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COVERING A FUNCTION ON THE PLANE BY TWO CONTINUOUS FUNCTIONS ON AN UNCOUNTABLE SQUARE - THE CONSISTENCY

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Abstract. It is consistent that for every function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ there is an uncountable set $A \subseteq \mathbb{R}$ and two continuous functions $f_0, f_1 : D(A) \to \mathbb{R}$ such that $f(\alpha, \beta) \in \{f_0(\alpha, \beta), f_1(\alpha, \beta)\}$ for every $(\alpha, \beta) \in A^2$, $\alpha \neq \beta$.

1. Introduction

Suppose that X is a topological space and $f : X \to \mathbb{R}$ is a real-valued function on X. Is there a "large" subset of X such that the restriction $f \restriction X$ is continuous? Obviously, if $A \subseteq X$ is a discrete subspace, then $f \restriction A$ is continuous. Hence in the case when $\mathrm{dom}(f) = \mathbb{R}$, we can always find an infinite subset on which f is continuous. The problem whether there is such "large" set has been investigated by Abraham, Rubin and Shelah in [ARSh]. They proved that it is consistent that every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous on some uncountable set. Later Shelah [Sh 473] showed that every function may be continuous on a non-meager set.

In this paper we consider functions on the plane, $\mathbb{R} \times \mathbb{R}$. The reasonable question to ask in this case is: is there a "large" set $A \subseteq \mathbb{R}$ such that on $A \times A$ the function f can be cover by two continuous functions? Note that we could not hope for f to be just continuous on $A \times A$, e.g., if g is a Sierpinski partition, then for every uncountable set A, g is not continuous on $A \times A$. The main result of this paper is the following theorem. For technical reasons we consider squares without the diagonal, i.e. for a set A we consider $D(A) = \{(x, y) : x, y \in A,\ x \neq y\}$.

Theorem . *Assume $2^{\aleph_l} = \aleph_{l+1}$ for $l < 4$, and $\diamondsuit_s(\aleph_4, \aleph_1, \aleph_0)$, see below. Then there is a forcing notion P which preserves cardinals and cofinalities and such that in V^P, $2^{\aleph_0} = \aleph_4$ and for every function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ there is an uncountable set $A \subseteq \mathbb{R}$ and two continuous functions $f_0, f_1 : D(A) \to \mathbb{R}$ such that $f(\alpha, \beta) \in \{f_0(\alpha, \beta), f_1(\alpha, \beta)\}$ for every $(\alpha, \beta) \in D(A)$.*

The proof breaks down into two parts. In Section 2, we prove the consistency of a guessing principle, diamond for systems. Then, is Section 3, we give the proof of the theorem.

Remark . (1) We can replace $\aleph_0$ by any $\mu = \mu^{<\mu}$.
(2) Our main goal was to prove the consistency of the statement in the theorem with $2^{\aleph_0} < \aleph_\omega$. We get $2^{\aleph_0} = \aleph_4$ naturally from the proof, but the values $\aleph_3$ or $\aleph_2$ may be possible.

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1.1. **Notation.** We use standard set-theoretic notation. Below we list some frequently used symbols.

- For A, B subsets of ordinals of the same order type, $OP_{B,A}$ is the order preserving isomorphism from A to B.
- If C is a set of ordinals, then $(C)'$ denotes the set of accumulation points.
- Let λ, χ be cardinals, χ regular. $S^\lambda_\chi = \{\alpha \in \lambda : \mathrm{cf}(\alpha) = \chi\}$.
- For a statement ϕ we define $TV(\phi) = 0$ if ϕ is true, otherwise $TV(\phi) = 1$.
- $\mathbb{R} = {}^\omega 2$.
- If M is a model, $X \subseteq M$, then $Sk(X)$ is the Skolem hull of X in M.
- $\mathcal{L}[\kappa, \theta)$ is a 'universal' vocabulary of cardinality $\kappa^{<\theta}$, arity $< \theta$.

2. Diamond for Systems

In this section we prove the consistency of a guessing principle, diamond for systems $\Diamond_s$.

Definition 2.1. *A sequence $\bar{M} = \langle M_u : u \in [B]^{\leq 2}\rangle$ is a system of models (of some fixed language) if:*

(1) $M_u \subseteq \mathrm{Ord}$, $B \subseteq \mathrm{Ord}$,
(2) $B \cap M_u = u$ *for every* $u \in [B]^{\leq 2}$,
(3) *for every* $u, v \in [B]^{\leq 2}$, $|u| = |v|$, *the models* M_u *and* M_v *are isomorphic and* OP_{M_u,M_v} *is the isomorphism from* M_v *onto* M_u, $OP_{M_u,M_v}(v) = u$,
(4) *for every* $u, v \in [B]^{\leq 2}$, $M_u \cap M_v \subseteq M_{u\cap v}$,
(5) *if* $|u| = |v|$, $u' \subseteq u$, $v' = \{\alpha \in v : (\exists \beta \in u')(|\beta \cap u| = |\alpha \cap v|)\}$, *then* $OP_{M_{u'},M_{v'}} \subseteq OP_{M_u,M_v}$, *and* $OP_{M_u,M_u} = id_{M_u}$, *and if* $|w| = |u|$, *then* $OP_{M_u,M_v} \circ OP_{M_v,M_w} = OP_{M_u,M_w}$.

Remark . See [Sh 289] on the existence of "nice" systems of models for λ a sufficiently large cardinal, e.g., measurable. Here we do not use large cardinals, and try to get a model in which the continuum is small, i.e., less than $\aleph_\omega$. For this we need a suitable guessing principle.

Definition 2.2 (Diamond for systems $\Diamond_s(\lambda, \sigma, \kappa, \theta)$). *Let $\{C_\alpha : \alpha \in \lambda\}$ be a square sequence on λ. $\langle \bar{M}^\alpha : \alpha \in W\rangle$ is a $\Diamond_s(\lambda, \sigma, \kappa, \theta)$ sequence, (or $\Diamond_s(\lambda, \sigma, \kappa, \theta)$-diamond for systems) if:*

(A) $W \subseteq \lambda$ *and for every* $\alpha \in W$, $\bar{M}^\alpha = \langle M^\alpha_u : u \in [B_\alpha]^{\leq 2}\rangle$ *is a system of models,* M^α_u *is a model of cardinality* κ, *universe* $\subseteq \alpha$, *vocabulary of cardinality* $\leq \kappa$, *arity* $< \theta$, *a subset of* $\mathcal{L}[\kappa, \theta)$.
(B) $B_\alpha \subseteq \alpha = \sup(B_\alpha)$, $\mathrm{otp}(B_\alpha) = \sigma$, *so* $\sigma = \mathrm{cf}(\alpha)$.
(C) *if* M *is a model with universe* λ, *vocabulary of cardinality* $\leq \kappa$, *arity* $< \theta$, *a subset of* $\mathcal{L}[\kappa, \theta)$, *then for stationarily many* $\alpha \in W$ *for all* $u \in [B_\alpha]^{\leq 2}$, $M^\alpha_u \prec M$,
(D) *if* $\alpha, \beta \in W$ *and* $\mathrm{otp}(C_\alpha) < \mathrm{otp}(C_\beta)$, *then*
  (i) *for some* $\zeta \in B_\beta$, $\bigcup\{M^\beta_u : u \in [B_\beta]^{\leq 2}\} - \bigcup\{M^\beta_u : u \in [B_\beta \cap \zeta]^{\leq 2}\}$ *is disjoint from* $\bigcup\{M^\alpha_u : u \in [B_\alpha]^{\leq 2}\}$,
(E) *if* $\alpha \neq \beta$ *in* W, $\mathrm{otp}(C_\alpha) = \mathrm{otp}(C_\beta)$, *then there is a one-to-one map* h *from* $\bigcup_{u\in[B_\alpha]^{\leq 2}} M^\alpha_u$ *onto* $\bigcup_{u\in[B_\beta]^{\leq 2}} M^\beta_u$, *order preserving, mapping* B_α *onto* B_β, M^α_u *onto* $M^\beta_{h(u)}$ *which is the identity on the intersection of these sets and the intersection is an initial segment of* $\bigcup_{u\in[B_\alpha]^{\leq 2}} M^\alpha_u$ *and* $\bigcup_{u\in[B_\beta]^{\leq 2}} M^\beta_u$.
(F) *if* $\sigma = \kappa$ *we may omit* σ.

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Lemma 2.3. *Assume: $\kappa < \mu < \lambda$ are uncountable cardinals, $\lambda = \chi^+$, $2^\mu = \chi$, $\square_\lambda$, $\diamondsuit_{S^\chi_\sigma}$, $\kappa = \kappa^{<\theta}$, $\mu^\kappa = \mu$, σ, χ, κ regular cardinals.*

Then there exists a diamond for systems on λ, $\diamondsuit_s(\lambda, \sigma, \kappa, \theta)$.

Proof Let $\bar{C} = \langle C_\gamma : \gamma \in \lambda \rangle$ be a square sequence on λ. We assume that each C_γ is closed unbounded in γ, if γ is a limit. Let $C_\gamma = \{\alpha^\gamma_\zeta : \zeta < \mathrm{otp}(C_\gamma)\}$. First choose a sequence $\langle b^\alpha_i : i < \chi \rangle$ for every $\alpha < \lambda$ such that $b^\alpha_i \subseteq \alpha$, $|b^\alpha_i| < \chi$, b^α_i increasing, continuous in i, $\alpha = \bigcup\{b^\alpha_i : i < \chi\}$. Next choose a_α for $\alpha < \lambda$ such that

(1) $a_\alpha \subseteq \alpha$,
(2) if $\mathrm{cf}(\alpha) < \chi$, then $|a_\alpha| < \chi$,
(3) if $\beta \in (C_\alpha)'$, then $a_\beta \subseteq a_\alpha$,
(4) if $\beta \in C_\alpha$ and $i = \mathrm{otp}(C_\alpha)$, then $b^\beta_i \subseteq a_\alpha$,
(5) if $\mathrm{otp}(C_\alpha)$ is a limit of limit ordinals, then $a_\alpha = \bigcup_{\beta \in (C_\alpha)'} a_\beta$.

Note that if $\alpha \in S^\lambda_\chi$, then there is a club $C'_\alpha \subseteq C_\alpha$ such that $\langle a_\beta : \beta \in C'_\alpha \rangle$ is an increasing, continuous sequence of subsets of α of cardinality $< \chi$ with union α. Let H_0, H_1 be functions which witness that $\lambda = \chi^+$, i.e., H_0, H_1 are two place functions, for every $\alpha \in [\chi, \lambda)$, $H_0(\alpha, -)$ is a one-to-one functions from α onto χ and $H_1(\alpha, H_0(\alpha, i)) = i$ for every $\alpha \in [\chi, \lambda)$ and $i < \alpha$.

Now by induction on $\alpha < \lambda$ we define the truth value of '$\alpha \in W$', and if we declare it to be true, then we also define $\bar{M}^\alpha$. Suppose we have defined $W \cap \alpha$ and $\bar{M}^\beta$ for $\beta \in W \cap \alpha$. Now consider the following properties of an ordinal $\alpha \in \lambda$.

(a) $a_\alpha \cap \chi = \mathrm{otp}(C_\alpha)$,
(b) a_α is closed under H_0 and H_1,
(c) for every $\gamma \in a_\alpha$ we have:
  (i) if $\mathrm{cf}(\gamma) < \chi$, then $a_\alpha \cap \gamma = b^\gamma_{\mathrm{otp}(C_\alpha)}$ and $C_\gamma \subseteq a_\alpha$ and $\mathrm{otp}(C_\gamma) \leq \mathrm{otp}(C_\alpha)$,
  (ii) if $\mathrm{cf}(\gamma) = \chi$, then $\sup(a_\alpha \cap \gamma) = \alpha^\gamma_{\mathrm{otp}(C_\alpha)}$ and $C_{\alpha^\gamma_{\mathrm{otp}(C_\alpha)}} \subseteq a_\alpha$,
(d) $\mathrm{cf}(\alpha) = \sigma$.

If α does not satisfy one of the conditions (a), (b), (c), and (d), then we declare that $\alpha \notin W$. So suppose that α satisfies (a), (b), (c), and (d). Let $\langle M_\zeta : \zeta \in \chi \rangle$ be the diamond sequence for S^χ_σ, i.e., each M_ζ is a model on ζ, vocabulary as above, and for every model M on χ, there are stationarily many $\zeta \in S^\chi_\sigma$, such that $M \cap \zeta = M_\zeta$. We say that M_ζ is suitable if it is of the form $(\zeta, <^*_\zeta, M^*_\zeta)$, where $<^*_\zeta$ is a well-ordering of ζ. For each ζ such that M_ζ is suitable, let $\xi_\zeta = \mathrm{otp}(\zeta, <^*_\zeta)$. Let $h_\zeta : \zeta \to \xi_\zeta$ be the isomorphism between $(\zeta, <^*_\zeta)$ and $(\xi_\zeta, <)$. Let $M^\oplus_\zeta$ be the model with universe ξ_ζ, such that h_ζ is the isomorphism between M^*_ζ and $M^\oplus_\zeta$. For $\alpha \in \lambda$ let $\zeta(\alpha) = \mathrm{otp}(C_\alpha)$. Consider the following properties of $\alpha \in \lambda$.

(e) there is a system $\bar{N}^{\zeta(\alpha)} = \langle N^{\zeta(\alpha)}_s : s \in [\bar{B}_{\zeta(\alpha)}]^{\leq 2} \rangle$, $N^{\zeta(\alpha)}_s \prec M^\oplus_{\zeta(\alpha)}$, $||N^{\zeta(\alpha)}_s|| = \kappa$, $\bar{B}_{\zeta(\alpha)}$ cofinal in $\xi_{\zeta(\alpha)}$, $\mathrm{otp}(\bar{B}_{\zeta(\alpha)}) = \sigma$,
(f) $\mathrm{otp}(a_\alpha) = \xi_{\zeta(\alpha)}$.

If α does not satisfy (e), and (f), then declare $\alpha \notin W$. So assume that α satisfies (e) and (f). Let $g_\alpha : \xi_{\zeta(\alpha)} \to a_\alpha$ be the order preserving isomorphism. Let $\bar{M}^\alpha = \langle M^\alpha_u : u \in [B_\alpha]^{\leq 2} \rangle$ be the system of models on a_α, which is isomorphic to $\bar{N}^{\zeta(\alpha)}$ and the isomorphism is g_α. If this system satisfies:

(g) for every $\beta \in (C_\alpha)'$ there is $\nu \in B_\alpha$ such that $a_\beta \cap \bigcup\{M^\alpha_u : u \in [B_\alpha]^{\leq 2}\} \subseteq \bigcup\{M^\alpha_u : u \in [B_\alpha \cap \nu]^{\leq 2}\}$,

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then we declare $\alpha \in W$. This finishes the definition of the diamond for systems sequence, $\langle \bar{M}^\alpha : \alpha \in W \rangle$.

We have to prove that it is as required. Clauses (A) and (B) are clear.

Proof of clause (C). We need the following fact, it is proved essentially in [Sh 300F], but for completeness we give the proof at the end of the section.

Lemma 2.4. *Assume:*

(1) $\lambda = (2^\mu)^+$, $\mu = \mu^\kappa$, $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, $\kappa^{<\theta} = \kappa$,
(2) *M is a model with universe λ, at most κ functions each with $< \theta$ places and $\leq \kappa$ relations including the well-ordering of λ.*

Then: for some club E of λ for every $\delta \in E$ of cofinality $\geq \mu^+$ we can find $I \subseteq \delta = \sup(I)$ and $\langle N_t : t \in [I]^{\leq 2}, s \in I \rangle$ such that:

(α) *$\langle N_t : t \in [I]^{\leq 2} \rangle$ is a system of elementary submodels of M, $||N_t|| = \kappa$.*

Suppose that $\mathcal{A}$ is a model on λ, C a club on λ. We have to find $\alpha \in C \cap W$ such that $M^\alpha_u \prec \mathcal{A}$ for every $u \in [B_\alpha]^{\leq 2}$. Let $E \subseteq \lambda$ be the club given by Lemma 2.4. W.l.o.g. we can assume that $E \subseteq C'$, where C' is the set of limit points of C, (so if $\delta \in E$, then $C \cap \delta$ is a club in δ). Fix $\delta \in S^\lambda_\chi \cap E$. Let $f_\delta : \delta \to \chi$ be a bijection and let

$$D_1 = \{\zeta < \chi : \zeta \text{ is a limit, } f_\delta \text{ maps } a_{\alpha^\delta_\zeta} \text{ onto } \zeta\}.$$

D_1 is a σ-club, i.e., unbounded, closed under σ-sequences. Let $\mathcal{A}^{[\delta]}$ be $(\chi, f''_\delta(<\restriction \delta), f''_\delta(\mathcal{A} \restriction \delta))$. Note that by Lemma 2.4 we have a system of submodels on $\mathcal{A} \restriction \delta$, we transfer this system on $\mathcal{A}^{[\delta]}$ by the bijection f_δ and, choosing a subsystem if necessary, we can assume that we have an end-extension system on $\mathcal{A}^{[\delta]}$ which is cofinal in χ, i.e., we have $\bar{N}^* = \langle N^*_u : u \in I \rangle$, $I \subseteq \chi$, $\sup(I) = \chi$, $N^*_u \prec \mathcal{A}^{[\delta]}$ and if $\xi < \zeta$ in I, then $\min(N^*_{\{\zeta\}} \setminus N^*_\emptyset) > \sup(N^*_{\{\xi\}})$, and if u is an initial segment of v, then N^*_u is an initial segment of N^*_v. Hence the set

$$D_2 = \{\zeta < \chi : \bigcup_{u \in [\zeta \cap I]^{\leq 2}} N^*_u \subseteq \zeta\}$$

is a club of χ and such that for every $\zeta \in D_2$ there is a system of models on ζ, $(\langle N^*_u : u \in [\zeta \cap I]^{\leq 2} \rangle)$.
Note that the set

$$D_3 = \{\zeta < \chi : \alpha^\delta_\zeta \in C \text{ and } \alpha^\delta_\zeta \text{ satisfies conditions } (a) - (d)\}$$

is a σ-club of χ. Note that $\mathcal{A}^{[\delta]}$ is a model on χ. Hence by $\diamondsuit_{S^\chi_\sigma}$, for stationary many $\zeta \in S^\chi_\sigma$ we have guessed it, i.e., the set

$$S = \{\zeta \in S^\chi_\sigma : M_\zeta = \mathcal{A}^{[\delta]} \restriction \zeta\}$$

is stationary. Now if $\zeta \in S \cap (D_1)' \cap D_2 \cap D_3$ then $\alpha^\delta_\zeta \in C$, and α^δ_ζ satisfies conditions (a)-(d). Note that $\zeta(\alpha^\delta_\zeta) = \mathrm{otp}(C_{\alpha^\delta_\zeta}) = \zeta$. Moreover, as $\zeta \in D_1 \cap S$ we have $\xi_\zeta = \mathrm{otp}(a_{\alpha^\delta_\zeta})$, i.e., condition (f) holds. By the construction it follows that condition (e) holds, (the system of submodels on ξ_ζ is isomorphic to the system on $a_{\alpha^\delta_\zeta}$ given by Lemma 2.4). Finally, (g) holds, as $\zeta \in (D_1)'$ and the system of models of $\mathcal{A}^{[\delta]}$ is end-extending.

Hence $\alpha^\delta_\zeta \in W \cap C$, and $\bar{M}^{\alpha^\delta_\zeta}$ is a system of models as required.

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Proof of clause (E). Suppose $\alpha, \beta \in W$, $\xi = \mathrm{otp}(C_\alpha) = \mathrm{otp}(C_\beta)$. By the construction, both a_α and a_β are isomorphic to $M^{\oplus}_\xi$ and the isomorphisms are order preserving functions. Hence a_α is order isomorphic to a_β. Note that $a_\alpha \cap \chi = a_\beta \cap \chi = \xi$. Since both a_α and a_β are closed under H_0 and H_1 it follows that $a_\alpha \cap a_\beta$ is an initial segment of both a_α and a_β.

Proof of clause (D). Suppose that $\alpha, \beta \in W$ and $\mathrm{otp}(C_\alpha) < \mathrm{otp}(C_\beta)$. As above, since a_α and a_β are closed under H_0 and H_1, it follows that $a_\alpha \cap a_\beta$ is an initial segment of a_α. Let $\gamma = \sup(a_\alpha \cap a_\beta)$. We have four cases, we will show that the first three never occur.
Case 1. $\gamma \in a_\alpha \cap a_\beta$. We can assume that each a_α is closed under successor, so this case can never happen.
Case 2. $\gamma \in a_\alpha - a_\beta$. Note that $C_\gamma \subseteq a_\alpha$. Let $\gamma_1 = \min(a_\beta - \gamma)$. By (c)(i) for a_β it follows that we must have $\mathrm{cf}(\gamma_1) = \chi$. Now by (c)(ii), $\gamma = \sup(a_\beta \cap \gamma_1) = \alpha^{\gamma_1}_{\mathrm{otp}(C_\beta)}$. So $\gamma \in C_{\gamma_1}$ and $\mathrm{otp}(C_\gamma) = \mathrm{otp}(C_\beta)$. Note that $\mathrm{cf}(\gamma) < \chi$. Hence by (c)(i) for a_α we have $\mathrm{otp}(C_\gamma) \le \mathrm{otp}(C_\alpha)$, a contradiction.
Case 3. $\gamma \notin (a_\alpha \cup a_\beta)$. Let $\gamma_0 = \min(a_\alpha - \gamma)$ and $,\gamma_1 = \min(a_\beta - \gamma)$. As above we have $\mathrm{otp}(C_\gamma) = \mathrm{otp}(C_\alpha)$ and $\mathrm{otp}(C_\gamma) = \mathrm{otp}(C_\beta)$, a contradiction.
Case 4. $\gamma \in a_\beta - a_\alpha$. Let $\gamma_0 = \min(a_\alpha - \alpha)$. We have $\mathrm{cf}(\gamma_0) = \chi$ and $\mathrm{otp}(C_\gamma) = \mathrm{otp}(C_\alpha)$, so $C_\gamma \subseteq a_\alpha$. Note that $a_\alpha \cap \gamma = \bigcup_{\zeta \in C_\gamma}(a_\alpha \cap \zeta)$. But for $\zeta \in a_\alpha$ with $\mathrm{cf}(\zeta) < \chi$ we have $a_\alpha \cap \zeta = b^\zeta_{\mathrm{otp}(C_\alpha)}$. Hence $a_\alpha \cap \gamma = \bigcup_{\zeta \in (C_\gamma)'} b^\zeta_{\mathrm{otp}(C_\alpha)} \subseteq a_{\beta_1}$, for some $\beta_1 \in (C_\beta)'$ large enough. Hence by (g) in the definition of the diamond for systems sequence, the conclusion follows.

Proof of Lemma 2.4 We prove slightly more. In addition to the sequence $\langle N_t : t \in [I]^{\le 2}\rangle$ there is a sequence $\langle N'_{\{\alpha\}} : \alpha \in I\rangle$ such that:

- (β) $N_{\{\alpha\}}, N'_{\{\alpha\}}$ realize the same $L_{\theta,\theta}$-type over M, for $\alpha \in I$,
- (γ) we have $N'_{\{\alpha\}} \prec N_{\{\alpha\}}$ for $\alpha \in I$ and for $\alpha < \beta$ in I we have $N_{\{\alpha,\beta\}} = Sk(N_{\{\alpha\}} \cup N'_{\{\beta\}})$,

Remark . (1) Note that for $\alpha < \beta$, $N_{\{\beta\}}$ is not necessarily a subset of $N_{\{\alpha,\beta\}}$.
(2) The idea of the proof is to define $N^*_{\{0\}}$, ${N^*_{\{1\}}}'$ and $N^*_{\{0,1\}}$ (and more, see definition of a witness below). Then we use it as a blueprint and "copy" it many times using elementarity, to obtain a suitable system.

We can assume that M has Skolem functions, even for $L_{\theta,\theta}$. Let χ^* be large enough. Let for $i < \lambda$, $\mathcal{B}_i \prec (H(\chi^*), \in, <^*_{\chi^*})$ such that $||\mathcal{B}_i|| = 2^\mu < \lambda$, and $M \in \mathcal{B}_i$, $\mathcal{B}_i$ increasing continuous with i, and if $\mathrm{cf}(i) \ge \mu^+$ or i non-limit, then $\mathcal{B}_i \prec_{L_{\mu^+,\mu^+}} (H(\chi^*), \in, <^*_{\chi^*})$. Let $E = \{\delta < \lambda : \delta$ is a limit and $\mathcal{B}_\delta \cap \lambda = \delta\}$, it is a club of λ. Fix $\delta \in E \cap S^\lambda_{\ge\mu^+}$. Note that $\mathcal{B}_\delta \prec_{L_{\mu^+,\mu^+}} (H(\chi^*), \in, <^*_{\chi^*})$.
We say that $(N^*_\emptyset, N^*_{\{0\}}, {N^*_{\{1\}}}', N^*_{\{0,1\}}, \alpha_0, \alpha_1)$ is a witness if:

1. $N^*_u \prec M$, $|N^*_u| = \kappa$, $N^*_{\{0\}} \cap {N^*_{\{1\}}}' = N^*_\emptyset$, $N^*_\emptyset, N^*_{\{0\}} \prec M \restriction \mathcal{B}_\delta$, $N^*_{\{0,1\}} = Sk({N^*_{\{1\}}}' \cup N^*_{\{0\}})$,
2. $N^*_{\{1\}} \cap \mathcal{B}_\delta = N^*_\emptyset$, $\alpha_0 \in N^*_{\{0\}} - N^*_\emptyset$, $\alpha_1 \in {N^*_{\{1\}}}' - N^*_\emptyset$,
3. if $\alpha \in N^*_{\{0,1\}} \setminus {N^*_{\{1\}}}'$, $\beta = \min({N^*_{\{1\}}}' \setminus \alpha)$, then $\mathrm{cf}(\beta) \ge \mu^+$,
4. for every $A \subseteq \mathcal{B}_\delta$, $|A| \le \mu$ there are $N'_{\{1\}} \prec N_{\{1\}}$ and $N_{\{0,1\}}$ such that
   (a) $N'_{\{1\}}, N_{\{0,1\}} \prec M \cap \mathcal{B}_\delta$,

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(b) $N'_{\{1\}}$ is order isomorphic to ${N^*_{\{1\}}}'$,
(c) $N_{\{1\}}$ is order isomorphic to $N^*_{\{0\}}$,
(d) $OP_{N_{\{0,1\}},N^*_{\{0,1\}}}$ is an isomorphism from $N^*_{\{0,1\}}$ onto $N_{\{0,1\}}$ which is the identity on ${N^*_{\{1\}}}'$, maps $N^*_{\{0\}}$ onto $N_{\{0\}}$,
(e) for $\alpha \in N^*_{\{0,1\}} \setminus {N^*_{\{1\}}}'$, if $\beta = \min(N'_{\{1\}} - \alpha)$, then $OP_{N_{\{0,1\}},N^*_{\{0,1\}}}(\alpha) \in \sup(A \cap \beta, \beta)$,

Claim 2.5. *There is a witness.*

We can find $\mathcal{C} \prec_{L_\mu,L_\mu} (H(\chi^*), \in, <^*_{\chi^*})$ such that $||\mathcal{C}|| = \mu$, ${}^\kappa\mathcal{C} \subseteq \mathcal{C}$, $\mu+1 \subseteq \mathcal{C}$ and $(M, \mathcal{B}_\delta, \delta) \in \mathcal{C}$. As $\mathcal{B}_\delta \prec_{L_{\mu^+,\mu^+}} (H(\chi^*), \in, <^*_{\chi^*})$ it follows that there is a function f, $\mathrm{dom}(f) = \mathcal{C}$, $\mathrm{rang}(f) \subseteq \mathcal{B}_\delta$, $f \restriction \mathcal{C} \cap \mathcal{B}_\delta$ is the identity, f preserves satisfaction of L_{μ^+,μ^+} formulas, i.e. f is an isomorphism.

Let $\mathcal{N} \prec (H(\chi^*), \in, <^*_{\chi^*})$ be such that $\{\mathcal{B}_\delta, \mathcal{C}, f, \delta\} \in \mathcal{N}$, $||\mathcal{N}|| = \kappa$. Let $\mathcal{N}_1 = \mathcal{N} \cap \mathcal{C}$, $\mathcal{N}_0 = \mathcal{N} \cap \mathcal{B}_\delta$. Let $\mathcal{N}'_0 = f(\mathcal{N}_1)$, note that $\mathcal{N}'_0 \subseteq \mathcal{N}_0$. Let $\delta_0 = f(\delta_1)$. W.l.o.g. we can assume that $\mathcal{N} = Sk(\mathcal{N}_0, \mathcal{N}_1)$. Let $\mathcal{N}_\emptyset = \mathcal{B}_\delta \cap \mathcal{C} \cap \mathcal{N}$. We claim that $(\mathcal{N}_\emptyset, \mathcal{N}_0, \mathcal{N}'_1, \mathcal{N}, \delta_0, \delta_1)$ is a witness. Note that

($*$) if $\alpha \in \mathcal{N} \cap (\delta+1)$, then $\min(\mathcal{C} - \alpha) \in \mathcal{N}_1$.

Let us check condition (3). Suppose that $\alpha \in \mathcal{N} - \mathcal{N}_1$ and let $\beta = \min(\mathcal{N}_1 - \alpha)$. Note that by ($*$) we have $\beta = \min(\mathcal{C} - \alpha)$. But as $\mu+1 \subseteq \mathcal{C}$ and $\mathcal{C} \prec (H(\chi^*), \in, <^*_{\chi^*})$ we must have $\mathrm{cf}(\beta) \geq \mu^+$.

Now to verify (4), suppose that there is a set A such that the conclusion of (4) fails. Then A is definable from: $\mathcal{N}_1$, the isomorphism type of $\mathcal{N}$ over $\mathcal{N}_1$ and the isomorphism type of $\mathcal{N}_0$ over $\mathcal{N}'_0$. As $\mathcal{N}_1, \mathcal{N}_\emptyset$ are in $\mathcal{C}$ and $\mathcal{C} \prec_{L_\mu,L_\mu} (H(\chi^*), \in, <^*_{\chi^*})$ and $\kappa < \mu$ it follows that such set A is in $\mathcal{C}$. But now the witness itself is a counterexample. Note that clause (e) follows from ($*$).

Claim 2.6. *If there is a witness, then there is a system as required, (for our $\delta \in E \cap S^\lambda_{\geq \mu^+}$).*

By induction on $\alpha < \mu^+$ we define $\delta_\alpha < \delta$ and a system $\langle N'_{\{\alpha\}}, N_{\{\alpha\}}, N_{\{\alpha,\beta\}} \rangle$, for $\beta < \alpha$.

Suppose that we have defined the system for all $\beta < \alpha$. Let $A = \bigcup\{N_u : u \in [\{\delta_\beta : \beta < \alpha\}]^{\leq 2}\}$. Let $N'_{\{\alpha\}}$ and $N_{\{\alpha\}}$, $N_{\{0,\alpha\}}$ be as in the definition of a witness, for the above A. For $\beta < \alpha$ let $N_{\{\beta,\alpha\}} = Sk(N_{\{\beta\}}, N'_{\{\alpha\}})$. It follows that N_α is isomorphic to $\mathcal{N}_0$ and $N_{\{\beta,\alpha\}}$ is isomorphic to $\mathcal{N}$. Let $\delta_\alpha = OP_{N_{\{0,\alpha\}},N^*_{\{0,1\}}}(\alpha_0)$. Note that $I = \{\delta_\alpha : \alpha < \mu^+\}$ is such that $\sup(I) = \delta$ and $N_u \cap I = u$ for every $u \in [I]^{\leq 2}$. This finishes the proof.

3. Proof of the Theorem

Start with a model satisfying the assumptions of the theorem, i.e., we have $2^{\aleph_l} = \aleph_{l+1}$ for $l < 4$, $\{C_\alpha : \alpha \in \omega_4\}$ is a square sequence and $\langle \bar{M}^i : i \in W \rangle$ is a diamond for systems, $\diamondsuit_s(\aleph_4, \aleph_1, \aleph_1, \aleph_0)$. Let $\bar{M}^i = \langle M^i_u : u \in [\bar{B}_i]^{\leq 2} \rangle$ and let $\bar{B}_i = \{\alpha^i_\epsilon : \epsilon < \omega_1\}$ be the increasing enumeration.

Definition 3.1. *(1) A set $b \subseteq \alpha$ is $\bar{Q} \restriction \alpha$-closed, i.e. $\alpha \in b \Rightarrow a_\alpha \subseteq b$.*
(2) $\mathcal{K} = \mathcal{K}_\mu$ is the family of FS-iterations $\bar{Q} = \langle P_\alpha, Q_\alpha, a_\alpha, : \alpha < \alpha^* \rangle$ such that:

(a) $a_\alpha \subseteq \alpha$,
(b) $|a_\alpha| \leq \mu$,

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(c) $\beta \in a_\alpha \Rightarrow a_\beta \subseteq a_\alpha$,
(d) *for* $b \subseteq \alpha$, $P^*_b = \{p \in P_\alpha : \mathrm{dom}(p) \subseteq b$ and $(\forall \beta \in \mathrm{dom}(p)) p(\beta)$ is a $P^*_{b\cap\alpha}$ name $\}$,
(e) Q_α *is a* $P^*_{a_\alpha}$*-name, (see 3.2 below),*
(f) $P^*_{\alpha^*}$ *has the property K, (= Knaster).*

Remark . The above definition proceeds by induction on α^*, so part (d) is not circular.

Lemma 3.2. *Suppose* $\bar{Q} = \langle P_\alpha, Q_\alpha, a_\alpha, : \alpha < \alpha^*\rangle \in \mathcal{K}$. *If* $b \subseteq \alpha^*$ *is* $\bar{Q}$*-closed, then* $P^*_b \lessdot P^*_{\alpha^*}$.

Proof Straightforward, see [Sh 288] and [Sh 289].

Let $f : {}^{\omega_1 >}2 \to \aleph_1$ be one-to-one, such that if $\eta \lhd \nu$, then $f(\eta) \lhd f(\nu)$. For $\rho \in {}^{\omega_1}2$ let $w_\rho = \{f(\rho \restriction i) : i < \aleph_1\} \in [\aleph_1]^{\aleph_1}$. Note that if $\rho_1 \neq \rho_2$ in ${}^{\omega_1}2$, then $|w_{\rho_1} \cap w_{\rho_2}| < \aleph_1$. Let R be the countable support forcing adding $\aleph_4$ many Cohen subsets of ω_1, ρ_i $(i < \omega_4)$. Note that in V^R, $\{w_{\rho_i} : i \in \omega_4\}$ is a family of almost disjoint, uncountable subsets of ω_1. Let $B_i = \{\alpha^i_\epsilon : \epsilon \in w_{\rho_i}\}$. Note that $\{M^i_u : u \in [B_i]^{\leq 2}\}$ is still a system of models on i, hence without loss of generality we can assume that $w_{\rho_i} = \omega_1$. For $\zeta \in \omega_1$ define $B_i(\zeta) = \{\alpha^i_\epsilon : \epsilon < \zeta\}$. In V^R we shall define an iteration $\langle P_i, Q_i, a_i : i < \chi\rangle \in \mathcal{K}_{\aleph_4}$. Working in V^R, we define $\bar{Q} \restriction i$, by induction on $i < \omega_4$, and we prove that it is as in 3.1 (in V^R).

We call i good if it satisfies: $i \in W$, each M^i_u has a predetermined predicate describing $\bar{Q} \restriction M^i_u$ (as an R-name, with the limit $\underset{\sim}{P}{}^i_u$) and an $R \restriction M^i_u * \underset{\sim}{P}{}^i_u$-name $\underset{\sim}{f}$ for a function from ${}^\omega 2 \times {}^\omega 2$ into ${}^\omega 2$ and each M^i_u is $\bar{Q}$-closed. (Recall that we do not distinguish between the model M^i_u Nan its universe). In this case we put $a_i = \bigcup\{M^i_u : u \in [B_i]^{\leq 2}\}$ and define Q_i below.

If i is not good we put $a_i = \emptyset$ and define Q_i to be the Cohen forcing, i.e., $Q_i = ({}^{\omega >}2, \lhd)$. We can assume that if $\alpha \in B_i$, then Q_α is Cohen, (or just replace B_i by $\{\alpha + 1 : \alpha \in B_i\}$). For $\alpha \in B_i$, let r_α be the Cohen real forced by Q_α.

Remark . The reason we add $\aleph_4$ almost disjoint subsets of ω_1 is that, in V^R, if $i \neq j$ are good and $\mathrm{otp}(C_i) = \mathrm{otp}(C_j)$, then the systems associated with i and j are almost disjoint, i.e., there is $\zeta \in \omega_1$ such that

$$\Big(\bigcup\{M^i_u : u \in [B_i]^{\leq 2}\}\Big) \cap \Big(\bigcup\{M^j_u : u \in [B_j]^{\leq 2}\}\Big) \subseteq$$
$$\Big(\bigcup\{M^i_u : u \in [B_i(\zeta)]^{\leq 2}\}\Big) \cap \Big(\bigcup\{M^j_u : u \in [B_j(\zeta)]^{\leq 2}\}\Big)$$

Note that if $\mathrm{otp}(C_i) \neq \mathrm{otp}(C_j)$ then we have almost disjointness by 2.2(D)(i).

Notation For $\xi, \zeta \in \omega_1$ let $Z^i_{\xi,\zeta} = M^i_{\{\alpha^i_\xi, \alpha^i_\zeta\}} \cup M^i_{\{\alpha^i_\xi\}} \cup M^i_{\{\alpha^i_\zeta\}}$, $Z^i_\xi = M^i_{\{\alpha^i_\xi\}}$.

Now we fix a good i. Our goal is to define Q_i.

Definition 3.3. *For* $p, q \in R$ *(or in* $P^*_{\omega_4}$*),* $\mathrm{dom}(p), \mathrm{dom}(q) \subseteq Z^i_{0,1}$ *we say that* p *and* q *are dual if* $OP_{Z^i_1, Z^i_0}(p \restriction Z^i_0) = q \restriction Z^i_1$ *and* $OP_{Z^i_1, Z^i_0}(q \restriction Z^i_0) = p \restriction Z^i_1$.

Using $G_{R \restriction M^i_\emptyset}$ we choose, by induction on $k < \omega$, conditions r^i_η, $r^{i,l}_\eta \in R$ for $\eta \in {}^k 2$, $l < 2$, such that:

(a) $r^i_\eta \in (R \restriction Z^i_0)/G_{R \restriction M^i_\emptyset}$.
(b) $\nu \lhd \eta \Rightarrow r^i_\nu \leq r^i_\eta$.

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(c) if $l = m+1$, if $\eta \in {}^m2$, $l < 2$, then $r^{i,l}_\eta \in (R \restriction Z^i_{0,1})/G_{R\restriction M^i_\emptyset}$ and $r^i_\eta \leq r^{i,l}_\eta \restriction Z^i_0 \leq r^i_{\eta\frown<l>}$ and $OP_{Z^i_1,Z^i_0}(r^i_\eta) \leq r^{i,l}_\eta \restriction Z^i_1 \leq OP_{Z^i_1,Z^i_0}(r^i_{\eta\frown<1-l>})$, and $r^{i,0}_\eta$ and $r^{i,1}_\eta$ are dual.
(d) $r^{i,l}_\eta$ forces that $A^{\eta,l}_k = \{\underset{\sim}{p}^{\eta,l}_{k,n} : n \in \omega\}$ is a predense subset of $P^*_{Z^i_{0,1}}$, such that each $\underset{\sim}{p}^{\eta,l}_{k,n}$ forces the value $f^{\eta,l}_{k,n}$ of $\underset{\sim}{f}(r_{\alpha^i_0}, r_{\alpha^i_1}) \restriction k$.
(e) $A^{\eta,0}_k$ and $A^{\eta,1}_k$ are dual, i.e. for every $m \in \omega$, $\underset{\sim}{p}^{\eta,0}_{k,m}$ and $\underset{\sim}{p}^{\eta,1}_{k,m}$ are dual. Moreover if $k_1 < k_2$, then $A^{\eta,l}_{k_2}$ refines $A^{\eta,l}_{k_1}$.

Suppose we have r^i_η. We define $r^{i,0}_\eta$, $r^{i,1}_\eta$ and $A^{\eta,0}_k$, $A^{\eta,0}_k$ as follows.

1. Let $r_1 = r^i_\eta \cup OP_{Z^i_1,Z^i_0}(r^i_\eta)$.

2. Let $r_{1,0} \geq r_1$, $r_{1,0} \in R \restriction Z_{0,1}$, forces a maximal antichain $A_{1,0}$ of $P^*_{Z_{0,1}}$, such that each element of $A_{1,0}$ forces a value of $\underset{\sim}{f}(r_{\alpha^i_0}, r_{\alpha^i_1}) \restriction k$.

3. Let $r_2 = OP_{Z^i_1,Z^i_0}(r_{1,0} \restriction Z^i_0) \cup OP_{Z^i_0,Z^i_1}(r_{1,0} \restriction Z^i_1)$. Let $r_{2,1} \geq r_2$, $r_{2,1} \in R \restriction Z_{0,1}$ forces $A_{2,1}$ to be a predense subset of $P^*_{Z_{0,1}}$ such that each element of $A_{2,1}$ forces a value of $\underset{\sim}{f}(r_{\alpha^i_0}, r_{\alpha^i_1}) \restriction k$. Moreover, $A_{2,1} = \bigcup\{A_p : p \in A_{1,0}\}$, where for every $q \in A_p$ we have $q \geq OP_{Z^i_1,Z^i_0}(p \restriction Z^i_0) \cup OP_{Z^i_0,Z^i_1}(p \restriction Z^i_1)$.

4. Let $r_3 = OP_{Z^i_1,Z^i_0}(r_{2,1} \restriction Z^i_0) \cup OP_{Z^i_0,Z^i_1}(r_{2,1} \restriction Z^i_1)$.

5. Let $r_{3,0} = r_3 \cup r_{1,0}$ (note: $r_{3,0}$ is dual to $r_{2,1}$). Let $A_{3,0} = \{p \cup OP_{Z^i_1,Z^i_0}(q \restriction Z^i_0) \cup OP_{Z^i_0,Z^i_1}(q \restriction Z^i_1) : q \in A_p\}$.

6. Let $r^{i,0}_\eta = r_{3,0}$, $r^{i,1}_\eta = r_{2,1}$, $A^{\eta,0}_k = A_{3,0}$ and $A^{\eta,1}_k = A_{2,1}$.

Let for $\eta \in {}^\omega 2$, $r^i_\eta = \bigcup_{k<\omega} r^i_{\eta\restriction k}$. In V choose $\langle \eta^*_\epsilon : \epsilon < \omega_1\rangle$, distinct members of ${}^\omega 2$. Recall that ρ_j ($j < \aleph_4$) are the Cohen subsets of ω_1 forced by R In $V[\langle \rho_j : j \in \{i\} \cup a_i\rangle]$ we can find $w^i \in [\omega_1]^{\omega_1}$ such that

(α) if $\epsilon \in w^i$ then $OP_{Z^i_\epsilon,Z^i_0}(r_{\eta^*_\epsilon}) \in G_{R\restriction Z^i_\epsilon}$,
(β) if $\epsilon_0 < \epsilon_1$ are in w^i, $l = TV(\eta^*_{\epsilon_0} <_{lx} \eta^*_{\epsilon_1})$, then
$$OP_{Z^i_{\epsilon_0,\epsilon_1},Z^i_{0,1}}(r^{i,l}_{\eta^*_{\epsilon_0}\cap\eta^*_{\epsilon_1}}) \in G_{R\restriction Z^i_{\epsilon_0,\epsilon_1}}.$$

We choose the members of w^i inductively using the fact that R has $(< \aleph_1)$-support.

Notation For $\xi \in w^i$ denote $r^i_\xi = r_{\alpha^i_\xi}$.

Let H be R-generic and G be $P^*_{a_i}$-generic. In $V[H][G]$ we define Q_i. A condition in Q_i is $(u, v, \bar{\nu}, \bar{m}, F_0, F_1)$, where:

(1) u is a finite subset of w^i.
(2) v is a finite set of elements of the form (η, ρ), where
  (a) $\eta, \rho \in {}^{\omega>}2$, $\mathrm{lh}(\eta) = \mathrm{lh}(\rho)$, $\rho \neq \eta$,
  (b) $\eta \lhd r^i_\alpha$, $\rho \lhd r^i_\beta$ for some $\alpha, \beta \in u$ and if $\nu = \eta^*_\alpha \cap \eta^*_\beta$ then for every $\gamma \in u$ we have: if $\eta \lhd r^i_\gamma$, then $\eta^*_\gamma \restriction (\mathrm{lh}(\nu)+1) = \eta^*_\alpha \restriction (\mathrm{lh}(\nu)+1)$, and if $\rho \lhd r^i_\gamma$, then $\eta^*_\gamma \restriction (\mathrm{lh}(\nu)+1) = \eta^*_\beta \restriction (\mathrm{lh}(\nu)+1)$.
(3) $\bar{\nu}$ is a function from v into ${}^{\omega>}2$ such that for $(\eta, \rho) \in v$ we have: $\bar{\nu}(\eta, \rho)$ is such that there is $\alpha, \beta \in u$ such that $\eta \lhd r^i_\alpha$, $\rho \lhd r^i_\beta$ and $\bar{\nu}(\eta, \rho) = \eta^*_\alpha \cap \eta^*_\beta$, ($\bar{\nu}$ is well defined by (2)).
(4) $\bar{m}$ is a function from v to ω. For $(\eta, \rho) \in v$, $\bar{m}(\eta, \rho)$ is such that for every $\alpha, \beta \in u$ such that $\eta \lhd r^i_\alpha$, $\rho \lhd r^i_\beta$, we have $OP_{Z^i_{\alpha,\beta},Z^i_{0,1}}(p^{\nu,l}_{lh(\eta),\bar{m}(\eta,\rho)}) \in G$, where $l = TV(\eta^*_\alpha <_{lx} \eta^*_\beta)$ and $\nu = \eta^*_\alpha \cap \eta^*_\beta$.

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(5) For $l = 0, 1$, F_l is a function from v into ${}^{\omega >}2$, defined by: for $(\eta, \rho) \in v$, $F_l(\eta, \rho)$ is the value of $\underset{\sim}{f}(r_0, r_1) \restriction lh(\eta)$ forced by $p^{\bar{\nu}(\eta,\rho),l}_{lh(\eta),\bar{m}(\eta,\rho)}$.
(6) For $(\eta, \rho), (\eta_1, \rho_1) \in v$, if $\eta \lhd \eta_1$ and $\rho \lhd \rho_1$, then $F_l(\eta, \rho) \lhd F_l(\eta_1, \rho_1)$, for $l = 0, 1$.

Order: $(u, v, \bar{\nu}, \bar{m}, F_0, F_1) \le (u^1, v^1, \bar{\nu}^1, \bar{m}^1, F^1_0, F^1_1)$ if

(7) $u \subseteq u^1$,
(8) $v \subseteq v^1$,
(9) $F_l = F^1_l \restriction v$, $\bar{\nu} = \bar{\nu}^1 \restriction v$, $\bar{m} = \bar{m}^1 \restriction v$, $l = 0, 1$.

Lemma 3.4. *Suppose (q_α, p_α), (for $\alpha \in \omega_1$), are in $P^*_{a_i} * Q_i$, q_α forces p_α to be a real 6-tuple in Q_i, not just a $P^*_{a_i}$-name of such a tuple, $\mathrm{dom}(q_\alpha)$ $(\alpha \in \omega_1)$ form a delta system with the root Δ, $\zeta \in \omega_1$. Let $b = \bigcup\{M^i_u : u \in [B_i(\zeta)]^{\le 2}\}$. Suppose $\Delta - \{i\} \subseteq b$ and $\mathrm{dom}(q_\alpha) \cap b = \Delta$ for $\alpha \in \omega_1$.*

Then there is an uncountable set $E \subseteq \omega_1$ such that for every $\alpha, \beta \in E$, (q_α, p_α) and (q_β, p_β) are compatible, moreover if $q \in P^*_b$, $q \ge q_\alpha \restriction b, q_\beta \restriction b$, then $q, (q_\alpha, p_\alpha)$ and (q_β, p_β) are compatible.

Proof By thinning out we can find an uncountable set $E \subseteq \omega_1$ such that:

(a) For $\alpha \in E$ let $w_\alpha = \bigcup\{u \in [B_i]^{\le 2} : \mathrm{dom}(q_\alpha) \cap M^i_u \ne \emptyset\}$, (each w_α is finite). The sets w_α, $(\alpha \in E)$ form a delta system with the root w and if $\alpha < \beta$, $\xi \in w_\alpha, \zeta \in w_\beta$, then $\xi \le \zeta$.
(b) u^{p_α} $(\alpha \in E)$ form a delta system with the root u and $\alpha < \beta$, $\xi \in u^{p_\alpha}, \zeta \in u^{p_\beta}$, then $\xi \le \zeta$, $|u^{p_\alpha}| = n^*$.
(c) $v^{p_\alpha} = v^*$ for $\alpha \in E$ and the structures $(u^{p_\alpha}, \{q_\alpha(\xi) : \xi \in u^{p_\alpha}\}, v^*, \{\eta^*_\xi \restriction m^* : \xi \in u^{p_\alpha}\})$ are isomorphic, (isomorphism given by the order preserving bijection between respective u^{p_α}'s), where m^* is such that $\mathrm{lh}(\eta^*_\xi \cap \eta^*_\zeta) < m^*$ for every $\xi \ne \zeta$ in u^{p_α}.

Lemma 3.5. *P_{i+1} has the property K.*

Proof Let $\{p_\alpha : \alpha \in \omega_1\}$ be an uncountable subset of P_{i+1}. W.l.o.g. we can assume that $\mathrm{dom}(p_\alpha)$, $(\alpha \in \omega_1)$ form a delta system with the root Δ. We have to find an uncountable subset $E \subseteq \omega_1$ such that for any $\alpha, \beta \in E$, p_α and p_β are compatible. We prove it by induction on $k = |\Delta|$.

For $k = 0$, trivial. For the induction step assume that $\Delta = \{i_0, \dots, i_k\}$ ordered by $\lhd$, where for $\alpha, \beta < \omega_4$, we define $\alpha \lhd \beta$ iff $\mathrm{otp}(C_\alpha) < \mathrm{otp}(C_\beta)$ or $\mathrm{otp}(C_\alpha) = \mathrm{otp}(C_\beta)$ and $\alpha < \beta$.

By the induction hypothesis there is an uncountable set $E' \subseteq \omega_1$ such that for $\alpha, \beta \in E'$, $p_\alpha \restriction \bigcup_{l<k} a_{i_l}$ and $p_\beta \restriction \bigcup_{l<k} a_{i_l}$ are compatible. Note that there is $\zeta \in \omega_1$ such that $a_{i_k} \cap (\bigcup_{l<k} a_{i_l}) \subseteq \bigcup\{M^{i_k}_u : u \in [B_{i_k}(\zeta)]^{\le 2}\}$, (see 2.2(D)). Now use the previous lemma.

Now suppose that $G(i)$ is Q_i-generic. Let

$$A' = \bigcup\{u : \exists (v, \bar{\nu}, \bar{m}, F_0, F_1), (u, v, \bar{\nu}, \bar{m}, F_0, F_1) \in G(i)\}.$$

In $V[G]$ let $A = \{r^i_\alpha : \alpha \in A'\}$ and let $f_l : [A]^2 \to {}^\omega 2$ be defined by:

$$f_l(r^i_\alpha, r^i_\beta) = \bigcup\{F_l(\eta, \rho) : \exists (u, v, \bar{\nu}, \bar{m}, F_0, F_1) \in G(i),$$
$$\alpha, \beta \in u, (\eta, \rho) \in v, \eta \lhd r^i_\alpha, \rho \lhd r^i_\beta\}.$$

Let $\mathcal{V} = \bigcup\{v : \exists (u, \bar{\nu}, \bar{m}, F_0, F_1) : (u, v, \bar{\nu}, \bar{m}, F_0, F_1) \in G(i)\}$.

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Lemma 3.6. (1) *For every $\alpha, \beta \in A'$ and $n \in \omega$ there is $(\eta, \rho) \in \mathcal{V}$ such that $\mathrm{lh}(\eta) = \mathrm{lh}(\rho) \geq n$ and $\eta \lhd r_\alpha$ and $\rho \lhd r_\beta$,*
(2) *A is uncountable,*
(3) *f_0, f_1 are continuous,*
(4) *for every $(\alpha, \beta) \in [A]^2$, if $l = TV(\eta^*_\alpha <_{lx} \eta^*_\beta)$, then $f(r^i_\alpha, r^i_\beta) = f_l(r^i_\alpha, r^i_\beta)$.*

Proof (1) and (2) follow by a density argument. To prove (1) suppose that $(p, q) \in P_i * Q_i$, p forces that $\alpha, \beta \in u^q$. W.l.o.g. $\alpha, \beta \in \mathrm{dom}(p)$. Let $p_1 \in P_i$ be such that $\mathrm{dom}(p) = \mathrm{dom}(p_1)$, $p(\zeta) = p_1(\zeta)$ for $\zeta \in \mathrm{dom}(p) \setminus \{\alpha, \beta\}$, $p(\alpha) \lhd p_1(\alpha)$, $p(\beta) \lhd p_1(\beta)$, $\mathrm{lh}(p_1(\alpha)) = \mathrm{lh}(p_1(\beta)) \geq n$, (remember that Q_α, Q_β are Cohen). Let $\eta = p_1(\alpha)$, $\rho = p_1(\beta)$, $\nu = \eta^*_\alpha \cap \eta^*_\beta$, $l = TV(\eta^*_\alpha <_{lx} \eta^*_\beta)$. Let $m \in \omega$ be such that $OP_{Z_{\alpha,\beta}, Z_{0,1}}(p^{\nu,l}_{\mathrm{lh}(\eta),m})$ is compatible with p_1, and let p_2 be the common upper bound. Now define $q_1 \geq q$ as follows. $u^{q_1} = u^q$, $v^{q_1} = v^q \cup \{(\eta, \rho)\}$, $\bar{\nu}^{q_1}(\eta, \rho) = \nu$, $\bar{m}^{q_1}(\eta, \rho) = m$, $F^{q_1}_l(\eta, \rho)$ is the value forced by $p^{\nu,l}_{\mathrm{lh}(\eta),m}$. Hence $(p_2, q_1) \geq (p, q)$ and it forces what is required.

To prove (2) it is enough to show, in V^R, that for every $\alpha \in \omega_1$ and $(p, q) \in P_i * Q_i$ there is $\beta > \alpha$ and $(p_1, q_1) \geq (p, q)$, such that $\beta \in u^{q_1}$. Let $\beta > \alpha$ be such that $\mathrm{dom}(p) \cap Z^i_{\gamma,\beta} \subseteq M^i_\emptyset$ and $\beta > \gamma$ for every $\gamma \in u^q$. Let $\gamma \in u^q$ be such that $(\eta^*_{\gamma_1} \cap \eta^*_\beta) \lhd (\eta^*_\gamma \cap \eta^*_\beta)$ for every $\gamma_1 \in u^q$. Define condition $q_1(\beta) = q(\gamma)$ and let p_1 be a condition extending p and each of conditions $OP_{Z^i_{\gamma_1,\beta}, Z^i_{0,1}}(p^{\bar{\nu}(\eta,\rho),l}_{\mathrm{lh}(\eta),\bar{m}(\eta,\rho)})$ such that $(\eta, \rho) \in v$, $\eta \lhd q(\gamma_1)$, $\rho \lhd q(\gamma)$ and $l = TV(\eta^*_{\gamma_1} < \eta^*_\beta)$. Finally extend q to q_1 such that $u^{q_1} = u^q \cup \{\beta\}$.

Condition (3) follows from (1) and (5) and (6) in the definition of Q_i.

To prove (4) it is enough to show that for every $n \in \omega$, $f(r^i_\alpha, r^i_\beta) \restriction n = f_l(r^i_\alpha, r^i_\beta) \restriction n$. By condition (1) there is $(\eta, \rho) \in V$ such that $k = \mathrm{lh}(\eta) \geq n$ and $\eta \lhd r^i_\alpha$ and $\rho \lhd r^i_\beta$. Recall that $p = p^{\bar{\nu}(\eta,\rho),l}_{\mathrm{lh}(\eta),\bar{m}(\eta,\rho)}$ forces that $f(r^i_0, r^i_1) \restriction k = h$ for some fixed h. Now working in V consider $(r^{i,l}_{\eta^*_\alpha \cap \eta^*_\beta}, p) \in R * P_i \restriction Z^i_{0,1}$. By the construction the condition $(r', p) = OP_{Z^i_{\alpha,\beta}, Z^i_{0,1}}(r^{i,l}_{\eta^*_\alpha \cap \eta^*_\beta}, p) \in H * G$, and forces that $f(r^i_\alpha, r^i_\beta) = h$. On the other hand, by definition $F_l(\eta, \rho) = h$ and $F_l(\eta, \rho) \lhd f_l(r^i_\alpha, r^i_\beta)$ This finishes the proof.

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