DOP and FCP in Generic Structures

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1 Context

1.1 Context. We work throughout in a finite relational language L. This paper is built on [2] and [3]. We repeat some of the basic notions and results from these papers for the convenience of the reader but familiarity with the setup in the first few sections of [3] is needed to read this paper. Spencer and Shelah [6] constructed for each irrational α between 0 and 1 the theory T^α as the almost sure theory of random graphs with edge probability $n^{-\alpha}$. In [2] we proved that this was the same theory as the theory T_α built by constructing a generic model in [3]. In this paper we explore some of the

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more subtle model theoretic properties of this theory. We show that T^α has the dimensional order property and does not have the finite cover property.

We work in the framework of [3] so probability theory is not needed in this paper. This choice allows us to consider a wider class of theories than just the T_α. The basic facts cited from [3] were due to Hrushovski [4]; a full bibliography is in [3]. For general background in stability theory see [1] or [5].

We work at three levels of generality. The first is given by an axiomatic framework in Context 1.11. Section 2 is carried out in this generality. The main family of examples for this context is described Examples 1.4. Sections 3 and 4 depend on a function δ assigning a real number to each finite L-structure as in these examples. Some of the constructions in Section 3 (labeled at the time) use heavily the restriction of the class of examples to graphs. The first author acknowledges useful discussions on this paper with Sergei Starchenko.

1.2 Notation. Let $\mathbf{K}_0$ be a class of finite structures closed under substructure and isomorphism and containing the empty structure. Let $\overline{\mathbf{K}}_0$ be the universal class determined by $\mathbf{K}_0$.

1.3 Notation. Let $B \cap C = A$. The *free amalgam* of B and C over A, denoted $B \bigotimes_A C$, is the structure with universe BC but no relations not in B or C.

We write $A \subseteq_\omega B$ to mean A is a finite subset of B. A structure A is called *discrete* if there are no relations among the elements of A. Let $\delta : \mathbf{K}_0 \mapsto \Re^+$ (the nonnegative reals) be an arbitrary function with $\delta(\emptyset) = 0$. Extend δ to $d : \overline{\mathbf{K}}_0 \times \mathbf{K}_0 \mapsto \Re^+$ by for each $N \in \overline{\mathbf{K}}_0$,

$$d(N, A) = \inf\{\delta(B) : A \subseteq B \subseteq_\omega N\}.$$

We usually write $d(N, A)$ as $d_N(A)$. We only use this definition when δ is defined on every finite subset of N. We will omit the subscript N if it is clear from context.

For $g = \delta$ or d_N and finite A, B, we define *relative dimension* by $g(A/B) = g(AB) - g(B)$. For infinite B and finite A, $d(A/B) = \inf\{d(A/B_0) : B_0 \subset_\omega B\}$. This definition is justified in e.g. Section 3 of [3]. For any finite sequence $\overline{a} \in N$, $d_N(\overline{a})$ is the same as $d_N(A)$ where $\overline{a}$ enumerates A.

Consider a finite structure B for a finite relational language L. We assume that each relation of L holds of a tuple $\overline{a}$ only if the elements $\overline{a}$ are distinct and if $R(\overline{a})$ holds, $R(\overline{a}')$ holds for any permutation $\overline{a}'$ of $\overline{a}$.

$R(B)$ denotes the collection of subsets $B_0 = \{b_1, \dots b_n\}$ of B such that for some (any) ordering $\overline{b}$ of B_0, $B \models R(\overline{b})$ for some relation symbol R of L; $e(B) = |R(B)|$. Let A, B, C be disjoint sets. We write $R(A, B)$ for the collection of subsets from AB that satisfy some relation of L (counting with multiplicity if a set satisfies more than one relation) and contain at least one member of A and one of B. Write $e(A, B)$ for $|R(A, B)|$. Similarly, we write $R(A, B, C)$ for the collection of subsets from ABC that satisfy some relation of L and contain at least one member of A and one of C. Write $e(A, B, C)$ for $|R(A, B, C)|$.

1.4 Example. The most important examples arise by defining δ as follows. In the last section of [3] we enumerated several other examples to which this axiomatization applies. Let

$$\delta_{\beta,\alpha}(A) = \beta|A| - \alpha e(A).$$

We may write δ_α for $\delta_{1,\alpha}$. The class $\mathbf{K}_\alpha$ is the collection of finite L-structures A such that for any $A' \subseteq A$, $\delta_\alpha(A') \geq 0$. We denote by T_α the theory of the generic model of $\mathbf{K}_\alpha$.

1.5 Axioms. Let N be in $\overline{\mathbf{K}}_0$ and let A, B, $C \in \mathbf{K_0}$ be substructures of N.

1. If A, B, and C are disjoint then $\delta(C/A) \geq \delta(C/AB)$.
2. For every n there is an $\epsilon_n > 0$ such that if $|A| < n$ and $\delta(A/B) < 0$ then $\delta(A/B) \leq -\epsilon_n$.
3. There is a real number ϵ independent of N, A, B, C such that if A, B, C are disjoint subsets of a model N and $\delta(A/B) - \delta(A/BC) < \epsilon$ then $R(A, B, C) = \emptyset$ and $\delta(A/B) = \delta(A/BC)$.
4. For each $A \in \mathbf{K}_0$, and each $A' \subseteq A$, $\delta(A') \geq 0$.

We call a function $d = d_N$ derived from δ satisfying Axioms 1.5 a *dimension function.*

1.6 Lemma. *If δ is a dimension function satisfying the properties of Axiom 1.5 and $\leq_s$ (read* strong submodel *) is defined by $A \leq_s N$ if $d_N(A) = d_A(A)$, then $\leq_s$ satisfies the following propositions. Let $M, N, N' \in \overline{\mathbf{K}}_0$.*

A1. *$M \leq_s M$.*
A2. *If $M \leq_s N$ then $M \subseteq N$.*
A3. *$M \leq_s N' \leq_s N$ implies $M \leq_s N'$.*
A4. *If $M \leq_s N$, $N' \subseteq N$ then $M \bigcap N' \leq_s N$.*
A5. *For all $M \in \overline{\mathbf{K}}_0$, $\emptyset \leq_s M$.*

We need to analyze extensions which are far from being strong.

1.7 Definition. For $A, B \in S(\mathbf{K}_0)$, $A \leq_i B$ if $A \subseteq B$ but there is no A' properly contained in B with $A \subseteq A' \leq_s B$. If $A \leq_i B$, we say B is an *intrinsic* extension of A.

1.8 Definition. The intrinsic closure of A in M, $\mathrm{icl_M(A)}$ is the union of B with $A \subseteq B \subseteq M$ and $A \leq_i B$. When M is clear from context, we write $\overline{A}$ for $\mathrm{icl_M(A)}$. The intrinsic closure can be more finely analyzed as follows.

1. For any $M \in \mathbf{K}$, any $m \in \omega$, and any $A \subseteq M$,
$$\mathrm{icl^m_M(A)} = \cup\{B : A \leq_i B \subseteq M \& |B - A| < m\}.$$
2. $$\mathrm{icl_M(A)} = \cup_m \mathrm{icl^m_M(A)}.$$
3. M has *finite closures* if for each finite $A \subseteq M$, $\mathrm{icl(A)}$ is finite.
4. $\mathbf{K}$ has *finite closures* if each $M \in \mathbf{K}$ has finite closures.

Using **A4**, note that the intrinsic closure of A in M is the intersection of the strong substructures of M which contain A. Thus, when finite, $\mathrm{icl_M(A)} \in \mathbf{K}_0$ and is a strong substructure of M. Moreover, a countable M has finite closures if and only if M can be written as an increasing union of finite strong substructures.

1.9 Definition. The countable model $M \in \overline{\mathbf{K}}_0$ is $(\mathbf{K}_0, \leq_s)$-*generic* if

1. If $A \leq_s M, A \leq_s B \in \mathbf{K}_0$, then there exists $B' \leq_s M$ such that $B \cong_A B'$,

2. M has finite closures.

1.10 Fact. If $(\mathbf{K}_0, \leq_s)$ satisfies the properties of Lemma 1.6 and the amalgamation property with respect to $\leq_s$ then there is a countable $\mathbf{K}_0$-generic model.

1.11 Context. Henceforth, $(\mathbf{K}_0, \leq_s)$ is class of finite structures closed under isomorphism and substructure with $\leq_s$ induced by a function δ obeying Axioms 1.5. Moreover, we assume $(\mathbf{K}_0, \leq_s)$ satisfies the amalgamation property and $\mathbf{K}$ is the class of models of the theory of the generic model M of $(\mathbf{K}_0, \leq_s)$. $\mathcal{M}$ is a large saturated model of $T = \mathrm{Th(M)}$. In the absence of other specification, the dimension function d is the function induced on $\mathcal{M}$ by δ and we work with substructures of $\mathcal{M}$.

2 Independence and Orthogonality

As indicated in Context 1.11, the following definitions take place in a suitably saturated model elementarily equivalent to the generic. We work in that context throughout this section.

2.1 Definition. We say the finite sets A and B are d-independent over C and write

1. $A \downarrow^d_C B$ if
   (a) $d(A/C) = d(A/CB)$.
   (b) $\overline{AC} \cap \overline{BC} \subseteq \overline{C}$.
2. We say the (arbitrary) sets A and B are d-independent over C and write $A \downarrow^d_C B$ if for every finite $A' \subseteq A$ and $B' \subseteq B$, $A' \downarrow^d_C B'$

The compatibility of the two definitions is shown, e.g., in Section 3 of [3]. The following is well known (cf. 3.31 of [3]).

2.2 Lemma. *Suppose A, B and $C = A \cap B$ are closed and $A \downarrow^d_C B$. Then AB is closed, i.e. $\overline{AB} = \overline{A} \cup \overline{B}$.*

The equivalence of d-independence and stability theoretic independence was first proved in this generality in [3] but the basic setup comes from [4].

2.3 Fact. Suppose T satisfies Context 1.11. If C is intrinsically closed then for any A and B, $A \downarrow_C B$ if and only if $A \downarrow^d{}_C B$.

We give a different proof that is not as involved with the intricacies of amalgamation in the case without finite closures as the one in [3].

Suppose for contradiction that $R(A, C, B) \neq \emptyset$. Then for ϵ chosen according to Axiom 1.5, $\delta(A/B) - \delta(A/BC) > \epsilon$. Now, construct a nonforking sequence $\langle A_i, B_i\rangle$ in $\mathrm{tp(AB/C)}$. Since A is not in the algebraic closure of BC, no A_j is in the algebraic closure of the union of B_i for $i < j$. We will use this fact to show that the types $p_i = \mathrm{tp(A_i/CB_i)}$ are n-contradictory for some n. If not, for each n there is an A^* which is common solution for, say $p_1, \ldots, p_n$. Fix n such that $n \cdot \epsilon > \delta(A/C)$. But $\delta(A^*/B_1, \ldots B_n) \leq \delta(A/C) - n \cdot \alpha$ so this implies $A \subseteq \mathrm{acl}(CB_1 \ldots, B_n)$ and this contradiction yields the result. The extension property for nonforking types and uniqueness suffice to deduce the converse from d-dependence implies forking dependence so we finish as in Lemma 3.35 of [3].

We extend our notion of dimension to a global real-valued rank on types.

2.4 Definition. Let $p \in S(A)$. Define $d(p)$ as $d(\overline{a}/A)$ for some (any) $\overline{a}$ realizing p.

2.5 Definition. Let $p_1, p_2 \in S(A)$.

1. p_1 and p_2 are disjoint if for any $\overline{a}_1$, $\overline{a}_2$ realizing p_1, p_2, $\mathrm{icl(A\overline{a}_1)} \cap \mathrm{icl(A\overline{a}_2)} \subseteq \mathrm{icl(A)}$.
2. $p_1 \in S(A)$ and $p_2 \in S(B)$ are disjoint if any pair of nonforking extensions of p_1 and p_2 to AB are disjoint.

2.6 Lemma. *Let $A \subset B$, $p \in S(B)$ and $p|A = q$ and suppose A is intrinsically closed.*

1. *If $d(p) < d(q)$ then p forks over A.*
2. *q is stationary.*

Proof. 1) follows immediately from Fact 2.3; 2) is also proved in [3] (Lemma 3.38).

2.7 Lemma. *Let A be intrinsically closed, $p_1, p_2 \in S(A)$. If p_1 and p_2 are disjoint and $d(p_1) = 0$ then p_1 and p_2 are orthogonal.*

Proof. If not, there exist sequences $a_1 \ldots a_k$ and $b_1 \ldots b_m$ of realizations of p_1 and p_2 respectively, which are independent over A, such that $\overline{a} \not\downarrow_A \overline{b}$. Since $d(p_1) = 0$, $d(\overline{a}/A) = 0$ and $\mathrm{icl}(A\overline{a}) \cap \mathrm{icl}(A\overline{b}) \not\subseteq A$. By Lemma 2.2, intrinsic closure is a trivial dependence relation. Since the a_i and the b_j are independent, this implies that for some i, j, $\mathrm{icl}(Aa_i) \cap \mathrm{icl}(Ab_j) \not\subseteq A$. But this contradicts the disjointness of p_1 and p_2 and we finish.

The *dimensional order property* (DOP) and *dimensional discontinuity property* DIDIP are defined in [5]. Either of these conditions implies T has many models in uncountable powers. T has the eventually non-isolated dimensional order property (eni-dop) if some type witnessing the dimension order property is not isolated. This condition implies that T has the maximal number of countable models. Since T_α is not small, this is not new information. However, the eni-dop seems to be a much more intrinsic feature of the construction than the smallness. (For precise definition see e.g. [1].)

2.8 Theorem. *Let $\mathbf{K}_0$ be a class satisfying Context 1.11. Let T be the theory of the generic model for $(\mathbf{K}_0, \leq_s)$. Suppose further that there is a pair of independent points $B = \{x, y\}$ and a nonalgebraic type p with $d(p/B) = 0$ but $d(p/x) > 0$ and $d(p/y) > 0$.*

1. *The theory T has the dimensional order property.*
2. *If p is not isolated the theory T has the eni dimensional order property.*
3. *The theory T has the dimensional discontinuity property.*

Proof. i) Let $A = \{a, b\}$ where a and b are independent over the empty set. It suffices to show that there is a type $p \in S(A)$ with $d(p) = 0$ and such that if $\overline{c}$ realizes p, $\overline{c} \not\downarrow_a b$ and $\overline{c} \not\downarrow_b a$. For then we can construct an independent sequence of points a_i and disjoint copies $p_{i,j}$ over $\{a_i, a_j\}$ which will be pairwise orthogonal by Lemma 2.7. The required type is constructed in Theorem 3.6. ii) follows by the same argument if p is not isolated.

For iii) it suffices to find an independent sequence of sets B_n for $n < \omega$ and $p \in S(B)$ where $B = \cup B_n$ such that $p \dashv \cup_{n<j} B_n$ for each j. Choose B_n and C_n as described at the beginning of the proof of Theorem 3.6. Let B be the union for $n < \omega$ of $B_n = \{x_n, y_n\}$ with no relations on B. For each n, let f_n map c_n to c, x to x_n and y to y_n. Then $B \cup \{c\}$ is as required. That is, $d(t(c/B)) = 0$ but $d(t(c/\cup_{n<m} B_n)) > 0$.

3 Constructing types of d-rank 0

We construct a nonalgebraic type p over a two element set with $d(p) = 0$.

3.1 Context. We work with a class $\mathbf{K}_0$ of finite structures as in Example 1.4. Thus, $(\mathbf{K}_0, \leq_s)$ witnesses Contex 1.11. Recall that $\mathbf{K}$ is the class of models of the theory of the generic M, $\mathcal{M}$ is a saturated model of this theory, and $S(\mathbf{K})$ is the universal class it determines.

Finally, the α parameterizing the dimension function may be rational or irrational. This distinction affects only the question of whether the type with rank 0 is isolated and we discuss that when it arises.

3.2 Definition. $(\mathbf{K}_0, \leq_s)$ has the *full amalgamation property* if $B \cap C = A$ and $A \leq_s B$ imply $B \bigotimes_A C \in \mathbf{K}_0$ and $C \leq_s B \bigotimes_A C$.

It is easy to check (Section 4 of [3])that if $(\mathbf{K}_0, \leq_s)$ is closed under free amalgamation then it has full amalgamation.

3.3 Assumption. $(\mathbf{K}_0, \leq_s)$ has the *full amalgamation property.*

3.4 Examples. Each of the following classes is closed under free amalgamation.

1. The class $(\mathbf{K}_\alpha, \leq_s)$ of all finite L-structures A with $\delta_{1,\alpha}(A)$ hereditarily positive. The resulting theory is ω-stable if α is rational and stable if α is irrational.

2. The class yielding the stable $\aleph_0$-categorical pseudoplane of [4].

The main aim of this section is to establish the following result which leads easily by Theorem 2.8 to showing the theory of the generic model $\mathcal{M}$ has DOP and DIDIP.

3.5 Definition. We say C is a *primitive* extension of B if $B \leq_s C$ but there is no B' properly between B and C with $B' \leq_s C$.

3.6 Theorem. *There exists a triple $\{x, y, c\} \in \mathcal{M}$ such that $B = \{x, y\}$ is an independent pair over $\emptyset$ and $d(c/xy) = 0$ but $d(c/x) > 0$, $d(c/y) > 0$ and $c \notin \mathrm{acl}(x, y)$.*

Proof. Fix a discrete structure B with universe $\{x, y\}$. We will construct a family $\langle (C_n, x_n, y_n, c_n) : n < \omega \rangle$ of structures in $\mathbf{K}_0$ which satisfy the following conditions. Let $B_n = \{x_n, y_n\}$. The inequalities in the following discussion automatically become strict inequalities if α is irrational.

1. $0 \leq \delta(C_n/B_n) < 1/n$.
2. (x_n, y_n, c_n) is a discrete substructure of C_n.
3. C_n is a primitive extension of B_n.

Now map each B_n to B and amalgamate the images of the C_n disjointly over B. Then identify all the c_n as c to form a structure A. Without loss of generality we can assume A is strongly embedded in $\mathcal{M}$. Thus, $\mathrm{icl}_{\mathcal{M}}(\mathrm{cB}) = \mathrm{A}$. Then $d(c/B) = 0$ but $d(c/x)$ and $d(c/y)$ are both at least one. Thus $c \not\downarrow_x xy$ and $c \not\downarrow_y xy$. Since $\delta(C_n/B_n) \geq 0$, for every n, $c \notin \mathrm{acl}(B)$.

3.7 Remark. If α is irrational, all the C_n are necessary and tp(c/xy) is nonprincipal. If α is rational, for some n, $\delta(C_n/B_n) = 0$. (We expand on this remark after Observation 3.9.) The type is principal but still not algebraic since in this context there are infinitely many copies (in a generic) of a primitive extension with relative dimension 0.

The construction of the C_n follows a rather tortured path. We first need to consider structures with negative dimension over B.

3.8 Definition. Let $\mathcal{A} = \mathcal{A}_\alpha$ be the class of structures of the form (A, a, b, e) which satisfy the following conditions. Let B be the structure with universe $\{a, b\}$ and no relations.

1. $A \in \mathbf{K}_0$.
2. $\{a, b, e\}$ is the universe of a discrete substructure of A.
3. For each A' with $B \subseteq A'$ and A' properly contained in A, $\delta(A') > \delta(A)$.
4. $-1 < \delta(A/B) \leq 0$.

3.9 Observation. 1. The choice of δ as δ_α makes $\mathcal{A}$ depend on α.

2. If the last three conditions are satisfied, the first is as well.

3. The last condition implies that $\delta(A/a) > 0$ and $\delta(A/b) > 0$.

We first show that the set

$$X = X_\alpha = \{\beta : \beta = \delta(A/\{a,b\}) \text{ for some } (A,a,b,e) \in \mathcal{A}\}$$

is not bounded away from zero. If α is irrational, each element of X is irrational so this implies X is infinite. If $\alpha = p/q$ is rational, every element of X has the form $(mq - np)/q$ so there cannot be an infinite sequence of members of X tending to 0. That is, there will be an A with $\delta(A/B) = 0$. As indicated X depends on α (through $\delta = \delta_\alpha$ and $\mathcal{A} = \mathcal{A}_\alpha$.) But the bulk of the proof is uniform in α, so to enhance readability we keep track of α only for that part of the proof where the dependence is not uniform.

3.10 Construction. There are two elementary steps in the construction. It is easy to check that if the constituent models described here are in $\mathbf{K}_0$, then so is the result.

1. If $\delta(A/B) = \beta$ and $\beta \in X$, and A^* is the free amalgam over B of k copies of A, then $\delta(A^*/B) = k\beta$.

2. Let (A_1, a_1, b_1, c_1) and (A_2, a_2, b_2, c_2) be in $\mathcal{A}$. Let A^* be formed by identifying b_1 and a_2 and freely amalgamating over that point.

3.11 Lemma. *If $\beta > -1/k$ and $\beta \in X$ then $k\beta \in X$.*

Proof. Use Construction 3.10 i).

It is straightforward to determine the following properties of the second construction.

3.12 Lemma. *Suppose $\delta(A_1/\{a_1, b_1\}) = \beta_1$, $\delta(A_2/\{a_2, b_2\}) = \beta_2$ and $\beta_1, \beta_2 \in X$. Let A^* be formed as in Construction 3.10 ii).*

1. *$\delta(A^*/\{a_1, b_2\}) = \beta_1 + \beta_2 + 1$.*

2. *If $-2 < \beta_1 + \beta_2 \leq -1$ then $\beta_1 + \beta_2 + 1 \in X$ and $\langle A^*, a_1, b_2, c_1\rangle \in \mathcal{A}$.*

3. *If* $-1 \leq \beta_1 + \beta_2 < -1 + 1/n$ *then*

   (a) $0 \leq \delta(A^*/\{a_1, b_2\}) < 1/n$.

   (b) $\delta(A^*/a_1) \geq 1$ *and* $\delta(A^*/b_2) \geq 1$.

Proof. The key observations for 1)and thus 2) and 3a) is that for any $B \subseteq A_1 \subseteq A^*$,

$$\delta(A'/\{a_1, b_2\}) = \delta(A' \cap A_1/\{a_1, b_1\}) + \delta(A' \cap A_2/\{a_2, b_2\}) + 1.$$

For 3b) we need the further remark:

$$\delta(A'/a_1) = \delta(A'/b_2) = \delta(A'/\{a_1, b_2\}) + 1.$$

3.13 Lemma. *If L contains a single binary relation and $\mathbf{K}_0 = \mathbf{K}_\alpha$, then X is not empty.*

Proof. It suffices to show that each $\mathcal{A}_\alpha$ is nonempty for $0 < \alpha \leq 1$. The construction is somewhat ad hoc and proceeds by a number of cases depending on α. Thus to establish Lemma 3.13 we will use the notations $\mathcal{A}_\alpha, \delta_\alpha$. These constructions are very specific to graphs. The second author has an alternative argument which avoids the dependence on α. However, it passes through hypergraphs and has it own computational complexities.

3.14 Case 1. $3/4 < \alpha < 1$: Let A_1 be the structure obtained by adding to $\{a, b, e\}$ two points b_1, b_2 such that b_1 is connected to a and e while b_2 is connected to b and e. Then

$$-1 < \delta_\alpha(A_1/B) = 3 - 4\alpha < 0$$

for the indicated α and $(A_1, a, b, e) \in \mathcal{A}_\alpha$.

3.15 Case 2. $2/3 \leq \alpha < 4/5$: Let A_2 be the structure obtained by adding to $\{a, b, e\}$ two points b_1, b_2 such that b_1 is connected to a, b, and e while b_2 is connected to b and e. Then

$$-1 < \delta_\alpha(A_2/B) = 3 - 5\alpha < 0$$

for the indicated α and $(A_2, a, b, e) \in \mathcal{A}_\alpha$.

3.16 Case 3. $0 < \alpha < 2/3$: Let $A_{n,k}$ be the structure obtained by adding to $\{a, b, e\}$ both n points $a_1, \ldots, a_n$ such that each a_i is connected to a, b, and e and k points $b_1, \ldots, b_k$ such that each b_i is connected to all the a_i.

Then $\delta_\alpha(A_{n,k}/B) = n + k + 1 - (nk + 3n)\alpha$. We say α is *acceptable* for n and k if the following inequality is satisfied.

$$\ell_{n,k} = \frac{n+k+1}{nk+3n} < \alpha < \frac{n+k+2}{nk+3n} = u_{n,k}.$$

To show that if α is acceptable for n and k, then $(A_{n,k}, a, b, e) \in \mathcal{A}_\alpha$ we need several claims.

3.17 Claim 1. For each k,

1. $u_{n+1,k} > \ell_{n,k}$,
2. $\ell_{n+1,k} < \ell_{n,k}$,
3. $\lim_{n\to\infty} \ell_{n,k} = 1/(k + 3)$.

Claim 1 is established by routine computations.

3.18 Claim 2. For every α that is acceptable for n and k, if $B \subseteq A' \subseteq A_{n,k}$, $\delta_\alpha(A'/B) \geq \delta_\alpha(A_{n,k}/B)$.

To see this, note that any such A', for some $m \leq n$ and $\ell \leq k$, either A' has the form $A_{m,\ell}$ or the form $B_{m,\ell}$, where $B_{m,\ell}$ is the structure obtained by omitting the element e from $A_{m,\ell}$. Now note that if $\delta_\alpha(B_{m,\ell}/B) < 0$ then $\delta_\alpha(B_{m,\ell}/B) \geq \delta_\alpha(B_{m+1,\ell}/B)$ and $\delta_\alpha(B_{m,\ell}/B) \geq \delta_\alpha(B_{m,\ell+1}/B)$. The same assertion holds when $A_{m,\ell}$ is substituted for $B_{m,\ell}$. Finally, $\delta_\alpha(B_{n,k}/B) \geq \delta_\alpha(A_{n,k}/B)$. These three observations yield the second claim.

From these two claims we see that for each α, there is a pair n, k with $A_{n,k} \in \mathcal{A}_\alpha$. The remainder of the argument does not depend on α so we return to the use of the notation X and $\mathcal{A}$.

3.19 Lemma. *For every n there is an element β of X with $\beta > -1/n$.*

Proof. If not, fix the least n such that all elements of X are at most $-1/(n+1)$ and fix $\beta_0 \in X$ with $-1/n < \beta_0 \leq -1/(n+1)$. (If $\beta_0 = -1/(n+1)$, $\beta_1 = 0$ and we finish.) Define by induction $\beta_{\ell+1} = (n+1)\beta_\ell + 1$. Combining the two elementary steps we see that each $\beta_\ell \in X$. Let β'_ℓ be the distance between

$-1/n$ and β_ℓ. That is, $\beta'_\ell = |-1/n - \beta_\ell| = 1/n + \beta_\ell$. Now $\beta_\ell \leq -1/(n+1)$ if and only if $\beta'_\ell \leq 1/(n)(n+1)$.

But

$$\beta'_{\ell+1} = 1/n + (n+1)\beta_\ell + 1 = (n+1)\beta'_\ell.$$

So

$$\beta'_\ell = (n+1)^\ell \beta'_0.$$

As $\beta'_0 > 0$, for sufficiently large ℓ, $\beta'_\ell > 1/(n)(n+1)$ so $\beta_\ell < -1/(n+1)$ as required.

With a few more applications of our fundamental constructions, we can find the C_n needed for Theorem 3.6.

By applying Construction 3.10 i) and Lemma 3.19 for any n, and $i = 1, 2$ we can find $(A_1^n, x_1^n, y_1^n, c_1^n)$ and $(A_2^n, x_2^n, y_n^n, c_2^n)$ containing $B_i^n = \{x_i^n, y_i^n\}$ such that $\{x_i^n, y_i^n, c_i^n\}$ is discrete and $\delta(A_i^n/B_i^n) = \beta_i^n$ with $-1 < \beta_1^n + \beta_2^n < -1 + 1/n$.

To construct A_1^n, choose using Lemma 3.19 a $(D^n, x_1^n, y_1^n, c_1^n) \in \mathcal{A}$ with $-1/n < \delta(D^n/B_1^n) \leq 0$. Take an appropriate number, k, of copies of D^n over B_1^n and apply Construction 3.10 i) to form A_1^n with

$$-1 < k\delta(D^n/B_1^n) = \delta(A_1^n/B_1^n) = \beta_1^n < -1 + 1/n$$

and choose $c_1^n \in A_1^n$ so that (x_1^n, y_1^n, c_1^n) is discrete. By Lemma 3.19 again choose $(A_2^n, x_2^n, y_2^n, c_2^n) \in \mathcal{A}$ with

$$-(\beta_1^n + 1)/2 < \delta(A_2^n/B_2^n) = \beta_2^n < 0.$$

Now apply Construction 3.10 ii) to $(A_1^n, x_1^n, y_1^n, c_1^n)$ and $(A_2^n, x_2^n, y_n^n, c_2^n)$ to form (C_n, x_n, y_n, c_n) where $x_n = x_1^n$, $y_n = y_2^n$, and $c_n = c_1^n$. Denote $\{x_n, y_n\}$ by B_n. Then $0 < \delta(C_n/B_n) = 1 + \beta_1^n + \beta_2^n < 1/n$. Each C_n contains a discrete set $\{x_n, y_n, c_n\}$ and the third property of the C_n follows using the second part of Lemma 3.12. This completes the construction of the type of d-rank 0.

Using the argument for constructing A_1^n, we easily show the following density result.

3.20 Corollary. *For any γ, δ with $-1 \leq \gamma < \delta < 0$ there is a $(D, a, b, e) \in \mathcal{A}$ with $\gamma < \delta(D/\{a, b\}) < \delta$.*

The restriction to one-types in the following lemma is solely for ease of presentation.

3.21 Lemma. *Suppose $A \subseteq M \models T_\alpha$ is intrinsically closed and $p_1, p_2 \in S_1(A)$ are disjoint. If $0 < d(p_i)$ for $i = 1, 2$ then $p_1 \not\perp p_2$.*

Proof. Clearly if p_1 and p_2 are not disjoint or if there is an edge between realizations of the two types, they are not orthogonal. Let a_1, a_2 realize p_1, p_2 and suppose for contradiction that p_1 and $P - 2$ are orthogonal and $d(a_1a_2/A) = d(a_1/A) + d(a_2/A) = \beta > 0$. In particular, there is no edge linking a_1 and a_2. By Lemma 3.25 of [3] there are finite $A_1 \supseteq a_1a_2$ and $A_0 \subset A$ with $\beta \leq \gamma = \delta(A_1/A_0) < \beta + 1$. Lemma 3.20 allows us to choose a finite $B \supseteq \{a_1, a_2\}$ with

$$-1 < \delta(B/\{a_1, a_2\}) < \beta - \gamma < 0.$$

Then Ba_1a_2 is in $\mathbf{K}_0$. B y full amalgamation we can freely amalgamate B with AA_1 over $\{a_1, a_2\}$ inside $\mathcal{M}$. Then $d(a_1a_2/A) \leq \delta(A_1B/A_0)$. Note $\delta(B/A_1A_0) = \delta(B/\{a_1, a_2\}) < \beta - \gamma$. So

$$\delta(A_1B/A_0) = \delta(B/A_1A_0) + \delta(A_1/A_0) < \beta.$$

This contradicts $d(a_1a_2/A) = \beta$ so we conclude $p_1 \not\perp p_2$.

Using the Lemmas 2.7 and 3.20 it is easy to see

3.22 Corollary. *In T_α,*

1. *For disjoint p_1, p_2, $p_1 \perp p_2$ if and only if $d(p_1) = 0$ or $d(p_2) = 0$.*
2. *Every regular type satisfies $d(p) = 0$.*

Our construction yields some further information.

3.23 Definition. The type $p \in S(A)$ is *minimal* if p is not algebraic but for any formula $\phi(x, \overline{b})$ either $p \cup \{\phi(x, \overline{b})\}$ or $p \cup \{\neg\phi(x, \overline{b})\}$ is algebraic.

3.24 Definition. The type $p \in S(A)$ is i-minimal if for every $\overline{a}$ realizing p, if $c \in \mathrm{icl}(\mathrm{A}\overline{\mathrm{a}})$, $\mathrm{icl}(\mathrm{Ac}) = \mathrm{icl}(\mathrm{A}\overline{\mathrm{a}})$.

3.25 Theorem. *If p is constructed as in Lemma 3.6 then p is minimal and trivial.*

Proof. If $d(p) = 0$ and p is i-minimal then p is minimal. We constructed p so that $d(p) = 0$ but the fact that each C_n is primitive over B and A is intrinsically closed guarantees that p is i-minimal and we finish.

Clearly, $d(p) = 0$ does not imply p is minimal. For, if $d(a/A) = d(b/A) = 0$ then $d(ab/A) = 0$ but if, for example a and b are independent $\mathrm{tp(ab/A)}$ is not minimal.

4 The Finite Cover Property

In this section we show that for classes as described in Example 1.4 with the full amalgamation property, and in particular for $(\mathbf{K}_\alpha, \leq_s)$, the theory of the generic does not have the finite cover property. We rely on the following characterization due to Shelah [5, II.2.4].

4.1 Fact. If T is a stable theory with the finite cover property then there is a formula $\phi(\overline{x}, \overline{y}, \overline{z})$ such that

1. For every $\overline{c}$, $\phi(\overline{c}, \overline{y}, \overline{z})$ defines an equivalence relation. We call this relation $\overline{c}$-equivalence.
2. For arbitrarily large n, there exists $\overline{c}_n$ such that the equivalence relation defined by $\phi(\overline{c}_n, \overline{y}, \overline{z})$ has exactly n equivalence classes.

Here is some necessary notation.

4.2 Definition. Let A, B be finite substructures of M with $A \subseteq B$ then

1. $\chi_M(B/A)$ is the number of distinct copies of B over A in M.
2. $\chi^*_M(B/A)$ is the supremum of the cardinalities of maximal families of disjoint (over A) copies of B over A in M.

4.3 Definition. (A, B) is a *minimal pair* if $\delta(B/A) < 0$ and for every B', with $A \subseteq B' \subseteq B$, $\delta(B/A) < \delta(B'/A)$.

The next result is proved in [3].

4.4 Fact. There is a function t taking pairs of integers to integers such that if $A \leq_i B$ then for any $N \in \mathbf{K}$ and any embedding f of A into N, $\chi_N(fB/fA) \leq t(|A|, |B|)$.

There is an easy partial converse to this result.

4.5 Lemma. *For any $M \in \mathbf{K}_0$, if $\chi^*_M(B/A) > t(|A|, |B|)$ then $A \leq_s B$.*

Proof. Suppose some B' with $A \subseteq B$ satisfies $A \leq_i B'$. Then there are more than $t(|A|, |B|)$ disjoint copies of B' over A in M contradicting Fact 4.4.

We also need the finer analysis of the intrinsic closure carried out in [2]. In fact, this argument depends on the slightly finer notion of a *semigeneric* which is defined in [2]. The crucial facts from [3] and [2] are the following.

4.6 Fact. If $(\mathbf{K}_0, \leq_s)$ satisfies Context 1.11 and has the full amalgamation property then the theory of the generic T satisfies

1. All models of T are semigeneric.
2. T is stable. For any formula $\phi(x_1, \ldots, x_r)$ there is an integer $\ell = \ell_\phi$, such that for any semigeneric $M \in \mathbf{K}$ and any r-tuples $\overline{a}$ and $\overline{a}'$ from M if $\mathrm{icl}^{\ell_\phi}_{\mathrm{M}}(\overline{\mathrm{a}}) \approx \mathrm{icl}^{\ell_\phi}_{\mathrm{M}}(\overline{\mathrm{a}}')$ then $M \models \phi(\overline{a})$ if and only if $M \models \phi(\overline{a}')$.

4.7 Theorem. *If $(\mathbf{K}_0, \leq_s)$ satisfies Context 1.11 and has the full amalgamation property then the theory of the generic T does not have the finite cover property.*

Proof. Suppose not. We know T is stable so there is a formula ϕ and sequences $\langle \overline{c}_m : m < \omega \rangle$ satisfying the conditions of Fact 4.1. Each model of T is semigeneric. Choose $\ell = \ell_\phi$ as in Fact 4.6 so that the isomorphism type of $\mathrm{icl}^{\ell}_{\mathrm{M}}(\overline{\mathrm{c}}, \overline{\mathrm{a}}, \overline{\mathrm{b}})$ determines the truth of $\phi(\overline{c}, \overline{a}, \overline{b})$ for any triple of $\overline{c}, \overline{a}, \overline{b}$ of appropriate length. Let p bound the cardinality of $\mathrm{icl}^{\ell}_{\mathrm{M}}(\overline{\mathrm{c}}, \overline{\mathrm{a}})$ for any $\overline{c}$ and $\overline{a}$ and any semigeneric M.

Fix a semigeneric model M.

4.8 Claim. There exist $C \subseteq \hat{C} \leq_s A \in \mathbf{K}$ and f mapping A into the semigeneric model M such that $C = \mathrm{icl}^{\ell}_{\mathrm{A}}(\overline{\mathrm{c}})$, $A = \mathrm{icl}^{\ell}_{\mathrm{A}}(\overline{\mathrm{c}}, \overline{\mathrm{a}})$, $fC = \mathrm{icl}^{\ell}_{\mathrm{M}}(\mathrm{f}\overline{\mathrm{c}})$, and $fA = \mathrm{icl}^{\ell}_{\mathrm{M}}(\mathrm{f}\overline{\mathrm{c}}, \mathrm{f}\overline{\mathrm{a}})$.

Proof. Choose m sufficiently large with respect to the maximal cardinality of $\mathrm{icl}^{\ell}_{\mathrm{M}}(\overline{\mathrm{c}}, \overline{\mathrm{a}}, \overline{\mathrm{b}})$ so that there is a subsequence of representatives of the equivalence classes of $\phi(\overline{c}_m, \overline{y}, \overline{z})$, $\langle \overline{a}_i : i < n \rangle$ with $n \geq t(p, p)$ such that for some $C^m \subseteq \hat{C}^m \subseteq A^m$ where $C^m = \mathrm{icl}^{\ell}_{\mathrm{M}}(\overline{\mathrm{c}})$ and $A^m_i = \mathrm{icl}^{\ell}_{\mathrm{M}}(\overline{\mathrm{c}}, \overline{\mathrm{a}}_{\mathrm{i}})$:

1. for all i, j, $A^m_i \approx_{\hat{C}^m} A^m_j$
2. for all $i < j$, $A^m_i \cap A^m_j = \hat{C}^m$.

(We first apply the pigeonhole principle to get isomorphic representatives and then the finite Δ-system lemma to get the disjointness.) Since $n \geq t(p, p)$ by Lemma 4.5, $\hat{C}^m \leq_s A^m_0$. Now choose a finite structure $A \in \mathbf{K}$ with $C \subseteq \hat{C} \leq_s A$ isomorphic by a map f to $C^m, \hat{C}^m, A^m_0$ respectively to complete the proof.

4.9 Notation. Fix $C \subseteq \hat{C} \leq_s A$ from Claim 4.8. For $q \geq 1$, let B_q be $A_1 \otimes_{\hat{C}} A_2 \ldots \otimes_{\hat{C}} A_q$ where each $A_i \approx \mathrm{icl}^{\ell}_{\mathrm{B_q}}(\bar{\mathrm{c}}, \bar{\mathrm{a}}_{\mathrm{i}}) \approx \mathrm{A}$ and in fact fix $A_1 = A$.

4.10 Claim. For each $1 \leq q < \omega$, there exist maps g_q from B_q into M such that: All the g_q agree on $\hat{C}$, $g_q C = \mathrm{icl}^{\ell}_{\mathrm{M}}(\mathrm{g_q}\bar{\mathrm{c}})$, and $\mathrm{icl}^{\ell}_{\mathrm{M}}(\mathrm{g_q}\bar{\mathrm{c}}, \mathrm{g_i}\bar{\mathrm{a}}_{\mathrm{i}}, \mathrm{g_i}\bar{\mathrm{a}}_{\mathrm{j}}) \approx \mathrm{A_i} \otimes_{\hat{\mathrm{C}}} \mathrm{A_j}$.

Proof. Let $B_0 \approx \mathrm{icl}^{\ell}_{\mathrm{M}}(\hat{\mathrm{C}})$. Now for each q, $\hat{C} \leq_s B_q$ so by semigenericity there are maps g_q such that the universe of $\mathrm{icl}^{\ell}_{\mathrm{M}}(\mathrm{g_q B_q})$ is the free amalgam of $\mathrm{icl}^{\ell}_{\mathrm{M}}\mathrm{g_q C}$ (which is always isomorphic to B_0) and an isomorphic copy of B_q over $\hat{C}$. For each i, j, this implies $\mathrm{icl}^{\ell}_{\mathrm{M}}(\mathrm{g_q}\bar{\mathrm{c}}, \mathrm{g_q}\bar{\mathrm{a}}_{\mathrm{i}}, \mathrm{g_q}\bar{\mathrm{a}}_{\mathrm{j}}) \approx \mathrm{A_i} \otimes_{\hat{\mathrm{C}}} \mathrm{A_j}$ since $\mathrm{icl}^{\ell}_{\mathrm{B_0 \otimes_{\hat{C}} B_q}}(\bar{\mathrm{c}}, \bar{\mathrm{a}}_{\mathrm{i}}, \bar{\mathrm{a}}_{\mathrm{j}}) = \mathrm{A_i} \otimes_{\hat{\mathrm{C}}} \mathrm{A_j}$.

We need one further fact.

4.11 Claim. In M, $g_q\bar{a}_0$ and $g_q\bar{a}_1$ are not $\bar{c}$-equivalent.

Proof. If they are, any $\bar{d}, \bar{f}$ with $\mathrm{icl}^{\ell}_{\mathrm{M}}(\bar{\mathrm{c}}, \bar{\mathrm{d}}, \bar{\mathrm{f}}) \approx \mathrm{icl}^{\ell}_{\mathrm{M}}(\bar{\mathrm{c}}, \bar{\mathrm{a}}_0, \bar{\mathrm{a}}_1)$ are also $\bar{c}$-equivalent. Take two sequences $\bar{d}_0, \bar{d}_1$ which are not $\bar{c}$-equivalent but $D_i = \mathrm{icl}^{\ell}_{\mathrm{M}}(\bar{\mathrm{c}}, \bar{\mathrm{d}}_{\mathrm{i}}) \approx \mathrm{A}$ for each i and $D_1 \cap D_2 = \hat{C}$. (These were shown to exist in the proof of Claim 4.8.) Now by semigenericity embed another copy of A over $\hat{C}$ as $\mathrm{icl}^{\ell}_{\mathrm{M}}(\bar{\mathrm{c}}, \bar{\mathrm{f}})$ which is freely amalgamated with $D_1 D_2$ and thus each of $\mathrm{icl}^{\ell}_{\mathrm{M}}(\bar{\mathrm{c}}, \bar{\mathrm{d}}_{\mathrm{i}})$ over $\hat{C}$; then $\mathrm{icl}^{\ell}_{\mathrm{M}}(\bar{\mathrm{c}}, \bar{\mathrm{d}}_{\mathrm{i}}, \bar{\mathrm{f}}) \approx \mathrm{A} \otimes_{\hat{\mathrm{C}}} \mathrm{D_i}$. So, $\bar{f}$ is equivalent to both $\bar{d}_i$ which is impossible.

Now we observe that $\bar{c}_m$ contradicts the finite cover property. The equivalence relation indexed by $\bar{c}_m$ has m classes. However, we have $C = \mathrm{icl}^{\ell}_{\mathrm{M}}(\bar{\mathrm{c}}_{\mathrm{m}}) \subseteq \hat{\mathrm{C}} \leq_{\mathrm{s}} \mathrm{icl}^{\ell}_{\mathrm{M}}(\bar{\mathrm{c}}_{\mathrm{m}}, \bar{\mathrm{a}})$ (for an appropriate $\bar{a}$). Now in Claim 4.10, we constructed arbitrarily long sequences of copies of $\bar{a}$ such that $\mathrm{icl}^{\ell}_{\mathrm{M}}(\bar{\mathrm{c}}, \bar{\mathrm{a}}_{\mathrm{i}}, \bar{\mathrm{a}}_{\mathrm{j}}) \approx \mathrm{A_i} \otimes_{\hat{\mathrm{C}}} \mathrm{A_j}$. By Claim 4.11, these represent different $\bar{c}_m$-equivalence classes and this contradiction completes the proof.

4.12 Conclusion. The arguments in the paper are fully worked out only for languages with binary relation symbols. This restriction does not apply to Section 4 which holds for any 'determined theory' with full amalgamation. The combinatorial arguments in Section 3 are sufficiently complicated that the proof is the general case is less clear. But it would be quite surprising if the restriction to a binary language is actually necessary. We thank Eric Rosen and Mike Benedikt for forcing us to clarify the proof in Section 4.

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