In the random graph $G(n,p), p = n^{-a}$: if ψ has probability $O(n^{-\varepsilon})$ for every $\varepsilon > 0$ then it has probability $O(e^{-n^{\varepsilon}})$ for some $\varepsilon > 0$

Sh551

SAHARON SHELAH

Institute of Mathematics
The Hebrew University Jerusalem, Israel

Rutgers University
Department of Mathematics
New Brunswick, NJ USA

We thank Alice Leonhardt for the beautiful typing.
Latest Revision 95/Dec/22
The author would like to thank the United States Israel Binational Science Foundation for partially supporting this research.
We thank Joel Spencer for telling us the problem

§0 Introduction

Shelah Spencer [ShSp 304] proved the $0-1$ law for the random graphs $G(n,p_n)$, $p_n = n^{-\alpha}$, $\alpha \in (0,1)$ irrational (set of nodes in $[n] = \{1,\dots,n\}$, the edges are drawn independently, probability of edge is p_n). One may wonder what can we say on sentences ψ for which $\mathrm{Prob}(G(n,p_n) \models \psi)$ converge to zero, Lynch [L] asked the question and did the analysis, getting (for every ψ):

(α) $\mathrm{Prob}[G(n,p_n) \models \psi] = cn^{-\beta} + O(n^{-\beta-\varepsilon})$ for some ε such that $\beta > \varepsilon > 0$

or

(β) $\mathrm{Prob}(G(n,p_n) \models \psi) = O(n^{-\varepsilon})$ for every $\varepsilon > 0$.

Lynch conjectured that in case (β) we have

(β^+) $\mathrm{Prob}(G(n,p_n) \models \psi) = O(e^{-n^{\varepsilon}})$ for some $\varepsilon > 0$.

We prove it here.

Notation Let ℓ, m, n, k be natural numbers.
Let $\varepsilon, \zeta, \alpha, \beta, \gamma$ be positive reals.
$[n] = \{1,\dots,n\}$.
$\mathbb{R}$ is the set of reals.
$\mathbb{R}^+$ is the set of reals > 0.

§1

1.1 Theorem. *1) For any first order sentence ψ in the language of graphs and irrational $\alpha \in (0,1)_{\mathbb{R}}$ we have (where $p_n = n^{-\alpha}$ and $Prob(G_{n,p_n} \models \psi) \to 0$):*

$$\underline{\text{either}}\ Prob(G_{n,p_n} \models \psi) \text{ is } cn^{-\beta} + O(n^{\beta-\varepsilon}) \text{ for some reals } \beta > \varepsilon > 0 \text{ and } c > 0$$

$$\underline{\text{or}}\ Prob(G_{n,p_n} \models \psi) \text{ is } O(e^{-n^{\varepsilon}}) \text{ for some real } \varepsilon > 0.$$

2) However, this is not recursive.

Proof. We change the context generalizing it.

1.2 Definition of the Probability Context.

(a) $Q_n \subseteq \{1,\dots,n\}, G^*_{Q_n}$ a graph on Q_n.
(b) We consider first order sentences or formulas with vocabulary $\subseteq \tau = \{=, Q, R\}$, ($=$ is equality, Q is a monadic predicate, R is a symmetric irreflexive binary relation (will be "being an edge").)
(c) $G = G_{n,p_n}[G^*_Q]$ a graph on $[n]$, $G \restriction Q = G^*_Q$, and except this, G is random with edge probability p_n (i.e. for every edge not included in Q we flip a coin with probability p_n and do it independently for the set of edges). We consider G a τ-model with $Q^M = Q, R$ the edge relation.

Remark. The point is that $|Q|$ will be required to be just $< n^{\varepsilon}$ not say $<\ \log(n)$.

Proof. We consider only graphs H in $\{H : H$ a graph whose set of nodes include Q, moreover $H \restriction Q = G^*_Q\}$. First, we repeat the proof in Shelah Spencer [ShSp 304], section 4, starting in p.105. In our context we define "$[H_0, H_1)$ has type (v,e)", it holds if $v = |H_1 \backslash H_0 \backslash Q|$, and

$$e = \left|\left\{\{x,y\} \in E(G_{n,p}) : \{x,y\} \subseteq H_1 \cup Q, \{x,y\} \not\subseteq H_0 \cup Q\right\}\right|,$$

(where for a graph G, $E(G)$ is the set of edges of G).
Then define dense, sparse, safe, rigid, hinged as there adding "over Q and/or inside G" for definiteness. We also define $cl_\ell(H_0; H_1)$ as in p.107, line 7. Later we write $cl_\ell(H_0; Q)$. All claims hold, but arriving to Theorem 1.3 (bottom of p.107) we should be careful. We consider only embeddings which are the identity on Q.

1.3 Lemma. *1) Let $\ell^* \in \mathbb{N}$. For every small enough $\varepsilon > 0$, for some $\xi > 0$, for every n large enough, if $|Q| \le n^\xi, Q \subseteq [n]$ we have: if (H_0, H_1) is safe of type (v, e) and f embeds H_0 into G (and f is the identity on Q) and $|H_1 \backslash Q| \le \ell^*$ then:*

$$Prob\Big(\neg[n^{v-\alpha\, e-\varepsilon} < N(f, H_0, H_1) < n^{v-\alpha\, e+\varepsilon}]\Big) < e^{-n^\xi}$$

(where $N(f, H_0, H_1)$ is the number of extensions $g : H_1 \to G$ satisfying: $x \in H_0 \Rightarrow g(x) = f(n)$ and $\{x, b\} \in E(H_1), b \notin H_0 \Rightarrow \{g(x), g(y)\} \in E(G)$).
2) Let $\varepsilon \in \mathbb{R}^+$ and $\ell^* \in \mathbb{N}$ be given, then for some $\xi > 0$ for every n large enough and any $Q \subseteq [n], |Q| \le n^\varepsilon$ and graph G^*_Q on Q we consider only embeddings which are the identity on Q. Then

- (*) *if H_1 is a graph with $|H_1 \backslash Q| \le \ell^*$, $H_0 \subseteq H_1$, we assume f embeds H_0 into Q, f is the identity on H_0 and (H_0, H_1) is rigid then:*

$$Prob\Big(N(f, H_0, H_1, G_{n,p_n}) > 0\Big) < n^{-\varepsilon}.$$

Proof. 1) As in [ShSp 304, Theorem 3,p.107] + extra computation by the central limit theorem or see [Sh 550, §5] for more.
2) As in [ShSp 304].

4 Lemma. *For any $k, m \in \mathbb{N}$ there are ℓ^* and $\varepsilon^* > 0$ depending on k only such that the following holds:*

- (*) *For any formula $\psi = \psi(x_1, \ldots, x_m)$ of quantifier depth $\le k$ in the vocabulary $\{=, Q, R\}$ there is a formula $\theta_\psi = \theta_\psi(x_1, \ldots, x_m)$ in the vocabulary $\{=, Q, R\}$ such that:*
- (**) *For every n large enough, $Q \subseteq \{1, \ldots, n\}, |Q| \le n^{\varepsilon^*}$, and graph G^*_Q on Q and $G = G_{n,p_n}[G^*_Q]$ such that the small probability cases from Lemma ?,? (for (H_1, H_2) of type $(v, e), v \le 2\ell^*$), or just $\otimes^1_{\ell^*} + \otimes^2_{\ell^*}$ below do not occur, we have:*
- (***) *for every $a_1, \ldots, a_m \in \{1, \ldots, m\}$ we have*

$$(\{1, \ldots, n\}, Q, R) \models \psi[a_1, \ldots, a_m] \ \underline{\text{iff}}$$
$$\Big(Q \cup \{a_1, \ldots, a_m\}, Q, R \restriction (Q \cup \{a_1, \ldots, a_m\})\Big) \models \theta_\psi[a_1, \ldots, a_m].$$

where

- $\bigotimes^1_{\ell^*}$ *if (H_0, H_1) is safe (so $Q \subseteq H_0$) $|H_1 \backslash Q| \le \ell^*, H_0 \subseteq G_{n,p_n}[G^*_Q]$ then we can extend id_{H_0} to an embedding g of H_1 into $G_{n,p_n}[G^*_Q]$ such that $c\ell_{\ell^*}\left(g(H_1), G_{n,p}[G^*_q]\right) = g(H_1) \cup c\ell_{\ell^*}\left(f(H_0, G_{n,p_n}[G^*_Q]\right)$*
- $\bigotimes^2_{\ell^*}$ *if (H_0, H_1) is rigid, $|H_1 \backslash Q| \le \ell^*, H_0 = G^*_Q$ then there is no extension of f of id_{H_0} to an embedding of H_1 into $G_{n,p_n}[G^*_Q]$.*

Proof. Similar to the proof in [ShSp 304], (and is a particular case of [Sh 467, §2] (see related)).

Proof of Theorem 1.1. Part (1) Let θ_ψ be from the analysis (i.e. Lemma ? for the ψ from Theorem 1.1) for the original sentence ψ.

Case A. For some finite graph G^* on say $\{1,\dots,m^*\}$ we have $G^* \models \theta_\psi$.

In this case the probability that G^* can be embedded into G_{n,p_n} is $\geq O(n^{-\beta})$ for some $\beta \in (0,\infty)$ if $n \geq m^*$ of course; so this means that one of the $\leq n^{m^*}$ possible mapping is an embedding, but more convenient is to consider the event $G \restriction [m^*] = G^*$ which also has probability $\geq n^{-\beta}$ for some β. Now modulo this event the probability that the conclusion of Lemma ? fails is (for n large enough) much smaller than n^{-m^*}. So we can assume that for $G \restriction [m^*] \cong G^*$ and that the conclusion of Lemma ? holds for this. Now check and if we succeed by Lemma ?, we are done, i.e. the probability that $G_{n,p_n} \models \psi$ is quite high.

Case B. For no finite graph $G^*, G^* \models \theta_\psi$.

Choose $\ell^* \in \mathbb{N}$ large enough as needed for our sentence ψ in Lemma 4.

Let $\zeta \in \mathbb{R}^+$ be such that:
$v \in \{0,\dots,2\ell^*\}, e \in \mathbb{N} \Rightarrow |v - \alpha e| \geq \zeta$ and it satisfies the requirements on ζ in Lemma ? (for $2\ell^*$ (readily follows).)
(The $2\ell^*$ rather than ℓ^* is for the bound on $\text{Prob}(\mathcal{E}_2)$.) Clearly ζ exists and if (H_0,H_1) is hinged and $|H_1 \backslash H_0| \leq \ell^*$ and (H_0,H_1) is of type (v,e) then $v - \alpha e < -\zeta$.

Let $\varepsilon(\ell^*), \xi$ be such that:

(*a*) $\varepsilon(\ell^*) \in \mathbb{R}^+$ and $\varepsilon(\ell^*) < \zeta/2, \xi < \zeta/2$
(*b*) in Lemma ? $\varepsilon(\ell^*), \xi$ satisfies the requirements of ε, ξ respectively.

We shall prove that for n large enough $\text{Prob}(G_{n,p_n} \models \psi)$ is $\leq e^{-(n^\xi)}$, this is enough.

For any $G = G_{n,p_n}$, we define by induction on $j \leq n$, a subset $P_j = P_j[G]$ of $\{1,\dots,n\}$ as follows:

$$P_0 = \emptyset$$

$$P_{j+1} = P_j \cup \{H : P_j \subseteq H \subseteq G, |H \backslash P_j| \leq \ell^*, H \neq P_j \text{ and } (P_j, H) \text{ is rigid in } G\}.$$

For some $j(*) < n$ we have $P_{j(*)} = P_{j(*)+1}$ (hence $P_{j(*)+1} = P_{j(*)+2}$, etc).
If $|P_{j(*)}| \leq n^{\varepsilon(\ell^*)}$ and $\otimes^1_{\ell^*}$ holds then, (as $P_{j(*)} = P_{j(*)+1}$) this implies $\otimes^2_{\ell^*}$ and then by Lemma ? we are done ($P_{j(*)}$ is Q). So it is enough to give an upper bound of the form e^{-n^ε} to the probability $\text{Prob}(\mathcal{E}_1) + \text{Prob}(\mathcal{E}_2)$ were $\mathcal{E}_1$ is the event $|P_{j(*)}| > n^{\varepsilon(\ell^*)}$ and $\mathcal{E}_2$ is the event $|P_{j(*)}| \leq n^{\varepsilon(\ell^*)}$ & $[\otimes^1_{\ell^*}$ fails].

<u>On $\text{Prob}(\mathcal{E}_1)$</u>. If $|P_{j(*)}| \geq n^{\varepsilon(\ell^*)}$ then we can find $a_{j,\ell}$ for $j < [n^{\varepsilon(\ell^*)}/\ell^*]$ and $\ell < \ell_j \leq \ell^*$ such that

$\Big(H_i \cap \{a_{i,\ell} : \ell < \ell_i\}, \{a_{i,\ell} : \ell < \ell_i\}\Big)$ (in G) is rigid of type (v_i, e_i) where $H_i =: \{a_{j,\ell} : j < i \text{ and } \ell < \ell_j\}$ (so we may have not used all $P_{j(*)}$). Clearly there is a real $\zeta > 0$ depending on ℓ^*, α only such that $v_i - e_i\alpha \leq -\zeta$, (simply, there are only finitely many possible pairs (v, e)).

Let I be a sequence describing this situation, i.e. it contains

$$\langle \ell_i : i < [n^{\varepsilon(\ell^*)}/\ell^*]\rangle$$

$$\{((i_1, m_1), (i_2, m_2)) : a_{\ell_1,m_1} = a_{i_2,m_2}\}$$

$$\{(i, m_1, m_2) : a_{i,m_1} R^G a_{i,m_2}\}.$$

There are $\prod\limits_{i<[n^{\varepsilon(\ell^*)}/\ell^*]} (\ell^* \times (\ell^* \times i)^{\ell^*} \times 2^{2\ell^*})$ possible such sequences I (an overkill). [Why? The ith term in the product is an upper bound on the number of choices in stage i, there ℓ^* is the number of possible $\ell_i, \ell^* \times i$ is an upper bound on the number $|\{a_{j,\ell} : j < i, \ell < \ell_j\}|$, $(\ell^* \times i)^{\ell^*}$ is an upper bound to the number of choices of $\langle a_{i,\ell} : \ell < \ell^*, a_{i,\ell} \in \{a_{j,s} : j < i, s < \ell_j\}\rangle$, and $2^{2\ell^*}$ is an upper bound to the number of possible $G \restriction \{a_{i,\ell} : \ell < \ell_i\}$].

Now for some constants c_0, c_1 depending only on ℓ^* (i.e. ψ) this number is $\leq c_0^{n^{\varepsilon(\ell^*)}/\ell^*} \times [(n^{\varepsilon(k^*)}/\ell^*)!]^{\ell^*} \leq n^{\varepsilon(\ell^*)n^{\varepsilon(\ell^*)}}$. For each I the number of possibilities for the $a_{i,\ell}$ is $\leq \prod\limits_i n^{v_i}$, and the probability it holds in G is $\prod\limits_i n^{-\alpha e_i}$, hence the expected value is

$$\prod_i n^{(v_i - \alpha e_i)} \leq \prod_i n^{-\zeta} = n^{-\zeta(n^{\varepsilon(\ell^*)}/\ell^*)}.$$

So the expected number of number of such $\langle a_{i,\ell} : i < n^{\varepsilon(\ell^*)}/\ell^* \text{ and } \ell < \ell_i\rangle$ for some I is $\leq n^{(\varepsilon(\ell^*)-\zeta)n^{\varepsilon(\ell^*)}}$ and as we have $\varepsilon(\ell^*) < \zeta$ the conclusion should be clear.

<u>Probability of $\mathcal{E}_2$</u>. Should be clear by Lemma ?; i.e. except suitably small probability the number of extensions of f to embedding of H_1 is much larger than the number of such extensions failing the requirement in $\otimes^1_{\ell^*}$.

Proof of Theorem 1.1-part (2). In non-trivial cases for some ℓ and pair (H_0, H_1) we have $H_1 \neq H_0$ and $H_1 \subseteq cl_\ell(H_0)$.
Now for n large enough (if $|cl_\ell(H_0)| \ll \log n$),
on $cl_\ell(H_0)$ in G_{n,p_n} we can interpret arithmetic on $cl_\ell(H_0)$ (with parameters) and all subsets and all second place relations. Fix H_0, ℓ.

For a sentence ψ speaking on $\mathbb{N} \restriction k$, (or 2^k) we can compute ψ^* in the vocabulary of graphs saying

(∗) there is a copy H_0' of H_0 such that

$$\mathbb{N} \restriction |cl_\ell(H_0')| \models \psi^*.$$

So for every function $h : \mathbb{N} \to \mathbb{N}$ converging to infinity

$$\text{Lim inf}_n \left(\text{Prob}(G_{n,p_n} \models \psi^*)/n^{-h(n)} \right) \geq 1 \underline{\text{ iff }} \bigvee_k [\mathbb{N} \restriction k \models \psi].$$

But the set $\{\psi : (\exists k)[|N \restriction k| \models \psi]\}$ is like the set of sentences having a finite model (i.e. same Turing degree) so is not recursive.

Concluding Remarks. 1) In fact, we have to consider P_j (in case B during the proof of Theorem 1) only for $j \leq 2^r$, where r is the quantifier depth of the sentence ψ (for which we are proving Theorem 1.1).

REFERENCES.

[L] Richard Laver. Making the supercompactness of κ indestructible under κ-directed closed forcing. *Israel J. of Math.*, **29**:385–388, 1978.

[Sh 550] Saharon Shelah. 0–1 laws. *Preprint.*

[Sh 467] Saharon Shelah. Zero-one laws for graphs with edge probabilities decaying with distance. Part I. *Fundamenta Mathematicae*, **175**:195–239, 2002.

[ShSp 304] Saharon Shelah and Joel Spencer. Zero-one laws for sparse random graphs. *Journal of the American Mathematical Society*, **1**:97–115, 1988.