More on Entangled Orders

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Introduction.

This paper grew as a continuation of [Sh462] but in the present form it can serve as a motivation for it as well. We deal with the same notions, all defined in 1.1, and use just one simple lemma from there whose statement and proof we repeat as 2.1. Originally entangledness was introduced, in [BoSh 210] for example, in order to get narrow boolean algebras and examples of the nonmultiplicativity of c.c-ness. These applications became marginal when other methods were found and successfully applied (especially Todorčeric walks) but after the pcf constructions which made their début in [Sh-g] and were continued in [Sh 462] it seems that this notion gained independence.

Generally we aim at characterizing the existence of strong and weak entangled orders in cardinal arithmetic terms. In [Sh462 §6] necessary conditions were shown for strong entangledness which in a previous version was erroneously proved to be equivalent to plain entangledness. In §1 we give a forcing counterexample to this equivalence and in §2 we get those results for entangledness (certainly the most interesting case). A new construction of an entangled order ends this section. In §3 we get weaker results for positively entangledness, especially when supplemented with the existence of a separating point (definition 2.2). An antipodal case is defined in 3.10 and completely characterized in 3.11. Lastly we outline in 3.12 a forcing example showing that these two subcases of positive entangledness comprise no dichotomy. The work was done during the fall of 1994 and the winter of 1995. The second author proved theorems 1.2, 2.14, the result that is mentioned in remark 2.11 and what appears in this version as theorem 2.10(a) with the further assumption $den(I)^{\theta} < \mu$. The first author is responsible for waving off this

*This is publication no. 553 for the second author assumption (actually by proving it per se), for theorems 2.12 and 2.13 in section 2 and for the work which is presented in section 3.

Our only notational idiosyncrasy is obeying the Jerusalem convention – the stronger forcing condition is the greater one. We also abuse the logic by writing the assumption "There is a such-and-such order" as "I is such-and-such". We thank Shani Ben David for the beautiful typing.

§1. Entangledness is not strong Entangledness.

Definition 1.1: (a) A linear order $(I,<)$ is called (μ,σ)-entangled if for any matrix of distinct elements from it $\langle t^\varepsilon_i | i<\mu,\quad \varepsilon<\sigma_1\rangle(\sigma_1<\sigma)$ and $u\subset\sigma_1$ there are $\alpha<\beta<\mu$ satisfying $\forall\varepsilon<\sigma_1(t^\varepsilon_\alpha<t^\varepsilon_\beta\leftrightarrow\varepsilon\in u)$.

(b) A linear order $(I,<)$ is called (μ,σ)-strongly entangled if for any matrix $\langle t^\varepsilon_i|i<\mu\ \ \varepsilon<\sigma_1\rangle$ $(\sigma_1<\sigma)$ and $u\subset\sigma$, s.t. $\forall\alpha<\mu\forall\varepsilon_0\in u\forall\varepsilon_1\in\sigma_1\backslash u(t^{\varepsilon_0}_\alpha\neq t^{\varepsilon_1}_\alpha)$ there are $\alpha<\beta<\mu$ satisfying $\forall\varepsilon<\sigma_1(t^\varepsilon_\alpha\leq t^\varepsilon_\beta\leftrightarrow\varepsilon\in u)$.

(c) A linear order is called (μ,σ) positively [positively*] entangled if for every $\sigma_1<\sigma$ and any matrix $\langle t^\varepsilon_\alpha|\varepsilon<\sigma_1,\alpha<\mu\rangle$ s.t. $\forall\varepsilon<\sigma_1\forall\alpha,\beta<\mu(t^\varepsilon_\alpha\neq t^\varepsilon_\beta)$ and $u\in\{\phi,\sigma_1\}$ there are $\alpha<\beta[\alpha\neq\beta]$ satisfying $\forall\varepsilon<\sigma_1(t^\varepsilon_\alpha<t^\varepsilon_\beta\leftrightarrow\varepsilon\in u)$.

(d) The phrase "I is (μ,σ) entangled with minimal μ" stands for "I is (μ,σ) entangled but not (μ',σ) entangled for any $\mu'<\mu$".

Theorem 1.2. *For any cardinals $\lambda=\lambda^{<\lambda}>\theta, cf\mu=\kappa>\lambda$ there is a cardinal preserving forcing adding a (μ,θ^+)-entangled order with minimal μ. In particular, it is not (μ,θ^+) strongly entangled.*

Proof: Fix $\langle\mu_i|i<cf\mu\rangle$ increasing to μ and define $\mathbb{P}=\{p|\mathrm{dom}\,p\in[\mu]^{<\lambda}$, for some $\alpha<\lambda$ ran $p\subseteq^\alpha 2$, p is $1-1$, $\forall\alpha(2\alpha\in\mathrm{dom}\,p\leftrightarrow 2\alpha+1\in\mathrm{dom}\,p)\forall\alpha,\beta\in[\mu_i,\mu_i+\mu_i)$ $(p(2\alpha)<p(2\beta)\leftrightarrow p(2\alpha{+}1)<p(2\beta{+}1))\}$ where $<$ is the lexicographic order. $p\leq q$ iff $\mathrm{dom}\,p\subset\mathrm{dom}\,q$ and $\forall\alpha\in\mathrm{dom}\,p\,(p(\alpha)\lhd q(\alpha))$. Easily $\mathbb{P}$ is λ-closed. In order to see that it is also λ^+-c.c. (hence cardinals preserving) note that $\forall\alpha<\lambda\forall p_0,p_1,p'_0,p'_1\in^\alpha 2(p_0\lhd p'_0\wedge p_1\lhd p'_1\wedge p_0\neq p_1)\rightarrow(p_0<p_1\leftrightarrow p'_0<p'_1)$, so that if $\langle p_a|\alpha<\lambda^+\rangle$ are from $\mathbb{P}$ wlog $\{\mathrm{dom}\,p_\alpha|\alpha<\lambda^+\}$ is a Δ system and for some $p^*\in\mathbb{P}\forall\alpha<\lambda^+\forall i\in\mathrm{dom}\,p_\alpha(p_\alpha(i)=p^*(otp\,\mathrm{dom}\,p_\alpha\cap i))$.

Now define for $\alpha < \beta < \lambda^+$

$$q(x) = \begin{cases} p_\alpha(x)^\wedge 0 & x \in \operatorname{dom} p_\alpha \\ p_\beta(x)^\wedge 1 & x \in \operatorname{dom} p_\beta \backslash \operatorname{dom} p_\alpha \end{cases}$$

and this element from $\mathbb{P}$ satisfies $p_\alpha, p_\beta < q$. Any $\mathbb{P}$-generic set induces $A = \langle e_\alpha | \alpha < \mu \rangle \subset {}^\lambda 2$ which are distinct and satisfy $\forall \alpha, \beta \in [\mu_i, \mu_i + \mu_i)$ $(e_{2\alpha} < e_{2\beta} \leftrightarrow e_{2\alpha+1} < e_{2\beta+1})$. Again, A is ordered lexicographically. This shows that $(A, <)$ is not $(\mu_1, 2)$-entangled for all $\mu_1 < \mu$. Suppose by contradiction that A is not (μ, λ)-entangled. In that case there is $p \in \mathbb{P}$ and $p \Vdash$ "$\langle \underset{\sim}{t}{}^\varepsilon_i | i < \mu, \varepsilon < \lambda_1 \rangle$, $\lambda_1 < \lambda$, $u = \{2\rho | \rho < \lambda_1\}$ is a counterexample". For $i < \mu$ pick $p < p_i$ and $\langle \alpha(\varepsilon, i) | \varepsilon < \lambda_1 \rangle \subset \operatorname{dom} p_i$ such that $p_i \Vdash$ " $\bigwedge_{\varepsilon < \lambda_1} \underset{\sim}{t}{}^\varepsilon_i = e_{\alpha(\varepsilon,i)}$". Wlog for some $p^* \in \mathbb{P} \forall i < \mu \forall j \in \operatorname{dom} p_i (p_i(j) = p^*(otp \operatorname{dom} p_i \cap j))$.

We can assume also that for every $i < \mu$ $\{e_{\alpha(\varepsilon,i)} | \varepsilon < \lambda_1\} \subset \operatorname{dom} p_i$ and that $\langle otp$ $\operatorname{dom} p_i \cap \alpha(\varepsilon, i) | \varepsilon < \lambda_1 \rangle$ does not depend on i (here we use the inequality $\kappa > \lambda$). By the $\lambda^+ - c.c.$ we have two comparable elements, call them p_i, p_j. Now define

$$q(\gamma) = \begin{cases} p_i(\gamma)^\wedge 0 & \gamma = \alpha(i, 2p) \\ p_i(\gamma)^\wedge 1 & \gamma = \alpha(i, 2p+1) \\ p_j(\gamma)^\wedge 1 & \gamma = \alpha(j, 2p) \\ p_j(\gamma)^\wedge 0 & \gamma = \alpha(j, 2p+1) \\ p_i(\gamma)^\wedge 0 & \gamma \in \operatorname{dom} p_i \text{ not as above} \\ p_j(\gamma)^\wedge 0 & \gamma \in \operatorname{dom} p_j \text{ not as above} \end{cases}$$

But $p < q \Vdash$ "$\underset{\sim}{t}$ is not a counterexample by looking at i, j", a contradiction.
For the second claim of the theorem apply [Sh 462, 6.24(2)]. $\square$

§2 Positive results on entangled orders.

First we quote the useful lemma [Sh462 1.2(4)].

LEMMA 2.1. *If I is $(\mu, 2)$ entangled then the density of I is smaller than μ.*

PROOF: Otherwise define inductively a sequence of intervals $\langle (a^0_\alpha, a^1_\alpha) | \alpha < \mu \rangle$ s.t.(a^0_α, a^1_α) exemplifies the nondensity of $\{a^0_\beta, a^1_\beta | \beta < \alpha\}$ in I, i.e. disjoint to this set. Now the matrix $\{a^i_\alpha | i < 2, \alpha < \mu\}$ contradicts the entangledness with respect to $u = \{0\}$. $\square$

DEFINITION 2.2: For a linear order I and $x, y \in I$ we define $\langle x, y \rangle := (x, y)_I \cup (y, x)_I$. We call the point x μ-separative if $|\{y \in I | y < x\}|$, $|\{y \in I | y > x\}| \geq \mu$. Let $f(x) =$

$\min_{y \in I \setminus \{x\}} |\langle x, y \rangle|$.

The following theorem states a basic property of entangled orders.

THEOREM 2.3. *If $\lambda = \theta^+$ is infinite and $(I, <)$ is (μ, λ)-entangled linear order with minimal μ then $|\{x \in I | f(x) < \mu\}| < \mu$.*

PROOF: Suppose first μ is limit. We assume that $|I| = \mu$–if $|I| < \mu$ the conclusion is trivial, and if $|I| > \mu$ we take any subset of I of cardinality μ. Fix a strictly increasing sequence of successor cardinals converging to μ, $\langle \mu_\alpha | \alpha < cf\mu \rangle$. Define on I the equivalence relation $xEy \leftrightarrow |\langle x, y \rangle| < \mu$. We look for disjoint intervals $\langle I_\alpha | \alpha < cf\mu \rangle$ satisfying $|I_\alpha| \geq \mu_\alpha$. If the conclusion of the theorem fails then the union of the equivalent classes which have more than one element is of power μ. If there are μ many such classes choose $\langle a^0_\beta, a^1_\beta | \beta < \mu \rangle$ with no repetitions s.t. $\forall \beta, \alpha < \mu \forall i, j \in \{0, 1\}(a^i_\alpha E a^j_\beta \leftrightarrow \alpha = \beta)$ which contradicts $(\mu, 2)$-entangledness with respect to $u = \{0\}$. So there are less than μ classes. If any equivalence class has power smaller than μ then choose by induction for I_α any sufficiently large but yet unchosen class. Otherwise fix one class J, $|J| = \mu$. Pick $x \in J$ and wlog $|\{y \in I | y > x\}| = \mu$ (inversion of the order does not affect the entangledness). Choose inductively $\langle x_\alpha | \alpha < cf\mu \rangle$ where x_α will be taken as any element above the previous ones and if $\alpha = \beta + 1$ for some β then $|(x_\beta, x_\alpha)_I| > \mu_\beta$. Set $I_\alpha = (x_\alpha, x_{\alpha+1})_I$. Next choose counterexamples for (μ_α, λ)-entangledness $\langle t^\varepsilon_i | \varepsilon < \theta \;\; i \in [\sum_{\beta<\alpha} \mu_\beta, \mu_\alpha)\rangle, u = \langle 2\varepsilon | \varepsilon < \theta \rangle$. For any $\alpha < cf\mu$ choose different elements $\langle t^\varepsilon_i | \varepsilon \in \{\theta, \theta + 1\},\ i \in [\sum_{\beta<\alpha} \mu_\beta, \mu_\alpha)\rangle$ in $I_\alpha \setminus \langle t^\varepsilon_i | \varepsilon < \theta, i \in [\sum_{\beta<\alpha} \mu_\beta, \mu_\alpha)\rangle$ (this is possible as by the entangledness $\theta < 2^\theta \leq \mu$). Wlog all the t^ε_i above are with no repetitions. This contradicts the (μ, λ)-entangledness with respect to $u' = u \cup \{\theta\}$. For a successor cardinal the proof is simpler since we may disregard the counterexamples. $\square$

DEFINITION 2.4: For a linear order I $c.c.(I)$ is the first cardinality in which there is no family of disjoint nonempty open intervals. We define $h.c.c.(I) = \min\{c.c.(J) | J \in [I]^{|I|}\}$.

LEMMA 2.5. *If $\lambda = \theta^+$ and I is (μ, λ)-entangled linear order with minimal μ then for any $\{\sigma_i | i < \theta\} \subset hc.c.(I)$ we have $\prod_{i<\theta} \sigma_i < cf\mu$.*

PROOF: Assume not. After throwing away less than μ points of I we ensure that $\forall x \in I(f(x) = \lambda)$ (by lemma 2.3). Suppose the theorem fails for $\{\sigma_i | i < \theta\}$. Choose for every

$i < \theta$ a collection of disjoint intervals $\{I^i_\alpha | \alpha < \sigma_i\}$ and distinct functions in $\prod\limits_{i<\theta} \sigma_i$, $\langle f_\alpha | \alpha < cf\mu \rangle$. Fix counterexamples t^ε_i, u and cardinals $\langle \mu_\alpha | \alpha < cf\mu \rangle$ as above. For any $\varepsilon < \theta$ choose $\langle t^{\theta+2\varepsilon}_i, t^{\theta+2\varepsilon+1}_i | i \in [\sum\limits_{\beta<\alpha} \mu_\beta, \mu_\alpha)\rangle$ different elements from $I^\varepsilon_{f_\alpha(\varepsilon)} \backslash \langle t^\varepsilon_i | \varepsilon < \theta, i \in [\sum\limits_{\beta<\alpha} \mu_\beta, \mu_\alpha)\rangle$ (remember that $|I^i_\alpha| = \mu$). Wlog all the t^ε_i are with no repetitions and so $\langle t^\varepsilon_i | \varepsilon < 2\theta,\ \ i < \mu\rangle$ contradicts the (μ, λ) entangledness with respect to $u' = u \cup \langle \theta + 2\varepsilon | \varepsilon < \theta\rangle$. $\square$

Lemma 2.6. *If $\lambda = \theta^+$ and I is (μ, λ) entangled linear order with minimal μ then $\kappa = hc.c.(I)$ satisfies $\kappa^\theta \leq cf\mu$.*

Proof: Choose $\langle \sigma_i | i < cf\kappa\rangle$ unbounded in κ. If $cf\kappa \leq \theta$ then $\kappa^\theta = \prod\limits_{i<cf\kappa}\sigma_i < cf\mu$ by lemma 2.5. Otherwise by the same lemma $\forall \sigma < \kappa(\sigma^\theta < cf\mu)$ so that $\kappa^\theta = \kappa \cdot \sum\limits_{i<cf\kappa} \sigma^\theta_i \leq cf\mu$ (remember that $\kappa \leq cf\mu$). $\square$

The next corollary strengthens [Sh 462, 6.17(a)] where I was assumed to be (μ, λ) strongly entangled and we got only $\forall \theta < \lambda(2^\theta < \mu)$.

Corollary 2.7. *If I is (μ, λ) entangled linear order with density χ then $\forall \theta < \lambda(\chi^\theta < \mu)$.*

Proof: Fix $\theta < \lambda$. Wlog $\lambda = \theta^+$ and μ is minimal for which I is (μ, λ) entangled. Let $\kappa = h.c.c.(I)$. We know that $\kappa \in \{\chi, \chi^+\}$. By lemma 2.6 we have to consider only the case $\kappa^\theta = \mu$. By the proof above it follows that $cf\kappa > \theta$ and $\mu = \kappa \cdot \sum\limits_{i<cf\kappa} \sigma^\theta_i$ (we keep the same notation) so that $\kappa = \mu = cf\mu$. $\chi < \mu = \kappa$ holds by 2.1 and we can use lemma 2.5 to get the desired conclusion.

Remark 2.8 The case $\chi < \kappa = \mu (= \chi^+$ follows) occurs for example in the construction from [BoSh 210] if we assume CH (here $\chi = \aleph_0$ and $\mu = \aleph_1$).

Conclusion 2.9: If I is (μ, λ) entangled, μ is regular, $\theta < \lambda$ and $\langle t^i_\varepsilon | i < \mu, \varepsilon < \theta\rangle$ is a matrix of different elements from I then for $A \in [\mu]^\mu$ and a sequence of mutually disjoint intervals $\langle I_\varepsilon | \varepsilon < \theta\rangle$ $\forall i \in A\, \forall \varepsilon < \theta(t^i_\varepsilon \in I_\varepsilon)$.

Proof: This is immediate from corollary 2.7 and [Sh 462, 1.2(3)]. $\square$

Theorem 2.10. *(a) If I is a (μ, λ) entangled order with minimal μ and $\lambda = \theta^+$ then $\lambda < h.c.c.(I) \leq cf\mu$. (b) If I is a (μ, λ) entangled order with minimal μ and $cf\mu \neq cf\lambda < \lambda$ then its density χ satisfies $\lambda < \chi$.*

PROOF: (a) Assume the contrary. By 2.6 $h.c.c.(I) \leq cf\mu$ so here $\lambda \geq h.c.c.(I)$. For any $x \in I$ choose a strictly increasing sequence converging to it with minimal (hence a regular cardinal) length $\langle a^x_\alpha | \alpha < r(x) \rangle$. By the assumption $\forall x(r(x) < \lambda)$. As $|rang\, r| \leq |\theta \cup \{\theta\}| = \theta < \lambda \leq cf\mu$ (by lemma 3.1) for some $a = \langle x_i | i < \mu \rangle \subset I$ and some $\sigma < \lambda$ $\forall x \in A(r(x) = \sigma)$. Wlog $\forall x \in I(f(x) = \mu)$. Define $\langle t^\varepsilon_i | i < \mu, \varepsilon < \sigma \rangle$ by induction on i: for any $\varepsilon < \sigma$ choose $t^\varepsilon_i \in (a^{x_i}_\varepsilon, a^{x_i}_{\varepsilon+1})$ different from previously chosen t's. This contradicts the (μ, λ) entangledness with respect to $u = \langle 2\varepsilon | \varepsilon < \sigma \rangle$. (b) Assume not. Since $cf\mu \neq cf\lambda$ for some $\theta^+ < \lambda$ I is (μ, θ^+) entangled with minimal μ hence we get the conclusion of Theorem 2.3. Now in the proof of (a) r is into λ since $h.c.c.(I) \leq \chi^+$ and we can ensure only its boundedness on a large $A \subset I$. Now take t^ε_i to be in $(a^{x_i}_{\varepsilon_1}, a^{x_i}_{\varepsilon_1+1})$ where ε_1 is ε modulo $r(x_i)$. $\square$

REMARK 2.11 Note that the conclusion of 2.10(a) $(\lambda < cf\mu)$ is tight in view of theorem 1.2. For inaccessible λ we have a forcing example of a (μ, λ) entangled order with minimal μ and $cf\mu = \lambda$.

Next we give a weakening of conclusion 2.9 which is valid also for singular μ.

THEOREM 2.12. *If I is (μ, λ)-entangled then for any $\theta < \lambda$ and for any matrix $\langle t^\varepsilon_i | \varepsilon < \theta,\ i < \mu \rangle$ of distinct elements there is a sequence of disjoint intervals $\langle I_\varepsilon | \varepsilon < \theta \rangle$ such that all the α's and β's in the definition of entangledness can be chosen to satisfy $\forall \varepsilon < \theta(t^\varepsilon_\alpha, t^\varepsilon_\beta \in I_\varepsilon)$.*

PROOF: Suppose the theorem fails for I with density χ. By conclusion 2.9 μ is singular. Let $\langle \mu_i | i < cf\mu \rangle$ be a strictly increasing sequence of successor cardinals and $\langle t^\varepsilon_i | \varepsilon < \theta, i < \mu \rangle$ a counterexample to the theorem. As $\chi^\theta < \mu$ wlog $\mu_0 > \chi^\theta$ and by induction on i we can choose $\langle I^i_\varepsilon | \varepsilon < \theta, i < cf\mu \rangle$ such that $\langle I^i_\varepsilon | \varepsilon < \theta \rangle$ are disjoint for all $i < cf\mu$, for $i < cf\mu$ $|\{v \in [\sum_{\alpha<i} \mu_\alpha, \mu_i) | \forall \varepsilon < \theta(t^\varepsilon_v \in I^i_\varepsilon)\}| = \mu_i$ — wlog this set is $[\sum_{a<i} \mu_\alpha, \mu_i)$ — and $\forall i < cf\mu \exists j(i) < cf\mu \forall j > j(i) \exists \varepsilon < \theta \, |I^i_\varepsilon \cap \{t^\varepsilon_v | v \in [\sum_{\alpha<j} \mu_\alpha, \mu_j)\}| < \mu_j$ hence wlog $\forall i, j < cf\mu \left(i \neq j \rightarrow \exists \varepsilon < \theta(I^i_\varepsilon \cap I^j_\varepsilon = \emptyset)\right)$. As $\langle t^\varepsilon_i | i < \mu, \varepsilon < \theta \rangle$ is a counterexample, for any $i < cf\mu$ there is $u_i \subset \theta$ such that $\forall \alpha, \beta \in [\sum_{\alpha<i} \mu_\alpha,\ \mu_i) \exists \varepsilon < \theta\ \ t^\varepsilon_\alpha < t^\varepsilon_\beta \leftrightarrow \varepsilon \notin u_i$. By a previous lemma $2^\theta < cf\mu$ so wlog the u_i's are the same u. Now $\langle s^\varepsilon_i | i < \mu, \varepsilon < 3\theta \rangle$ defined by $s^\varepsilon_i = t^\varepsilon_{3i}, s^{\theta+\varepsilon}_i = t^\varepsilon_{3i+1}, s^{2\theta+\varepsilon}_i = t^\varepsilon_{3i+2} (i < \mu, \varepsilon < \theta)$ contradicts the (μ, λ)-entangledness with respect to $u' = u \cup [\theta, 2\theta)$. $\Box$

THEOREM 2.13. *If I is (μ, θ^+)-entangled with minimal μ then there are two θ^+-closed $\mu - c.c.$ posets whose product is not $\mu - c.c.$*

PROOF: Let $\langle x_\alpha | \alpha < \mu \rangle$ be distinct elements of I. Denote by $\prec$ the partial order on $E = \{(x_{2\alpha}, x_{2\alpha+1}) | \alpha < \mu\}$ which is the product of $<_I$ with itself. Let $A = \{a \in [E]^{\leq\theta} | a$ is $\prec -$chain$\}$ and $B = \{a \in [E]^{\leq\theta} | \neg\exists x, y \in a(x \prec y)\}$. A and B are θ^+-closed when ordered by inclusion and $A \times B$ is not $\mu - c.c.$ since $\{((x_{2\alpha}, x_{2\alpha+1}), (x_{2\alpha}, x_{2\alpha+1})) | \alpha < \mu\}$ is an antichain in it. If $\langle a_\alpha | \alpha < \mu \rangle \subseteq [E]^{\leq\theta}$ then look at any matrix $\langle t^i_\varepsilon | i < \mu, \varepsilon < \theta \rangle$ satisfying $\forall \alpha < \mu \{(t^\alpha_{2\varepsilon}, t^\alpha_{2\varepsilon+1}) | \varepsilon < \theta\} \supset a_\alpha$ and apply theorem 2.11 with respect to $u = \phi$ to see that it is not an A-antichain and with respect to $u = \{2\beta | \beta < \theta\}$ to see that it is not a B-antichain. This proves the theorem. $\Box$

By previous theorems the existence of a (λ^+, λ) entangled order implies that $\lambda^{<\lambda} = \lambda$. Below we give sufficient conditions.

THEOREM 2.14. *If $\lambda^{<\lambda} = \lambda > \beth_w$ and $2^\lambda = \lambda^+$ then there is a (λ^+, λ) entangled order, (also strongly as $\lambda = \lambda^{>\lambda}$).*

PROOF: Fix an enumeration of all the triples $(\gamma, \bar\eta, \varepsilon)$ where $\varepsilon, \gamma < \lambda$ and $\bar\eta = \langle \eta^\alpha | \alpha < \gamma \rangle \subset {}^\varepsilon\lambda$ is a sequence of different functions, $\langle (\gamma_\alpha, \bar\eta_\alpha, \varepsilon_\alpha) | \alpha < \lambda \rangle$ (remember that $\lambda^{<\lambda} = \lambda$). By [Sh 460 3.5] $\lambda^{<\lambda} = \lambda > \beth_w$ implies that there are λ disjoint stationary subsets of λ $\langle S_\alpha | \alpha < \lambda \rangle$ s.t. for each $\alpha < \lambda$ $D\ell(S_\alpha)$ holds. We remind the reader that $D\ell(S_\alpha)$ is a weakening of diamond and here we use the following form of it: there is a sequence $\langle P_\beta | \beta \in S_\alpha \rangle$ s.t. P_β is a family of less than λ sequences of length γ_α of functions from ${}^\alpha\lambda$ and given any sequence of length γ_α of functions from ${}^\lambda\lambda$, $\langle f_i | i < \gamma_\alpha \rangle$, for stationary many $\beta \in S_\alpha \langle f_i \restriction \beta | i < \gamma_\alpha \rangle \in P_\beta$. Since $2^\lambda = \lambda^+$ there is a cofinal and increasing sequence of functions $\langle f_\alpha | \alpha < \lambda^+ \rangle$ in $({}^\lambda\lambda, <^*)$ where $<^*$ means eventual dominance. Now set $A = \{f \in {}^\alpha\lambda | \exists \beta, \delta < \lambda (2\delta < \gamma_\beta \wedge \alpha \in S_\beta \wedge \eta^{2\delta}_\beta \triangleleft f)\}$ and define $I = \langle f_\alpha | \alpha < \lambda^+ \rangle$ and $f <_I g$ iff $f \restriction \alpha \in A \leftrightarrow f(\alpha) < g(\alpha)$ where $\alpha = min\{\beta < \lambda | f(\beta) \neq g(\beta)\}$. To prove that I is as required let $\gamma < \lambda, u \subset \gamma$ and $\langle f_{\alpha^\beta_\nu} | \beta < \lambda^+, \nu < \gamma \rangle$ be as in definition 1.1. To simplify the notation we write f^β_ν for $f_{\alpha^\beta_\nu}$. Wlog $\langle \alpha^\beta_\nu | \beta < \lambda^+ \rangle$ is increasing for all $\nu < \gamma$, (for this replace the set of indices by an inductively chosen sequence of length λ^+) γ is an infinite cardinal and $u = \langle 2\alpha | \alpha < \gamma \rangle$. For every $\beta < \lambda^+$ there is $\varepsilon(\beta) < \lambda$ s.t. $\langle f^\beta_\nu \restriction \varepsilon(\beta) | \nu < \gamma \rangle$ are distinct so that on $B \in [\lambda^+]^{\lambda^+}$ all $\varepsilon(\beta)$ are equal to some ε^* and all $\langle f^\beta_\nu \restriction \varepsilon(\beta) | \nu < \gamma \rangle$ are the same, to be denoted by $\bar\eta^*$. Let β be s.t. $(\gamma, \bar\eta^*, \varepsilon^*) = (\gamma_\beta, \bar\eta_\beta, \varepsilon_\beta)$. If some $\varepsilon_0 < \lambda$ and $\bar\eta = \langle \bar\eta_\alpha | \alpha < \gamma \rangle \subset {}^{\varepsilon_0}\lambda$ satisfy that for all $i < \lambda^+$ and $\delta < \lambda$ there is $\zeta \in B$ s.t. $\bar\eta = \langle f^\zeta_\nu \restriction \varepsilon_0 | r < \gamma \rangle$ and $\min\{f^\zeta_\nu(\varepsilon_0) | r < \gamma\} > \delta$ then we are clearly done (take such ζ with respect to $(0,0)$ then such ζ' with respect to $(\zeta, sup\{f^\zeta_\nu(\varepsilon, 0) | v < \gamma\})$). Otherwise for every $\bar\eta$ as above there are witnesses for its failure, $i(\bar\eta)$ and $\delta(\bar\eta)$. Since $\lambda^{<\lambda} = \lambda$ the supremum of $i(\bar\eta)$ over all relevant $\bar\eta$ is less than λ', denote it by i^*. Define $\delta \colon S_\beta \to \lambda$ by $\delta(\alpha) = sup\{\delta(\bar\eta) | \bar\eta \in P_\alpha\} < \lambda$ and using the cofinality of the f_α's find $\zeta \in B \backslash i^*$ for which $\delta <^* f_\zeta \restriction S_\beta$. Now using $D\ell(S_\beta)$ there is $\alpha \in S_\beta$ s.t. $\langle f^\zeta_v \restriction \alpha | v < \gamma \rangle \in P_\alpha$, moreover we can get $\alpha > sup\ min\{\varepsilon \in S_\beta | \delta(\varepsilon) > f^\zeta_v(\varepsilon)\}$ so $min f^\zeta_v(\alpha) > \delta(\alpha) \geq \delta(\langle f^\zeta_v \restriction \alpha \rangle)$, a contradiction. $\square$

REMARK 2.15: Notice that for λ as in the theorem the construction in [BoSh 210] gives only a $(\lambda^+, \aleph_0)$ entangled order. However, their proof gives also a $(\aleph_1, \aleph_0)$ entangled order and that is done assuming only cf $2^{\aleph_0} = \aleph_1$. Remember that under $MA + 2^{\aleph_0} > \aleph_1$, there is no such an order at all.

§3 Results on Positively entangled orders.

THEOREM 3.1. *If I is a (μ, λ) [positively*] [positively] entangled linear order with minimal μ then $cf\mu \geq cf\lambda$.*

PROOF: Suppose not. We deal with positive entangledness (the other cases are similar). Fix $\langle \mu_\alpha | \alpha < cf\mu \rangle$ increasing to μ and $\langle \lambda_\alpha | \alpha < cf\mu \rangle$ s.t. for every $\alpha < cf\mu$ $\lambda_\alpha < \lambda$ and I is not $(\mu_\alpha, \lambda^+_\alpha)$ positively entangled and counterexamples $\langle t^\varepsilon_i | i < \mu_\alpha,\ \varepsilon \in [\sum\limits_{j<i} \lambda_j, \sum\limits_{j\leq i} \lambda_j) \rangle$, wlog all with respect to $u = \emptyset$ (here $\sum$ stands for ordinal summation). In each row ε choose fillers $\langle t^\varepsilon_i | \mu_\alpha \leq i < \mu \rangle$ different from $\langle t^\varepsilon_i | i < \mu_\alpha \rangle$. As $\sum\limits_{i<cf\mu} \lambda_i < \lambda$ this contradicts the (μ, λ)-positively entangledness with respect to $u = \phi$. $\square$

LEMMA 3.2. *If a (μ, λ) positively* entangled linear order I has a μ-separative point then $\forall \theta < \lambda (2^\theta < \mu)$.*

PROOF: Let x be such a point and suppose by contradiction $\theta < \lambda,\ 2^\theta \geq \mu$. Fix distinct functions $\langle f_\alpha | \alpha < \mu \rangle \subset^\theta 2$. Define $\langle t_i^\varepsilon | \varepsilon < \theta,\ i < \mu \rangle$ inductively on i : choose any $x_0 < x < x_1$ different from previously chosen t's and put $t_i^{2\varepsilon+\ell} = x_\ell$ for $\ell \in \{0,1\}$ if $f_i(\varepsilon) = 0$ and $t_i^{2\varepsilon+\ell} = x_{1-\ell}$ else. This contradicts the (μ,λ)-positively* entangledness. $\square$

Corollary 3.3. *(a) If I is (μ,λ) positively* entangled then $\chi = den\, I \geq \lambda$. (b) If I is (μ,λ) positively* entangled then it is not $(\lambda, 2)$ entangled.*

Proof: Assume I is a counterexample for (a). Wlog μ is the minimal cardinal s.t. I is (μ, χ^+) positively* entangled. If there is no μ-separating point in I we can define inductively a monotone sequence in I of length $cf\mu$ which is greater than χ by theorem 3.1, a contradiction. If there is a μ-separating point then by lemma 2.3 $2^\chi < \mu$, a contradiction. (b) follows from (a) and lemma 2.1 [Sh462 1.2(4)]. $\square$

Theorem 3.4. *If I is (μ,λ) positively entangled then $\forall\theta < \lambda(2^\theta < \mu)$.*

Proof: Suppose this fails for some θ. Wlog $\lambda = \theta^+$. In view of lemma 3.2 we can assume that I has no μ-separating point. It follows that $cf\mu < \mu$. For any $\mu_1 < \mu$ there is a μ_1-separating point, otherwise wlog $\forall x \in I\ |\{y \in I | y < x\}| < \mu_1$, so we can define an increasing sequence of length $\mu_1 + 1$ and pick the last element of it. By lemma 3.2 I is not (μ_1,λ) positively* entangled for any $\mu_1 < \mu$. But now if $\langle \mu_\alpha | \alpha < cf\mu \rangle$ are increasing to μ, $\langle \langle t_i^\varepsilon | \varepsilon < \theta,\quad i \in [\mu_\alpha, \mu_{\alpha+1}) \rangle | \alpha < cf\mu \rangle$ are counterexamples for (μ_α, λ) positively* entangledness and $\langle I_\alpha | \alpha < cf\mu \rangle$ is an inductively chosen monotone sequence of intervals s.t. $|I_\alpha| \geq \mu_\alpha$ (here we use the nonexistence of a μ-separating point) then pick for every $\alpha < cf\mu$ different $\langle t_i^\theta | i \in [\mu_\alpha, \mu_{\alpha+1}) \rangle$ from I_α to contradict the (μ,λ) positively entangledness with $\langle t_i^\varepsilon | \varepsilon \leq \theta,\ i < \mu \rangle$. $\square$

Theorem 3.5. *If I is (μ,λ) positively entangled with minimal μ which has a μ-separative point and $\lambda = \theta^+$ then $2^\theta < cf\mu$. In particular $\lambda \leq cf\mu$.*

Proof: Let $x \in I$ be μ-separating and assume that $2^\theta \geq cf\mu$. Fix distinct $\langle f_\alpha | \alpha < cf\mu \rangle \subset^\theta 2$ and choose $\langle t_i^\varepsilon | \varepsilon < \theta\quad i \in [\mu_\alpha, \mu_{\alpha+1}) \rangle$ counterexamples for (μ_α, λ) positively entangledness, wlog all with respect to $u = \phi$. For every $\varepsilon < \theta$ choose by induction on α $x_0 < x < x_1$ different from previously chosen elements and put $t_\alpha^{\theta+\varepsilon} = x_\ell$ for $\ell \in \{0,1\}$ if $f_\beta(\varepsilon) = 0$ and $t_\alpha^{\theta+\varepsilon} = x_{1-\ell}$ else (here β is s.t. $\alpha \in [\mu_\beta, \mu_{\beta+1})$). $\langle t_\alpha^\varepsilon | \alpha < \mu\ \varepsilon < \theta + \theta \rangle$ contradicts the (μ,λ) positively entangledness. $\square$

Notice that below one cannot wave off the assumption $cf\mu \neq cf\lambda$(see remark 2.11).

COROLLARY 3.6. *If I is (μ, λ) positively entangled with minimal μ which has a μ-separative point and $cf\mu \neq cf\lambda$ then $\forall\theta < \lambda(2^\theta < cf\mu)$ and $\lambda < cf\mu$.*

PROOF: As $cf\mu \neq cf\lambda$ there is $\theta_1 < \lambda$ such that I is (μ, θ^+) entangled with minimal μ for every $\theta_1 \leq \theta < \lambda$ so we can use theorem 3.5. Note that the possibility $\lambda = cf\mu$ is excluded by the assumption.

DEFINITION 3.7 A linear order I is called hereditarily separative if every $A \in [I]^{|I|}$ has a $|I|$-separative point. The assumption below (λ is singular strong limit $\Rightarrow pp\lambda =^+ 2^\lambda$) is not known to be independent of ZFC. see [Sh-g].

THEOREM 3.8. *If I is hereditarily separative (μ, λ)-positively entangled with minimal μ, $cf\mu \neq cf\lambda$ and (λ is singular strong limit $\Rightarrow pp\lambda =^+ 2^\lambda$) then $\lambda^{<\lambda} < cf\mu$.*

PROOF: If λ is not strong limit them for some $\theta_1 < \lambda$ $\lambda \leq 2^{\theta_1}$ and by theorem 3.5 $\forall\theta < \lambda(\lambda^\theta \leq 2^{\theta+\theta_1} < cf\mu)$. If λ is inaccessible $\lambda^{<\lambda} = \lambda$ so we can apply corollary 3.6. We are left with the case λ is strong limit singular, $pp\lambda =^+ 2^\lambda$. Fix $\theta < \lambda$. By the trivial direction of [Sh410, 3.7] there are functions $\langle f\alpha | \alpha < \lambda^\theta\rangle \subset^\theta\lambda$ s.t. $\forall\alpha < \beta < \lambda^\theta \exists\varepsilon < \theta(f_\alpha < f_B(\varepsilon))$. Assume that $\lambda^\theta \geq cf\mu.|$

If A is an equivalence class of the equivalence relation $xEy \leftrightarrow |\langle x, y\rangle_I| < \mu$ and is of cardinality μ then pick any $x \in A$. Wlog $|\{y \in A\colon y > x\}| = \mu$. Since I is hereditarily separative $\{y \in A | y > x\}$ has μ-separative point, call it z. In particular $|(x, z)_I| = \mu$ so $x\not{E}z$, a contradiction. We conclude that any equivalence class of E is of size less than μ which implies that there are at least $cf\mu$ many such classes. By corollary 3.6 $\lambda < cf\mu$ and as λ is strong limit $(2^\theta)^+ < \lambda$. Choosing any $(2^\theta)^+$ distinct equivalence classes of E they inherit the order I since they are convex subsets of it so by the Erdös-Rado theorem θ many from them form a monotone sequence, call it $\langle J_\alpha | \alpha < \theta\rangle$. Replacing it by $\langle J'_\alpha | \alpha < \theta\rangle$ where $J'_\alpha =$ convex $(J_{2\alpha} \cup J_{2\alpha+1})$ we ensure also $\forall\alpha(|J'_\alpha| = \mu)$, (this is as J'_α contains an interval between two nonequivalent points). Of course, this can be done for any $\tau < \lambda$ instead of θ. Starting from any such, wlog, increasing sequence $\langle J_\alpha | \alpha < cf\lambda\rangle$ (remember that $cf\lambda < \lambda$) we fix a strictly monotone sequence of cardinals converging to λ, $\langle\lambda_\alpha | \alpha < cf\lambda\rangle$. Any J_α

is also hereditarily separative so it contains by the same argument monotone sequence of length λ_α of intervals of power μ $\langle J^\beta_\alpha|\beta<\lambda_\alpha\rangle$. If in one J_α there is no increasing sequence of length λ_α then starting from decreasing intervals $\langle J_{\alpha'}|\alpha<cf\mu\rangle$ inside this J_α we can take all the sequences decreasing. Otherwise we take them all increasing. Concatenating them yields a monotone sequence of intervals $\langle I_\alpha|\alpha<\lambda\rangle, \forall\alpha(|I_\alpha|=\mu)$. Now choose $\langle\mu_\alpha|\alpha<cf\mu\rangle$ $\langle t^\alpha_\varepsilon|\alpha<\mu,\varepsilon<\theta\rangle$ as in the proof of theorem 3.5. For all $\varepsilon<\theta$ choose by induction on α $t^\alpha_{\theta+\varepsilon}\in I_{f_\beta(\varepsilon)}\backslash\{t^\gamma_{\theta+\varepsilon}|\gamma<\alpha\}$ where $\alpha\in[\mu_\beta,\mu_{\beta+1})$. This is always possible because $\forall\alpha(|I_\alpha|=\mu)$. Now check that $\langle t^\alpha_\varepsilon|\alpha<\mu,\varepsilon<\theta+\theta\rangle$ contradicts the (μ,λ)-positively entangledness. We conclude that $\forall\theta<\lambda(\lambda^\theta<cf\mu)$. As $\lambda<cf\mu$ this gives the desired inequality. $\square$

Compare the following with theorem 2.10(b).

COROLLARY 3.9. *If I is (μ,λ) positively entangled hereditarily separative linear order with minimal μ and with density χ, $cf\mu\neq cf\lambda<\lambda$ and (λ is strong limit singular $\rightarrow pp\lambda=^+2^\lambda$) then $\chi>\lambda$.*

PROOF: Assume that I is a counterexample and deduce by corollary 3.3 (a) that $\chi=\lambda$. Fix $A\in[I]^\lambda$ dense in I. For every $x\in I$ find a well ordered sequence of elements from A converging to x of minimal length $\langle a^x_\alpha|\alpha<r(x)\rangle$. By minimality $r(x)$ is always a regular cardinal hence smaller than λ. By theorem 3.8 $\lambda^{<\lambda}<cf\mu$ so there are two distinct points in I with the same sequences, a clear contradiction. $\square$

Below we deal with a typical example of orders I that (usually) have no $|I|$-separative points.

DEFINITION 3.10: If μ is a singular cardinal then a linear order I is called "of type s_μ" if it contains for some (equivalently any) sequence of cardinals converging to μ $\langle\mu_\alpha|\alpha<cf\mu\rangle$ an isomorphic copy of $\bigcup\limits_{\alpha<cf\mu}\{\mu_\alpha\}\times\mu_\alpha$ ordered by $(\alpha,\beta)<(\alpha_1,\beta_1)$ iff $\alpha<\alpha_1$ or $\alpha=\alpha_1$ and $\beta>\beta_1$. We say that "s_μ is (μ,λ) positively entangled" if some (equivalently any) order of type s_μ has this property.

THEOREM 3.11. *s_μ is (μ,θ^+)-positively entangled iff $\theta<cf\mu$ and $(cf\mu)^\theta<\mu$.*

PROOF: Throughout the proof fix a sequence of successor cardinals $\langle\mu_\alpha|\alpha<cf\mu\rangle$ strictly increasing to μ. First assume $(cf\mu)^\theta<\mu$ and $\theta<cf\mu$. Given any $\langle t^\alpha_\varepsilon|\alpha>\mu,\varepsilon<\theta\rangle$

as in definition 1.1(c) then, as $(cf\mu)^\theta < \mu$, there is $A < \mu$ of cardinality $(2^\theta)^+$ for which if $\alpha, \beta \in A$ and $\varepsilon < \theta$ then t^α_ε and t^β_ε have the same first coordinate. Now we can find $\alpha, \beta \in A$ satisfying $\forall \varepsilon < \theta (t^\alpha_\varepsilon > t^\beta_\varepsilon)$ and $\alpha < \beta$. Otherwise color $[A]^2$ with $f(\{\alpha, \beta\}) = \min\{\varepsilon < \theta | t^\alpha_\varepsilon < t^\beta_\varepsilon\}$ (here $\alpha < \beta$) and using Erdös-Rado get a homogeneous set of size θ giving rise to a decreasing sequence of ordinals of this length, a contradiction. To get the other condition observe that $\bigcup_{\varepsilon < \theta}\{\alpha < \mu | t^\theta_\varepsilon > t^\alpha_\varepsilon\}$ is of cardinality less than μ as it is a union of size less than $cf\mu$ of initial segments of s_μ, which is of order type μ. For any α in its complements we have $\forall \varepsilon < \theta (t^0_\varepsilon < t^\alpha_\varepsilon)$. We conclude that s_μ is (μ, θ^+)-positively entangled.

Suppose $(cf\mu)^\theta \geq \mu$, hence there are distinct $\langle f_\alpha | \alpha < \mu \rangle \subset {}^\theta(cf\mu)$. Wlog $\forall \alpha \geq \mu_\alpha (\min f_\alpha > \alpha)$. For $\varepsilon < \theta$ $\beta = \mu_\alpha + \gamma < \mu_{\alpha+1}$ define $t^\beta_\varepsilon = (f_\beta(\varepsilon), \gamma) \in s_\mu$. Now fix any $\alpha < cf\mu$ and choose a partition of $\mu_{\alpha+2}$ to $\mu_{\alpha+1}$ unbounded sets $\langle A_\delta | \delta < \mu_{\alpha+1}\rangle$. For any $\varepsilon < \theta$ look at the relation on $\mu_{\alpha+1} \backslash \mu_\alpha$ defined by $\beta <_\varepsilon \gamma \leftrightarrow f_\beta(\varepsilon) < f_\gamma(\varepsilon)$. $\prec_\varepsilon$ is a partial order with no infinite decreasing sequences so we can define a rank function g_ε into $\mu_{\alpha+2}$ satisfying $\beta \prec_\varepsilon \gamma \to g_\varepsilon(\beta) < g_\varepsilon(\gamma)$ by $\prec_\varepsilon$-recursion: $g_\varepsilon(\beta) = minA_\beta \backslash \sup\{g_\varepsilon(\gamma) | \gamma <_\varepsilon \beta\}$. For $\beta \in \mu_{\alpha+1} \backslash \mu_\alpha$ set $t^\beta_{\theta+\varepsilon} = \big(\alpha + 2, g_\varepsilon(\beta)\big)$. By the construction the t's are different in each μ-row. If $\beta < \gamma < \mu$ then either $\exists \alpha < cf\mu (\mu_\alpha \leq \beta < \gamma < \mu_{\alpha+1})$ in this case since the f_α's are distinct there is $\varepsilon < \theta$ for which $f_\beta(\varepsilon) \neq f_\gamma(\varepsilon)$; or $f_\beta(\varepsilon) < f_\gamma(\varepsilon)$ so $t^\beta_\varepsilon < t^\gamma_\varepsilon$ or $f_\beta(\varepsilon) > f_\gamma(\varepsilon)$ which implies $\beta >_\varepsilon \gamma, g_\varepsilon(\beta) > g_\varepsilon(\gamma)$ and $t^\beta_{\theta+\varepsilon} < t^\gamma_{\theta+\varepsilon}$. We summarize that $\forall \beta < \gamma < \mu \exists \varepsilon < \theta + \theta (t^\beta_\varepsilon < t^\gamma_\varepsilon)$ which means that s_μ is not (μ, θ^+)-positively entangled.

Finally we show that s_μ cannot be $(\mu, (cf\mu)^+)$-positively entangled. For this partition $cf\mu$ into $cf\mu$ mutually disjoint stationary sets $\langle A_\alpha | \alpha < cf\mu\rangle$ and enumerate their elements $A_\alpha = \langle a^\alpha_i | i < cf\mu\rangle$. Wlog $\forall \alpha (a^\alpha_0 > \alpha)$. For any $\varepsilon < f\mu$ $\beta = \mu_\alpha + \gamma < \mu_{\alpha+1}$ set $t^\beta_\varepsilon = (a^\alpha_\varepsilon, \gamma) \in s_\mu$. These t's are different in each μ-row. Now if for some $\beta < \gamma < \mu$ $\forall \varepsilon < cf\mu$ $(t^\beta_\varepsilon < t^\gamma_\varepsilon)$ holds then necessarily there are distinct $\alpha, \bar\alpha < cf\mu$ s.t. $\beta \in [\mu_\alpha, \mu_{\alpha+1}), \gamma \in [\mu_{\bar\alpha}, \mu_{\bar\alpha+1})$. The function $f = \{(a^{\bar\alpha}_\varepsilon, a^\alpha_\varepsilon) | \varepsilon < cf\mu\}$ is a one to one regressive function with domain $A_{\bar\alpha}$ which is stationary - a contradiction. $\square$

By the above theorem one can see that theorem 3.5 does not hold generally (for any θ take $\mu = (2^\theta)^{+\theta^+}$. Now s_μ is (μ, θ^+) positively entangled but $cf\mu = \theta^+ \leq 2^\theta$).

Theorem 3.12. *There is a c.c.c. forcing adding a* $(\aleph_\omega, \aleph_0)$ *positively entangled linear* *order of density $\aleph_0$ (in particular not of type $s_{\aleph_\omega}$) which has no $\aleph_\omega$-separative point.*

Proof: Fix any $n < \omega$ and define $\mathbb{P} = \{f$ is a function, $\mathrm{dom}\, f \in [n \times \aleph_\omega]^{<\omega}, ran f \subset 2^{<\omega}$, if $\aleph_m \leq \alpha < \beta < \aleph_{m+1}$ are in $\mathrm{dom}\, f$ then $\exists i < n\big((i,\alpha),(i,\beta) \in \mathrm{dom}\, f \wedge f(i,\alpha) <_{\ell x} f(i,\beta)\big)\}$. The order is $f \leq g$ iff $\mathrm{dom}\, f \supseteq \mathrm{dom}\, g$ and $\forall x \in \mathrm{dom}\, g\big(g(x) \triangleleft f(x)\big)$. If G is $\mathbb{P}$ generic we define $I = \bigcup_{m<\omega} m + \{x \in 2^\omega | \forall i < \omega \exists f \in G \exists y \in n \times [\aleph_m, \aleph_{m+1})\big(f(y) = x \restriction i\big)\}$ after identifying 2^ω with Cantor set. The rest is almost identical to the proof of theorem 1.1. $\square$

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