Sticks and clubs

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April 3, 1997

Abstract

We study combinatorial principles known as stick and club. Several variants of these principles and cardinal invariants connected to them are also considered. We introduce a new kind of side-by-side product of partial orderings which we call pseudo-product. Using such products, we give several generic extensions where some of these principles hold together with $\neg$CH and Martin's Axiom for countable p.o.-sets. An iterative version of the pseudo-product is used under an inaccessible cardinal to show the consistency of the club principle for every stationary subset of limits of ω_1 together with $\neg$CH and Martin's Axiom for countable p.o.-sets.

Keywords: stick principle, club principle, weak Martin's axiom, preservation theorem.
1991 Mathematics Subject classification: 03 *E* 35, 03 *E* 05.

1 Beating with sticks and clubs

In this paper, we study combinatorial principles known as 'stick' and 'club', and their diverse variants which are all weakenings of $\diamondsuit$. Hence some of the consequences of $\diamondsuit$ still hold under these principles. On the other hand, they are weak enough to be consistent with the negation of the continuum hypothesis or even with a weak version of Martin's axiom in addition. See e.g. [2], [4], [10] for applications of these principles. We shall begin with introducing the principles and some cardinal numbers connected to them.

($\stackrel{\bullet}{|}$) (read "stick") is the following principle introduced in S. Broverman, J. Ginsburg, K. Kunen and F. Tall [2]:

($\stackrel{\bullet}{|}$): *There exists a sequence $(x_\alpha)_{\alpha<\omega_1}$ of countable subsets of ω_1 such that for any $y \in [\omega_1]^{\aleph_1}$ there exists $\alpha < \omega_1$ such that $x_\alpha \subseteq y$.*

Of course the sequence $(x_\alpha)_{\alpha<\omega_1}$ above is a bluff. What is essential here is that there exists an $X \subseteq [\omega_1]^{\aleph_0}$ of cardinality $\aleph_1$ such that for any $y \in [\omega_1]^{\aleph_1}$ there is an $x \in X$ with $x \subseteq y$. The formulation above is chosen here merely to make the connection to the principle ($\clubsuit$) introduced later, more apparent.

Note that ($\stick$) follows from CH.

The principle ($\stick$) suggests the following cardinal number:

$$\stick = \min\{\,|\,X\,|\,:\, X \subseteq [\omega_1]^{\aleph_0},\ \forall y \in [\omega_1]^{\aleph_1}\ \exists x \in X\ x \subseteq y\,\}.$$

We have $\aleph_1 \leq \stick \leq 2^{\aleph_0}$ and ($\stick$) holds if and only if $\stick = \aleph_1$. We also consider the following variants of $\stick$:

$$\begin{aligned}\stick' = \min\{\,\kappa\,:\kappa \geq \aleph_1, &\text{ there is an } X \subseteq [\kappa]^{\aleph_0}\\ &\text{such that } |\,X\,| = \kappa \text{ and } \forall y \in [\kappa]^{\aleph_1}\ \exists x \in X\ x \subseteq y\,\};\end{aligned}$$

$$\begin{aligned}\stick'' = \min\{\,\kappa\,:\kappa \geq \aleph_1, &\text{ there is an } X \subseteq [\kappa]^{\aleph_0}\\ &\text{such that } |\,X\,| = \kappa \text{ and } \forall y \in [\kappa]^{\kappa}\ \exists x \in X\ x \subseteq y\,\};\end{aligned}$$

$$\begin{aligned}\stick_\lambda = \min\{\,|\,X\,|\,: &X \subseteq [\lambda]^{\aleph_0}\\ &\text{such that } \forall y \in [\lambda]^{\aleph_1}\ \exists x \in X\ x \subseteq y\,\}.\end{aligned}$$

We have $\aleph_1 \leq \stick'' \leq \stick' \leq 2^{\aleph_0}$ and $\lambda \leq \stick_\lambda \leq \lambda^{\aleph_0}$. ($\stick$) holds if and only if $\stick = \stick' = \stick'' = \aleph_1$. Let us call X as in the definition of $\stick$ ($\stick'$, $\stick''$ and $\stick_\lambda$ respectively) a $\stick$-set ($\stick'$-set, $\stick''$-set and $\stick_\lambda$-set respectively).

Lemma 1.1

a) $\stick \leq \stick'$.

b) *If* $\stick < \aleph_{\omega_1}$ *then* $\stick = \stick'$. *In particular, we have then* $\stick'' \leq \stick$.

c) *If* $\lambda \leq \lambda'$ *then* $\stick_\lambda \leq \stick_{\lambda'}$.

d) $\stick \leq \stick_{\stick} \leq \stick'$.

Proof a): Let $X \subseteq [\kappa]^{\aleph_0}$ be a $\stick'$-set of cardinality $\stick'$. Then $X_0 = X \cap [\omega_1]^{\aleph_0}$ is a $\stick$-set of cardinality $\leq \stick'$.

b): By a), it is enough to show $\stick' \leq \stick$. We show inductively that, for every uncountable $\kappa \leq \stick$,

$(*)_\kappa$ *there exists an* $X_\kappa \subseteq [\kappa]^{\aleph_0}$ *such that* $|\,X_\kappa\,| \leq \stick$ *and* $\forall y \in [\kappa]^{\aleph_1}\ \exists x \in X_\kappa\ (x \subseteq y)$.

For $\kappa = \aleph_1$ this is clear.

Assume that we have shown $(*)_\lambda$ for all $\lambda < \kappa$. If κ is a successor then by induction hypothesis, we can find $X_\alpha \subseteq [\alpha]^{\aleph_0}$ for all $\alpha < \kappa$ such that $|\,X_\alpha\,| \leq \stick$ and $\forall y \in [\alpha]^{\aleph_1}\ \exists x \in X_\alpha\ x \subseteq y$. Let $X_\kappa = \bigcup_{\alpha<\kappa} X_\alpha$. Then X_κ has the desired

property: $|\, X_\kappa \,| \leq \text{⫯}$ is clear. If $y \in [\kappa]^{\aleph_1}$, there is some $\alpha < \kappa$ such that $y \in [\alpha]^{\aleph_1}$. Hence there is an $x \in X_\alpha \subseteq X$ such that $x \subseteq y$.
Suppose now that κ is a limit. By assumption, we have $\mathrm{cof}(\kappa) = \omega$. Let $(\kappa_n)_{n\in\omega}$ be an increasing sequence of cardinals below κ such that $\kappa = \bigcup_{n\in\omega} \kappa_n$. For each n, let $X_{\kappa_n} \subseteq [\kappa_n]^{\aleph_0}$ be as in $(*)_{\kappa_n}$ and let $X_\kappa = \bigcup_{n\in\omega} X_{\kappa_n}$. Then X_κ is as desired: clearly $|\, X_\kappa \,| \leq \text{⫯}$. If $y \in [\kappa]^{\aleph_1}$ there is an $n \in \omega$ such that $y \cap \kappa_n$ is uncountable. Hence there exists an $x \in X_{\kappa_n} \subseteq X_\kappa$ such that $x \subseteq y \cap \kappa_n \subseteq y$.

In particular we have shown that $(*)_{\text{⫯}}$ holds and hence $\text{⫯}' \leq \text{⫯}$.

c): Similarly to a).

d): By a) and c), we have $\text{⫯} = \text{⫯}_{\aleph_1} \leq \text{⫯}_{\text{⫯}} \leq \text{⫯}_{\text{⫯}'} = \text{⫯}'$. ❑ (Lemma 1.1)

The question, whether $\text{⫯} < \text{⫯}'$ is consistent, turned out to be a very delicate one: the problem is connected with some natural weakenings of GCH whose status (i.e. whether they are theorems in ZFC) is still open. One of them implies that $\text{⫯} = \text{⫯}'$ (this is essentially stated in [14, 1.2, 1.2A] in the light of [13, 6.1 [D]]; for more see [15]) while the negation of the other implies that the inequality is consistent. In this paper, we shall treat the latter consistency proof (Proposition 3.4). In contrast, the consistency of the inequality $\text{⫯}'' < \text{⫯}$ can be shown without any such additional set-theoretic assumptions (Proposition 3.5).

The principle $(\clubsuit)$ ('club'), a strengthening of (⫯), was first formulated in Ostaszewski [10]. Let $Lim(\omega_1) = \{\gamma < \omega_1 : \gamma \text{ is a limit}\}$. For a stationary $E \subseteq Lim(\omega_1)$,

$\clubsuit(E)$: *There exists a sequence $(x_\gamma)_{\gamma\in E}$ of countable subsets of ω_1 such that for every $\gamma \in E$, x_γ is a cofinal subset of γ with $otp(x_\gamma) = \omega$ and for every $y \in [\omega_1]^{\aleph_1}$ there is $\gamma \in E$ such that $x_\gamma \subseteq y$.*

Let us call $(x_\gamma)_{\gamma\in E}$ as above a $\clubsuit(E)$-sequence. For $E = Lim(\omega_1)$ we shall simply write $(\clubsuit)$ in place of $\clubsuit(Lim(\omega_1))$. Clearly (⫯) follows from $(\clubsuit)$. Unlike (⫯), $(\clubsuit)$ does not follow from CH since $(\clubsuit)$ + CH is known to be equivalent to $\diamondsuit$ (K. Devlin, see [10]). This equivalence holds also in the version argumented with a stationary $E \subseteq Lim(\omega_1)$.

Fact 1.2 *For any stationary $E \subseteq Lim(\omega_1)$, $\clubsuit(E)$ + CH is equivalent to $\diamondsuit(E)$.*

Proof The proof in [10] argumented with E works. ❑ (Fact 1.2)

S. Shelah [11] proved the consistency of $\neg$CH + $(\clubsuit)$ in a model obtained from a model of GCH by making the size of $\wp(\omega_1)$ to be $\aleph_3$ by countable conditions and then collapsing $\aleph_1$ to be countable. Soon after that, in an unpublished note, J. Baumgartner gave a model of $\neg$CH + $\clubsuit$ where collapsing of cardinals is not involved: his model was obtained from a model of $V = L$ by adding many Sacks reals by side by side product. I. Juhász then proved in an unpublished note that "$\neg$CH + MA(*countable*) + ($\clubsuit$)" is consistent. Here MA(*countable*) stands for Martin's axiom restricted to countable partial orderings. Later P. Komjáth [7] cited a remark by Baumgartner that Shelah's model mentioned above also satisfies $\neg$CH + MA(*countable*) + ($\clubsuit$). In Section 3, we shall give yet another model of $\neg$CH + MA(*countable*) + ($\clubsuit$) in which collapsing of cardinals is not involved (Theorem 3.8). In section 5, we construct a model of $\neg$CH + MA(*countable*) + "$\clubsuit(E)$ for every stationary $E \subseteq \mathit{Lim}(\omega_1)$ " starting from a model of ZFC with an inaccessible cardinal (Theorem 5.6).

These results are rather optimal in the sense that a slight strengthening of MA(*countable*) implies the negation of ($\clubsuit$). Let MA(*Cohen*) denote Martin's axiom restricted to the partial orderings of the form $\mathrm{Fn}(\kappa, 2)$ for some κ where, as in [8], $\mathrm{Fn}(\kappa, 2)$ is the p.o.-set for adding κ Cohen reals, i.e. the set of functions from some finite subset of κ to 2 ordered by reverse inclusion.

Fact 1.3 MA *for the partial ordering* $\mathrm{Fn}(\omega_1, 2)$ *implies* $\stick = \stick' = 2^{\aleph_0}$. *Further, if* MA(*Cohen*) *holds, then we have also* $\stick'' = 2^{\aleph_0}$.

Proof Both equations can be proved similarly. For the first equation, it is enough to show $\stick = 2^{\aleph_0}$ by Lemma 1.1. Suppose that $X \subseteq [\omega_1]^{\aleph_0}$ is of cardinality less than $2^{\aleph_0}$. We show that X is not a $\stick$-set. Let $P = \mathrm{Fn}(\omega_1, 2)$. Then for each $x \in X$ the set

$$D_x = \{\, q \in \mathrm{Fn}(\omega_1, 2) \,:\, \exists\alpha \in \mathrm{dom}(q) \cap x \; q(\alpha) = 0 \,\}$$

is dense in P. For each $\alpha < \omega_1$,

$$E_\alpha = \{\, q \in \mathrm{Fn}(\omega_1, 2) \,:\, \exists\beta > \alpha \, (\beta \in \mathrm{dom}(q) \;\wedge\; q(b) = 1) \,\}$$

is also a dense subset of P. Let $\mathcal{D} = \{\, D_x \,:\, x \in X \,\} \cup \{\, E_\alpha \,:\, \alpha < \omega_1 \,\}$ and G be a $\mathcal{D}$-generic filter over P. Then the uncountable set

$$Y = \{\, \alpha < \omega_1 \,:\, q(\alpha) = 1 \text{ for some } q \in G \,\}$$

contains no $x \in X$ as a subset. ❏ (Fact 1.3)

We shall see in Proposition 3.5 that MA for the partial ordering $\mathrm{Fn}(\omega_1, 2)$ is not enough for the last assertion in Fact 1.3.

$\clubsuit(E)$ is equivalent to the following seemingly much stronger statement. Let $E \subseteq \mathit{Lim}(\omega_1)$ be a stationary set.

$\clubsuit^\dagger(E)$: *There exists a sequence* $(x_\gamma)_{\gamma \in E}$ *of countable subsets of* ω_1 *such that for every* $\gamma \in E$, x_γ *is a cofinal subset of* γ *with* $otp(x_\gamma) = \omega$ *and for every* $X \in [\omega_1]^{\aleph_1}$, $\{\, \alpha \in E \,:\, x_\alpha \subseteq X \,\}$ *is stationary.*

Fact 1.4 *For any stationary $E \subseteq Lim(\omega_1)$, $\clubsuit(E)$ and $\clubsuit^\dagger(E)$ are equivalent.*

Proof Like Fact 1.2, an easy modification of the corresponding proof in [10] will work. Nevertheless we give here a proof for convenience of the reader.

Clearly it is enough to show $\clubsuit(E) \Rightarrow \clubsuit^\dagger(E)$. Suppose that $(x_\gamma)_{\gamma\in E}$ is a $\clubsuit(E)$-sequence. We claim that $(x_\gamma)_{\gamma\in E}$ is then also a $\clubsuit^\dagger(E)$-sequence. Otherwise there would be a $Y \in [\omega_1]^{\aleph_1}$ and a club $C \subseteq Lim(\omega_1)$ such that $x_\gamma \not\subseteq Y$ for every $\gamma \in C \cap E$. By thinning out C if necessary, we may assume that $Y \cap \alpha$ is cofinal in α for each $\alpha \in C$. For $\alpha \in C$, denoting by α^+ the next element to α in C, let $y_\alpha \subseteq [\alpha, \alpha^+) \cap Y$ be a cofinal subset in α^+ with $otp(y_\alpha) = \omega$. Now let $Y' = \bigcup_{\alpha\in C} y_\alpha$. Then $Y' \in [\omega_1]^{\aleph_1}$ and $Y' \subseteq Y$. We show that $\{\, \gamma \in E \,:\, x_\gamma \subseteq Y' \,\} = \emptyset$ which is a contradiction: if $\gamma \in E \cap C$ then $x_\gamma \not\subseteq Y'$ follows from $Y' \subseteq Y$. If $\gamma \in E \setminus C$ then there is $\alpha \in C$ such that $\alpha < \gamma < \alpha^+$. By the choice of y_α, $Y' \cap \gamma$ is not cofinal in γ. Hence again $x_\gamma \not\subseteq Y'$. ❑ (Fact 1.4)

Now, let us consider the following variants of the $(\clubsuit)$-principle:

$(\clubsuit_{\mathrm{w}})$: *There exists a sequence $(x_\gamma)_{\gamma\in Lim(\omega_1)}$ of countable subsets of ω_1 such that for every $\gamma \in Lim(\omega_1)$, x_γ is cofinal subset of γ, $otp(x_\gamma) = \omega$ and for every $y \in [\omega_1]^{\aleph_1}$, there is $\gamma < \omega_1$ such that $x_\gamma \setminus y$ is finite.*

$(\clubsuit_{\mathrm{w}^2})$: *There exists a sequence $(x_\gamma)_{\gamma\in Lim(\omega_1)}$ of countable subsets of ω_1 such that for every $\gamma \in Lim(\omega_1)$, x_γ is cofinal subset of γ, $otp(x_\gamma) = \omega$ and for every $y \in [\omega_1]^{\aleph_1}$*

$$\{\, \alpha < \omega_1 \,:\, x_\alpha \cap y \textit{ is finite} \,\} \cup \{\, \alpha < \omega_1 \,:\, x_\alpha \setminus y \textit{ is finite} \,\}$$

is stationary in ω_1.

Clearly $(\clubsuit)$ implies $(\clubsuit_{\mathrm{w}})$. Similarly to Fact 1.4, we can prove the equivalence of $(\clubsuit_{\mathrm{w}})$ with $(\clubsuit^\dagger_{\mathrm{w}})$ which is obtained from $(\clubsuit_{\mathrm{w}})$ by replacing "there is an $\alpha < \omega_1 \ldots$" with "there are stationary may $\alpha < \omega_1 \ldots$". Hence $(\clubsuit_{\mathrm{w}})$ implies $(\clubsuit_{\mathrm{w}^2})$. It is also easy to see that $(\clubsuit_{\mathrm{w}})$ implies $(\stackrel{\bullet}{\mid})$: if $(x_\gamma)_{\gamma\in Lim(\omega_1)}$ is a sequence as in the definition of $(\clubsuit_{\mathrm{w}})$, then $\{\, x_\gamma \setminus u \,:\, \gamma \in Lim(\omega_1),\, u \in [\omega_1]^{<\aleph_0} \,\}$ is a $\stackrel{\bullet}{\mid}$-set of cardinality $\aleph_1$. Džamonja and Shelah [3] gave a model of $\neg$CH + $(\clubsuit_{\mathrm{w}})$ + $\neg(\clubsuit)$. By the remark above this model also shows the consistency of non-equivalence of $(\stackrel{\bullet}{\mid})$ and $(\clubsuit)$ under $\neg$CH. In this paper we prove that $(\clubsuit_{\mathrm{w}^2})$ is strictly weaker than $(\clubsuit_{\mathrm{w}})$ by showing the consistency of $\neg(\stackrel{\bullet}{\mid})$ + $(\clubsuit_{\mathrm{w}^2})$ (Corollary 3.12). The partial ordering used in Corollary 3.12 does not force MA(*countable*) hence the following problem remains open:

Problem 1.5 *Is* MA(*countable*) + $\neg(\stackrel{\bullet}{\mid})$ + $(\clubsuit_{\mathrm{w}^2})$ *consistent?*

2 Pseudo product of partial orderings

In this section, we introduce a new kind of side-by-side product of p.o.'s which will be used in the next section to prove various consistency results. Let X be any set and $(P_i)_{i\in X}$ be a family of partial orderings. For $p \in \Pi_{i\in X}P_i$ the support of p is defined by $supp(p) = \{\, i \in X \,:\, p(i) \neq 1_{P_i} \,\}$. For a cardinal κ, let $\Pi^*_{\kappa,i\in X}P_i$ be the set

$$\{\, p \in \Pi_{i\in X}P_i \,:\, |\, supp(p) \,| < \kappa \,\}$$

with the partial ordering

$$p \leq q \quad\Leftrightarrow\quad \begin{array}{l} p(i) \leq q(i) \text{ for all } i \in X \text{ and} \\ \{\, i \in X \,:\, p(i) \lneqq q(i) \lneqq 1_{P_i} \,\} \text{ is finite .} \end{array}$$

For $\kappa = \aleph_0$ this is just a finite support product. We are mainly interested in the case where $\kappa = \aleph_1$. In this case we shall drop the subscript $\aleph_1$ and write simply $\Pi^*_{i\in X}P_i$. Further, if $P_i = P$ for some partial ordering P for every $x \in X$, we shall write $\Pi^*_{\kappa,X}P$ (or even $\Pi^*_X P$ when $\kappa = \aleph_1$) to denote this partial ordering.

For p, $q \in \Pi^*_{\kappa,i\in X}P_i$ the relation $p \leq q$ can be represented as a combination of the two other distinct relations which we shall call horizontal and vertical, and denote by $\leq_h$ and $\leq_v$ respectively:

$$p \leq_h q \quad\Leftrightarrow\quad supp(p) \supseteq supp(q) \text{ and } p\restriction supp(q) \subseteq q;$$

$$p \leq_v q \quad\Leftrightarrow\quad \begin{array}{l} supp(p) = supp(q),\ p(i) \leq q(i) \text{ for all } i \in X \text{ and} \\ \{\, i \in X \,:\, p(i) \lneqq q(i) \lneqq 1_{P_i} \,\} \text{ is finite .} \end{array}$$

For $p \in \Pi^*_{\kappa,i\in X}P_i$ and $Y \subseteq X$ let $p\lceil Y$ denote the element of $\Pi^*_{\kappa,i\in X}P_i$ defined by $p\lceil Y(i) = 1_{P_i}$ for every $i \in X \setminus Y$ and $p\lceil Y(i) = p(i)$ for $i \in Y$.

The following is immediate from definition:

Lemma 2.1 *For p, $q \in \Pi^*_{\kappa,i\in X}P_i$, the following are equivalent:*

a) $p \leq q$;

b) *There is an $r \in \Pi^*_{\kappa,i\in X}P_i$ such that $p \leq_h r \leq_v q$;*

c) *There is an $s \in \Pi^*_{\kappa,i\in X}P_i$ such that $p \leq_v s \leq_h q$.*

Proof b) $\Rightarrow$ a) and c) $\Rightarrow$ a) are clear. For a) $\Rightarrow$ b), let $r = p\lceil supp\, q$; for a) $\Rightarrow$ c), $s = q\restriction supp(q) \cup p\restriction (X \setminus supp(q))$. ❑ (Lemma 2.1)

Lemma 2.2

1) *If P_i has the property K for all $i \in X$ then $P = \Pi^*_{i\in X}P_i$ preserves $\aleph_1$.*

2) *Suppose that $\lambda \leq \kappa$. If P_i has the strong λ-cc (i.e. for every $C \in [P_i]^\lambda$ there is pairwise compatible $D \in [C]^\lambda$), then $P = \Pi^*_{\kappa,i\in X}P_i$ preserves λ.*

Proof This proof is a prototype of the arguments we are going to apply repeatedly. 1) and 2) can be proved similarly. For 1), assume that there would be $p \in P$ and a P-name $\dot{f}$ such that

*) $p \Vdash_P$ "$\dot{f} : (\omega_1)^V \to \omega$ and $\dot{f}$ is 1-1".

Then, let $(p_\alpha)_{\alpha<\omega_1}$ and $(q_\alpha)_{\alpha<\omega_1}$ be sequences of elements of P such that

a) $p_0 \leq p$ and $(p_\alpha)_{\alpha<\omega_1}$ is a descending sequence with respect to $\leq_h$;

b) $q_\alpha \leq_v p_\alpha$ and q_α decides $\dot{f}(\alpha)$ for all $\alpha < \omega_1$;

c) $p_\alpha \restriction S_\alpha = q_\alpha \restriction S_\alpha$ for every $\alpha < \omega_1$ where

$$S_\alpha = supp(q_\alpha) \setminus (supp(p) \cup \bigcup_{\beta<\alpha} supp(q_\beta)).$$

For $\alpha < \omega_1$ let $d_\alpha = \bigcup_{\beta<\alpha} supp(q_\beta)$. Then $(d_\alpha)_{\alpha<\omega_1}$ is a continuously increasing sequence in $[X]^{<\omega_1}$. Let $u_\alpha = \{\, \beta \in supp(q_\alpha) \; : \; q_\alpha(\beta) \neq p_\alpha(\beta) \,\}$ for $\alpha < \omega_1$. By b), u_α is finite and by c) we have $u_\alpha \subseteq d_\alpha$. Hence by Fodor's lemma, there exists an uncountable (actually even stationary) $Y \subseteq \omega_1$ such that $u_\alpha = u^*$ for all $\alpha \in Y$, for some fixed $u^* \in [X]^{<\aleph_0}$. Since $\Pi_{i\in u^*} P_i$ has the property K, there exists an uncountable $Y' \subseteq Y$ such that $\{\, q_\alpha \restriction u^* \; : \; \alpha \in Y' \,\}$ is pairwise compatible. It follows that q_α, $\alpha \in Y'$ are pairwise compatible. For each $\alpha \in Y'$ there exists an $n_\alpha \in \omega$ such that $q_\alpha \Vdash_P$ "$n_\alpha = \dot{f}(\alpha)$" by b). By *), n_α, $\alpha \in Y'$ must be pairwise distinct. But this is impossible as Y' is uncountable.

For 2), essentially the same proof works with sequences of elements of P of length λ, using the Δ-system lemma argument in place of Fodor's lemma.

❏ (Lemma 2.2)

Lemma 2.3 *If* $|\, P_i \,| \leq 2^{<\kappa}$ *for all* $i \in X$*, then* $\Pi^*_{\kappa,i\in X} P_i$ *has the* $(2^{<\kappa})^+$*-cc.*

Proof By the usual Δ-system lemma argument. ❏ (Lemma 2.3)

Corollary 2.4 a) *Under* CH*, if* P_i *satisfies the property* K *and* $|\, P_i \,| \leq \aleph_1$ *for every* $i \in X$*, then* $P = \Pi^*_{i\in X} P_i$ *preserves* $\aleph_1$ *and has the* $\aleph_2$*-cc. In particular* P *preserves every cardinals.*

b) *Suppose that* $2^{<\kappa} = \kappa$*. If* P_i *satisfies the strong* λ*-cc for every* $\aleph_1 \leq \lambda \leq \kappa$ *and* $|\, P_i \,| \leq \kappa$ *then* $\Pi^*_{\kappa,i\in X} P_i$ *preserves every cardinalities* $\leq \kappa$ *and has the* κ^+*-cc. In particular* $\Pi^*_{\kappa,i\in X} P_i$ *preserves every cardinals.*

Proof By Lemmas 2.2, 2.3. ❏ (Corollary 2.4)

Lemma 2.5 *For any* $Y \subseteq X$ *and* $x \in X \setminus Y$*, we have*

$$\Pi^*_{\kappa,i\in X} P_i \cong \Pi^*_{\kappa,i\in Y} P_i \times P_x \times \Pi^*_{\kappa,i\in X\setminus(Y\cup\{x\})} P_i.$$

Proof The mapping from $\Pi^*_{\kappa,i\in X}P_i$ to $\Pi^*_{\kappa,i\in Y}P_i \times P_x \times \Pi^*_{\kappa,i\in X\setminus(Y\cup\{x\})}P_i$ defined by

$$p \mapsto (p\restriction Y, p(x), p\restriction (X \setminus (Y \cup \{x\})))$$

is an isomorphism. ❑ (Lemma 2.5)

In the following we mainly use the partial orderings of the form $\mathrm{Fn}(\lambda, 2)$ for some λ as P_i in $\Pi^*_{\kappa,i\in X}P_i$. Note that $\mathrm{Fn}(\lambda, 2)$ has the property K and strong κ-cc in the sense above for every regular κ.

For a pseudo product of the form $\Pi^*_{i\in X}\mathrm{Fn}(\kappa_i, 2)$, Lemma 2.2 can be still improved:

Theorem 2.6 (T. Miyamoto) *For any set X, and sequence $(\kappa_i)_{i\in X}$, the partial ordering $P = \Pi^*_{i\in X}\mathrm{Fn}(\kappa_i, 2)$ satisfies the Axiom A.*

Proof The sequence of partial orderings $(\leq_n)_{n\in\omega}$ defined by: $p \leq_0 q \;\Leftrightarrow\; p \leq q$ and $p \leq_n q \;\Leftrightarrow\; p \leq_h q$ for every $n > 0$ witnesses the Axiom A of P. We omit here the details of the proof since this assertion is never used in the following. The idea of the proof needed here is to be found in the proof of Lemmas 2.7 and 5.2.

❑ (Theorem 2.6)

Lemma 2.7 *Suppose that $|\, P_i \,| \leq \kappa$ for every $i \in X$ and $P = \Pi^*_{\kappa^+,i\in X}P_i$. Then*

1) *If $\dot{x}$ is a P-name with $\Vdash_P$ "$\dot{x} \in V$", then for any $p \in P$ there is $q \in P$ such that $q \leq_h p$ and*

> (†) *for any $r \leq q$, if r decides $\dot{x}$ then $r\lceil supp(q)$ already decides $\dot{x}$.*

2) *Let G be P-generic. If $u \in V[G]$ is a subset of V of cardinality $< \kappa^+$, then there is a ground model set $X' \subseteq X$ of cardinality $\leq \kappa$ (in the sense of V) such that $u \in V[G \cap (\Pi^*_{\kappa^+,i\in X'}P_i)]$.*

Proof 1): Let $\Phi : \kappa \to \kappa \times \kappa;\ \alpha \mapsto (\varphi_1(\alpha), \varphi_2(\alpha))$ be a surjection such that $\varphi_1(\alpha) \leq \alpha$ for every $\alpha < \kappa$. Let $(p_\alpha)_{\alpha<\kappa}$, $(p'_\alpha)_{\alpha<\kappa}$ and $(r_{\alpha,\beta})_{\alpha<\kappa,\beta<\kappa}$ be sequences of elements of P defined inductively by:

a) $p_0 = p$; $(p_\alpha)_{\alpha<\kappa}$ is a descending sequence with respect to $\leq_h$;

b) for a limit $\gamma < \kappa$, p_γ is such that $supp(p_\gamma) = \bigcup_{\alpha<\gamma} supp(p_\alpha)$ and, for $i \in supp(p_\gamma)$, $p_\gamma(i) = p_\alpha(i)$ for some $\alpha < \gamma$ such that $i \in supp(p_\alpha)$;

c) $(r_{\alpha,\beta})_{\beta<\kappa}$ is an enumeration of $\{\, r \in P \,:\, r \leq_v p_\alpha \,\}$;

d) let $r = r_{\varphi_1(\alpha),\varphi_2(\alpha)}$ and

$$p'_\alpha = r\restriction\ supp(r) \cup p_\alpha\restriction (X \setminus supp(r)).$$

If there is $s \leq_h p'_\alpha$ such that s decides $\dot{x}$, then let

$$p_{\alpha+1} = p_\alpha{\restriction}\ supp(p_\alpha) \cup s{\restriction}\ (X \setminus supp(p_\alpha)).$$

Otherwise let $p_{\alpha+1} = p_\alpha$.

Let $q \in \Pi^*_{\kappa, i \in X} P_i$ be defined by $supp(q) = \bigcup_{\alpha<\kappa} supp(p_\alpha)$ and, for $i \in supp(q)$, $q(i) = p_\alpha(i)$ for some $\alpha < \kappa$ such that $i \in supp(P_\alpha)$. We show that this q is as desired: suppose that $r \leq q$ decides $\dot{x}$. Then there is some $\alpha < \kappa$ such that

$$r\lceil supp(q) = p'_\alpha{\restriction}\ supp(p'_\alpha) \cup q{\restriction}\ (X \setminus supp(p'_\alpha)).$$

By d), it follows that $r\lceil supp(q) \leq r\lceil supp(p_{\alpha+1})$ decides $\dot{x}$.

2): Let $\dot{u}$ be a P-name for u and let $\dot{x}_\alpha$, $\alpha < \kappa$ be P-names such that $\Vdash_P$ "$\dot{x}_\alpha \in V$" for every $\alpha < \kappa$ and $\Vdash_P$ "$u = \{\, \dot{x}_\alpha : \alpha < \kappa \,\}$". By 1), for each $p \in P$, we can build a sequence $(p_\alpha)_{\alpha<\kappa}$ of elements of P decreasing with respect to $\leq_h$ such that $p_0 \leq_h p$ and

$(\dagger)_\alpha$ for any $r \leq p_\alpha$, if r decides $\dot{x}_\alpha$, then $r\lceil supp(p_\alpha)$ already decides $\dot{x}_\alpha$.

Let $q \in P$ be defined by $supp(q) = \bigcup_{\alpha<\kappa} supp(p_\alpha)$ and, for $i \in supp(q)$, $q(i) = p_\alpha(i)$ for some $\alpha < \kappa$ such that $i \in supp(p_\alpha)$. Then q satisfies:

$(\dagger\dagger)$ for any $r \leq q$, if r decides $\dot{x}_\alpha$ for some $\alpha < \kappa$, then $r\lceil supp(q)$ already decides $\dot{x}_\alpha$.

The argument above shows that q's with the property $(\dagger\dagger)$ are dense in P. Hence, by genericity, there is such $q \in G$. Clearly, $G \cap \Pi^*_{\kappa, i \in supp(q)} P_i$ contains every information needed to construct u. ❏ (Lemma 2.7)

3 Consistency results

Proposition 3.1 (CH) *For any infinite cardinal λ, let $P = \Pi^*_\lambda \mathrm{Fn}(\omega_1, 2)$. Then* $\Vdash_P$ "$\pin = \lambda$".

Proof

Claim 3.1.1 $\Vdash_P$ "$\pin \geq \lambda$".

$\vdash$ If $\lambda = \aleph_1$ this is clear. So assume that $\lambda \geq \aleph_2$. For $\xi < \lambda$, let $\dot{f}_\xi$ be the P-name of the generic function from ω_1 to 2 added by the ξ-th copy of $\mathrm{Fn}(\omega_1, 2)$ in P. Let G be a P-generic filter over V. In $V[G]$ let $X \subseteq [\omega_1]^{\aleph_0}$ be such that $|\, X \,| < \lambda$. Then by $\aleph_2$-cc of P there exists $\xi < \lambda$ such that $X \in V[G']$ for $G' = G \cap \Pi^*_{\lambda \setminus \{\xi\}} \mathrm{Fn}(\omega_1, 2)$. Since $(\dot{f}_\xi)[G]$ is $\mathrm{Fn}(\omega_1, 2)$-generic over $V[G']$ by Lemma 2.5, we have $x \not\subseteq ((\dot{f}_\xi)[G])^{-1}\{0\}$ for every $x \in X$. $\dashv$ (Claim 3.1.1)

Claim 3.1.2 $\Vdash_P$ "$\text{⫯} \leq \lambda$".

$\vdash$ For $u \in [\lambda]^{<\aleph_0}$, let $\dot{\mathcal{P}}_u$ be a P-name such that

$$\Vdash_P \text{“}\dot{\mathcal{P}}_u = ([\omega_1]^{\aleph_0})^{V[(\dot{f}_\xi)_{\xi\in u}]}\text{”}$$

where $\dot{f}_\xi$ is as in the proof of the previous claim. Let $\dot{\mathcal{P}}$ be a P-name such that

$$\Vdash_P \text{“}\dot{\mathcal{P}} = \bigcup\{\, \dot{\mathcal{P}}_u \,:\, u \in [\lambda]^{<\aleph_0}\,\}\text{”}.$$

For each $u \in [\lambda]^{<\aleph_0}$, $(\dot{f}_\xi[G])_{\xi\in u}$ corresponds to a generic filter over $\Pi^*_u\mathrm{Fn}(\omega_1, 2) \approx \mathrm{Fn}(\omega_1, 2)$. Hence, by CH, we have $\Vdash_P$ "$|\,\dot{\mathcal{P}}_u\,| = \aleph_1$". It follows that $\Vdash_P$ "$|\,\dot{\mathcal{P}}\,| = \lambda$". Thus it is enough to show that $\Vdash_P$ "$\dot{\mathcal{P}}$ is a ⫯-set".

Let $p \in P$ and $\dot{A}$ be a P-name such that $p \Vdash_P$ "$\dot{A} \in [\omega_1]^{\aleph_1}$". We show that there is an $r \leq p$ such that $r \Vdash_P$ "$\exists x \in \dot{\mathcal{P}}\ x \subseteq \dot{A}$".

Now we proceed as in the proof of Lemma 2.2. Let $(p_\alpha)_{\alpha<\omega_1}$, $(q_\alpha)_{\alpha<\omega_1}$ be sequences of elements of P and $(\xi_\alpha)_{\alpha<\omega_1}$ be a strictly increasing sequence of ordinals $< \omega_1$ such that

a) $p_0 \leq p$ and $(p_\alpha)_{\alpha<\omega_1}$ is a descending sequence with respect to $\leq_h$;

b) $q_\alpha \leq_v p_\alpha$ and $q_\alpha \Vdash_P$ "$\xi_\alpha \in \dot{A}$" for all $\alpha < \omega_1$;

c) $p_\alpha\restriction S_\alpha = q_\alpha\restriction S_\alpha$ for every $\alpha < \omega_1$ where

$$S_\alpha = \mathit{supp}(q_\alpha) \setminus (\mathit{supp}(p) \cup \bigcup_{\beta<\alpha} \mathit{supp}(q_\beta)).$$

For $\alpha < \omega_1$ let $u_\alpha = \{\, \beta \in \mathit{supp}(q_\alpha) \,:\, q_\alpha(\beta) \neq p_\alpha(\beta)\,\}$. As in the proof of Lemma 2.2, there exists $u^* \in [\lambda]^{<\aleph_0}$ such that $S = \{\, \alpha \in \omega_1 \,:\, u_\alpha = u^*\,\}$ is stationary. Now $(q_\alpha\restriction u)_{\alpha\in S}$ is an infinite sequence of elements of $P_{u^*} = \Pi_{u^*}\mathrm{Fn}(\omega_1, 2)$. Since P_{u^*} satisfies the ccc, there exists an $\varepsilon \in S$ and $\zeta < \omega_1$ such that $q_\varepsilon\restriction u^* \Vdash_{P_{u^*}}$ "$\{\, \xi \in S \cap \zeta \,:\, p_\xi\restriction u^* \in \dot{G}\,\}$ is infinite". Let $r = q_\varepsilon \cup p_\zeta\restriction(\mathit{supp}(p_\zeta) \setminus \mathit{supp}(p_\varepsilon))$. Let $\dot{b}$ be a P-name such that

$$r \Vdash_P \text{“}\dot{b} = \{\, \xi \in S \cap \zeta \,:\, q_\xi\restriction u^* \in \{\, p\restriction u^* \,:\, p \in \dot{G}\,\}\,\}\text{”}.$$

Let $\dot{x}$ be a P-name such that $r \Vdash_P$ "$\dot{x} = \{\, \xi_\alpha \,:\, \alpha \in \dot{b}\,\}$". Then $r \Vdash_P$ "$|\,\dot{x}\,| = \aleph_0$". Since $\dot{b}$ can be computed in $V[(\dot{f}_\xi[G])_{\xi\in u^*}]$ we have $r \Vdash_P$ "$\dot{x} \in \dot{\mathcal{P}}_{u^*}$". It is also clear by definition of $\dot{x}$ that $r \Vdash_P$ "$\dot{x} \subseteq \dot{A}$". $\dashv$ (Claim 3.1.2)

❑ (Proposition 3.1)

Proposition 3.1 shows that ⫯ can be practically every thing. In particular we obtain:

Corollary 3.2 *The assertion* '$\mathrm{cof}(\text{⫯}) = \omega$' *is consistent with* ZFC. ❑

Actually, $\mathrm{Fn}(\lambda, 2)$ forces almost the same situation:

Lemma 3.3 *Suppose that λ is a cardinal such that $\mu^{\aleph_0} \leq \lambda$ for every $\mu < \lambda$. Then, for $P = \mathrm{Fn}(\lambda, 2)$, we have $\Vdash_P$ "$\stick = \lambda$".*

Proof $\Vdash_P$ "$\stick \geq \lambda$" can be proved similarly to Claim 3.1.1. For $\Vdash_P$ "$\stick \leq \lambda$", let G be a P-generic filter and let $G_\alpha = G \cap \mathrm{Fn}(\alpha, 2)$ for $\alpha < \lambda$. In $V[G]$, let $X = \bigcup\{\, V[G_\alpha] \cap [\omega_1]^{\aleph_0} \,:\, \alpha < \lambda \,\}$. Then $|\, X \,| = \lambda$ (here we need SCH in general). We show that X is a $\stick$-set. For this, it is enough to show the following:

Claim 3.3.1 *In $V[G]$, if $y \subseteq [\omega_1]^{\aleph_1}$, then there is $\alpha^* < \lambda$ and infinite $y' \in V[G_{\alpha^*}]$ such that $y' \subseteq y$.*

$\vdash$ In V, let $\dot{y}$ be a P-name of y which is nice in the sense of [8]. For $\alpha < \lambda$, let $\dot{y}_\alpha = \dot{y} \cap \{\, \check{\beta} \,:\, \beta < \omega_1 \,\} \times \mathrm{Fn}(\alpha, 2)$. Then $\Vdash_P$ "$\dot{y} = \bigcup_{\alpha<\lambda} \dot{y}_\alpha$". Hence $\Vdash_P$ "$\exists \alpha < \lambda\ \dot{y}_\alpha$ is infinite". It follows that there is some $\alpha^* < \lambda$ such that $y' = \dot{y}_{\alpha^*}[G]$ is infinite. Since $\dot{y}_{\alpha^*}$ is an $\mathrm{Fn}(\alpha^*, 2)$-name, $\dot{y}_{\alpha^*}[G] \in V[G_{\alpha^*}]$. Thus these α^* and y' are as desired. $\dashv$ (Claim 3.3.1)

❏ (Lemma 3.3)

Proposition 3.4 (CH) *Suppose that*

$(*)_{\lambda,\mu}$ *There is a sequence $(A_i)_{i<\mu}$ of elements of $[\lambda]^{\aleph_1}$ such that $|\, A_i \cap A_j \,| < \aleph_0$ for every i, $j < \mu$, $i \neq j$*

holds for some $\mu > \lambda \geq 2^{\aleph_0}$. Then there exists a partial ordering P such that

a) *P preserves $\aleph_1$ and and has the $\aleph_2$-cc;*
b) *$\Vdash_P$ "$\stick = \lambda$" and*
c) *$\Vdash_P$ "$\stick_\lambda \geq \mu$".*

In particular, if $(*)_{\lambda,\mu}$ is consistent with ZFC *for some $\mu > \lambda \geq 2^{\aleph_0}$, then so is $\stick < \stick'$.*

Remark. By [12, §6], $(**)_\mu$ and $(*)_{\lambda,\mu}$ for some $\lambda < \mu$ are equivalent, where

$(**)_\mu$ there are finite $a_i \subseteq Reg \setminus \aleph_2$ for $i < \omega_1$ such that, for any $A \in [\omega_1]^{\aleph_0}$, $\max pcf(\cup_{i \in A} a_i) \geq \mu$.

For more see [15].

Proof Let P be as in Proposition 3.1. We claim that P is as desired: a) follows from Corollary 2.4 and b) from Proposition 3.1. For d), if $X \subseteq [\lambda]^{\aleph_0}$ is a $\stick_\lambda$-set then for each $i < \mu$ there is an $x_i \in X$ such that $x_i \subseteq A_i$. Since A_i, $i < \mu$ are almost disjoint x_i, $i < \mu$ must be pairwise distinct.

The last assertion follows from Lemma 1.1, d). ❏ (Proposition 3.4)

Now we show the consistency of the inequality $\stick'' < \stick$:

Proposition 3.5 *Assume $2^{\aleph_1} = \aleph_2$. Then for any cardinal $\lambda \geq \aleph_2$ there exists a partial ordering P such that*

a) *P satisfies the $\aleph_3$-cc;*
b) *P preserves $\aleph_1$ and $\aleph_2$;*
c) *if $\lambda^{\aleph_0} = \lambda$ in addition, then $\Vdash_P$ "$\mathrm{MA}(\mathrm{Fn}(\omega_1, 2))$";*
d) *$\Vdash_P$ "$⫯ = \lambda$" and*
e) *$\Vdash_P$ "$⫯'' = \aleph_2$".*

Proof Without loss of generality let λ be regular and let $P = \Pi^*_{\aleph_2,\lambda}\mathrm{Fn}(\omega_1, 2)$. Then a) and b) follow from Corollary 2.4. For c), note that $\Vdash_P$ "$2^{\aleph_0}$" $= \lambda$ under $\lambda^{\aleph_0} = \lambda$. Hence, by Lemma 2.7 and Lemma 2.5, we see easily that $\Vdash_P$ "$\mathrm{MA}(\mathrm{Fn}(\omega_1, 2))$". An argument similar to the proof of of Proposition 3.1 shows that $\Vdash_P$ "$⫯ = \lambda$". For e), we prove first the following:

Claim 3.5.1 *Let $X = [\aleph_2]^{\aleph_0}$. Then we have $\Vdash_P$ "X is a $⫯''$-set". In particular $\Vdash_P$ "$⫯'' \leq \aleph_2$".*

$\vdash$ Suppose that, for some $p \in P$ and a P-name $\dot{y}$ we have $p \Vdash_P$ "$\dot{y} \in [\omega_2]^{\aleph_2}$". Let $\dot{f}$ be a P name such that $p \Vdash_P$ "$\dot{f} : \omega_2 \to \dot{y}$ and $\dot{f}$ is 1-1". Let $(p_\alpha)_{\alpha<\omega_2}$ and $(q_\alpha)_{\alpha<\omega_2}$ be sequences of elements of P such that

f) $p_0 \leq p$ and $(p_\alpha)_{\alpha<\omega_2}$ is a descending sequence with respect to $\leq_h$;
g) $q_\alpha \leq_v p_\alpha$ and q_α decides $\dot{f}(\alpha)$ for all $\alpha < \omega_2$;
h) $p_\alpha \restriction S_\alpha = q_\alpha \restriction S_\alpha$ for every $\alpha < \omega_2$ where

$$S_\alpha = supp(q_\alpha) \setminus (supp(p) \cup \bigcup_{\beta<\alpha} supp(q_\beta)).$$

For $\alpha < \omega_2$, let $\xi_\alpha \in \omega_2$ be such that $q_\alpha \Vdash_P$ "$\dot{f}(\alpha) = \xi_\alpha$". Let $u_\alpha = \{\beta \in supp(q_\alpha) : q_\alpha(\beta) \neq p_\alpha(\beta)\}$ for $\alpha < \omega_2$. Just like in the proof of Lemma 2.4, we can find $u^* \in [\lambda]^{<\aleph_0}$ such that $S = \{\alpha < \omega_2 : u_\alpha = u^*\}$ is stationary in ω_2. Since $|\mathrm{Fn}(\omega_1, 2)| = \aleph_1$, there exists $T \subseteq S$ of cardinality $\aleph_2$ such that $q_\alpha \restriction u^*$, $\alpha \in T$ are all the same. Let α_n, $n \in \omega$ be ω elements of T and let $q = \bigcup_{n\in\omega} q_{\alpha_n}$. Then $q \leq p$ and $q \Vdash_P$ "$\{\xi_{\alpha_n} : n \in \omega\} \subseteq \dot{y}$". $\dashv$ (Claim 3.5.1)

Now by d), we have $\Vdash_P$ "$⫯'' > \aleph_1$". Hence, by the claim above, it follows that $\Vdash_P$ "$⫯'' = \aleph_2$". ❑ (Proposition 3.5)

Modifying the proofs of Propositions 3.1 and 3.5 slightly, we can also blow up the continuum while setting $⫯$ strictly between $\aleph_1$ and $2^{\aleph_0}$. For example:

Proposition 3.6 *Assume CH and $2^{\aleph_1} = \aleph_2$. Then for any cardinals λ, μ such that $\aleph_2 \leq \lambda \leq \mu$ and $\mu^{\aleph_1} = \mu$, there exists a partial ordering P such that*

a) *P satisfies the $\aleph_3$-cc;*
b) *P preserves $\aleph_1$ and $\aleph_2$;*
c) $\Vdash_P$ "MA(*countable*)";
d) $\Vdash_P$ "$\text{⫯} = \lambda$";
e) $\Vdash_P$ "$\text{⫯}'' = \aleph_2$" *and*
f) $\Vdash_P$ "$2^{\aleph_0} = \mu$".

Proof For $i < \mu$ let

$$P_i = \begin{cases} \mathrm{Fn}(\omega_1, 2), & \text{if } i < \lambda, \\ \mathrm{Fn}(\omega, 2), & \text{otherwise.} \end{cases}$$

Then $P = \Pi^*_{\aleph_2, i<\mu} P_i$ is as desired. e) can be proved by almost the same proof as that of Claim 3.5.1. a), b), c) can be shown just as in Proposition 3.5. Since P adds (at least) μ many Cohen reals over V and $|\, P \,| = \mu$, f) follows from a). d) is proved similarly to Claims 3.1.1 and 3.1.2. For $\Vdash_P$ "$\text{⫯} \le \lambda$" we need the following modification of Claim 3.1.2: let $\dot{\mathcal{P}}$ be defined as in the proof of Claim 3.1.2. As there, we can show easily that $\Vdash_P$ "$|\, \dot{\mathcal{P}} \,| = \lambda$". To show that $\Vdash_P$ "$\dot{\mathcal{P}}$ is a ⫯-set", let $p \in P$ and $\dot{A}$ be a P-name such that $p \Vdash_P$ "$\dot{A} \in [\omega_1]^{\aleph_1}$". Now let $(p_\alpha)_{\alpha<\omega_1}$, $(q_\alpha)_{\alpha<\omega_1}$, $(\xi_\alpha)_{\alpha<\omega_1}$, $u^* \in [\mu]^{<\aleph_0}$ and S be just as in the proof of Claim 3.1.2. Let $v^* = u^* \setminus \lambda$. Since $P_{v^*} = \Pi_{i \in v^*} P_i$ is countable, we may assume without loss of generality that $q_\alpha \restriction v^*$, $\alpha \in S$ are all the same. Now we can proceed just like in the proof of Claim 3.1.2 with u^* replaced by $u^* \setminus v^*$. ❑

The following Lemmas 3.7 and 3.9 show that, in spite of typographical similarity, $\Pi^*_\lambda \mathrm{Fn}(\omega_1, 2)$ and $\Pi^*_\lambda \mathrm{Fn}(\omega, 2)$ are quite different forcing notions: while the first one destroys (♣) or even (⫯) by Lemma 3.1, the second one not only preserves a (♣)-sequence in the ground model but also creates such a sequence generically.

Lemma 3.7 *Let $S = (x_\gamma)_{\gamma \in E}$ be a ♣(E)-sequence for a stationary $E \subseteq Lim(\omega_1)$. Let $P = \Pi^*_\kappa \mathrm{Fn}(\omega, 2)$ for arbitrary κ. Then we have $\Vdash_P$ "S is a ♣(E)-sequence".*

Proof Let $p \in P$ and $\dot{A}$ be a P-name such that $p \Vdash_P$ "$\dot{A} \in [\omega_1]^{\aleph_1}$". We show that there is $q \le p$ and $\gamma \in E$ such that $q \Vdash_P$ "$x_\gamma \subseteq \dot{A}$". Let $\dot{f}$ be a P-name such that that $p \Vdash_P$ "$\dot{f} : \omega_1 \to \dot{A}$ and $\dot{f}$ is 1-1". Let $(p_\alpha)_{\alpha<\omega_1}$ and $(q_\alpha)_{\alpha<\omega_1}$ be sequence of elements of P satisfying the conditions a) – c) in the proof of Lemma 2.2. Also, let u_α, $\alpha < \omega_1$ be as in the proof of Lemma 2.2. As there, we can find an uncountable $Y \subseteq \omega_1$ and $u^* \in [\kappa]^{<\aleph_0}$ such that $u_\alpha = u^*$ for all $\alpha \in Y$. Since $\Pi_{u^*} \mathrm{Fn}(\omega, 2)$ is countable we may assume that $q_\alpha \restriction u^*$ are all the same for $\alpha \in Y$. Now for each $\alpha \in Y$ let β_α be such that $q_\alpha \Vdash_P$ "$\dot{f}(\alpha) = \beta_\alpha$" and let $Z = \{\, \beta_\alpha : \alpha \in Y \,\}$. Since q_α, $\alpha \in Y$ are pairwise compatible, β_α, $\alpha \in Y$ are pairwise distinct and so Z is uncountable. Note that Z is a ground model set. Hence there exists $\gamma \in E$ such that $x_\gamma \subseteq Z$. Let $q = \bigcup_{\alpha\in Y\cap\gamma} q_\alpha$. Then $q \le p$. Since $\sup\{\,\beta_\alpha : \alpha < \gamma\,\} \ge \gamma$ and $\Vdash_P$ "$\{\,\beta_\alpha : \alpha < \gamma\,\}$ is an initial segment of Z", we have $q \Vdash_P$ "$Z \cap \gamma \subseteq \dot{A}$". Hence $q \Vdash_P$ "$x_\gamma \subseteq \dot{A}$". ❑ (Lemma 3.7)

Theorem 3.8 *"*$\neg$CH+ MA(*countable*) + *there exists a constructible* ♣ *-sequence" is consistent.*

Proof We can obtain a model of the statement by starting from a model of $V = L$ and force with $P = \Pi^*_\kappa \mathrm{Fn}(\omega, 2)$ for a regular κ. By Corollary 2.4, every cardinal of V is preserved in $V[G]$. Since P adds κ many Cohen reals over V while $|\,P\,| = \kappa$ and P has the $\aleph_2$-cc, we have $V[G] \models$ "$2^{\aleph_0} = \kappa$". By Lemma 2.5, $V[G] \models$ "MA(*countable*)". By Lemma 3.7, the $\diamondsuit$-sequence in V remains a ♣-sequence in $V[G]$. ❑ (Theorem 3.8)

In fact, we do not need a ♣-sequence in the ground model to get (♣) in the generic extension by $\Pi^*_\kappa \mathrm{Fn}(\omega, 2)$:

Lemma 3.9 *Let κ be uncountable and $P = \Pi^*_\kappa \mathrm{Fn}(\omega, 2)$. Then for any stationary $E \subseteq Lim(\omega_1)$ we have $\Vdash_P$ "♣(E) holds".*

Proof For $\gamma \in E$ let

$$f_\gamma : [\gamma,\ \gamma + \omega) \to \gamma$$

be a bijection and let

$$S_\gamma = \{\, x \subseteq \gamma \,:\, x \text{ is a cofinal subset of } \gamma,\ otp(x) = \omega \,\}.$$

For each $x \in S_\gamma$ let $p_x \in P$ be defined by

$$p_x = \{\, (\gamma + n, \{\, (0, i)\,\}) \,:\, n \in \omega,\ i \in 2,\ i = 1 \Leftrightarrow f_\gamma(\gamma + n) \in x \,\}.$$

For distinct $x, x' \in S_\gamma$, p_x and $p_{x'}$ are incompatible. Hence, for each $\gamma \in E$, we can find a P-name $\dot{x}_\gamma$ such that

$$\Vdash_P \text{"} \dot{x}_\gamma \text{ is a cofinal subset of } \gamma \text{ and } otp(\dot{x}_\gamma) = \omega \text{"}$$

and

$$p_x \Vdash_P \text{"} \dot{x}_\gamma = x \text{"} \quad \text{for each } x \in S_\gamma.$$

We show that $\Vdash_P$ "$(\dot{x}_\gamma)_{\gamma\in E}$ is a ♣(E)-sequence". For this, it is enough to show that, for any $p \in P$ and a P-name $\dot{A}$, if $p \Vdash_P$ "$\dot{A} \in [\omega_1]^{\aleph_1}$", then there is $q \le p$ and $\gamma \in E$ such that $q \Vdash_P$ "$\dot{x}_\gamma \subseteq \dot{A}$". Let $\dot{f}$ be such that

$$p \Vdash_P \text{"} \dot{f} : \omega_1 \to \dot{A} \text{ and } \dot{f} \text{ is 1-1"}.$$

Now let $(p_\alpha)_{\alpha<\omega_1}$, $(q_\alpha)_{\alpha<\omega_1}$, $(u_\alpha)_{\alpha\in\omega_1}$, Y and u^* be as in the proof of Lemma 2.2. For each $\alpha \in Y$ let β_α be such that $q_\alpha \Vdash_P$ "$\dot{f}(\alpha) = \beta_\alpha$" and let $Z = \{\, \beta_\alpha \,:\, \alpha \in Y \,\}$. Let

$$C = \{\, \gamma \in Lim(\omega_1) \,:\, \bigcup_{\alpha\in Y\cap\gamma}(supp(q_\alpha) \cap \omega_1) \subseteq \gamma \text{ and } Z \cap \gamma \text{ is unbounded in } \gamma \,\}.$$

Then C is closed unbounded in ω_1 and hence there exists a $\gamma^* \in C \cap E$. Let $q' = \bigcup_{\alpha\in Y\cap\gamma^*} q_\alpha$. Then we have $q' \le q$ and $q' \Vdash_P$ "$Z \cap \gamma^* \subseteq \dot{A}$". Now let $x \in S_{\gamma^*}$ be such that $x \subseteq Z \cap \gamma^*$. Finally let $q = q' \cup q_x$. Then we have $q \le p$ and $q \Vdash_P$ "$\dot{x}_\alpha = x \subseteq Z \cap \gamma^* \subseteq \dot{A}$". ❑ (Lemma 3.9)

Note that E's in Lemmas 3.7 and 3.9 are ground model sets. To force $\clubsuit(E)$ for every stationary $E \subseteq Lim(\omega_1)$ which may be also added generically, we need a sort of iteration described in the next section.

Toward the consistency of $\neg(\clubsuit_{\mathrm{w}}) + (\clubsuit_{\mathrm{w}^2})$, we consider first the following lemma which should be a well-known fact. Nevertheless, we include here a proof:

Lemma 3.10 *Assume that there is a sequence $(C_\beta)_{\beta<\kappa}$ of elements of $[\omega_1]^{\aleph_1}$ such that $|\, C_\beta \cap C_\gamma \,| \le \aleph_0$ for all $\beta < \gamma < \kappa$. Then there exists a partial ordering P with the property K such that in V^P there is a sequence $(B_\beta)_{\beta<\kappa}$ of elements of $[\omega_1]^{\aleph_1}$ such that $B_\beta \subseteq C_\beta$ and $|\, B_\beta \cap B_\gamma \,| < \aleph_0$ for all $\beta < \gamma < \kappa$.*

Proof Let

$$P = \{\, (D, f) \,:\, D \in [\kappa]^{<\aleph_0},\ f : D \to \mathrm{Fn}(\omega_1, 2),\ f(\delta) \in \mathrm{Fn}(C_\delta, 2) \text{ for all } \delta \in D \,\}.$$

For (D, f), $(D', f') \in P$, let

$$(D', f') \le (D, f) \quad\Leftrightarrow\quad D \subseteq D',\ f(\delta) \subseteq f'(\delta) \text{ for all } \delta \in D \text{ and } (f'(\delta))^{-1}[\{1\}] \setminus (f(\delta))^{-1}[\{1\}],\ \delta \in D \text{ are pairwise disjoint.}$$

By the usual Δ-system lemma argument, we can show that P has the property K. Since C_β, $\beta < \kappa$ are pairwise disjoint modulo countable, the set

$$\mathcal{D}_{\beta,\delta} = \{\, (D, f) \in P \,:\, \beta \in D,\ \delta \in \mathrm{dom}(f(\beta)) \text{ and } \exists \eta > \delta\, (\eta \in \mathrm{dom}(f(\beta)) \ \wedge\ f(\beta)(\eta) = 1) \,\}$$

is dense in P for every $\beta < \kappa$ and $\delta < \omega_1$. Hence if G is a V-generic filter over P, then

$$B_\beta = \{\, \alpha < \omega_1 \,:\, f(\beta)(\alpha) = 1 \text{ for some } (D, f) \in G \,\}$$

is cofinal in ω_1 and hence uncountable. Also by the definition of $\leq$ on P, we have $|\ B_\beta \cap B_\gamma\ | < \aleph_0$ for every $\beta < \gamma < \kappa$. ❑ (Lemma 3.10)

Note that if there is a sequence $(B_\beta)_{\beta<\kappa}$ as in Lemma 3.10 then by the argument in the proof of Proposition 3.4, we have $\mathfrak{stick} \geq \kappa$.

Lemma 3.11 *There is a partial ordering Q with the property K such that* $\Vdash_Q$ "$(\clubsuit_{\mathrm{w}^2})$".

Proof Let $(Q_\alpha, \dot{R}_\alpha)_{\alpha\leq\omega_1}$ be the finite support iteration of partial orderings with the property K such that for each $\gamma \in Lim(\omega_1)$, there is a Q_γ name $\dot{U}_\gamma$ such that Q_γ forces:

> $\dot{U}_\gamma$ is an ultrafilter over γ, $\gamma \setminus \beta \in \dot{U}_\gamma$ for all $\beta < \gamma$, $\dot{R}_\alpha$ is a p.o.-set with the property K and there is an $\dot{R}_\gamma$-name $\dot{x}_\gamma$ such that
>
> $$\Vdash_{\dot{R}_\gamma} \text{“}\dot{x}_\gamma \text{ is a cofinal subset of } \gamma \text{ of ordertype } \omega \text{ and } |\ \dot{x}_\gamma \setminus a\ | < \aleph_0 \text{ for all } x \in \dot{U}_\gamma\text{”}.$$

For example, we can take the Mathias forcing for the ultrafilter $\dot{U}_\gamma$ as $\dot{R}_\gamma$. For successor $\alpha < \omega_1$ let $\Vdash_{Q_\alpha}$ "$\dot{R}_\alpha = \{1\}$".

Let $Q = Q_{\omega_1}$. As $(Q_\alpha, \dot{R}_\alpha)_{\alpha\leq\omega_1}$ is a finite support iteration of property K p.o.s, Q satisfies also the property K (see e.g. [9]). Now let G be a V-generic filter over Q. In $V[G]$, if $X \in [\omega_1]^{\aleph_1}$ then the set $\{\, \alpha < \omega_1 \ :\ X \cap \alpha \in V[G_\alpha]\,\}$ contains a club subset C of $Lim(\omega_1)$. Let $S_0 = \{\, \alpha \in C \ :\ X \cap \alpha \in \dot{U}_\alpha[G]\,\}$ and $S_1 = \{\, \alpha \in C \ :\ \alpha \setminus X \in \dot{U}_\alpha[G]\,\}$. Since $\dot{U}_\alpha[G]$ is an ultrafilter over α in $V[G_\alpha]$ for every $\alpha \in C$, we have $C = S_0 \dot{\cup} S_1$. We have $|\ \dot{x}_\alpha[G] \setminus X\ | < \aleph_0$ for $\alpha \in S_0$ and $|\ \dot{x}_\alpha[G] \cap X\ | < \aleph_0$ for $\alpha \in S_1$. Thus $(\dot{x}_\alpha[G])_{\alpha\in Lim(\omega_1)}$ is a $(\clubsuit_{\mathrm{w}^2})$-sequence in $V[G]$.

Actually this proof shows that $(\dot{x}_\alpha[G])_{\alpha\in Lim(\omega_1)}$ is even a $(\clubsuit_{\mathrm{w}^2})$-sequence in the stronger sense that it satisfies the assertion of the definition of $(\clubsuit_{\mathrm{w}^2})$ with "is stationary" replaced by "contains a club". ❑ (Lemma 3.11)

Corollary 3.12 *There is a partial ordering R with property K such that* $\Vdash_R$ "$\mathfrak{stick} \geq \aleph_2$ *but* $(\clubsuit_{\mathrm{w}^2})$ *holds". In particular* $\neg\,(\mathfrak{stick}) + (\clubsuit_{\mathrm{w}^2})$ *is consistent with* ZFC. *Further if* CH *holds then for any cardinal κ, there exists a cardinals preserving proper partial ordering R_κ such that* $\Vdash_{R_\kappa}$ "$\mathfrak{stick} \geq \kappa$ *but* $(\clubsuit_{\mathrm{w}^2})$ *holds".*

Proof Let $R = P_1 * \dot{P}_2$ where P_1 is as P in Lemma 3.10 for $\kappa = \aleph_2$ and $\dot{P}_2$ as Q in Lemma 3.11 in V^{P_1}.

For the second assertion, we let $R_\kappa = \mathrm{Fn}(\kappa, 2, \omega_1) * \dot{P}_1 * \dot{P}_2$. Note that under CH, $\mathrm{Fn}(\kappa, 2, \omega_1)$ is cardinals preserving and forces that $2^{\aleph_1} = \kappa$. Hence there is a sequence $(C_\beta)_{\beta<\kappa}$ as in Lemma 3.10 in the generic extension. Thus in $V^{\mathrm{Fn}(\kappa,2,\omega_1)}$, $\dot{P}_1$ can be taken as in Lemma 3.10 for our κ. Finally, in $V^{\mathrm{Fn}(\kappa,2,\omega_1)*\dot{P}_1}$ let $\dot{P}_2$ be as in Lemma 3.11. ❑ (Corollary 3.12)

4 CS*-iteration

In this section, we introduce an iterative construction of p.o.s which is closely related to the pseudo product we introduced in section 2. We adopt here the conventions of [5] on forcing. In particular, a p.o. (or forcing notion) P is a *pre-ordering* with a greatest element 1_P. In the following, we just try to develop a minimal theory needed for Theorem 5.6. More general treatment of the iterations like the one described below should be found in [16].

We call a sequence of the form $(P_\alpha, \dot{Q}_\alpha)_{\alpha\leq\varepsilon}$ a *CS*-iteration* if the following conditions hold for every $\alpha \leq \varepsilon$:

*0) P_α is a p.o. and, if $\alpha < \varepsilon$, then $\dot{Q}_\alpha$ is a P_α name such that $\Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ is a p.o. with a greatest element $1_{\dot{Q}_\alpha}$".

*1) $P_\alpha = \{\, p :$ p is a function such that $\mathrm{dom}(p) \in [\alpha]^{\leq\aleph_0}$; $p\restriction\beta \in P_\beta$ for any $\beta < \alpha$ and, if $\beta \in \mathrm{dom}(p)$ then $p]restr\beta \Vdash_{P_\beta}$ "$p(\beta) \in \dot{Q}_\beta$" $\}$.

*2) For p, $q \in P_\alpha$, $p \leq_{P_\alpha} q$ if and only if

   i) for any $\beta < \alpha$, $p\restriction\beta \Vdash_{P_\beta}$ "$p(\beta) \leq q(\beta)$";

   ii) *diff*$(p, q) = \{\, \beta \in \mathrm{dom}(p)\cap\mathrm{dom}(q) : p\restriction\beta \not\Vdash_{P_\beta}$ "$p(\beta) = q(\beta)$" $\}$ is finite.

We first show that such a sequence $(P_\alpha, \dot{Q}_\alpha)_{\alpha\leq\varepsilon}$ is really an iteration in the usual sense. In the following we assume always that $(P_\alpha, \dot{Q}_\alpha)_{\alpha\leq\varepsilon}$ is a CS*-iteration as defined above.

Lemma 4.1 *Suppose that $\alpha \leq \beta \leq \varepsilon$. Then*

0) *if $p \in P_\beta$, then $p\restriction\alpha \in P_\alpha$;*

1) *$P_\alpha \subseteq P_\beta$;*

2) *for p, $q \in P_\alpha$, we have $p \leq_{P_\alpha} q \Leftrightarrow p \leq_{P_\beta} q$;*

3) *for p, $q \in P_\beta$, if $p \leq_{P_\beta} q$ then $p\restriction\alpha \leq_{P_\alpha} q\restriction\alpha$.*

Proof 1) can be proved by induction on β. Other assertions are clear from the definition of CS*-iteration. ❑ (Lemma 4.1)

Lemma 4.2 *Suppose that $\alpha \leq \beta \leq \varepsilon$ and p, $q \in P_\alpha$. Then $p\perp_{P_\alpha} q \Leftrightarrow p\perp_{P_\beta} q$.*

Proof Suppose that p and q are compatible in P_α, say $r \leq_{P_\alpha} p, q$ for some $r \in P_\alpha$. Then $r \in P_\beta$ by Lemma 4.1, 1) and $r \leq_{P_\beta} p, q$ by Lemma 4.1, 2). Hence p and q are compatible in P_β.

Conversely, suppose that p and q are compatible in P_β, say $s \leq_{P_\beta} p$, q for some $s \in P_\beta$. Then we have $s\restriction\alpha \in P_\alpha$ by Lemma 4.1, 0), $s\restriction\alpha \leq_{P_\alpha} p\restriction\alpha = p$ and

$s\restriction \alpha \leq_{P_\alpha} q\restriction \alpha = q$. Hence p and q are compatible in P_α. ❑ (Lemma 4.2)

Suppose that $\alpha \leq \beta \leq \varepsilon$, $p \in P_\beta$. By Lemma 4.1, 0), we have $p\restriction \alpha \in P_\alpha$. For $r \leq_{P_\alpha} p\restriction \alpha$, let

$$p^\frown r = p\restriction (\mathrm{dom}(p) \setminus \alpha) \ \cup \ r.$$

For p, $q \in P_\varepsilon$, $p \leq^h_{P_\varepsilon} q \Leftrightarrow p \leq_{P_\varepsilon} q$ and $p\restriction \mathrm{dom}(q) = q$; $p \leq^v_{P_\varepsilon} q \Leftrightarrow p \leq_{P_\varepsilon} q$ and $\mathrm{dom}(p) = \mathrm{dom}(q)$ (h and v stand for 'horizontal' and 'vertical' respectively).

Lemma 4.3 1) *Let α, β, p, r be as above. Then $p^\frown r \in P_\beta$ and $p^\frown r \leq_{P_\beta} r$, p.*

2) *For p, $q \in P_\varepsilon$, $r = q\restriction (\mathrm{dom}(q)\setminus \mathrm{dom}(p)) \cup p$ is an element of P_ε and $r \leq^h_{P_\varepsilon} p$.*

3) *If $p_n \in P_\varepsilon$ for $n \in \omega$ and $p_{n+1} \leq^h_{P_\varepsilon} p_n$ for every $n \in \omega$, then $q = \bigcup\{ p_n : n \in \omega \}$ is an element of P_ε and $q \leq^h_{P_\varepsilon} p_n$ for every $n \in \omega$*

Proof 1): By induction on β. If $\beta = \alpha$ then $p^\frown r = r \leq p\restriction \alpha = p$. Suppose that we have shown the inequality for every $\beta' < \beta$. Let p and r be as above. If β is a limit then we obtain easily $p^\frown r \in P_\beta$ and $p^\frown r \leq_{P_\beta} r$, p by checking *1) and *2) of the definition of CS*-iteration. In particular, *2), ii) holds for the inequality $p^\frown r \leq_{P_\beta} r, p$ since $\mathit{diff}(p^\frown r, p) = \mathit{diff}(r, p\restriction \alpha)$ and $\mathit{diff}(p^\frown r, r) = \emptyset$. If $\beta = \gamma + 1$ for some $\gamma \geq \alpha$, then $p\restriction \gamma^\frown r \in P_\gamma$, $p\restriction \gamma^\frown r \leq_{P_\gamma} r$, $p\restriction \gamma$ by induction hypothesis. If $\gamma \notin \mathrm{dom}(p)$ then it follows $p = p\restriction \gamma \in P_\beta$ and $p^\frown r \leq_{P_\beta} r$, p. Otherwise $(p^\frown r)\restriction \gamma \Vdash_{P_\gamma}$ "$p(\gamma) \leq_{\dot{Q}_\gamma} p(\gamma)$". Hence again it follows that $p^\frown r \in P_\beta$ and $p^\frown r \leq_{P_\beta} r$, p.

2) and 3) are trivial. ❑ (Lemma 4.3)

Lemma 4.4 *Suppose that $\alpha \leq \beta \leq \varepsilon$, $p \in P_\alpha$ and $q \in P_\beta$. If p and q are incompatible in P_β then p and $q\restriction \alpha$ are incompatible in P_α.*

Proof Suppose that p and $q\restriction \alpha$ are compatible in P_α. Then there is $r \in P_\alpha$ such that $r \leq_{P_\alpha} p$, $q\restriction \alpha$. Let $s = q^\frown r$. By Lemma 4.3, we have $s \leq_{P_\beta} q$, r. Hence p and q are compatible in P_β. ❑ (Lemma 4.4)

Lemma 4.5 *Suppose that $\alpha \leq \beta \leq \varepsilon$ and that A is a maximal antichain in P_α. Then A is also a maximal antichain in P_β.*

Proof By Lemma 4.1, 1), we have $A \subseteq P_\beta$. By Lemma 4.2, A is an antichain in P_β. Suppose that A were not a maximal antichain in P_β. Then there is some $q \in P_\beta$ such that q is incompatible with each of $p \in A$. By Lemma 4.4, it follows that $q\restriction \alpha$ is incompatible with each of $p\restriction \alpha = p$, $p \in A$. This is a contradiction to the assumption that A is a maximal antichain in P_α. ❑ (Lemma 4.5)

5 CS*-iteration of Cohen reals

In the rest, we consider the CS*-iteration $(P_\alpha, \dot{Q}_\alpha)_{\alpha\le\kappa}$ for a cardinal κ such that

$$\Vdash_{P_\alpha} \text{“}\dot{Q}_\alpha = \mathrm{Fn}(\omega, 2)\text{”}$$

for every $\alpha < \kappa$.

Lemma 5.1 *Let p, $q \in P_\kappa$ be such that $p \le q$. Then there is $r \in P_\kappa$ such that $r \le p$ and for any $\alpha \in \mathit{diff}(r, q)$, there is $t \in \mathrm{Fn}(\omega, 2)$ such that $r{\restriction}\,\alpha \Vdash_{P_\alpha}$ “$r(\alpha) = \check{t}$”.*

Proof We define inductively a decreasing sequence $(\alpha_n)_{n<\omega}$ of ordinals and a decreasing sequence $(p_n)_{n\in\omega}$ of elements of P_κ as follows: Let $\alpha_0 = \max \mathit{diff}(p, q)$. Choose $p'_0 \in P_{\alpha_0}$ so that $p'_0 \le p{\restriction}\,\alpha_0$ and that p'_0 decides $p(\alpha_0)$. Let $p_0 = p \frown p'_0$. If α_n and p_n have been chosen, let $D_n = \mathit{diff}(p_n, q) \cap \alpha_n$. If $D_n = \emptyset$ we are done. Otherwise, let $\alpha_{n+1} = \max D_n$. Choose $p'_{n+1} \in P_{\alpha_{n+1}}$ such that $p'_{n+1} \le p_n{\restriction}\,\alpha_{n+1}$ and p'_{n+1} decides $p_n(\alpha_{n+1})$. Let $p_{n+1} = p_n \frown p'_{n+1}$. This process terminates after m steps for some $m \in \omega$, since otherwise we would obtain an infinite decreasing sequence of ordinals. Clearly $r = p_m$ is as desired. ❏ (Lemma 5.1)

Lemma 5.2 *P_κ satisfies the axiom A.*

Proof Let $\le_n$, $n \in \omega$ be the relations on P_κ defined by $p \le_n q \Leftrightarrow p \le^h_{P_\kappa} q$ for p, $q \in P_\kappa$ and *every* $n \in \omega$ (in Ishiu [6] an axiom A p.o., for which the $\le_n$'s can be taken to be all the same, is called uniformly axiom A). $(\le_n)_{n\in\omega}$ has the fusion property by Lemma 4.3, 3). Hence it is enough to show the following:

Claim 5.2.1 *For any $p \in P_\kappa$ and maximal antichain $D \subseteq P_\kappa$, there is $q \le^h_{P_\kappa} p$ such that $\{\, r \in D : r$ is compatible with $q \,\}$ is countable.*

$\vdash$ Let $\Phi : \omega \to \omega \times \omega$; $n \mapsto (\varphi_1(n), \varphi_2(n))$ be a surjection such that $\varphi_1(n) < n$ for all $n > 0$ and, for any k, $l \in \omega$, there are infinitely many $n \in \omega$ such that $\Phi(n) = (k, l)$. We construct inductively p_k, t_k, $u_k \in P_\kappa$ and a sequence $(s_{k,l})_{l\in\omega}$ for $k \in \omega$ as follows: let $p_0 = p$. If p_k has been chosen then let $(s_{k,l})_{l\in\omega}$ be an enumeration of $\mathrm{Fn}(\mathrm{dom}(p_k), \mathrm{Fn}(\omega, 2))$. If there are $t \in D$ and $u \in P_\kappa$ such that $u \le t$, p_κ, $\mathit{diff}(u, p_k) = \mathrm{dom} s_{\varphi_1(k),\varphi_2(k)}$ and $u{\restriction}\,\mathit{diff}(u, p_k) = s_{\varphi_1(k),\varphi_2(k)}$ (of course we identify here elements t of $\mathrm{Fn}(\omega, 2)$ with corresponding P_α-name $\check{t}$), then let t_k and u_k be such t and u and let $p_{k+1} = p_k \cup u{\restriction}\,(\mathrm{dom}(u_k) \setminus \mathrm{dom}(p_k))$. By Lemma 4.3, 2), we have $p_{k+1} \in P$. Otherwise let $t_k = u_k = 1_{P_\kappa}$ and $p_{k+1} = k_k$.

Now, let $q = \bigcup_{k\in\omega} p_k$. Then by Lemma 4.3, 3), we have $q \in P_\kappa$ and $q \le_{P_\kappa} p$. We show that this q is as desired.

Suppose that $t \in D$ is compatible with q. Then by Lemma 5.1, there is $u \subseteq_{P_\kappa} t$, q such that $u{\restriction}\,\mathit{diff}(q, r)$ has its values in $\mathrm{Fn}(\omega, 2)$. Let $n \in \omega$ be such that *diff* $(q,r) \subseteq q_n$ and $k \geq n$ be such that $s_{\varphi_1(k),\varphi_2(k)} = u\restriction \textit{diff}(q,r)$. Clearly $t_k \in D$ by construction. We claim that $t = t_k$: otherwise t and t_k would be incompatible. Hence u_k and u should be incompatible. But this is a contradiction.

It follows that

$$\{\, r \in D \,:\, r \text{ is compatible with } q \,\} \subseteq \{\, t_k \,:\, k \in \omega \,\}.$$

$\dashv$ (Claim 5.2.1)

❏ (Lemma 5.2)

In particular, P_κ is proper and hence the following covering property holds:

Corollary 5.3 *Suppose that G is a P_κ-generic filter over V. Then for any $a \in V[G]$ such that $V[G] \models$ "a is a countable set of ordinals", there is a $b \in V$ such that $a \subseteq b$ and $V \models$ "b is a countable set of ordinals".* ❏

Lemma 5.4 *If κ is strongly inaccessible, then P_κ satisfies the κ-cc.*

Proof Suppose that $p_\beta \in P_\kappa$ for $\beta < \kappa$. We show that there are compatible conditions among them. Without loss of generality we may assume that $\{\, \mathrm{dom}(p_\beta) \,:\, \beta < \kappa \,\}$ is a Δ-system with the root $x \in [\kappa]^{\leq\aleph_0}$ Let $\alpha_0 = \sup\{\, \gamma+1 \,:\, \gamma \in x \,\}$. Then $\alpha_0 < \kappa$ and $p_\beta\restriction x \in P_{\alpha_0}$ for every $\beta < \kappa$. Since $|\, P_\alpha \,| < \kappa$ there are β, $\beta' < \kappa$, $\beta \neq \beta'$ such that $p_\beta\restriction x = p_{\beta'}\restriction x$. But then $q = p_\beta \cup p_{\beta'} \in P_\kappa$ and $q \leq_{P_\kappa} p_\beta$, $p_{\beta'}$.

❏ (Lemma 5.4)

Lemma 5.5 *Suppose that $E \subseteq Lim(\omega_1)$ is stationary. Then $\Vdash_{P_\kappa}$ "$\clubsuit(E)$".*

Proof For each $\gamma \in E$ let $f_\gamma : [\gamma, \gamma+\omega) \to \gamma$ be a bijection and let

$$S_\gamma = \{\, x \subseteq \gamma \,:\, x \text{ is a cofinal subset of } \gamma,\ \mathrm{otp}(x) = \omega \,\}.$$

For each $x \in S_\gamma$, let $p_x \in P_\kappa$ be defined by

$$p_x = \{\, (\gamma+n, \dot{q}^\gamma_{x,n}) \,:\, n \in \omega \,\}$$

where $\dot{q}^\gamma_{x,n}$ is the standard $P_{\gamma+n}$-name for $\{\, (0,i) \,\}$ with $i \in 2$ and $i = 1 \Leftrightarrow f_\gamma(\gamma+n) \in x$. For distinct x, $x' \in S_\gamma$, p_x and $p_{x'}$ are incompatible. Hence there is a P_κ-name $\dot{x}_\gamma$ such that $\Vdash_{P_\kappa}$ "$\dot{x}_\gamma$ is a cofinal subset of γ with $\mathrm{otp}(\dot{x}_\gamma) = \omega$" and $p_x \Vdash_{P_\kappa}$ "$\dot{x}_\gamma = x$" for every $x \in S_\gamma$.

We show that $\Vdash_{P_\kappa}$ "$(\dot{x}_\gamma)_{\gamma\in E}$ is a $\clubsuit(E)$-sequence". Suppose that $p \in P_\kappa$ and $\dot{A}$ is a P_κ-name such that $p \Vdash_{P_\kappa}$ "$\dot{A} \in [\omega_1]^{\aleph_1}$". We have to show that there is $q \leq_{P_\kappa} p$ and $\gamma \in E$ such that $q \Vdash_{P_\kappa}$ "$\dot{x}_\gamma \subseteq \dot{A}$".

Let $\dot{f}$ be a P_κ-name such that $p \Vdash_{P_\kappa}$ "$\dot{f} : \omega \to \dot{A}$ is 1-1". Choose p_α, q_α, u_α for $\alpha < \omega_1$ inductively such that

a) $p_0 \leq_{P_\kappa} p$ and $(p_\alpha)_{\alpha<\omega_1}$ is a decreasing sequence with respect to $\leq^h_{P_\kappa}$;

b) $q_\alpha \leq^v_{P_\alpha} p_\alpha$ and q_α decides $\dot{f}(\alpha)$;

c) $u_\alpha = \mathit{diff}(q_\alpha, p_\alpha) \subseteq \mathrm{dom}(p) \cup \bigcup_{\beta<\alpha} \mathrm{dom}(q_\beta)$;

d) $q_\alpha \restriction u_\alpha \in \mathrm{Fn}(\kappa, \mathrm{Fn}(\omega, 2))$.

The condition d) is possible because of Lemma 5.1. By Fodor's lemma, there is $Y \in [\omega_1]^{\aleph_1}$ and $r \in \mathrm{Fn}(\kappa, \mathrm{Fn}(\omega, 2))$ such that $q_\alpha \restriction u_\alpha = r$ for every $\alpha \in Y$. For each $\alpha \in Y$, there is $\beta_\alpha \in \omega_1$ such that $q_\alpha \Vdash_{P_\kappa}$ "$\dot{f}(\alpha) = \beta_\alpha$" by b). Let $Z = \{ \beta_\alpha : \alpha \in Y \}$. Let

$$C = \{ \gamma \in \mathit{Lim}(\omega_1) : \textstyle\bigcup_{\alpha \in Y \cap \gamma} (\sup(q_\alpha) \cap \omega_1) \subseteq \gamma \text{ and } Z \cap \gamma \text{ is unbounded in } \gamma \}.$$

Then C is closed unbounded in ω_1. Since E was stationary, there exists a $\gamma^* \in C \cap E$. Let $q' = \bigcup_{\alpha \in Y \cap \gamma^*} q_\alpha$. Then we have $q' \Vdash_{P_\kappa}$ "$Z \cap \gamma^* \subseteq \dot{A}$". Now let $x \in S_{\gamma^*}$ be such that $x \subseteq Z \cap \gamma^*$. Finally let $q = q' \cup p_x$. Then we have $q \leq^h_{P_\kappa} p$ and $q \Vdash_{P_\kappa}$ "$\dot{x}_\alpha = x \subseteq Z \cap \gamma^* \subseteq \dot{A}$". ❑ (Lemma 5.5)

Let $(P_\alpha, \dot{Q}_\alpha)_{\alpha \leq \kappa}$ be a CS*-iteration as above. For $\alpha < \kappa$ let $P_\kappa/\dot{G}_\alpha$ be a P_α-name such that $\Vdash_{P_\alpha}$ "$P_\kappa/\dot{G}_\alpha = \{ p \in \check{P}_\kappa : p \restriction \alpha \in \dot{G}_\alpha \}$ with the ordering $p \leq_{\kappa,\alpha} q \Leftrightarrow p \leq_{P_\alpha} q$". As in [5], we can show that $P_\kappa \approx P_\alpha * P_\kappa/\dot{G}_\alpha$. Also, by Corollary 5.3, practically the same proof as in [5] shows that

$$\Vdash_{P_\alpha} \text{“} P_\kappa/\dot{G}_\alpha \text{ is } \approx \text{ to a CS}^*\text{-iteration of } \mathrm{Fn}(\omega, 2)\text{”}.$$

Now we are ready to prove the main theorem of this section:

Theorem 5.6 *Suppose that* ZFC + *"there exists an inaccessible cardinal" is consistent. Then* ZFC + ¬CH + MA(*countable*) + *"*$\clubsuit(E)$ *for every stationary* $E \subseteq \mathit{Lim}(\omega_1)$*" is consistent as well.*

Proof Suppose that κ is strongly inaccessible. For P_κ as above, let G_κ be a P_κ-generic filter over V. We show that $V[G_\kappa]$ models the assertions. Let $E \subseteq \mathit{Lim}(\omega_1)$ be a stationary set in $V[G_\kappa]$. Since P_κ has the κ-cc by Lemma 5.4, there is some $\alpha < \kappa$ such that $E \in V[G_\alpha]$ where $G_\alpha = G_\kappa \cap P_\alpha$. Hence by the remark before this theorem, we may assume without loss of generality that $E \in V$. But then, by Lemma 5.5, we have $V[G_\kappa] \models$ "$\clubsuit(E)$".

Finally, we show that MA(*countable*) holds in $V[G_\alpha]$. Let $\mathcal{D}$ be a family of dense subsets of $\mathrm{Fn}(\omega, 2)$ in $V[G_\kappa]$ of cardinality $< \kappa$. Again by the κ-cc of P_κ, we can find an $\alpha < \kappa$ such that $\mathcal{D} \in V[G_\alpha]$. Since we have

$$P_\kappa \approx P_\alpha * \dot{Q}_\alpha * P_\kappa/\dot{G}_{\alpha+1},$$

the generic set over $V[G_\alpha]$ added by $\dot{Q}_\alpha[G_\alpha] = \mathrm{Fn}(\omega, 2)$ is $\mathcal{D}$-generic over $\mathrm{Fn}(\omega, 2)$ in $V[G_\kappa]$. ❑ (Theorem 5.6)

At the moment we — or more precisely the first and the third author — do not know if an inaccessible cardinal is really necessary in Theorem 5.6. As for CS-iteration, κ is collapsed to be of cardinality $\aleph_2$ in the model above, since the continuum of each of the intermediate models is collapsed to $\aleph_1$ in the following limit step of cofinality $\geq \omega_1$. Thus the following problem seems to be a rather hard one:

Problem 5.7 *Is the combination* MA(*countable*) + $\clubsuit(E)$ *for every stationary* $E \subseteq \mathit{Lim}(\omega_1)$ *consistent with* $2^{\aleph_0} > \aleph_2$ *?*

Acknowledgments

The research of this paper began when the first author (S.F.) was at the Hebrew University of Jerusalem. He would like to thank The Israel Academy of Science and Humanities for enabling his stay there. He also would like to thank T. Miyamoto for some quite helpful remarks.

The second author (S.S.) was partially supported by the Deutsche Forschungsgemeinschaft(DFG) grant Ko 490/7–1. He also gratefully acknowledges partial support by the Edmund Landau Center for research in Mathematical Analysis, supported by the Minerva Foundation (Germany). The present paper is the second author's Publication No. 544.

The third author (L.S.) is partially supported by the Hungarian National Foundation for Scientific Research grant No. 16391 and the Deutsche Forschungsgemeinschaft (DFG) grant Ko 490/7–1.

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