Embedding Cohen algebras using pcf theory

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Abstract

Using a theorem from pcf theory, we show that for any singular cardinal ν, the product of the Cohen forcing notions on κ, $\kappa < \nu$ adds a generic for the Cohen forcing notion on ν^+.

The following question (problem 5.1 in Miller's list [Mi91]) is attributed to Rene David and Sy Friedman:

> Does the product of the forcing notions ${}^{\aleph_n >}2$ add a generic for the forcing ${}^{\aleph_{\omega+1}>}2$?

We show here that the answer is yes in ZFC. Previously Zapletal [Za] has shown this result under the assumption $\square_{\aleph_{\omega+1}}$.

In fact, a similar theorem can be shown about other singular cardinals as well. The reader who is interested only in the original problem should read $\aleph_{\omega+1}$ for λ, $\aleph_\omega$ for μ and $\{\aleph_n : n \in (1, \omega)\}$ for $\mathfrak{a}$.

Definition 1 *1. Let $\mathfrak{a}$ be a set of regular cardinals. $\prod \mathfrak{a}$ is the set of all functions f with domain $\mathfrak{a}$ satisfying $f(\kappa) \in \kappa$ for all $\kappa \in \mathfrak{a}$.*

2. A set $\mathfrak{b} \subseteq \mathfrak{a}$ is bounded if $\sup \mathfrak{b} < \sup \mathfrak{a}$. $\mathfrak{b}$ is cobounded if $\mathfrak{a} \setminus \mathfrak{b}$ is bounded.

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3. *If J is an ideal on $\mathfrak{a}$, $f, g \in \prod \mathfrak{a}$, then $f <_J g$ means $\{\kappa \in \mathfrak{a} : f(\kappa) \not< g(\kappa)\} \in J$. We write $\prod \mathfrak{a}/J$ for the partial (quasi)order $(\prod \mathfrak{a}, <_J)$.*

4. *$\lambda = tcf(\prod \mathfrak{a}/J)$ (λ is the true cofinality of $\prod \mathfrak{a}/J$) means that there is a strictly increasing cofinal sequence of functions in the partial order $(\prod \mathfrak{a}, <_J)$.*

5. *$\mathrm{pcf}(\mathfrak{a}) = \{\lambda : \exists J\ \lambda = tcf(\prod \mathfrak{a}/J)\}$.*

We will use the following theorem from pcf theory:

Lemma 2 *Let μ be a singular cardinal. Then there is a set $\mathfrak{a}$ of regular cardinals below μ, $|\mathfrak{a}| = cf(\mu) < \min \mathfrak{a}$ and $\mu^+ \in \mathrm{pcf}(\mathfrak{a})$.*

Moreover, we can even have $tcf(\prod \mathfrak{a}/J^{bd}) = \mu^+$, where J^{bd} is the ideal of all bounded subset of $\mathfrak{a}$.

Proof See [Sh 355, theorem 1.5].

Theorem 3 *Let $\mathfrak{a}$ be a set of regular cardinals, $\mu = \sup \mathfrak{a} \notin \mathfrak{a}$, $2^{<\lambda} = 2^\mu$, $\lambda > \mu$, $\lambda \in \mathrm{pcf}(\mathfrak{a})$, and moreover:*

(∗) *There is an ideal J on $\mathfrak{a}$ containing all bounded sets such that $\lambda = tcf(\prod \mathfrak{a}/J)$.*

Then the forcing notion $\prod_{\kappa \in \mathfrak{a}} {}^{\kappa>}2$ adds a generic for ${}^{\lambda>}2$.

Corollary 4 *If ν is a singular cardinal, and P is the product of the forcing notions ${}^{\kappa>}2$ for $\kappa < \nu$, then P adds a generic for ${}^{\nu^+>}2$.*

Proof By lemma 2 and theorem 3

Remark 5 1. *The condition (∗) in the theorem is equivalent to:*

(∗∗) *For all bounded sets $\mathfrak{b} \subset \mathfrak{a}$ we have $\lambda \in \mathrm{pcf}(\mathfrak{a} \setminus \mathfrak{b})$.*

2. *Clearly the assumption $2^{<\lambda} = 2^\mu$ is necessary, because otherwise the forcing notion $\prod_{\kappa \in \mathfrak{a}} {}^{\kappa>}2$ would be too small to add a generic for ${}^{\lambda>}2$.*

Proof of the theorem: By our assumption we have some ideal J containing all bounded sets such that $tcf(\prod \mathfrak{a}/J) = \lambda$.

We will write $\forall^J \kappa \in \mathfrak{a}\ \varphi(\kappa)$ for

$$\{\kappa \in \mathfrak{a} : \neg\varphi(\kappa)\} \in J$$

So we have a sequence $\langle f_\alpha : \alpha < \lambda \rangle$ such that

(a) $f_\alpha \in \prod \mathfrak{a}$

(b) If $\alpha < \beta$, then $\forall^J \kappa \in \mathfrak{a}\ f_\alpha(\kappa) < f_\beta(\kappa)$

(c) $\forall f \in \prod \mathfrak{a}\ \exists \alpha\ \forall^J \kappa \in \mathfrak{a}\ f(\kappa) < f_\alpha(\kappa)$.

The next lemma shows that if we allow these fucntions to be defined only almost everywhere, then we can additionally assume that in each block of length μ these functions have disjoint graphs:

Lemma 6 *Assume that $\mathfrak{a}$, λ, μ are as above. Then there is a sequence $\langle g_\alpha : \alpha < \lambda \rangle$ such that*

(a) $\mathrm{dom}(g_\alpha) \subseteq \mathfrak{a}$ *cobounded (so in particular $\forall^J \kappa \in \mathfrak{a} : \kappa \in \mathrm{dom}(g_\alpha(\kappa))$).*

(b) If $\alpha < \beta$, then $\forall^J \kappa \in \mathfrak{a}\ g_\alpha(\kappa) < g_\beta(\kappa)$

(c) $\forall f \in \prod \mathfrak{a}\ \exists \alpha\ \forall^J \kappa \in \mathfrak{a}\ f(\kappa) < g_\alpha(\kappa)$. Moreover, we may choose α to be divisible by μ.

(d) If $\alpha < \beta < \alpha + \mu$, then $\forall \kappa \in \mathrm{dom}(g_\alpha) \cap \mathrm{dom}(g_\beta) : g_\alpha(\kappa) < g_\beta(\kappa)$.

Proof Let $\langle f_\alpha : \alpha < \lambda \rangle$ be as above. Now define $\langle g_\alpha : \alpha < \lambda \rangle$ by induction as follows:
If $\alpha = \mu \cdot \zeta$, then let g_α be any function that satisfies $g_\beta <_J g_\alpha$ for all $\beta < \alpha$, and also $f_\alpha <_J g_\alpha$. Such a function can be found because set of functions of size $< \lambda$ can be $<_J$-bounded by some f_β.
If $\alpha = \mu \cdot \zeta + i$, $0 < i < \mu$, then let

$$g_\alpha(\kappa) = \begin{cases} g_{\mu\cdot\zeta}(\kappa) + i & \text{if } i < \kappa \\ \text{undefined} & \text{otherwise} \end{cases}$$

It is easy to see that (a)–(d) are satisfied.

Definition 7 *1. Let P_κ be the set ${}^{\kappa>}2$, partially ordered by inclusion (= sequence extension). Let $P = \prod_{\kappa\in\mathfrak{a}} P_\kappa$. [We will show that P adds a generic for ${}^{\lambda>}2$]*

2. Assume that $\langle g_\alpha : \alpha < \lambda \rangle$ is as in lemma 6.

3. Let $H : {}^{\mu}2 \to {}^{\lambda>}2$ be onto.

4. For $\kappa \in \mathfrak{a}$, let $\underset{\sim}{\eta}_\kappa$ be the P_κ-name for the generic function from κ to 2. Define a P-name of a function $\underset{\sim}{h} : \lambda \to 2$ by

$$\underset{\sim}{h}(\alpha) = \begin{cases} 0 & \text{if } \forall^J \kappa \in \mathfrak{a}\ \underset{\sim}{\eta}_\kappa(g_\alpha(\kappa)) = 0 \\ 1 & \text{otherwise} \end{cases}$$

5. *For $\xi < \lambda$ let $\underset{\sim}{\rho}_\xi$ be a P-name for the element of ${}^{\mu}2$ that satisfies $\underset{\sim}{\rho}_\xi \simeq \underset{\sim}{h}\restriction[\mu\cdot\xi, \mu\cdot(\xi+1))$, i.e.,*

$$i < \mu \;\Rightarrow \Vdash_P \underset{\sim}{\rho}_\xi(i) = \underset{\sim}{h}(\mu\cdot\xi + i).$$

Define $\underset{\sim}{\rho} \in {}^{\lambda}2$ by

$$\underset{\sim}{\rho} = H(\underset{\sim}{\rho}_0)^\frown H(\underset{\sim}{\rho}_1)^\frown\cdots{}^\frown H(\underset{\sim}{\rho}_\xi)^\frown\cdots$$

Main Claim 8 *$\underset{\sim}{\rho}$ is generic for ${}^{\lambda>}2$.*

Definition 9 *For $\alpha < \lambda$ let $P^{(\alpha)}$ be the set of all conditions p satisfying $\forall^J \kappa : \mathrm{dom}(p_\kappa) = g_\alpha(\kappa)$.*

Remark 10 *$\bigcup_{\zeta<\lambda} P^{(\mu\cdot\zeta)}$ is dense in P.*

Proof By lemma 6(c).

Fact 11 *Let $\alpha = \mu\cdot\zeta$, $p \in P^{(\alpha)}$, $\sigma \in {}^{\mu}2$. Then there is a condition $q \in P^{(\alpha+\mu)}$, $q \geq p$ and*

$$\forall j < \mu\; \forall^J \kappa\; q_\kappa(g_{\alpha+j}(\kappa)) = \sigma(j)$$

Proof Let $p = (p_\kappa : \kappa \in \mathfrak{a})$. There is a set $\mathfrak{b} \in J$ such that: For all $\kappa \in \mathfrak{a}\setminus\mathfrak{b}$ we have $\mathrm{dom}(p_\kappa) = g_\alpha(\kappa)$. Define $q \in P^{(\alpha+\mu)}$, $q = (q_\kappa : \kappa \in \mathfrak{a})$ as follows:

$$q_\kappa(\gamma) = \begin{cases} p_\kappa(\gamma) & \text{if } \gamma \in \mathrm{dom}(p_\kappa) \\ \sigma(j) & \text{if } \gamma = g_{\alpha+j}(\kappa),\ \kappa \in \mathfrak{a}\setminus\mathfrak{b} \\ 0 & \text{otherwise} \end{cases}$$

We have to explain why q is well-defined: First note that the first and the second case are mutually exclusive. Indeed, if $\gamma = g_{\alpha+j}(\kappa)$, then $\gamma > g_\alpha(\kappa)$, whereas $\kappa \notin \mathfrak{b}$ implies that $\mathrm{dom}(p_\kappa) = g_\alpha(\kappa)$, so $\gamma \notin \mathrm{dom}(p_\kappa)$.
Next, by the property (d) from lemma 6 there is no contradiction between various instances of the second case.

Hence we get that for all $j < \mu$, whenever $\kappa \in \mathfrak{a}\setminus\mathfrak{b}$, and $\kappa > j$, then $q_\kappa(g_{\alpha+j}(\kappa)) = \sigma(j)$. Since J contains all bounded sets, this means that $\forall^J \kappa : q_\kappa(g_{\alpha+j}(\kappa)) = \sigma(j)$.

Remark 12 *Assume that $\alpha = \mu\cdot\zeta$, and p, q, σ are as above. Then $q \Vdash \underset{\sim}{\rho}_\zeta = \sigma$.*

Proof of the main claim: Let $p \in P$, and $D \subseteq {}^{\lambda>}2$ be a dense open set. We may assume that for some $\alpha^* < \lambda$, $\zeta^* < \lambda$ we have $\alpha^* = \mu \cdot \zeta^*$ and $p \in P^{(\alpha^*)}$, i.e., for some $\mathfrak{b} \in J$: $\forall \kappa \notin \mathfrak{b} : \mathrm{dom}(p_\kappa) = g_{\alpha^*}(\kappa)$

So p decides the values of $h{\restriction}\alpha^*$, and hence also the values of $\underset{\sim}{\rho}_\zeta$ for $\zeta < \zeta^*$. Specifically, for each $\zeta < \zeta^*$ we can define $\sigma_\zeta \in {}^\mu 2$ by

$$\sigma_\zeta(i) = \begin{cases} 0 & \text{if } \forall^J \kappa \; p_\kappa(g_{\mu \cdot \zeta + i}(\kappa)) = 0 \\ 1 & \text{otherwise} \end{cases}$$

(Note that for all $\zeta < \zeta^*$, for all $i < \mu$, for almost all κ the value of $p_\kappa(g_{\mu \cdot \zeta + i}(\kappa))$ is defined.)

Clearly $p \Vdash \underset{\sim}{\rho}_\zeta = \sigma_\zeta$. Since D is dense and H is onto, we can now find $\sigma_{\zeta^*} \in {}^\mu 2$ such that

$$H(\sigma_0)^\frown \cdots {}^\frown H(\sigma^*_\zeta) \in D$$

Using 11 and 12, we can now find $q \geq p$ such that $q \Vdash \rho_{\zeta^*} = \sigma_{\zeta^*}$.

Hence $q \Vdash \underset{\sim}{\rho} \in D$, and we are done.

References

[Mi91] Arnold Miller. Arnie Miller's problem list. In Haim Judah, editor, *Set Theory of the Reals*, volume 6 of *Israel Mathematical Conference Proceedings*, pages 645–654. Proceedings of the Winter Institute held at Bar–Ilan University, Ramat Gan, January 1991.

[Za] Jindrich Zapletal. *Some Results in Set Theory and Boolean Algebras.* PhD thesis, Penn State University, 1995.

[Sh 355] Saharon Shelah. $\aleph_{\omega+1}$ has a Jonsson Algebra. In *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides*, chapter II. Oxford University Press, 1994.