Can a Small Forcing Create Kurepa Trees[1]

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Abstract

In the paper we probe the possibilities of creating a Kurepa tree in a generic extension of a model of CH plus no Kurepa trees by an ω_1-preserving forcing notion of size at most ω_1. In the first section we show that in the Lévy model obtained by collapsing all cardinals between ω_1 and a strongly inaccessible cardinal by forcing with a countable support Lévy collapsing order many ω_1-preserving forcing notions of size at most ω_1 including all ω-proper forcing notions and some proper but not ω-proper forcing notions of size at most ω_1 do not create Kurepa trees. In the second section we construct a model of CH plus no Kurepa trees, in which there is an ω-distributive Aronszajn tree such that forcing with that Aronszajn tree does create a Kurepa tree in the generic extension. At the end of the paper we ask three questions.

0. INTRODUCTION

By a model we mean a model of *ZFC*. By a forcing notion we mean a separative partially ordered set $\mathbb{P}$ with a largest element $1_{\mathbb{P}}$ used for a corresponding forcing extension. Given a model V of CH, one can create a generic Kurepa tree by forcing with an ω_1-closed, ω_2-c.c. forcing notion no matter whether or not V contains Kurepa trees [Je1]. One can also create a generic Kurepa tree by forcing with a c.c.c. forcing notion provided V satisfies $\square_{\omega_1}$ in addition [V]. Both forcing notions mentioned here have size at least ω_2. The size being at least ω_2 seems necessary for guaranteeing the generic trees have at least ω_2 branches. On the other hand, a Kurepa tree has a base set of size ω_1, so it seems possible to create a Kurepa tree by a forcing notion of size $\leqslant \omega_1$. In this paper we discuss the following question: Given a model of CH plus no Kurepa tree, whether can we find an ω_1-preserving forcing notion of size $\leqslant \omega_1$ such that the forcing creates Kurepa trees?

This question is partially motivated by a parallel result about Souslin tree. Given a ground model V. A Souslin tree could be created by a c.c.c. forcing notion of size ω_1 [ST]. There is also an ω_1-closed forcing notion of size ω_1 which creates Souslin tree provided V satisfies CH [Je1]. The question whether a Souslin trees could be created

by a countable forcing notion (equivalent to adding a Cohen real) turns out to be much harder. It was answered positively by the second author [S1] ten years ago.

We call a forcing notion ω_1-preserving if ω_1 in the ground model is still a cardinal in the generic extension. In this paper we consider only ω_1-preserving forcing notion by the following reason. Let V be the Lévy model. In V there are no Kurepa trees and CH holds. Notice also that there is an ω_2-Kurepa tree in V. If we simply collapse ω_1 by forcing with the collapsing order $Coll(\omega, \omega_1)$, the set of all finite partial function from ω to ω_1 ordered by reverse inclusion, in V, then the ω_2-Kurepa tree becomes a Kurepa tree in $V^{Coll(\omega,\omega_1)}$. Notice also that $Coll(\omega, \omega_1)$ has size ω_1 in V. So we require the forcing notions under consideration be ω_1-preserving to avoid the triviality.

In the first section we show some evidence that in the Lévy model it is extremely hard to find a forcing notion, if it ever exists, of size $\leqslant \omega_1$ which could create a Kurepa tree in the generic extension. Assume our ground model V is the Lévy model. We show first an easy result that any forcing notion of size $\leqslant \omega_1$ which adds no reals could not create Kurepa trees. Then we prove two main results: (1) For any stationary set $S \subseteq \omega_1$, if $\mathbb{P}$ is an (S, ω)-proper forcing notion of size $\leqslant \omega_1$, then there are no Kurepa trees in the generic extension $V^{\mathbb{P}}$. Note that all axiom A forcing notions are (S, ω)-proper. (2) Some proper forcing notions including the forcing notion for adding a club subset of ω_1 by finite conditions do not create Kurepa trees in the generic extension.

In the second section we show that there is a model of CH plus no Kurepa trees, in which there is an ω-distributive Aronszajn tree T such that forcing with T does create a Kurepa tree in the generic extension. We start with a model V containing a strongly inaccessible cardinal κ. In V we define an ω_1-strategically closed, κ-c.c. forcing notion $\mathbb{P}$ such that forcing with $\mathbb{P}$ creates an ω-distributive Aronszajn tree T and a T-name $\dot{K}$ for a Kurepa tree K. Forcing with $\mathbb{P}$ collapses also all cardinals between ω_1 and κ so that κ is ω_2 in $V^{\mathbb{P}}$. Take $\bar{V} = V^{\mathbb{P}}$ as our ground model. Forcing with T in $\bar{V}$ creates a Kurepa tree in the generic extension of $\bar{V}$. So the model $\bar{V}$ is what we are looking for except that we have to prove that there are no Kurepa trees in $\bar{V}$, which is the hardest part of the second section.

We shall write V, $\bar{V}$, etc. for (countable) transitive models of ZFC. For a forcing notion $\mathbb{P}$ in V we shall write $V^{\mathbb{P}}$ for the generic extension of V by forcing with $\mathbb{P}$. Sometimes, we write also $V[G]$ instead of $V^{\mathbb{P}}$ for a generic extension when a particular generic filter G is involved. We shall fix a large enough regular cardinal λ throughout this paper and write $H(\lambda)$ for the collection of sets hereditarily of power less than λ

equipped with the membership relation. In a forcing argument with a forcing notion $\mathbb{P}$ we shall write $\dot{a}$ for a $\mathbb{P}$-name of a and $\ddot{a}$ for a $\mathbb{P}$-name of $\dot{a}$ which is again a $\mathbb{Q}$-name of a for some forcing notion $\mathbb{Q}$. If a is already in the ground model we shall write simply a for a canonical name of a. Let $\mathbb{P}$ be a forcing notion and $p \in \mathbb{P}$. We shall write $q \leqslant p$ to mean $q \in \mathbb{P}$ and q is a condition stronger than p. We shall often write $p \Vdash$ "..." for some $p \in \mathbb{P}$ instead of $p \Vdash_{\mathbb{P}}^{V}$ "..." when the ground model V and the forcing notion $\mathbb{P}$ in the argument is clear. We shall also write $\Vdash$ "..." instead of $1_{\mathbb{P}} \Vdash$ "...". In this paper all of our trees are subtrees of the tree $\langle 2^{<\omega_1}, \subseteq \rangle$. So if C is a linearly ordered subset of a tree T, then $\bigcup C$ is the only possible candidate of the least upper bound of C in T. In this paper all trees are growing upward. If a tree is used as a forcing notion we shall put the tree upside down. Let T be a tree and $x \in T$. We write $ht(x) = \alpha$ if $x \in T \cap 2^{\alpha}$. We write T_{α} or $(T)_{\alpha}$, the α-th level of T, for the set $T \cap 2^{\alpha}$ and write $T \restriction \alpha$ or $(T) \restriction \alpha$ for the set $\bigcup_{\beta < \alpha} T_{\beta}$. We write $ht(T)$ for the height of T, which is the smallest ordinal α such that T_{α} is empty. By a normal tree we mean a tree T such that (1) for any $\alpha < \beta < ht(T)$, for any $x \in T_{\alpha}$ there is an $y \in T_{\beta}$ such that $x < y$; (2) for any α such that $\alpha + 1 < ht(T)$ and for any $x \in T_{\alpha}$ there is $\beta < ht(T)$ and there are distinct $y_1, y_2 \in T_{\beta}$ such that $x < y_1$ and $x < y_2$. Given two trees T and T'. We write $T \leqslant_{end} T'$ for T' being an end-extension of T, i.e. $T' \restriction ht(T) = T$. By a branch of a tree T we mean a totally ordered set of T which intersects every non-empty level of T. By an ω_1-tree we mean a tree of height ω_1 with each of its levels at most countable. A Kurepa tree is an ω_1-tree with more than ω_1 branches. To see [J], [K] and [S2] for more information on forcing, iterated forcing, proper forcing, etc. and to see [T] for more information on trees.

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1. Creating Kurepa Trees By a Small Forcing Is Hard

First, we would like to state a theorem in [S2, 2.11] without proof as a lemma which will be used in this section.

4

Lemma 1. *In a model V let $\mathbb{P}$ be a forcing notion and let N be a countable elementary submodel of $H(\lambda)$. Suppose $G \subseteq \mathbb{P}$ is a V-generic filter. Then*

$$N[G] = \{\dot{a}_G : \dot{a} \text{ is a } \mathbb{P}\text{-name and } \dot{a} \in N\}$$

is a countable elementary submodel of $(H(\lambda))^{V[G]}$.

We choose the Lévy model $\bar{V} = V^{Lv(\kappa,\omega_1)}$ as our ground model throughout this section, where κ is a strongly inaccessible cardinal in V and $Lv(\kappa, \omega_1)$, the Levy collapsing order, is the set

$$\{p \subseteq (\kappa \times \omega_1) \times \kappa : \quad p \text{ is a countable function and}$$
$$(\forall (\alpha, \beta) \in dom(p))(p(\alpha, \beta) \in \alpha)\}$$

ordered by reverse inclusion. For any $A \subseteq \kappa$ we write $Lv(A, \omega)$ for the set of all $p \in Lv(\kappa, \omega_1)$ such that $dom(p) \subset A \times \omega_1$.

We now prove an easy result.

Theorem 2. *Let $\mathbb{P}$ be a forcing notion of size $\leqslant \omega_1$ in $\bar{V}$. If forcing with $\mathbb{P}$ does not add new countable sequences of ordinals, then there are no Kurepa trees in $\bar{V}^{\mathbb{P}}$.*

Proof: Since $\mathbb{P}$ has size $\leqslant \omega_1$, there is an $\eta < \kappa$ such that $\mathbb{P} \in V^{Lv(\eta,\omega_1)}$. Hence $\bar{V}^{\mathbb{P}} = V^{(Lv(\eta,\omega_1) * \dot{\mathbb{P}}) \times Lv(\kappa \smallsetminus \eta, \omega_1)}$. But $Lv(\kappa \smallsetminus \eta, \omega_1)$ in V is again a Levy collapsing order in $V^{Lv(\eta,\omega_1) * \dot{\mathbb{P}}}$ because $\mathbb{P}$ adds no new countable sequences of ordinals, so that the forcing notion $Lv(\kappa \smallsetminus \eta, \omega_1)$ is absolute between V and $V^{Lv(\eta,\omega_1) * \dot{\mathbb{P}}}$. Hence there is no Kurepa trees in $\bar{V}^{\mathbb{P}}$. $\square$

Next we prove the results about (S, ω)-proper forcing notions.

Definition 3. *A forcing notion $\mathbb{P}$ is said to satisfies property (†) if for any $x \in H(\lambda)$, there exists a sequence $\langle N_i : i \in \omega \rangle$ of elementary submodels of $H(\lambda)$ such that*

(1) $N_i \in N_{i+1}$ for every $i \in \omega$,

(2) $\{\mathbb{P}, x\} \subseteq N_0$,

(3) for every $p \in \mathbb{P} \cap N_0$ there exists a $q \leqslant p$ and q is $(\mathbb{P}, N_i)$-generic for every $i \in \omega$.

Lemma 4. *Let V be any model. Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcing notions in V such that $\mathbb{P}$ has size $\leqslant \omega_1$ and satisfies property (†), and $\mathbb{Q}$ is ω_1-closed (in V). Suppose T is an ω_1-tree in $V^{\mathbb{P}}$. Then T has no branches which are in $V^{\mathbb{P} \times \mathbb{Q}}$ but not in $V^{\mathbb{P}}$.*

Proof: Suppose, towards a contradiction, that there is a branch b of T in $V^{\mathbb{P}\times\mathbb{Q}}\smallsetminus V^{\mathbb{P}}$. Without loss of generality, we can assume that

$$\Vdash_{\mathbb{P}}\Vdash_{\mathbb{Q}} (\ddot{b} \text{ is a branch of } \dot{T} \text{ in } V^{\mathbb{P}\times\mathbb{Q}}\smallsetminus V^{\mathbb{P}}).$$

Claim 4.1   For any $p \in \mathbb{P}$, $q \in \mathbb{Q}$, $n \in \omega$ and $\alpha \in \omega_1$, there are $p' \leqslant p$, $q_j \leqslant q$ for $j < n$ and $\beta \in \omega_1 \smallsetminus \alpha$ such that

$$p' \Vdash ((\exists\{t_j : j < n\} \subseteq \dot{T}_\beta)((j \neq j' \to t_j \neq t_{j'}) \wedge \bigwedge_{j<n}(q_j \Vdash t_j \in \ddot{b}))).$$

Proof of Claim 4.1:   Since

$$p \Vdash_{\mathbb{P}} q \Vdash_{\mathbb{Q}} (\ddot{b} \text{ is a branch of } \dot{T} \text{ in } V^{\mathbb{P}\times\mathbb{Q}}\smallsetminus V^{\mathbb{P}}),$$

then p forces that q can't determine $\ddot{b}$. Hence

$$p \Vdash ((\exists\beta \in \omega_1 \smallsetminus \alpha)(\exists q_j \leqslant q \text{ for } j < n)(\exists t_j \in \dot{T}_\beta \text{ for } j < n)$$

$$((j \neq j' \to t_j \neq t_{j'}) \wedge \bigwedge_{j<n}(q_j \Vdash t_j \in \ddot{b}))).$$

Now the claim is true by a fact about forcing (see [K, pp.201]).

Claim 4.2   Let $\eta \in \omega_1$ and let $q \in \mathbb{Q}$. There exists a $\nu \leqslant \omega_1$, a maximal antichain $\langle p_\alpha : \alpha < \nu \rangle$ of $\mathbb{P}$, two decreasing sequences $\langle q_\alpha^j : \alpha < \nu \rangle$, $j = 0, 1$, in $\mathbb{Q}$ and an increasing sequence $\langle \eta_\alpha : \alpha < \nu \rangle$ in ω_1 such that $q_0^0, q_0^1 < q$, $\eta_0 > \eta$ and for any $\alpha < \nu$

$$p_\alpha \Vdash ((\exists t_0, t_1 \in \dot{T}_{\eta_\alpha})(t_0 \neq t_1 \wedge (q_\alpha^0 \Vdash t_0 \in \ddot{b}) \wedge (q_\alpha^1 \Vdash t_1 \in \ddot{b}))).$$

Proof of Claim 4.2:   We define those sequences inductively on α. First let's fix an enumeration of $\mathbb{P}$ in order type $\zeta \leqslant \omega_1$, say, $\mathbb{P} = \{x_\gamma : \gamma < \zeta\}$. For $\alpha = 0$ we apply Claim 4.1 for $p = 1_{\mathbb{P}}$ and $n = 2$ to obtain p_0, q_0^0, q_0^1 and η_0. Let α be a countable ordinal. Suppose we have found $\langle p_\beta : \beta < \alpha \rangle$, $\langle q_\beta^0 : \beta < \alpha \rangle$, $\langle q_\beta^1 : \beta < \alpha \rangle$ and $\langle \eta_\beta : \beta < \alpha \rangle$. If $\langle p_\beta : \beta < \alpha \rangle$ is already a maximal antichain in $\mathbb{P}$, then we stop and let $\nu = \alpha$. Otherwise choose a smallest $\gamma < \zeta$ such that x_γ is incompatible with all p_β's for $\beta < \alpha$. Pick $q^j \in \mathbb{Q}$ which are lower bounds of $\langle q_\beta^j : \beta < \alpha \rangle$ for $j = 0, 1$, respectively, and pick $\eta' \in \omega_1$ which is an upper bound of $\langle \eta_\beta : \beta < \alpha \rangle$. By applying Claim 4.1 twice we can find

$$p' \leqslant x_\gamma, \ q_0^0, q_1^0 \leqslant q^0, \ q_0^1, q_1^1 \leqslant q^1, \ \dot{t}_0^0, \dot{t}_1^0, \dot{t}_0^1, \dot{t}_1^1 \text{ and } \eta_\alpha > \eta'$$

such that

$$p' \Vdash (\dot{t}_0^0, \dot{t}_1^0 \in \dot{T}_{\eta_\alpha} \wedge \dot{t}_0^0 \neq \dot{t}_1^0 \wedge (q_0^0 \Vdash \dot{t}_0^0 \in \ddot{b}) \wedge (q_1^0 \Vdash \dot{t}_1^0 \in \ddot{b}))$$

6

and

$$p' \Vdash (\dot{t}_0^1, \dot{t}_1^1 \in \dot{T}_{\eta_\alpha} \wedge \dot{t}_0^1 \neq \dot{t}_1^1 \wedge (q_0^1 \Vdash \dot{t}_0^1 \in \ddot{b}) \wedge (q_1^1 \Vdash \dot{t}_1^1 \in \ddot{b})).$$

If $p' \Vdash \dot{t}_0^0 \neq \dot{t}_0^1$, then let $p_\alpha = p'$, $q_\alpha^0 = q_0^0$ and $q_\alpha^1 = q_0^1$. Otherwise we can find a $p_\alpha < p'$ such that $p_\alpha \Vdash \dot{t}_0^0 \neq \dot{t}_1^1$. Then let $q_\alpha^0 = q_0^0$ and $q_\alpha^1 = q_1^1$. If for any countable α, the set $\{p_\beta \in \mathbb{P} : \beta < \alpha\}$ has never been a maximal antichain, then the set $\{p_\beta \in \mathbb{P} : \beta < \omega_1\}$ must be a maximal antichain of $\mathbb{P}$ by the choice of p_β's according to the fixed enumeration of $\mathbb{P} = \{x_\gamma : \gamma < \zeta = \omega_1\}$. In this case we choose $\nu = \omega_1$.

The lemma follows from the construction. Let $n \in \omega$, $\delta_n = \omega_1 \cap N_n$ and let $\delta = \bigcup_{n \in \omega} \delta_n$. For each $s \in 2^n$ we construct, in N_n, a maximal antichain $\langle p_\alpha^s : \alpha < \nu_s \rangle$ of $\mathbb{P}$, two decreasing sequences $\langle q_\alpha^{s\hat{}j} : \alpha < \nu_s \rangle$ for $j = 0, 1$, and an increasing sequence $\langle \eta_\alpha^s : \alpha < \nu_s \rangle$ in δ_n such that $\nu_s \leqslant \delta_n$, $q_0^{s\hat{}j}$ are lower bounds of $\langle q_\alpha^s : \alpha < \nu_{s \restriction n-1} \rangle$ for $j = 0, 1$, $\eta_0^s = \delta^{n-1}$ and

$$p_\alpha^s \Vdash ((\exists t_0, t_1 \in \dot{T}_{\eta_\alpha^s})(t_0 \neq t_1 \wedge (q_\alpha^{s\hat{}0} \Vdash t_0 \in \ddot{b}) \wedge (q_\alpha^{s\hat{}1} \Vdash t_1 \in \ddot{b}))).$$

Each step of the construction uses Claim 4.2 relative to N_n for some $n \in \omega$. We can choose $q_0^{s\hat{}0}$ and $q_0^{s\hat{}1}$ to be lower bounds of $\langle q_\alpha^s : \alpha < \nu_{s \restriction n-1} \rangle$ because $\langle q_\alpha^s : \alpha < \nu_{s \restriction n-1} \rangle$ is constructed in N_{n-1} and hence, is countable in N_n. Here we use the fact $N_{n-1} \in N_n$.

Let $\bar{p} \leqslant 1_{\mathbb{P}}$ be $(\mathbb{P}, N_n)$-generic for every $n \in \omega$. Since $\mathbb{Q}$ is ω_1-closed in V, for every $f \in 2^\omega$ there is a q_f which is a lower bound of $\langle q_0^{f \restriction n} : n \in \omega \rangle$. Let $G \subseteq \mathbb{P}$ be a V-generic filter such that $\bar{p} \in G$. We claim that T_δ is uncountable in $V[G]$. This contradicts that T is an ω_1-tree in $V^{\mathbb{P}}$. Notice that $2^\omega \cap V$ is uncountable in $V[G]$. In $V[G]$ for each $f \in 2^\omega \cap V$ there is a $q_f' \leqslant q_f$ and a $t_f \in T_\delta$ such that $q_f' \Vdash t_f \in \ddot{b}$. Suppose $f, g \in 2^\omega \cap V$ are different and $n = \min\{i \in \omega : f(i) \neq g(i)\}$. If $t_f = t_g$, then there is a $p \in G$, $p \leqslant \bar{p}$ such that

$$p \Vdash ((\exists t \in \dot{T}_\delta)((q_f' \Vdash t \in \ddot{b}) \wedge (q_g' \Vdash t \in \ddot{b}))).$$

Suppose $f \restriction n = s = g \restriction n$, $f(n) = 0$ and $g(n) = 1$. Since p is $(\mathbb{P}, N_n)$-generic and $p \in G$, there is a $p_\alpha^s \in G$ for some $\alpha \leqslant \nu_s$. Let $p' \leqslant p, p_\alpha^s$. Then

$$p' \Vdash ((\exists t_0, t_1 \in \dot{T}_{\eta_\alpha^s})(t_0 \neq t_1 \wedge (q_f' \Vdash t_0 \in \ddot{b}) \wedge (q_g' \Vdash t_1 \in \ddot{b}))).$$

But this contradicts the following:

$$p' \Vdash (\dot{t}_0 \in \dot{T}_{\eta_\alpha^s} \wedge \dot{t} \in \dot{T}_\delta \wedge (q_f' \Vdash \dot{t}_0, \dot{t} \in \ddot{b}) \rightarrow \dot{t}_0 \leqslant \dot{t}),$$

$$p' \Vdash (\dot{t}_1 \in \dot{T}_{\eta_\alpha^s} \wedge \dot{t} \in \dot{T}_\delta \wedge (q_f' \Vdash \dot{t}_1, \dot{t} \in \ddot{b}) \rightarrow \dot{t}_1 \leqslant \dot{t}),$$

and

$$p' \Vdash (\dot{t}_0, \dot{t}_1 \in \dot{T}_{\eta_\alpha^s} \wedge \dot{t}_0 \leqslant \dot{i} \wedge \dot{t}_1 \leqslant \dot{i} \rightarrow \dot{t}_0 = \dot{t}_1).$$

Hence in $V[G]$ different f's in $2^\omega \cap V$ correspond to different t_f's in T_δ. Therefore T_δ is uncountable. $\square$

A forcing notion $\mathbb{P}$ is called ω-proper if for any ω-sequence $\langle N_n : n \in \omega \rangle$ of countable elementary submodels of $H(\lambda)$ such that $N_n \in N_{n+1}$ for every $n \in \omega$ and $\mathbb{P} \in N_0$, for any $p \in \mathbb{P} \cap N_0$ there is a $\bar{p} \leqslant p$ such that $\bar{p}$ is $(\mathbb{P}, N_n)$-generic for every $n \in \omega$. Let S be a stationary subset of ω_1. A forcing notion $\mathbb{P}$ is called S-proper if for any countable elementary submodel N of $H(\lambda)$ such that $\mathbb{P} \in N$ and $N \cap \omega_1 \in S$, and for any $p \in \mathbb{P} \cap N$ there is a $\bar{p} \leqslant p$ such that $\bar{p}$ is $(\mathbb{P}, N)$-generic. A forcing notion $\mathbb{P}$ is called (S, ω)-proper if for any ω-sequence $\langle N_n : n \in \omega \rangle$ of countable elementary submodels of $H(\lambda)$ such that $N_n \in N_{n+1}$ for every $n \in \omega$, $N_n \cap \omega_1 \in S$ for every $n \in \omega$, $N \cap \omega_1 \in S$, where $N = \bigcup_{n \in \omega} N_n$, and $\mathbb{P} \in N_0$, for any $p \in \mathbb{P} \cap N_0$ there is a $\bar{p} \leqslant p$ such that $\bar{p}$ is $(\mathbb{P}, N_n)$-generic for every $n \in \omega$.

Theorem 5. *Let S be a stationary subset of ω_1 and let $\mathbb{P}$ be an (S, ω)-proper forcing notion of size $\leqslant \omega_1$ in $\bar{V}$. Then there are no Kurepa trees in $\bar{V}^{\mathbb{P}}$.*

Proof: Choose an $\eta < \kappa$ such that S and $\mathbb{P}$ are in $V^{Lv(\eta, \omega_1)}$. Then

$$\bar{V}^{\mathbb{P}} = V^{(Lv(\eta, \omega_1) * \dot{\mathbb{P}}) \times Lv(\kappa \smallsetminus \eta, \omega_1)}$$

and $Lv(\kappa \smallsetminus \eta, \omega_1)$ is ω_1-closed in $V^{Lv(\eta, \omega_1)}$. By Lemma 4 it suffices to show that $\mathbb{P}$ satisfies property (†) in $V^{Lv(\eta, \omega_1)}$. Working in $V^{Lv(\eta, \omega_1)}$. Let $x \in H(\lambda)$. Since S is also stationary in $V^{Lv(\eta, \omega_1)}$, we can choose a sequence $\langle N_n : n \in \omega \rangle$ of countable elementary submodels of $H(\lambda)$ such that $N_n \in N_{n+1}$, $\{\mathbb{P}, x\} \subseteq N_0$ and $N_n \cap \omega_1 \in S$ for every $n \in \omega$. Since the forcing $Lv(\kappa \smallsetminus \eta, \omega_1)$ is countably closed, then we can choose a decreasing sequence $\langle q_n : n \in \omega \rangle$ in $Lv(\kappa \smallsetminus \eta, \omega_1)$ such that q_n is a $(Lv(\kappa \smallsetminus \eta, \omega_1), N_n)$-master condition ($q$ is a $(\mathbb{Q}, N)$-master condition iff for every dense open subset D of $\mathbb{Q}$ there exists a $d \in D$ such that $q \leqslant d$). Let q be a lower bound of $\langle q_n : n \in \omega \rangle$. Let $G \subseteq Lv(\kappa \smallsetminus \eta, \omega_1)$ be $V^{Lv(\eta, \omega_1)}$-generic such that $q \in G$. By Lemma 1 every $N_n[G]$ is a countable elementary submodel of $(H(\lambda))^{\bar{V}}$. It is also easy to see that $\{\mathbb{P}, x\} \subseteq N_0[G]$. Now we have $N_n[G] \in N_{n+1}[G]$ and $N_n[G] \cap \omega_1 \in S$ because $q \Vdash (N_n = N_n[\dot{G}])$.

Pick a $p \in \mathbb{P} \cap N_0$. Since $\mathbb{P}$ is (S, ω)-proper in $\bar{V}$, there exists a $\bar{p} \leqslant p$ such that $\bar{p}$ is $(\mathbb{P}, N_n[G])$-generic for every $n \in \omega$. It is easy to see that $\bar{p}$ is also $(\mathbb{P}, N_n)$-generic

8

because a maximal antichain of $\mathbb{P}$ in N_n is also a maximal antichain in $N_n[G]$. This shows that $\mathbb{P}$ satisfies property ($\dagger$) in $V^{Lv(\eta,\omega_1)}$. □

Remarks (1) If $\mathbb{P}$ satisfies Baumgartner's axiom A, then $\mathbb{P}$ is ω-proper or (ω_1, ω)-proper. Hence forcing with a forcing notion of size $\leqslant \omega_1$ satisfying axiom A in $\bar{V}$ does not create Kurepa trees. Notice also that all c.c.c. forcing notions, ω_1-closed forcing notions and the forcing notions of tree type such as Sack's forcing, Laver forcing, Miller forcing, etc. satisfy axiom A.

(2) The idea of the proof of Lemma 4 is originally from [D]. A version of Theorem 5 for axiom A forcing was proved in [J].

(3) The ω-properness implies the (S, ω)-properness and the (S, ω)-properness implies the property ($\dagger$).

Now we prove the results about some non-(S, ω)-proper forcing notions.

The existence of a Kurepa tree implies that there are no countably complete, $\aleph_2$-saturated ideals on ω_1. Therefore, one can destroy all those ideals by creating a generic Kurepa tree [V]. But one don't have to create Kurepa trees for this purpose. Baumgartner and Taylor [BT] proved that adding a club subset of ω_1 by finite conditions destroys all countably complete, $\aleph_2$-saturated ideals on ω_1. The forcing notion for adding a club subset of ω_1 by finite conditions has size $\leqslant \omega_1$ and is proper but not (S, ω)-proper for any stationary subset S of ω_1.. We are going to prove next that this forcing notion and some other similar forcing notions do not create Kurepa trees if our ground model is the Lévy model $\bar{V}$. Notice also that the ideal of nonstationary subsets of ω_1 could be $\aleph_2$-saturated in the Lévy model obtained by collapsing a supercompact cardinal down to ω_2 [FMS]. As a corollary we can have a ground model $\bar{V}$ which contains countably complete, $\aleph_2$-saturated ideals on ω_1 such that forcing with some small proper forcing notion $\mathbb{P}$ in $\bar{V}$ destroys all countably complete, ω_2-saturated ideals on ω_1 without creating Kurepa trees.

We first define a property of forcing notions which is satisfied by the forcing notion for adding a club subset of ω_1 by finite conditions.

Definition 6. *A forcing notion $\mathbb{P}$ is said to satisfy property (#) if for any $x \in H(\lambda)$ there exists a countable elementary submodel N of $H(\lambda)$ such that $\{\mathbb{P}, x\} \subseteq N$ and for any $p_0 \in \mathbb{P} \cap N$ there exists a $\bar{p} \leqslant p_0$, $\bar{p}$ is $(\mathbb{P}, N)$-generic, and there exists a countable subset C of $\mathbb{P}$ such that for any $\bar{p}' \leqslant \bar{p}$ there is a $c \in C$ and a $p' \in \mathbb{P} \cap N$, $p' \leqslant p_0$ such that*

(1) for any dense open subset D of $\mathbb{P}$ below p' in N there is an $d \in D \cap N$ such that d is compatible with c, and

(2) for any $r \in \mathbb{P} \cap N$ and $r \leqslant p'$, r is compatible with c implies r is compatible with $\bar{p}'$.

Let's call the pair (p', c) a related pair corresponding to $\bar{p}'$.

Examples 7. *Following three examples are the forcing notions which satisfy property (#).*

(1) Let

$$\mathbb{P} = \{p \subseteq \omega_1 \times \omega_1 : p \text{ is a finite function which can be extended to an increasing continuous function from } \omega_1 \text{ to } \omega_1.\}$$

and let $\mathbb{P}$ be ordered by reverse inclusion. $\mathbb{P}$ is one of the simplest proper forcing notion which does not satisfy axiom A [B2]. Forcing with $\mathbb{P}$ creates a generic club subset of ω_1 and destroys all $\aleph_2$-saturated ideals on ω_1 [BT]. It is easy to see that $\mathbb{P}$ satisfies property (#) defined above. For any $x \in H(\lambda)$ we can choose a countable elementary submodel N of $H(\lambda)$ such that $\{\mathbb{P}, x\} \subseteq N$ and $N \cap \omega_1 = \delta$ is an indecomposable ordinal. For any $p_0 \in \mathbb{P} \cap N$ let $\bar{p} = p_0 \cup (\delta, \delta)$ and let $C = \{\bar{p}\}$. Then for any $\bar{p}' \leqslant \bar{p}$ there is a $p' = \bar{p}' \upharpoonright \delta$ and a $c = \bar{p} \in C$ such that all requirements for the definition of property (#) are satisfied.

(2) Let S be a stationary subset of ω_1. If we define

$$\mathbb{P}_S = \{p : p \text{ is a finite function such that there is an increasing continuous function } f \text{ from some countable ordinal to } S \text{ such that } p \subseteq f.\}$$

and let $\mathbb{P}_S$ be ordered by reverse inclusion, then $\mathbb{P}_S$ is S-proper [B2]. Forcing with $\mathbb{P}_S$ adds a club set inside S. It is also easy to check that $\mathbb{P}_S$ satisfies (#). For any $x \in H(\lambda)$. Let N be a countable elementary submodel of $H(\lambda)$ such that $\{x, \mathbb{P}_S\} \subseteq N$, $N \cap \omega_1 = \delta$ is an indescomposable ordinal and $\delta \in S$. Then for any $p_0 \in \mathbb{P}_S \cap N$ the element $\bar{p} = p_0 \cup \{(\delta, \delta)\}$ is $(\mathbb{P}_S, N)$-generic. Now N, $\bar{p}$ and $C = \{\bar{p}\}$ witness that $\mathbb{P}_S$ satisfies property (#).

(3) Let T and U be two normal Aronszajn trees such that every node of T or U has infinitely many immediate successors. Let $\mathbb{P}$ be the forcing notion such that $p = (A_p, f_p) \in \mathbb{P}$ iff

(a) A_p is a finite subset of ω_1,

(b) f_p is a finite partial isomorphism from $T \upharpoonright A_p$ into $U \upharpoonright A_p$,

10

(c) $dom(f_p)$ is a subtree of $T \restriction A_p$ in which every branch has cardinality $|A_p|$.
$\mathbb{P}$ is ordered by $p \leqslant q$ iff $A_p \supseteq A_q$ and $f_p \supseteq f_q$. $\mathbb{P}$ is proper [T]. $\mathbb{P}$ is used in [AS] for generating a club isomorphism from T to U. For any $x \in H(\lambda)$, for any countable elementary submodel N of $H(\lambda)$ such that $\{\mathbb{P}, x\} \subseteq N$ and for any $p_0 \in \mathbb{P} \cap N$, let $\delta = N \cap \omega_1$, let $A_{\bar{p}} = A_{p_0} \cup \{\delta\}$ and let $f_{\bar{p}}$ be any extension of f_{p_0} such that $T_\delta \cap dom(f_{\bar{p}}) \neq \emptyset$. Then $\bar{p} = (A_{\bar{p}}, f_{\bar{p}})$ is a $(\mathbb{P}, N)$-generic condition. Let

$$C = \{d : d \text{ is a finite isomorphism from } T_\delta \text{ to } U_\delta\}.$$

Then C is countable. For any $\bar{p}' \leqslant \bar{p}$ let $c = (f_{\bar{p}'} \restriction \{\delta\}) \in C$, let $\alpha < \delta$, $\alpha > \max(A_{\bar{p}'} \cap \delta)$ and

$$g_\alpha = \{(t, u) \in T_\alpha \times U_\alpha : (\exists(t', u') \in (f_{\bar{p}'} \restriction \{\delta\}))(t < t' \wedge u < u')\}$$

be such that g_α and $f_{\bar{p}'} \restriction \{\delta\}$ have same cardinality, let $A_{p'} = (A_{\bar{p}'} \cap \delta) \cup \{\alpha\}$, let $f_{p'} = (f_{\bar{p}'} \restriction (A_{\bar{p}'} \cap \delta)) \cup g_\alpha$, and let $p' = (A_{p'}, f_{p'})$. Then (p', c) is a related pair corresponding to $\bar{p}'$ [AS] and $N, \bar{p}, C$ witness that $\mathbb{P}$ satisfies property ($\#$). For any stationary set S we can also define an S-proper version of this forcing notion.

Lemma 8. *Let V be a model. Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcing notions in V such that $\mathbb{P}$ has size $\leqslant \omega_1$ and satisfies property ($\#$), and $\mathbb{Q}$ is ω_1-closed (in V). Suppose T is an ω_1-tree in $V^{\mathbb{P}}$. Then T has no branches which are in $V^{\mathbb{P} \times \mathbb{Q}}$ but not in $V^{\mathbb{P}}$.*

Proof: Suppose, towards a contradiction, that there is a branch b of T in $V^{\mathbb{P} \times \mathbb{Q}} \setminus V^{\mathbb{P}}$. Without loss of generality, we assume that

$$\Vdash_{\mathbb{P}} \Vdash_{\mathbb{Q}} (\ddot{b} \text{ is a branch of } \dot{T} \text{ in } V^{\mathbb{P} \times \mathbb{Q}} \setminus V^{\mathbb{P}}).$$

Following the definition of property ($\#$), we can find a countable elementary submodel N of $H(\lambda)$ such that $\{\mathbb{P}, \mathbb{Q}, \dot{T}, \ddot{b}\} \subseteq N$, a $\bar{p} \leqslant 1_{\mathbb{P}}$ which is $(\mathbb{P}, N)$-generic and a countable set $C \subseteq \mathbb{P}$ such that $N, \bar{p}$ and C witness that $\mathbb{P}$ satisfies property ($\#$). Let $\langle (p_i, c_i) : i \in \omega \rangle$ be a listing of all related pairs in $(\mathbb{P} \cap N) \times C$ with infinite repetition, *i.e.* every related pair (p, c) in $(\mathbb{P} \cap N) \times C$ occurs infinitely ofter in the sequence.

We construct now, in V, a set $\{q_s \in \mathbb{Q} \cap N : s \in 2^{<\omega}\}$ and an increasing sequence $\langle \delta_n : n \in \omega \rangle$ such that
  (1) $s \subseteq t$ implies $q_t \leqslant q_s$,
  (2) $\delta_n \in \delta = N \cap \omega_1$,

(3) for every $n \in \omega$ there is a $p' \in \mathbb{P} \cap N, p' \leqslant p_n$ such that p' is compatible with c_n, and

$$p' \Vdash ((\exists \{t_s : s \in 2^n\} \subseteq \dot{T}_{\delta_n})((s \neq s' \to t_s \neq t_{s'}) \wedge \bigwedge_{s \in 2^n} (q_s \Vdash t_s \in \ddot{b}))).$$

The lemma follows from the construction. Let $G \subseteq \mathbb{P}$ be a V-generic filter and $\bar{p} \in G$. We want to show that

$$V[G] \models T_\delta \text{ is uncountable.}$$

For any $f \in 2^\omega \cap V$ let $q_f \in \mathbb{Q}$ be a lower bound of the set $\{q_{f \restriction n} : n \in \omega\}$ such that there is a $t_f \in T_\delta$ such that $q_f \Vdash t_f \in \dot{b}$. Suppose T_δ is countable. Then there are $f, g \in 2^\omega \cap V$ such that $t_f = t_g$. Let $\dot{t}_f, \dot{t}_g$ be $\mathbb{P}$-names for t_f, t_g and let $\bar{p}' \leqslant \bar{p}$ be such that

$$\bar{p}' \Vdash (\dot{t}_f = \dot{t}_g \wedge (q_f \Vdash \dot{t}_f \in \ddot{b}) \wedge (q_g \Vdash \dot{t}_g \in \ddot{b})).$$

Let $m = \min\{i \in \omega : f(i) \neq g(i)\}$. By the definition of property (#) we can find a related pair (p, c) corresponding to $\bar{p}'$. Choose an $n \in \omega$ such that $n \geqslant m$ and $(p, c) = (p_n, c_n)$. Since (1) of Definition 6 is true, there is a $p' \in \mathbb{P} \cap N$ such that $p' \leqslant p$, p' is compatible with c_n and

$$p' \Vdash ((\exists \{t_s : s \in 2^n\} \subseteq \dot{T}_{\delta_n})((s \neq s' \to t_s \neq t_{s'}) \wedge \bigwedge_{s \in 2^n} (q_s \Vdash t_s \in \ddot{b}))).$$

Since $q_f \leqslant q_{f \restriction n}$ and $q_g \leqslant q_{g \restriction n}$, then

$$p' \Vdash ((\exists t_0, t_1 \in \dot{T}_{\delta_n})(t_0 \neq t_1 \wedge (q_f \Vdash t_0 \in \ddot{b}) \wedge (q_g \Vdash t_1 \in \ddot{b}))).$$

But also

$$\bar{p}' \Vdash ((\exists t \in \dot{T}_\delta)((q_f \Vdash t \in \ddot{b}) \wedge (q_g \Vdash t \in \ddot{b}))).$$

By the fact that any two nodes in T_{δ_n} which are below a node in T_δ must be same, and that p' is compatible with $\bar{p}'$, we have a contradiction.

Now let's inductively construct $\{\delta_i : i \in \omega\}$ and $\{q_s : s \in 2^{<\omega}\}$. Suppose we have had $\{q_s : s \in 2^{\leqslant n}\}$ and $\{\delta_i : i \leqslant n\}$. let $D \subseteq \mathbb{P}$ be such that $r \in D$ iff

(1) $r \leqslant p_n$ (recall that (p_n, c_n) is in the enumeration of all related pairs in $(\mathbb{P} \cap N) \times C$),

(2) there exists $\eta > \delta_n$ and there exists $\{q_s \leqslant q_{s \restriction n} : s \in 2^{n+1}\}$ such that

$$r \Vdash ((\exists \{t_s : s \in 2^{n+1}\} \subseteq \dot{T}_\eta)((s \neq s' \to t_s \neq t_{s'}) \wedge \bigwedge_{s \in 2^{n+1}} (q_s \Vdash t_s \in \ddot{b}))).$$

It is easy to see that D is open and $D \in N$.

12

Claim 8.1 D is dense below p_n.

Proof of Claim 8.1: Suppose $r_0 \leqslant p_n$. It suffices to show that there is an $r \leqslant r_0$ such that $r \in D$. Applying Claim 4.1, for any $s \in 2^n$ we can find $r_s \leqslant r_0$, $\eta_s > \delta_n$ and $\{q_j^s \leqslant q_s : j < 2^{n+1}\}$ such that

$$r_s \Vdash ((\exists\{t_j : j < 2^{n+1}\} \subseteq \dot{T}_{\eta_s})((j \neq j' \to t_j \neq t_{j'}) \wedge \bigwedge_{j<2^{n+1}} (q_j^s \Vdash t_j \in \ddot{b}))).$$

Let $\{s_i : i < 2^n\}$ be an enumeration of 2^n. By applying Claim 4.1 2^n times as above we obtained $r_0 \geqslant r_{s_0} \geqslant r_{s_1} \geqslant \ldots r_{s_{2^n-1}}$ such that above arguments are true for any $s \in 2^n$. Pick $\eta = \max\{\eta_s : s \in 2^n\}$. Then we extend $r_{s_{2^n-1}}$ to r', and extend q_j^s to $\bar{q}_j^s$ for every such s and j such that for each $s \in 2^n$

$$r' \Vdash ((\exists\{t_j : j < 2^{n+1}\} \subseteq \dot{T}_\eta)((j \neq j' \to t_j \neq t_{j'}) \wedge \bigwedge_{j<2^{n+1}} (\bar{q}_j^s \Vdash t_j \in \ddot{b}))).$$

Now applying an argument in Claim 4.2 repeatedly we can choose $\{q_{s\hat{}0}, q_{s\hat{}1}\} \subseteq \{\bar{q}_j^s : j < 2^{n+1}\}$ for every $s \in 2^n$ and extend r' to r'' such that

$$r'' \Vdash ((\exists\{t_s : s \in 2^{n+1}\} \subseteq \dot{T}_\eta)((s \neq s' \to t_s \neq t_{s'}) \wedge \bigwedge_{s\in2^{n+1}} (q_s \Vdash t_s \in \ddot{b}))).$$

This showed that D is dense below p_n.

Notice that since N is elementary, then η exists in N and all those q_s' for $s \in 2^{n+1}$ exist in N. Choose $r \in D$ such that r, c_n are compatible and let δ_{n+1} be correspondent η. This ends the construction. $\square$

Theorem 9. *If $\mathbb{P}$ in $\bar{V}$ is a forcing notion defined in (1), (2) or (3) of Examples 7, then forcing with $\mathbb{P}$ does not create any Kurepa trees.*

Proof: Suppose T is a Kurepa tree in $\bar{V}^{\mathbb{P}}$. Let $\eta < \kappa$ be such that $\mathbb{P}, T \in V^{Lv(\eta,\omega_1)}$. Since the definition of $\mathbb{P}$ is absolute between $\bar{V}$ and $V^{Lv(\eta,\omega_1)}$, then $\mathbb{P}$ satisfies property (#) in $V^{Lv(\eta,\omega_1)}$. Since T has less than κ branches in $V^{Lv(\eta,\omega)*\dot{\mathbb{P}}}$, there exist branches of T in $\bar{V}^{\mathbb{P}}$ which are not in $V^{Lv(\eta,\omega_1)*\dot{\mathbb{P}}}$. This contradicts Lemma 8. $\square$

Remark: The forcing notions in Examples 7, (1), (2) and (3) are not (S,ω)-proper for any stationary S.

2. Creating Kurepa Trees By a Small Forcing Is Easy

In this section we construct a model of CH plus no Kurepa trees, in which there is an ω-distributive Aronszajn tree T such that forcing with T does create a Kurepa tree in the generic extension.

Let V be a model and κ be a strongly inaccessible cardinal in V. Let $\mathcal{T}$ be the set of all countable normal trees. Given a set A and a cardinal λ. Let $[A]^{<\lambda} = \{S \subseteq A : |S| < \lambda\}$ and $[A]^{\leqslant \lambda} = \{S \subseteq A : |S| \leqslant \lambda\}$. We define a forcing notion $\mathbb{P}$ as following:

Definition 10. *p is a condition in $\mathbb{P}$ iff*

$$p = \langle \alpha_p, t_p, k_p, U_p, B_p, F_p \rangle$$

where

(a) $\alpha_p \in \omega_1$,

(b) $t_p \in \mathcal{T}$ and $ht(t_p) = \alpha_p + 1$,

(c) k_p is a function from t_p to $\mathcal{T}$ such that for any $x \in t_p$, $ht(k_p(x)) = ht(x) + 1$, and for any $x, y \in t_p$, $x < y$ implies $k_p(x) \leqslant_{end} k_p(y)$,

(d) $U_p \in [\kappa]^{\leqslant \omega_1}$,

(e) $B_p = \{b_\gamma^p : \gamma \in U_p\}$ where b_γ^p is a function from $t_p \restriction (\beta_\gamma^p + 1)$ to $\omega_1^{<\omega_1}$ for some $\beta_\gamma^p \leqslant \alpha_p$ such that for any $x \in t_p \restriction (\beta_\gamma^p + 1)$, $b_\gamma^p(x) \in (k_p(x))_{ht(x)}$ and for any $x, y \in t_p \restriction (\beta_\gamma^p)$, $x \leqslant y$ implies $b_\gamma^p(x) \leqslant b_\gamma^p(y)$,

(f) $F_p = \{f_\gamma^p : \gamma \in U_p\}$ where f_γ^p is a function from δ_γ^p to γ for some $\delta_\gamma^p \leqslant \alpha_p$,

(g) for any $x \in t_p \restriction \alpha_p$, for any finite $U_0 \subseteq U_p$ and for any ϵ such that $ht(x) < \epsilon \leqslant \alpha_p$, there exists an $x' \in (t_p)_\epsilon$ such that $x' > x$ and for any $\gamma_1, \gamma_2 \in U_0$ either one of $\beta_{\gamma_1}^p, \beta_{\gamma_2}^p$ is less than ϵ or $b_{\gamma_1}^p(x) = b_{\gamma_2}^p(x)$ implies $b_{\gamma_1}^p(x') = b_{\gamma_2}^p(x')$.

In the condition (g) of the definition we call x' a conservative extension of x at level ϵ with respect to U_0 (or with respect to $\{b_\gamma^p : \gamma \in U_0\}$).

Generally we have the following notation. Suppose $t \in \mathcal{T}$ and B is a set of functions such that for each $b \in B$ there is a $\beta_b \leqslant ht(t)$ such that $domain(b) = t \restriction \beta$. We say t is consistent with respect to B if for any $x \in t \restriction ht(t)$, for any finite $B_0 \subseteq B$ and for any ϵ such that $ht(x) < \epsilon \leqslant ht(t)$, there exists an $x' \in t_\epsilon$ such that $x' > x$ and for any $b_1, b_2 \in B_0$ either one of β_{b_1}, β_{b_2} is less than ϵ or $b_1(x) = b_2(x)$ implies $b_1(x') = b_2(x')$. So $p \in \mathbb{P}$ implies that t_p is consistent with respect to B_p.

For any $p, q \in \mathbb{P}$ we define the order of $\mathbb{P}$ by letting $p \leqslant q$ iff

(1) $\alpha_q \leqslant \alpha_p$, $t_q \leqslant_{end} t_p$, $k_q \subseteq k_p$ and $U_q \subseteq U_p$,

14

(2) for any $\gamma \in U_q$, $b_\gamma^q \subseteq b_\gamma^p$ and $f_\gamma^q \subseteq f_\gamma^p$,

(3) $\{\gamma \in U_q : \beta_\gamma^p > \beta_\gamma^q\}$ is at most countable,

(4) $\{\gamma \in U_q : \delta_\gamma^p > \delta_\gamma^q\}$ is at most countable.

Remarks: In the definition of $\mathbb{P}$ the part t_p is used for creating an ω-distributive Aronszajn tree T. The part k_p is used for creating a T-name of an ω_1-tree K. The part B_p is used for adding κ branches to K so that K becomes a Kurepa tree in the generic extension by forcing with T. The part F_p is used for collapsing all cardinals between ω_1 and κ.

For any $\epsilon \in \omega_1$, $\gamma \in \kappa$ and $\eta \in \gamma$, let

$$D_\epsilon^1 = \{p \in \mathbb{P} : \alpha_p \geqslant \epsilon\},$$

$$D_\gamma^2 = \{p \in \mathbb{P} : \gamma \in U_p\},$$

$$D_{\eta,\gamma}^3 = \{p \in \mathbb{P} : \gamma \in U_p \text{ and } \eta \in range(f_\gamma^p)\},$$

$$D_{\epsilon,\gamma}^4 = \{p \in \mathbb{P} : \gamma \in U_p \text{ and } \beta_\gamma^p \geqslant \epsilon\}.$$

Lemma 11. *The sets D_ϵ^1, D_γ^2, $D_{\eta,\gamma}^3$ and $D_{\epsilon,\gamma}^4$ are open dense in $\mathbb{P}$.*

Proof: It is easy to see that all four sets are open. Let's show they are dense. The proofs of the denseness of the first three sets are easy.

Given $p_0 \in \mathbb{P}$. We need to find a $p \leqslant p_0$ such that $p \in D_\epsilon^1$. Pick an $\alpha_p \geqslant \epsilon$ and $\alpha_p \geqslant \alpha_{p_0}$. Let $t_p \in \mathcal{T}$ be such that $ht(t_p) = \alpha_p + 1$ and $t_{p_0} \leqslant_{end} t_p$. Let $k_p : t_p \mapsto \mathcal{T}$ be any suitable extension of k_{p_0}. Let $U_p = U_{p_0}$. For any $\gamma \in U_p$ let $b_\gamma^p = b_\gamma^{p_0}$ and $f_\gamma^p = f_\gamma^{p_0}$. Then $p \leqslant p_0$ and $p \in D_\epsilon^1$.

Given $p_0 \in \mathbb{P}$. We need to find a $p \leqslant p_0$ such that $p \in D_\gamma^2$. If $\gamma \in U_{p_0}$, let $p = p_0$. Otherwise, let

$$p = \langle \alpha_{p_0}, t_{p_0}, k_{p_0}, U_{p_0} \cup \{\gamma\}, B_{p_0} \cup \{b_\gamma^p\}, F_{p_0} \cup \{f_\gamma^p\}\rangle,$$

where b_γ^p and f_γ^p are empty functions. Then $p \leqslant p_0$ and $p \in D_\gamma^2$.

Given $p_0 \in \mathbb{P}$. We need to find a $p \leqslant p_0$ such that $p \in D_{\eta,\gamma}^3$. First, pick $p' \in D_{\alpha_0+1}^1$ such that $p' \leqslant p_0$ and $f_\gamma^{p'} = f_\gamma^{p_0}$. Then extend $f_\gamma^{p'}$ to f_γ^p on $\alpha_0 + 1$ arbitrary except assigning $f_\gamma^p(\alpha_0) = \eta$. Let everything else keep unchanged. Then $p \leqslant p'$ and $p \in D_{\eta,\gamma}^3$.

Proving the denseness of $D_{\epsilon,\gamma}^4$ is not trivial due to the condition (g) of Definition 10. Given $p_0 \in \mathbb{P}$. Without loss of generality we assume that $p_0 \in D_\epsilon^1 \cap D_\gamma^2$ and $\epsilon > \beta_\delta^{p_0}$ for all $\delta \in U_{p_0}$. We need to find a $p \leqslant p_0$ such that $p \in D_{\epsilon,\gamma}^4$. Choose

$\alpha_p = \alpha_{p_0}, t_p = t_{p_0}, k_p = k_{p_0}, U_p = U_{p_0}, b_\delta^p = b_\delta^{p_0}$ for all $\delta \in U_{p_0} \smallsetminus \{\gamma\}$ and $f_\delta^p = f_\delta^{p_0}$ for all $\delta \in U_{p_0}$. Let $\beta_\gamma^p = \epsilon$. We need to extend $b_\gamma^{p_0}$ to b_γ^p on $t_p \upharpoonright (\epsilon + 1)$ such that $p \in \mathbb{P}$.

For each $x \in t_p \upharpoonright (\epsilon + 1) \smallsetminus t_p \upharpoonright \beta_\gamma^{p_0}$ and for each $\mu \leqslant \epsilon$ Let $C_{x,\mu}$ be the cone above x up to level μ, *i.e.*

$$C_{x,\mu} = \{y \in t_p : x < y \text{ and } ht(y) \leqslant \mu\}.$$

We construct $t_0 \subseteq t_1 \subseteq \ldots$ with $t_0 = t_p \upharpoonright \beta_\gamma^{p_0}$ and define b_γ^p on t_n inductively. Suppose we have had t_n and $b_\gamma^p \upharpoonright t_n$. For any maximal node x of t_n we define a subset t_x^n above x. It will be self-clear from the construction that for any $n \in \omega$ and for any $x \in t_n$ there is a maximal node x' of t_n such that $x' \geqslant x$. Our t_{n+1} will be the union of t_n and those t_x^n's. Let x be a maximal node of t_n. Let

$$U_x = \{\beta_\delta^p : \delta \in U_p \smallsetminus \{\gamma\}, \beta_\delta^p > ht(x) \text{ and } b_\delta^p(x) = b_\gamma^p(x)\}.$$

Case 1: $U_x = \emptyset$. Let $t_x^n = \emptyset$. This means any choice of b_n^p above x will not violate the condition (g).

Case 2: U_x has a largest element, say $\beta_{\delta'}^p$. Let $t_x^n = C_{x,\beta_{\delta'}^p}$ and let $b_\gamma^p \upharpoonright t_x^n = b_{\delta'}^p \upharpoonright t_x^n$.

Case 3: $\bigcup U_x$ is a limit ordinal. Fix a strictly increasing sequence $\langle \nu_{x,m} : m \in \omega \rangle$ of ordinals such that $\bigcup_{m \in \omega} \nu_{x,m} = \bigcup U_x$. Let $x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n = x$ be such that x_i is a maximal node of t_i for $i = 0, 1, \ldots, n$. Notice that if $i < n$, then $\bigcup U_{x_i} \geqslant \bigcup U_x$, and if $\bigcup U_{x_i}$ is a limit ordinal, then $\langle \nu_{x_i,m} : m \in \omega \rangle$ has already been defined. Let

$$l = \min\{i : \bigcup_{m \in \omega} \nu_{x_i,m} = \bigcup_{m \in \omega} \nu_{x_n,m}\}$$

and let

$$\bar{\nu} = \max\{\nu_{x_i,n} : l \leqslant i \leqslant n\}.$$

Choose $\delta \in U_x$ such that $\beta_\delta^p \geqslant \bar{\nu}$ and let $b_\gamma^p \upharpoonright C_{x,\beta_\delta^p} = b_\delta^p \upharpoonright C_{x,\beta_\delta^p}$. Let $t_x^n = C_{x,\beta_\delta^p}$. Now we take

$$t_{n+1} = t_n \cup (\bigcup\{t_x^n : x \text{ is a maximal node of } t_n.\}$$

and define $b_\gamma^p \upharpoonright t_{n+1}$ accordingly. Let $t = \bigcup_{n \in \omega} t_n$. Notice that t may not be equal to $t_p \upharpoonright (\epsilon + 1)$. But it is no problem because any extension of $b_\gamma^p \upharpoonright t$ to $t_p \upharpoonright (\epsilon + 1)$ following the condition (e) will not violate the condition (g). Let b_γ^p be such an extension of $b_\gamma^p \upharpoonright t$.

Claim 11.1 $p \in \mathbb{P}$.

Proof of Claim 11.1: We need only to check that the condition (g) of Definition 10 is satisfied. Pick $x \in t_p \upharpoonright \epsilon$ and pick a finite subset U_0 of U_p. Pick also an ϵ' such

16

that $ht(x) < \epsilon' \leqslant \epsilon$. First, we assume that $x \in t_n \smallsetminus t_{n-1}$ for some $n \in \omega$ (let $t_{-1} = \emptyset$). Without loss of generality we assume that x is a maximal node of t_n.

Case 1: Every $\beta \in U_x$ is less than ϵ'. Then the condition (g) is trivially satisfied because any conservative extension of x at level ϵ' with respect to $U_0 \smallsetminus \{\gamma\}$ is a conservative extension of x with respect to U_0.

Case 2: There is a largest ordinal $\beta^p_{\delta'} \geqslant \epsilon'$ in U_x such that

$$b^p_\gamma \restriction C_{x,\beta^p_{\delta'}} = b^p_{\delta'} \restriction C_{x,\beta^p_{\delta'}}.$$

Then a conservative extension of x at level ϵ' with respect to $(U_0 \smallsetminus \{\gamma\}) \cup \{\delta'\}$ is a conservative extension of x with respect to U_0.

Case 3: $\bigcup U_x$ is a limit ordinal greater than ϵ'. First, choose $\beta^p_{\delta'} > \epsilon'$ in U_x. Suppose $\nu_{x,m} \leqslant \beta^p_{\delta'} < \nu_{x,m+1}$. Then choose a maximal node x_1 of t_{n+1} such that x_1 is a conservative extension of x with respect to $U_0 \cup \{\gamma, \delta'\}$. Now we have

$$\bigcup U_{x_1} \geqslant \beta^p_{\delta'} > \epsilon'.$$

Notice that $ht(x_1) \geqslant \nu_{x,n}$. We are done if U_{x_1} has a largest ordinal. Otherwise we repeat the same procedure to get x_2. Eventually, we can find an x_k such that x_k is a conservative extension of x with respect to $U_0 \cup \{\delta'\}$ and $ht(x_k) \geqslant \nu_{x,m+1} > \epsilon'$. Let $x'' \leqslant x'$ and $ht(x'') = \epsilon'$. It is easy to see that x'' is a conservative extension of x at level ϵ' with respect to U_0.

Suppose $x \notin t$. Then $U_x = \emptyset$. So every $x' \geqslant x$, $x' \in t_{\epsilon'}$ is a conservative extension of x with respect to U_0.

This ends the proof of the claim. It is easy to see that $p \in D^4_{\epsilon,\gamma}$. $\square$

Next we want to prove that $\mathbb{P}$ is ω_1-strategically closed. Let $\mathbb{Q}$ be a forcing notion. Two players, I and II, play a game $G(\mathbb{Q})$ by I choosing $p_n \in \mathbb{Q}$ and II choosing $q_n \in \mathbb{Q}$ alternatively such that

$$p_0 \geqslant q_0 \geqslant p_1 \geqslant q_1 \geqslant \ldots .$$

II wins the game $G(\mathbb{Q})$ if and only if the sequence $\langle p_0, q_0, p_1, q_1, \ldots \rangle$ has a lower bound in $\mathbb{Q}$. A forcing notion $\mathbb{Q}$ is called ω_1-strategically closed if II wins the game $G(\mathbb{Q})$. Note that any ω_1-strategically closed forcing notion does not add new countable sequences of ordinals to the generic extension.

Lemma 12. $\mathbb{P}$ *is ω_1-strategically closed.*

Proof: We choose q_n inductively for Player II after Player I choose any $p_n \leqslant q_{n-1}$. Suppose p_i, q_i have been chosen for $i < n$. Let $p_n \leqslant q_{n-1}$ be any element chosen by Player I. Player II want to choose $q_n \leqslant p_n$. Let

$$U_n = \{\gamma \in U_{p_n} : (\exists i < n)(\beta_\gamma^{p_i} \neq \beta_\gamma^{q_i}) \text{ or } (\exists i \leqslant n)(\beta_\gamma^{p_i} \neq \beta_\gamma^{q_{i-1}})\}.$$

Choose $q_n \leqslant p_n$ such that $\alpha_{q_n} > \alpha_{p_n}$ and for any $\gamma \in U_n$, $\beta_\gamma^{q_n} = \alpha_{q_n}$. This can be done by repeating the steps countably many times used in the proof of the denseness of $D_{\epsilon,\gamma}^4$ in Lemma 11. This finishes the inductive step of the construction. Let

$$\alpha_q = \bigcup_{n \in \omega} \alpha_{q_n}, \ t' = \bigcup_{n \in \omega}, \ k' = \bigcup_{n \in \omega} k_{q_n}, \ U_q = \bigcup_{n \in \omega} U_{q_n}$$

and for each $\gamma \in U_q$

$$b'_\gamma = \bigcup \{b_\gamma^{q_n} : n \in \omega, \gamma \in U_{q_n}\}$$

and

$$f_\gamma^q = \bigcup \{f_\gamma^{q_n} : n \in \omega, \gamma \in U_{q_n}\}.$$

We need now to add one more level on the top of t' and extend k' and b'_γ's accordingly. The main difficulty here is to make the condition (g) of Definition 10 true. Remember

$$U_\omega = \bigcup_{n \in \omega} U_n \subseteq U_p$$

is the set of all γ's such that $\beta_\gamma^{q_n}$ grows for some n. The set U_ω is at most countable due to the definition of the order of $\mathbb{P}$. Note that α_{q_n} is strictly increasing. Note also that for each $\gamma \in U_q \setminus U_\omega$ the sequence

$$\{b_\gamma^{q_n} : n \in \omega, \gamma \in U_{q_n}\}$$

is a constant sequence. So the top level we are going to add does not affect those b_γ^p's for $\gamma \in U_q \setminus U_\omega$.

Let $\{\langle x_m, \Gamma_m \rangle : m \in \omega\}$ be an enumeration of $t' \times [U_\omega]^{<\omega}$. For each $\langle x_m, \Gamma_m \rangle$ we choose an increasing sequence $\langle y_{m,i} : i \in \omega \rangle$ such that

$$x_m = y_{m,0} < y_{m,1} < \cdots,$$

$y_{m,i+1}$ is a conservative extension of $y_{m,i}$ with respect to Γ_m and

$$\bigcup_{i \in \omega} ht(y_{m,i}) = \alpha_q.$$

18

Now let $y_m = \bigcup_{i \in \omega} y_{m,i}$ and let $t_q = t' \cup \{y_m : m \in \omega\}$. It is easy to see that $t_q \in \mathcal{T}$. For each $\gamma \in U_\omega$ we define b_γ^q to be an extension of b_γ' on t_q such that

$$b_\gamma^q(y_m) = \bigcup_{i \in \omega} b_\gamma'(y_{m,i})$$

for all $m \in \omega$. We define also k_q to be an extension of k' on t_q such that for each $m \in \omega$, the tree $k_q(y_m)$ is in $\mathcal{T}$, $ht(k_q(y_m)) = \alpha_q + 1$, $k_q(y_m)$ is an end-extension of $\bigcup_{i \in \omega} k'(y_{m,i})$ and $b_\gamma^q(y_m) \in k_q(y_m)$ for all $\gamma \in U_\omega$. It is easy to see now that the element q is in $\mathbb{P}$ and is a lower bound of p_n's and q_n's. $\quad\square$

Lemma 13. *The forcing notion $\mathbb{P}$ satisfies κ-c.c..*

Proof: Let $\{p_\eta : \eta \in \kappa\} \subseteq \mathbb{P}$. By a cardinality argument and Δ-system lemma there is an $S \subseteq \kappa$, $|S| = \kappa$ and there is a triple $\langle \alpha_0, t_0, k_0 \rangle$ such that for every $\eta \in S$

$$\langle \alpha_{p_\eta}, t_{p_\eta}, k_{p_\eta} \rangle = \langle \alpha_0, t_0, k_0 \rangle,$$

and $\{U_{p_\eta} : \eta \in S\}$ forms a Δ-system with the root U_0. Furthermore, we can assume that for each $\gamma \in U_0$,

$$b_\gamma^{p_\eta} = b_\gamma^{p_{\eta'}} \text{ and } f_\gamma^{p_\eta} = f_\gamma^{p_{\eta'}}$$

for any $\eta, \eta' \in S$. Since there are at most $(|\omega_1^{\leqslant \alpha_0}|^{|t_0|})^{\omega_1} = 2^{\omega_1}$ sequences of length ω_1 of the functions from t_0 to $\omega_1^{\leqslant \alpha_0}$, there are $\eta, \eta' \in S$ such that

$$\{b_\gamma^{p_\eta} : \gamma \in U_{p_\eta} \smallsetminus U_0\} \text{ and } \{b_\gamma^{p_{\eta'}} : \gamma \in U_{p_{\eta'}} \smallsetminus U_0\}$$

are same set of functions. It is easy to see now that the element

$$p = \langle \alpha_0, t_0, k_0, U_{p_\eta} \cup U_{p_{\eta'}}, B_{p_\eta} \cup B_{p_{\eta'}}, F_{p_\eta} \cup F_{p_{\eta'}} \rangle$$

is a common lower bound of p_η and $p_{\eta'}$. $\quad\square$

Lemma 14. *All cardinals between ω_1 and κ in V are collapsed in $V^{\mathbb{P}}$.*

Proof: For any $\gamma \in \kappa$ let

$$f_\gamma = \bigcup \{f_\gamma^p : p \in G \text{ and } \gamma \in U_p\}$$

where $G \subseteq \mathbb{P}$ is a V-generic filter. It is easy to check that $range(f_\gamma) = \gamma$. Also $dom(f_\gamma) \subseteq \omega_1$. So in $V^{\mathbb{P}}$ we have $|\gamma| \leqslant \omega_1$. $\quad\square$

Remark: By Lemma 12, Lemma 13 and Lemma 14 we have

$$V^{\mathbb{P}} \models (2^\omega = \omega_1^V = \omega_1 \text{ and } 2^{\omega_1} = \kappa = \omega_2).$$

Lemma 15. *Let $G \subseteq \mathbb{P}$ be a V-generic filter and let $T_G = \bigcup\{t_p : p \in G\}$. Then T_G is an ω-distributive Aronszajn tree in $V[G]$.*

Proof: It is easy to see that T_G is an ω_1-tree. Suppose there is a $p_0 \in \mathbb{P}$ such that

$$p_0 \Vdash \dot{B} \text{ is a branch of } T_G.$$

We construct $p_0 \geqslant q_0 \geqslant p_1 \geqslant q_1 \geqslant \ldots$ similar to the construction in Lemma 12 such that

$$p_{n+1} \Vdash z_n \in \dot{B} \cap (t_{q_n})_{\alpha_{q_n}}$$

for some $z_n \in \omega_1^{\alpha_{q_n}}$. For constructing q_{n+1} we use almost same method as in Lemma 12 except that we require q_{n+1} satisfy the following condition (g'):

> For any $x \in t_{p_{n+1}}$ and $\Gamma \in [U_{n+1}]^{<\omega}$ (see Lemma 12 for the definition of U_{n+1}) there are infinitely many $x' \in (t_{q_{n+1}})_{\alpha_{q_{n+1}}}$ such that x' is a conservative extension of x with respect to Γ.

This can be done just by stretching $t_{q_{n+1}}$ a little bit higher and manipulating those $b_\gamma^{q_{n+1}} \restriction (t_{q_{n+1}} \smallsetminus t_{p_{n+1}})$ for $\gamma \in U_{n+1}$ more carefully. Let q be a lower bound of $\langle q_n : n \in \omega \rangle$ constructed same as in Lemma 12 except that for any $\langle x_m, \Gamma_m \rangle$ the sequence $\langle y_{m,i} : i \in \omega \rangle$ is chosen such that $\bigcup_{i \in \omega} y_{m,i}$ is different from $\bigcup_{n \in \omega} z_n$. This is guaranteed by the condition (g'). Now

$$\bigcup_{n \in \omega} z_n \notin (t_q)_{\alpha_q}.$$

Hence

$$q \Vdash \dot{B} \subseteq t_q.$$

This contradicts that B is a branch of T_G in $V[G]$.

Next we prove that T_G is ω-distributive. Let $\mathbb{Q} = \langle T_G, \leqslant' \rangle$ be the forcing notion by reversing tree order ($\leqslant' = \geqslant_{T_G}$). Given any $\tau \in 2^\omega$ in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$. It suffices to show that $\tau \in V$. We construct a decreasing sequence

$$\langle p_0, \dot{x}_0 \rangle \geqslant \langle q_0, \dot{x}_0 \rangle \geqslant \langle p_1, \dot{x}_1 \rangle \geqslant \langle q_1, \dot{x}_1 \rangle \geqslant \ldots$$

in $\mathbb{P} * \dot{\mathbb{Q}}$ such that

$$\langle p_0, \dot{x}_0 \rangle \Vdash \dot{\tau} \text{ is a function from } \omega \text{ to } 2,$$

$$p_n \Vdash \dot{x}_n \in \omega_1^{\alpha_{p_n}},$$

$$q_n \Vdash \dot{\tau}(n) = l_n$$

for some $l_n \in \{0, 1\}$ and

$$q_n \Vdash \dot{x}_n = \bar{x}_n.$$

20

for some $\bar{x}_n \in (t_{p_n})_{\alpha_{p_n}}$. In addition we can extend q_n so that the requirements for Player II to win the game are also satisfied. Now we can construct a lower bound q of q_n same as we did in Lemma 12 except that we put also $x = \bigcup_{n \in \omega} \bar{x}_n$ into the top level of t_q. It is easy to see that $\langle q, x \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$ and there is a $\sigma = \langle l_0, l_1, \ldots \rangle \in 2^\omega$ in V such that

$$\langle q, x \rangle \Vdash \dot{\tau} = \sigma. \quad \square$$

Lemma 16. *Let $G \subseteq \mathbb{P}$ be a V-generic filter and let $k_G = \bigcup\{k_p : p \in G\}$. Let T_G and $\mathbb{Q}$ be same as in Lemma 15. Suppose $H \subseteq \mathbb{Q}$ is a $V[G]$-generic filter. Then $K_H = \bigcup\{k_G(x) : x \in H\}$ is a Kurepa tree in $V[G][H]$.*

Proof: It is easy to see that K_H is an ω_1-tree. For any $\gamma \in \kappa$ let

$$b_\gamma = \bigcup\{b_\gamma^p : p \in G \text{ and } \gamma \in U_p\}.$$

Then b_γ is a function with domain T_G. Let

$$W_\gamma = \bigcup\{b_\gamma(x) : x \in H\}.$$

Then it is easy to see that W_γ is a branch of K_H. We need now only to show that W_γ and $W_{\gamma'}$ are different branches for different $\gamma, \gamma' \in \kappa$. Given distinct γ and γ' in κ. Let

$$D^5_{\gamma,\gamma'} = \{p \in \mathbb{P} : \beta_\gamma^p = \beta_{\gamma'}^p = \alpha_p \text{ and }$$
$$(\forall x \in t_p \restriction \alpha_p)(\exists y \in t_p)(y \geqslant x \text{ and } b_\gamma^p(y) \neq b_{\gamma'}^p(y))\}.$$

Claim 16.1   The set $D^5_{\gamma,\gamma'}$ is dense in $\mathbb{P}$.

Proof of Claim 16.1:   Given $p_0 \in \mathbb{P}$. Without loss of generality we assume that $p_0 \in D^2_\gamma \cap D^2_{\gamma'}$ and $\beta_\gamma^{p_0} = \beta_{\gamma'}^{p_0} = \alpha_{p_0}$. First, we extend t_{p_0} to $t_p \in \mathcal{T}$ such that

$$\alpha_p = ht(t_p) = \alpha_{p_0} + \omega + 1.$$

Then, we choose one extension k_p of k_{p_0} on t_p. Now we can easily extend $b_\gamma^{p_0}$ and $b_{\gamma'}^{p_0}$ to b_γ^p and $b_{\gamma'}^p$ on t_p while keeping other things unchanged such that the resulting element p is in $\mathbb{P}$ and for each $x \in t_p \restriction \alpha_p$ there is an $y \in (t_p)_{\alpha_p}$ and $y > x$ such that $b_\gamma^p(y) \neq b_{\gamma'}^p$. It is easy to see the element p is less than p_0 and is in $D^5_{\gamma,\gamma'}$. This ends the proof of the claim.

We need to prove W_γ and $W_{\gamma'}$ are different branches of K_H in $V[G][H]$. Suppose $x \in H$ and

$$x \Vdash \dot{W}_\gamma = \dot{W}_{\gamma'}$$

in $V[G]$. Let $p_0 \in G$ be such that $x \in t_{p_0}$. By the claim we can find a $p \leqslant p_0$ and $p \in G \cap D^5_{\gamma,\gamma'}$ such that $\alpha_p > ht(x)$. Then we can choose $y \in t_p$ and $y > x$ such that $b^p_\gamma(y) \neq b^p_{\gamma'}(y)$. Therefore

$$y \Vdash \dot{W}_\gamma \neq \dot{W}_{\gamma'},$$

which contradicts that

$$x \Vdash \dot{W}_\gamma = \dot{W}_{\gamma'}. \quad \square$$

The next lemma is probably the hardest part of this section.

Lemma 17. *There are no Kurepa trees in $V^{\mathbb{P}}$.*

Proof: Suppose

$$\Vdash_{\mathbb{P}} \dot{T} \text{ is a Kurepa tree with } \kappa \text{ branches } \dot{\mathcal{C}} = \{\dot{c}_\gamma : \gamma \in \kappa\}.$$

For each $\gamma \in \kappa$ such that $cof(\gamma) = (2^{\omega_1})^+$ we choose an elementary submodel $\mathfrak{A}_\gamma$ of $H(\lambda)$ such that

(a) $|\mathfrak{A}_\gamma| \leqslant 2^{\omega_1}$,

(b) $\{\dot{T}, \dot{\mathcal{C}}, \mathbb{P}, \gamma\} \subseteq \mathfrak{A}_\gamma$,

(c) $[\mathfrak{A}_\gamma]^{\leqslant \omega_1} \subseteq \mathfrak{A}_\gamma$.

By the Pressing Down Lemma we can find a set

$$S \subseteq \{\gamma \in \kappa : cof(\gamma) = (2^{\omega_1})^+\}$$

with $|S| = \kappa$ such that

(d) $\{\mathfrak{A}_\gamma : \gamma \in S\}$ forms a Δ-system with the common root $\mathfrak{B}$,

(e) there is a $\eta_0 \in \kappa$ such that $\eta_0 = \bigcup\{\eta \in \kappa : \eta \in \mathfrak{A}_\gamma \cap \gamma\}$ for every $\gamma \in S$,

(f) for any $\gamma, \gamma' \in S$ there is an isomorphism $h_{\gamma,\gamma'}$ from $\mathfrak{A}_\gamma$ to $\mathfrak{A}_{\gamma'}$ such that $h_{\gamma,\gamma'} \upharpoonright \mathfrak{B}$ is an identity map.

Notice that $\omega_1 \subseteq \mathfrak{B}$ and $\omega_1^{<\omega_1} \subseteq \mathfrak{B}$. So for any $x \in \omega_1^{<\omega_1}$ we have $h_{\gamma,\gamma'}(x) = x$. Let γ_0 be the minimal ordinal in S. For any $p, p' \in \mathbb{P}$ we write $p \upharpoonright \mathfrak{A}_\gamma = p'$ to mean $\langle \alpha_p, t_p, k_p \rangle = \langle \alpha_{p'}, t_{p'}, k_{p'} \rangle$, $U_p \cap \mathfrak{A}_\gamma = U_{p'}$, $b^p_\gamma = b^{p'}_\gamma$ and $f^p_\gamma = f^{p'}_\gamma$ for each $\gamma \in U_{p'}$. We write also $p \upharpoonright \mathfrak{B} = p'$ to mean the same thing as above except replacing $\mathfrak{A}_\gamma$ by $\mathfrak{B}$. Notice that for $p, p' \in \mathfrak{A}_\gamma$ the sentence $p \upharpoonright \mathfrak{B} = p'$ is first-order with parameters in $\mathfrak{A}_\gamma$, *i.e.* the term $\mathfrak{B}$ could be eliminated. Next we are going to do a complicated inductive construction of several sequences.

We construct inductively the sequences

$$\langle p_n \in \mathbb{P} : n \in \omega \rangle,$$

22

$$\langle p_s \in \mathbb{P} : s \in 2^{<\omega}\rangle,$$
$$\langle \eta_n \in \omega_1 : n \in \omega\rangle \text{ and}$$
$$\langle x_s \in \omega_1^{<\omega_1} : s \in 2^{<\omega}\rangle$$

in $\mathfrak{A}_{\gamma_0}$ such that

(1) $p_{n+1} < p_n$ and $\alpha_{p_n} < \alpha_{p_{n+1}}$ for every $n \in \omega$,

(2) $p_s \leqslant p_{s'}$ for any $s, s' \in 2^{<\omega}$ and $s' \subseteq s$,

(3) $p_s \restriction \mathfrak{B} = p_n$ for any $n \in \omega$ and $s \in 2^n$,

(4) $\eta_n < \eta_{n+1}$ for every $n \in \omega$,

(5) $x_{s'} \leqslant x_s$ for any $s, s' \in 2^{<\omega}$ and $s' \subseteq s$,

(6) $ht(x_s) = \eta_n$ for any $s \in 2^n$,

(7) $x_s \neq x_{s'}$ for any $s, s' \in 2^n$ and $s \neq s'$,

(8) $p_s \Vdash x_s \in \dot{c}_{\gamma_0}$ for every $s \in 2^{<\omega}$,

(9) t_{p_n} is consistent with respect to $\{b_\gamma^{p_s} : \gamma \in \bigcup_{s \in 2^n} U_{p_s}\}$ for each $n \in \omega$,

(10) $\beta_\gamma^{p_s} = \alpha_{p_s}$ for all $\gamma \in U_{p_s}$ such that $\beta_\gamma^{p_{s'}} \neq \beta_\gamma^{p_{s''}}$ for some $s' \subseteq s'' \subseteq s$,

(11) $\{b_\gamma^{p_s} : \gamma \in U_{p_s} \smallsetminus U_{p_n}\}$ and $\{b_\gamma^{p_{s'}} : \gamma \in U_{p_{s'}} \smallsetminus U_{p_n}\}$ are the same set of functions

for all $s, s' \in 2^n$.

We need to add more requirements for those sequences along the inductive construction.

For any $s \in 2^{<\omega}$ let

$$U_s = \{\gamma \in U_{p_s} : \exists s', s'' (s' \subseteq s'' \subseteq s \text{ and } \beta_\gamma^{p_{s'}} \neq \beta_\gamma^{p_{s''}})\}.$$

Let's fix an onto function $j : \omega \mapsto \omega \times \omega$ such that $j(n) = \langle a, b\rangle$ implies $a \leqslant n$. Let π_1, π_2 be projections from $\omega \times \omega$ to ω such that $\pi_1(\langle a, b\rangle) = a$ and $\pi_2(\langle a, b\rangle) = b$. Let

$$\xi_n : \omega \mapsto t_{p_n} \times ([\bigcup_{s \in 2^n} U_s]^{<\omega})$$

and

$$\zeta_n : \omega \mapsto \bigcup_{s \in 2^n} U_s$$

be two onto functions for each $n \in \omega$. Let e be a function with $domain(e) = \omega$ such that

$$e(n) = \xi_{\pi_1(j(n))}(\pi_2(j(n))).$$

The functions ξ_n's, ζ_n's and e are going to be used for bookkeeping purpose. For $s \in 2^m$ and $m < n$ let

$$C_{s,n} = \{s' \in 2^n : s \subseteq s'\}.$$

For any $m, n \in \omega$, $m \leqslant n$ let

$$Z_m^n = \begin{aligned} &\{b_\gamma^{p_{s'}} : s \in 2^{\pi_1(j(m))}, \gamma \in \pi_2(e(m)) \cap U_s \text{ and } s' \in C_{s,n}\} \cup \\ &\{b_\gamma^{p_{s'}} : s \in 2^{\pi_1(j(m))}, \gamma \in U_s \text{ and } \gamma = \zeta_{\pi_1(j(m))}(i) \text{ for some } i \leqslant n\}. \end{aligned}$$

Note that Z_m^n is finite and for each $b_\gamma^p \in Z_m^n$ we have $\beta_\gamma^p = \alpha_{p_n}$. For each $m, n \in \omega$ we need also construct another set

$$Y_m^n = \{y_{m,i} : m \leqslant i \leqslant n\}.$$

Then Z_m^n's and Y_m^n's and other four sequences should satisfy two more conditions.

(12) $y_{m,m} = \pi_1(e(m))$ and $y_{m,i} \in (t_{p_i})_{\alpha_{p_i}}$ for $m < i \leqslant n$,

(13) $y_{m,i+1}$ is a conservative extension of $y_{m,i}$ with respact to Z_m^{i+1}.

Next we do the inductive construction. Suppose we have had sequences

$\langle p_n \in \mathbb{P} : n < l \rangle$,

$\langle p_s \in \mathbb{P} : s \in 2^{<l} \rangle$,

$\langle \eta_n \in \omega_1 : n < l \rangle$,

$\langle x_s \in \omega_1^{<\omega_1} : s \in 2^{<l} \rangle$,

$\{Z_m^n : n < l, m \leqslant n\}$ and

$\{Y_m^n : n < l, m \leqslant n\}$.

We first choose distinct $\{\gamma_s : s \in 2^l\} \subseteq S$. For any $s \in 2^l$ let $p^s = h_{\gamma_0, \gamma_s}(p_{s\restriction l})$. Note that

$$p^s = \langle \alpha_{p_{s\restriction l}}, t_{p_{s\restriction l}}, k_{p_{s\restriction l}}, U_{p^s}, B_{p^s}, F_{p^s} \rangle$$

where

$$U_{p^s} = \{h_{\gamma_0, \gamma_s}(\gamma) : \gamma \in U_{p_{s\restriction l}}\},$$

$$B_{p^s} = \{b_{h_{\gamma_0, \gamma_s}(\gamma)}^{p^s} : b_{h_{\gamma_0, \gamma_s}(\gamma)}^{p^s} = h_{\gamma_0, \gamma_s}(b_\gamma^{p_{s\restriction l}}) \text{ and } \gamma \in U_{p_{s\restriction l}}\}$$

and

$$F_{p^s} = \{h_{\gamma_0, \gamma_s}(f_\gamma^{p_{s\restriction l}}) : \gamma \in U_{p_{s\restriction l}}\}.$$

Notice that $b_{h_{\gamma_0, \gamma_s}(\gamma)}^{p^s}$ and $b_\gamma^{p_{s\restriction l}}$ are same functions with different indices. Notice also that

$$\alpha_{p_{s\restriction l}} = \alpha_{p_{l-1}}, \quad t_{p_{s\restriction l}} = t_{p_{l-1}}, \quad k_{p_{s\restriction l}} = k_{p_{l-1}}$$

and

$$U_{p^s} = \{h_{\gamma_0, \gamma_s}(\gamma) : \gamma \in U_{p_{s\restriction l}} \smallsetminus U_{p_{l-1}}\} \cup U_{p_{l-1}}.$$

Let

$$\bar{p}_{l-1} = \langle \alpha_{p_{l-1}}, t_{p_{l-1}}, k_{p_{l-1}}, U_{\bar{p}_{l-1}}, B_{\bar{p}_{l-1}}, F_{\bar{p}_{l-1}} \rangle$$

24

where

$$U_{\bar{p}_{l-1}} = \bigcup_{s \in 2^l} U_{p^s},$$

$$B_{\bar{p}_{l-1}} = \{ b_\gamma^{p^s} : s \in 2^l, \gamma \in U_{p^s} \}$$

and

$$F_{\bar{p}_{l-1}} = \{ f_\gamma^{p^s} : s \in 2^l, \gamma \in U_{p^s} \}.$$

Since t_{p-1} is consistent with $\bigcup_{s \in 2^{l-1}} B_{p_s}$ by (9), then we have $\bar{p}_{l-1} \in \mathbb{P}$. Since

$$\bar{p}_{l-1} \Vdash \{ \dot{c}_{\gamma_s} : s \in 2^l \} \text{ is a set of distinct branches of } \dot{T},$$

then there exist $\bar{p}_l \leqslant \bar{p}_{l-1}$, $\eta_l \in \omega_1$ such that $\eta_l > \eta_{l-1}$, and there exist distinct

$$\{ x_s : s \in 2^l \} \subseteq \omega^{\eta_l}$$

such that

$$\bar{p}_l \Vdash x_s \in \dot{c}_{\gamma_s}$$

for all $s \in 2^l$. We can also require that $\alpha_{\bar{p}_l} > \alpha_{\bar{p}_{l-1}}$ and $\beta_\gamma^{\bar{p}_l} = \alpha_{\bar{p}_l}$ for all $\gamma \in U_{\bar{p}_l}$ such that $\beta_\gamma^{\bar{p}_l} > \beta_\gamma^{\bar{p}_{l-1}}$, or for all $\gamma \in \bigcup_{s \in 2^{l-1}} h_{\gamma_0, \gamma_s}[U_{s \restriction l}]$. For each $s \in 2^l$ let $\bar{U}_s$ be a set of ω_1 ordinals such that $\bar{U}_s \subseteq \mathfrak{A}_{\gamma_s} \smallsetminus \mathfrak{B}$ and $\bar{U}_s \cap U_{\bar{p}_l} = \emptyset$. Since $B_{\bar{p}_l}$ has only $\leqslant \omega_1$ functions, we can use the ordinals in $\bar{U}_s$ to re-index all functions in $B_{\bar{p}_l}$, say $B_{\bar{p}_l}$ and $\{ b_\gamma^{\bar{p}_l} : \gamma \in \bar{U}_s \}$ are same set of functions. Let $f_\gamma^{\bar{p}_l}$ be an empty function for each $\gamma \in \bar{U}_s$. We now construct a $\bar{p}$ such that

$$\bar{p} = \langle \alpha_{\bar{p}_l}, t_{\bar{p}_l}, k_{\bar{p}_l}, U_{\bar{p}}, B_{\bar{p}}, F_{\bar{p}} \rangle,$$

where

$$U_{\bar{p}} = U_{\bar{p}_l} \cup \Big(\bigcup_{s \in 2^l} \bar{U}_s \Big),$$

$$B_{\bar{p}} = B_{\bar{p}_l} \cup \Big(\bigcup_{s \in 2^l} \{ b_\gamma^{\bar{p}_l} : \gamma \in \bar{U}_s \} \Big)$$

and

$$F_{\bar{p}} = F_{\bar{p}_l} \cup \Big(\bigcup_{s \in 2^l} \{ f_\gamma^{\bar{p}_l} : \gamma \in \bar{U}_s \} \Big).$$

It is easy to see that $\bar{p} \in \mathbb{P}$ and $\bar{p} \leqslant \bar{p}_l$.

Claim 16.2 For each $s \in 2^l$ let $\bar{p}_s = \bar{p} \restriction \mathfrak{A}_{\gamma_s}$. Then $\bar{p}_s \Vdash x_s \in \dot{c}_{\gamma_s}$.

Proof of Claim 16.2: It is true that $\bar{p}_s \in \mathfrak{A}_{\gamma_s}$ because $(\mathfrak{A}_{\gamma_s})^{\leqslant \omega_1} \subseteq \mathfrak{A}_{\gamma_s}$. Suppose

$$\bar{p}_s \nVdash x_s \in \dot{c}_{\gamma_s}.$$

Then there is a $p'_s \leqslant \bar{p}_s$ such that

$$p'_s \Vdash x_s \notin \dot{c}_{\gamma_s}.$$

Since $\mathfrak{A}_{\gamma_s} \preceq H(\lambda)$, we can choose $p'_s \in \mathfrak{A}_{\gamma_s}$. It is now easy to see that p'_s and $\bar{p}$ are compatible (here we use the fact that every function in $B_{\bar{p}}$ is also in $B_{\bar{p}_s}$ with possibly different index). This derives a contradiction.

Let $p_l = \bar{p} \upharpoonright \mathfrak{B}$ and $p_s = h_{\gamma_s,\gamma_0}(\bar{p}_s)$. Then
$\langle p_n : n \leqslant l \rangle$,
$\langle p_s : s \in 2^{\leqslant l} \rangle$,
$\langle \eta_n : n \leqslant l \rangle$ and
$\langle x_s : s \in 2^{\leqslant l} \rangle$
satisfy conditions (1)—(11). For example, we have

$$p_s \Vdash x_s \in \dot{c}_{\gamma_0}$$

because $p_s = h_{\gamma_s,\gamma_0}(\bar{p}_s)$, $\gamma_0 = h_{\gamma_s,\gamma_0}(\gamma_s)$ and $x_s = h_{\gamma_s,\gamma_0}(x_s)$. We have also that t_{p_l} is consistent with $\{b_{\gamma}^{p_s} : s \in 2^l \text{ and } \gamma \in U_{p_s}\}$ because $\bar{p} \in \mathbb{P}$.

We need to deal with the conditions (12) and (13).

For each $m \leqslant l$ the set Z_m^l has been defined before. For $m < l$ since Z_m^l is finite, there exists a $y_{m,l} \in (t_{p_l})_{\alpha_{p_l}}$ such that $y_{m,l}$ is a consistent extension of $y_{m,l-1}$ with respect to Z_m^l. Let $y_{l,l} = \pi_1(e(l))$. It is not hard to see that those sequences up to stage l satisfy conditions (12) and (13). This ends the construction.

We want to draw the conclusion now.

For each $m \in \omega$ let $y_m = \bigcup_{i \in \omega} y_{m,i}$ and let

$$t_{p_\omega} = (\bigcup_{n \in \omega} t_{p_n}) \cup \{y_m : m \in \omega\}.$$

It is easy to see that $t_{p_\omega} \in \mathcal{T}$. Let $\alpha_{p_\omega} = \bigcup_{n \in \omega} \alpha_{p_n}$. Then $ht(t_{p_\omega}) = \alpha_{p_\omega} + 1$. Let

$$U = \bigcup_{s \in 2^{<\omega}} U_s = \{\gamma : \exists \tau \in 2^\omega \text{ such that } \bigcup\{\beta_\gamma^{p_{\tau \upharpoonright n}} : n \in \omega \text{ and } \gamma \in U_{p_{\tau \upharpoonright n}}\} = \alpha_{p_\omega}\}.$$

Then U is a countable set. Notice that for any $s \in 2^{<\omega}$ and $\gamma \in U_{p_s} \smallsetminus U$, for any $s' \supseteq s$ we have $\beta_\gamma^s = \beta_\gamma^{s'}$. Let $k' = \bigcup_{n \in \omega} k_{p_n}$. For each $\tau \in 2^\omega$ and $\gamma \in \bigcup_{s \in 2^{<\omega}} U_{p_s}$ let

$$b_\gamma^\tau = \bigcup\{b_\gamma^{p_{\tau \upharpoonright n}} : \gamma \in U_{p_{\tau \upharpoonright n}}, n \in \omega\}$$

and let

$$f_\gamma^\tau = \bigcup\{f_\gamma^{p_{\tau \upharpoonright n}} : \gamma \in U_{p_{\tau \upharpoonright n}}, n \in \omega\}.$$

26

For each $m \in \omega$ and $\gamma \in U$ we define

$$b_\gamma^\tau(y_m) = \bigcup \{b_\gamma^\tau(y_{m,i}) : m \leqslant i < \omega\}.$$

Since for each $\gamma \in U$ and $m \in \omega$ there exists an n such that for any $s, s' \in 2^l$ for $l \geqslant n$ and $s \upharpoonright n = s' \upharpoonright n$ we have $b_\gamma^{p_s}(y_m) = b_\gamma^{p_{s'}}(y_m)$. This is guaranteed by the construction of Z_m^n's and Y_m^n's. So for any $\tau, \tau' \in 2^\omega$,

$$\tau \upharpoonright n = \tau' \upharpoonright n \text{ implies } b_\gamma^\tau(y_m) = b_\gamma^{\tau'}(y_m).$$

Hence for each $m \in \omega$ the set

$$\{b_\gamma^\tau(y_m) : \tau \in 2^\omega, \gamma \in U\}$$

is countable. (This is why the condition (g) of Definition 10 is needed.) Let $k'(y_m)$ be in $\mathcal{T}$ such that

$$\bigcup_{i \in \omega} k'(y_{m,i}) \leqslant_{end} k'(y_m)$$

and

$$\{b_\gamma^\tau(y_m) : \tau \in 2^\omega, \gamma \in U\} \subseteq (k'(y_m))_{\alpha_{p_\omega}}.$$

Then let $k_{p_\omega} = k'$. For each $\tau \in 2^\omega$ let $x_\tau = \bigcup_{n \in \omega} x_{\tau \upharpoonright n}$. Then $x_\tau \in \omega_1^{\alpha_{p_\omega}}$. For each $\tau \in 2^\omega$ let p_τ be the lower bound of $\{p_{\tau \upharpoonright n} : n \in \omega\}$ constructed same as in Lemma 12. Then we have $p_\tau \in \mathbb{P}$ and

$$p_\tau \Vdash x_\tau \in \dot{c}_{\gamma_0}.$$

Choose distinct ordinals $\{\gamma_\tau : \tau \in O\} \subseteq S$ for some $O \subseteq 2^\omega$ and $|O| = \omega_1$. Let $p^\tau = h_{\gamma_0, \gamma_\tau}(p_\tau)$. Then

$$p^\tau \Vdash x_s \in \dot{c}_{\gamma_\tau}$$

for any $\tau \in O$. Let

$$q = \langle \alpha_{p_\omega}, t_{p_\omega}, k_{p_\omega}, U_q, B_q, F_q \rangle,$$

where

$$U_q = \bigcup_{\tau \in O} \{h_{\gamma_0, \gamma_\tau}(\gamma) : \gamma \in U_{p_\tau}\},$$

$$B_q = \bigcup_{\tau \in O} \{h_{\gamma_0, \gamma_\tau}(b_\gamma^\tau) : \gamma \in U_{p_\tau}\}$$

and

$$F_q = \bigcup_{\tau \in O} \{h_{\gamma_0, \gamma_\tau}(f_\gamma^\tau) : \gamma \in U_{p_\tau}\}.$$

Claim 16.3 The element q is in $\mathbb{P}$ and $q \leqslant p^\tau$ for all $\tau \in O$.

Proof of Claim 16.3: It is easy to see that $|U_q| \leqslant \omega_1$ (the condition $|U_p| \leqslant \omega_1$ for $p \in \mathbb{P}$ in Definition 10 is needed here since if we require only $|U_p| < \omega_1$, then q wouldn't be in $\mathbb{P}$). It is also easy to see that for each $\tau \in O$ we have $q \restriction \mathfrak{A}_{\gamma_\tau} = p^\tau$. Hence it suffices to show that t_{p_ω} is consistent with B_q. But this is guaranteed by condition (9) and the construction of y_m's.

Claim 16.4 $q \Vdash (\dot{T})_{\alpha_{p_\omega}}$ is uncountable.

Proof of Claim 16.4: This is because of the facts $x_\tau \neq x_{\tau'}$ for different $\tau, \tau' \in O$, $|O| = \omega_1$,

$$q \Vdash x_\tau \in \dot{c}_{\gamma_\tau}$$

and

$$q \Vdash \dot{c}_{\gamma_\tau} \subseteq \dot{T}.$$

By above claim we have derived a contradiction that

$$\Vdash (\dot{T} \text{ is a Kurepa tree})$$

but

$$q \Vdash (\dot{T} \text{ is not a Kurepa tree}). \quad \square$$

3. Questions

We would like to ask some questions.

Question 1. *Suppose our ground model is the Lévy model defined in the first section. Can we find a proper forcing notion such that the forcing extension will contain Kurepa trees? If the answer is 'no', then we would like to know if there are any forcing notions of size $\leqslant \omega_1$ which preserve ω_1 such that the generic extension contains Kurepa trees?*

Question 2. *Suppose the answer of one of the questions above is Yes. Is it true that given any model of CH there always exists an ω_1-preserving forcing notion of size $\leqslant \omega_1$ such that forcing with that notion creates Kurepa trees in the generic extension?*

Question 3. *Does there exist a model of CH plus no Kurepa trees, in which there is a c.c.c.-forcing notion of size $\leqslant \omega_1$ such that forcing with that notion creates Kurepa trees in the generic extension? If the answer is Yes, then we would like to ask the same question with c.c.c. replaced by one of some nicer chain conditions such as $\aleph_1$-caliber, Property K, etc.*

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