Assume that $(*)_{\mu,\lambda,\kappa}$ (as in Theorem 3.3) holds. Then $\oplus_{\mu,\kappa,\kappa}$ holds.*

Proof Since max pcf is always regular, we may assume that μ is regular. Let $\underset{\circ}{a} = \bigcup_{i<\kappa} \underset{\circ}{a}_i$. In place of $[\kappa]^{\aleph_0}$, we consider $[Z]^{\aleph_0}$ for $Z = \dot{\bigcup}_{i<\kappa} Z_i$ where $Z_i = \{\, i \,\} \times \prod \underset{\circ}{a}_i$. Hence we assume that $\mathcal{F} \subseteq [Z]^{\aleph_0}$ and $|\, \mathcal{F} \,| < \mu$.

For each $a \in \mathcal{F}$, let $g_a \in \prod \underset{\circ}{a}$ be defined by

$$g_a(\theta) = \sup\{\, \eta(\theta) \ :\ \eta \in a,\ \theta \in \mathrm{dom}(\eta) \,\}$$

for each $\theta \in \underset{\circ}{a}$, where we put $\sup \emptyset = 0$. Since $\prod \underset{\circ}{a} / J_{<\mu}[\underset{\circ}{a}]$ is μ-directed and $|\, \mathcal{F} \,| < \mu$, there is $f^* \in \prod \underset{\circ}{a}$ such that $g_a <_{J_{<\mu}[\underset{\circ}{a}]} f^*$ holds for all $a \in \mathcal{F}$. For $i < \kappa$, let $z_i = \{ (0,i) \} \cup (f^* \restriction \underset{\circ}{a}_i)$. Then $z_i \in Z_i$ for $i < \kappa$. We show that $Y = \{ z_i : i < \kappa \}$ is as required. Suppose not. Then $Y \cap a$ would be infinite for some $a \in \mathcal{F}$. By the assumption, it follows that $\bigcup_{z_i \in Y \cap a} \underset{\circ}{a}_i \notin J_{<\mu}[\underset{\circ}{a}]$. But for $z_i \in Y \cap a$ we have $\{ (0,i) \} \cup (f^* \restriction \underset{\circ}{a}_i) \in a$. It follows that for $\theta \in \underset{\circ}{a}_i$ we have $f^*(\theta) \leq g_a(\theta)$. This is a contradiction to $g_a <_{J_{<\mu}[\underset{\circ}{a}]} f^*$. ❑ (Lemma 3.4)

4 Independence of the theorem of Shapiro

The principle ($\stick$) suggests the following cardinal invariant $\stick$:

$$\stick = \min\{ \mid X \mid \,:\, X \subseteq [\omega_1]^{\aleph_0},\, \forall y \in [\omega_1]^{\aleph_1}\, \exists x \in X\ x \subseteq y \}.$$

Clearly $\aleph_1 \leq \stick \leq 2^{\aleph_0}$ and ($\stick$) holds if and only if $\stick = \aleph_1$. We can also consider the following variants of $\stick$:

$$\begin{aligned}\stick' = \min\{ \kappa : \kappa \geq \aleph_1, &\text{ there is an } X \subseteq [\kappa]^{\aleph_0} \\ &\text{such that } \mid X \mid = \kappa \text{ and } \forall y \in [\kappa]^{\aleph_1}\, \exists x \in X\ x \subseteq y \},\end{aligned}$$

$$\begin{aligned}\stick'' = \min\{ \kappa : \kappa \geq \aleph_1, &\text{ there is an } X \subseteq [\kappa]^{\aleph_0} \\ &\text{such that } \mid X \mid = \kappa \text{ and } \forall y \in [\kappa]^{\kappa}\, \exists x \in X\ x \subseteq y \}.\end{aligned}$$

We have $\aleph_1 \leq \stick'' \leq \stick' \leq 2^{\aleph_0}$ and ($\stick$) holds if and only if $\stick = \stick' = \stick'' = \aleph_1$ holds.

It can be easily shown that $\stick \leq \stick'$ holds. Moreover if $\stick < \aleph_{\omega_1}$, then $\stick = \stick'$ holds. The question, if $\stick < \stick'$ is consistent, is connected with some very fundamental unsolved problems on cardinal arithmetics while we can show that $\stick'' < \stick$ is consistent. For more, see [4] and [10].

Proposition 4.1 *There exists a Boolean algebra B such that $\mid B \mid = \stick'$, $\mathrm{Fr}\,\stick'$ is embeddable into B but there is no surjective Boolean mapping from B onto $\mathrm{Fr}\,\omega_1$.*

Proof Let $\Phi : \kappa \to \kappa;\ \alpha \mapsto \xi_\alpha$ be the continuously increasing function defined inductively by $\xi_0 = \omega$ and $\xi_{\alpha+1} = \xi_\alpha + \mid \xi_\alpha \mid$. Let $\kappa = \stick'$ and let $X \subseteq [\kappa \times \mathrm{Fr}\,\omega_1]^{\aleph_0}$ be such that $\mid X \mid = \kappa$, $\omega \times \mathrm{Fr}\,\omega \in X$ and $\forall y \in [\kappa \times \mathrm{Fr}\,\omega_1]^{\aleph_1}\, \exists x \in X\ x \subseteq y$ holds. Let $(x_\alpha)_{\alpha<\kappa}$ be an enumeration of X such that $x_\alpha \subseteq \xi_\alpha \times \mathrm{Fr}\,\omega_1$ for all $\alpha < \kappa$.

Now let $(B_\alpha)_{\alpha<\kappa}$ be a continuously increasing sequence of Boolean algebras such that for all $\alpha < \kappa$

1) the underlying set of B_α is ξ_α;

2) there exits a $b_\alpha \in B_{\alpha+1}$ such that b_α is free over B_α;

3) if x_α generates a Boolean mapping f_α from a subalgebra of B_α onto an infinite subalgebra of $\mathrm{Fr}\,\omega_1$ then $B_{\alpha+1}$ contains an element c_α of the form $\Sigma^{B_{\alpha+1}}_{n\in Z_\alpha} b^\alpha_n$ where $Z_\alpha \subseteq \omega$, b^α_n, $n<\omega$ are pairwise disjoint elements in $\mathrm{dom}(f_\alpha)$, $f_\alpha(b^\alpha_n) \neq 0$ for all $n<\omega$ and there is no $d \in \mathrm{Fr}\,\omega_1$ such that $d\cdot f_\alpha(b^\alpha_n) = f(b^\alpha_n)$ for all $n \in Z_\alpha$ and $d \cdot f_\alpha(b^\alpha_n) = 0$ for all $n < \omega \setminus Z_\alpha$ holds.

Let $B = \bigcup_{\alpha<\kappa} B_\alpha$. We show that this B is as desired. By *1*) the underlying set of B is κ. By *2*) $\{\, b_\alpha : \alpha < \kappa \,\}$ is an independent subset of B. Hence $\mathrm{Fr}\,\kappa$ is embeddable into B.

Suppose now that there would be a surjective Boolean mapping f from B onto $\mathrm{Fr}\,\omega_1$. Then there is a bijection $g \subseteq f$ from a subset of B onto $\mathrm{Fr}\,\omega_1$. Since g is uncountable there is an $\alpha < \kappa$ such that $x_\alpha \subseteq g$. Since $x_\alpha \subseteq f$, x_α satisfies the condition in *3*). Hence there is a $c_\alpha \in B_{\alpha+1}$ such that $c_\alpha = \Sigma^{B_{\alpha+1}}_{n\in Z_\alpha} b^\alpha_n$ for Z_α and b^α_n, $n < \omega$ as un *3*). But then $f(c_\alpha)\cdot f_\alpha(b^\alpha_n) = f(b^\alpha_n)$ for all $n \in Z_\alpha$ and $f(c_\alpha)\cdot f_\alpha(b^\alpha_n) = 0$ for all $n < \omega \setminus Z_\alpha$ holds. This is a contradiction to the choice of Z_α. ❑ (Proposition 4.1)

Corollary 4.2 $m_{\mathrm{Fn}(\omega_1,2)} \leq \stick'$.

Proof By Proposition 2.1 and Proposition 4.1. ❑ (Corollary 4.2)

With almost the same proof as in Proposition 4.1 we can also prove the following:

Proposition 4.3 *There exists a Boolean algebra B such that $|\, B \,| = \stick''$, $\mathrm{Fr}\,\stick''$ is embeddable into B but there is no surjective Boolean mapping from B onto $\mathrm{Fr}\,\stick''$.* ❑

Since we have $\stick' = \aleph_1$ under $(\stick)$, we obtain the following theorem:

Theorem 4.4 *If $(\stick)$ holds then there exists a Boolean algebra B of cardinality $\aleph_1$ such that $\mathrm{Fr}\,\omega_1$ is embeddable into B but there is no surjection from B onto $\mathrm{Fr}\,\omega_1$.* ❑

Hence if $\neg$CH and $(\stick)$ holds, by Theorem 4.4, there exists a counter-example to the theorem of Shapiro. This shows that we cannot just drop MA(*Cohen*) from Theorem 1.1. Since MA(*countable*) + $\neg$CH + $(\stick)$ is consistent (see e.g. [5] or [4]), we see that MA(*countable*) is not enough for Theorem 1.1.

Corollary 4.5 $m_{\mathit{Cohen}} \leq \stick''$.

Proof By Proposition 2.1 and Proposition 4.3. ❑ (Corollary 4.5)

If a Boolean algebra B is atomless then $\mathrm{Fr}\,\omega$ can be embdded into B. By Proposition 2.1, if MA(*countable*) holds and B is of cardinality $< 2^{\aleph_0}$, there exists a surjection from B onto $\mathrm{Fr}\,\omega$. Here again we cannot simply drop the assumption of MA(*countable*):

Proposition 4.6 *It is consistent that there is an atomless Boolean algebra B of cardinality $\aleph_1 < 2^{\aleph_0}$ such that there is no surjective Boolean mapping from B onto $\mathrm{Fr}\,\omega$.*

Proof By [9, Theorem 5.12], there is a model of ZFC+ $\neg$CH in which there is an endo-rigid atomless Boolean algebra B of cardinality $\aleph_1$. In particular there is no surjection from B onto $\mathrm{Fr}\,\omega$. ❑ (Proposition 4.6)

Note that, since (⫯) is consistent with $\neg$CH and MA(*countable*), (⫯) cannot supply such a Boolean algebra as in the proposition above.

References

[1] B. Balcar, F. Franěk: Independent families in complete Boolean algebras, Trans. Amer. Math. Soc., Vol. 274 (1982), 607–618.

[2] S. Broverman, J. Ginsburg, K. Kunen and F. Tall: Topologies determined by σ-ideals on ω_1, Can. J. Math., 30 No. 6 (1978), 1306–1312.

[3] K. Eda: Cardinality restrictions on preradicals, Contemporary Math. Vol. 87 (1989), 277–283.

[4] S. Fuchino, S. Shelah and L. Soukup: Sticks and Clubs, in preprint.

[5] P. Komjáth: Set systems with finite chromatic number, European Journal of Combinatorics, 10 (1989), 543–549.

[6] K. Kunen, F. Tall: Between Martin's axiom and Souslin's hypothesis, Fundamenta Mathematicae, Vol. 102 (1979), 173–181.

[7] A.J. Ostaszewski: On countably compact perfectly normal spaces, Journal of London Mathematical Society (2), 14 (1976), 505–516.

[8] L.B. Shapiro: On Šapirovskiĭ's Theorem, preprint.

[9] S. Shelah: Constructions of many complicated uncountable structures and Boolean algebras, Israel Journal of Mathematics, Vol. 45 (1983), 100–146.

[10] ____: PCF and infinite free subsets, in preparation.

[11] ____: Cardinal Arithmetic, Oxford University Press, in print.

Authors' addresses:

Institut of Mathematics, Hebrew University of Jerusalem
91904 Jerusalem, Israel
and
Institut für Mathematik II, Freie Universität Berlin
14195 Berlin, Germany

fuchino@math.fu-berlin.de

Institut of Mathematics, Hebrew University of Jerusalem
91904 Jerusalem, Israel
and
Department of Mathematics, Rutgers University
New Brunswick, NJ 08854, USA

shelah@math.huji.ac.il

Mathematical Institute of the Hungarian Academy of Sciences

soukup@math-inst.hu