THE PAIR $(\aleph_n, \aleph_0)$ MAY FAIL $\aleph_0$-COMPACTNESS
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Abstract. Let P be a distinguished unary predicate and $\mathbf{K} = \{M : M$ a model of cardinality $\aleph_n$ with P^M of cardinality $\aleph_0\}$. We prove that consistently, for $n = 4$, for some countable first order theory T we have: T has no model in $\mathbf{K}$ whereas every finite subset of T has a model in $\mathbf{K}$. We then show how we prove it for $n = 2$, too.

Key words and phrases. model theory, two cardinal theorems, compactness, partition theorems.

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Annotated Content

§0 Introduction

Interest in two cardinal models comes from the early days of model theory, as generalizations of the Lowenheim-Skolem theorem. Already Mostowski [Mo57] considered a related problem concerning generalized quantifiers. Let us introduce the problem. Throughout the paper λ, μ and κ stand for infinite cardinals and n, k for natural numbers.

We consider a countable vocabulary τ with a distinguished unary relation symbol P and models M for τ; i.e., τ-models.
<u>0.1 Notation</u>: We let

$$K_{(\lambda,\mu)} =: \{M : ||M|| = \lambda \ \& \ |P^M| = \mu\}.$$

0.2 Definition. 1) We say that $K_{(\lambda,\mu)}$ is $(< \kappa)$-compact when every first order theory T in the vocabulary τ (i.e., in the first order logic $\mathbb{L}(\tau)$) with $|T| < \kappa$, satisfies:

<u>if</u> every finite subset of T has a model in $K_{(\lambda,\mu)}$, <u>then</u> T has a model in $K_{(\lambda,\mu)}$.

We similarly give the meaning to $(\leq \kappa)$-compactness. We say that (λ, μ) is $(< \kappa)$-compact if $K_{(\lambda,\mu)}$ is.
2) We say that

$$(\lambda, \mu) \rightarrow'_\kappa (\lambda', \mu')$$

when for every first order theory T in $\mathbb{L}(\tau)$ with $|T| < \kappa$, <u>if</u> every finite subset T has a model in $K_{(\lambda,\mu)}$, <u>then</u> T has a model in $K_{(\lambda',\mu')}$. Instead "κ^+" we may write "$\leq \kappa$". Similarly in (3).
3) We say that

$$(\lambda, \mu) \rightarrow_\kappa (\lambda', \mu')$$

when for every first order theory T of $\mathbb{L}(\tau)$ with $|T| < \kappa$, <u>if</u> T has a model in $K_{(\lambda,\mu)}$, <u>then</u> T has a model in $K_{(\lambda',\mu')}$.
4) In both $\rightarrow'_\kappa$ and $\rightarrow_\kappa$ we omit κ if $\kappa = \aleph_0$.

<u>Note</u>: Note that $\rightarrow_\kappa$ is transitive and $\rightarrow'_\kappa$ is as well. Also note that $\rightarrow_{\aleph_0}$ and $\rightarrow'_{\aleph_0}$ are equivalent.

We consider the problem of $K_{(\lambda,\mu)}$ being compact. Before we start, we review the history of the problem. Note that a related problem is the one of completeness, i.e. is

$$\{\psi : \psi \text{ has a model in } K_{(\lambda,\mu)}\}$$

recursively enumerable? and other related problems, see in the end. We do not concentrate on those problems here.

We review some of the history of the problem, in an order which is not necessarily chronological.

Some early results on the compactness are due to Furkhen [Fu65]. He showed that

(A) if $\mu^\kappa = \mu$ and $\lambda \geq \mu$, <u>then</u> $K_{(\lambda,\mu)}$ is $(\leq \kappa)$-compact.

The proof is by using ultraproducts over regular ultrafilters on κ, generalizing the well known proof of compactness by ultrapowers. A related result of Morley is

(B) ([Mo68]) If $\mu^{\aleph_0} \leq \mu' \leq \lambda' \leq \lambda$, <u>then</u> $(\lambda,\mu) \rightarrow_{\leq\lambda} (\lambda',\mu')$.

Next result we mention is one of Silver concerning Kurepa trees,

(C) (Silver [Si71]) From the existence of a strongly inaccessible cardinal, it follows that the following is consistent with ZFC:

$$GCH + (\aleph_3,\aleph_1) \nrightarrow_{\aleph_0} (\aleph_2,\aleph_0).$$

Using special Aronszajn trees Mitchell showed

(D) (Mitchell [Mi72]) From the existence of a Mahlo cardinal, it follows that it is consistent with ZFC to have

$$(\aleph_1,\aleph_0) \nrightarrow_{\aleph_2} (\aleph_2,\aleph_1).$$

A later negative consistency result is the one of Schmerl

(E) (Schmerl [Sc74]) Con(if $n < m$ then $(\aleph_{n+1},\aleph_n) \nrightarrow (\aleph_{m+1},\aleph_m)$).

Earlier, Vaught proved two positive results

(F) (Vaught [MV62]) $(\lambda^+,\lambda) \rightarrow'_{\aleph_1} (\aleph_1,\aleph_0)$.

Keisler [Ke66] and [Ke66a] has obtained more results in this direction.

(G) (Vaught [Va65]) If $\lambda \geq \beth_\omega(\mu)$ and $\lambda' > \mu'$, <u>then</u> $(\lambda,\mu) \rightarrow'_{\leq\mu'} (\lambda',\mu')$.

In [Mo68] Morley gives another proof of this result, using Erdös-Rado Theorem and indiscernibles.
Another early positive result is the one of Chang:

(H) (Chang [Ch65]) If $\mu = \mu^{<\mu}$ then $(\lambda^+, \lambda) \to'_{\leq \mu} (\mu^+, \mu)$.

Jensen in [Jn] uses $\square_\mu$ to show

(I) (Jensen [Jn]) If $\mathbf{V} = \mathbf{L}$, then $(\lambda^+, \lambda) \to'_{\leq \mu} (\mu^+, \mu)$. (The fact that $0^{\#}$ does not exist suffices.)

Hence, Jensen's result deals with the case of μ is singular, which was left open after the result of Chang. For other early consistency results concerning gap-1 two cardinal theorems, including consistency, see [Sh 269], Cummings, Foreman and Magidor [CFM0x]].

In [Jn] there is actually a simplified proof of (I) due to Silver. A further result of Jensen, using morasses, is:

(J) (Jensen, see [De73] for $n = 2$) If $\mathbf{V} = \mathbf{L}$, then $(\lambda^{+n}, \lambda) \to'_{\leq \mu} (\mu^{+n}, \mu)$ for all $n < \omega$.

Note that by Vaught's result [MV62] stated in (F) we have: the statement in (I), in the result of Chang etc., (λ^+, λ) can be without loss of generality replaced by $(\aleph_1, \aleph_0)$.

(K) ([Sh 49]) $(\aleph_\omega, \aleph_0) \to'_{\aleph_0} (2^{\aleph_0}, \aleph_0)$.

Finally, there are many more related results, for example the ones concerning Chang's conjecture. A survey article on the topic was written by Schmerl in [Sc74]. Note that typically the positive results above (F)-(J), their proof also gives compactness of the pair, e.g., $(\aleph_1, \aleph_0)$ by [MV62].

We now mention some results of the author which will have a bearing to the present paper.

(α) (Shelah [Sh 8] and the abstract [Sh:E17]). If $K_{(\lambda,\mu)}$ is $(\leq \aleph_0)$-compact, then $K_{(\lambda,\mu)}$ is $(\leq \mu)$-compact and $(\lambda, \mu) \to_{\leq \mu'} (\lambda', \mu')$ when $\lambda \leq \lambda' \leq \mu' \leq \mu$.

More than $(\leq \mu)$-compactness cannot hold for trivial reasons. In the same work we have the analogous result on $\to'$ and:

(β) (Shelah [Sh 8] and the abstract [Sh:E17]) $(\lambda, \mu) \to'_{\aleph_1} (\lambda', \mu')$ is actually a problem on partition relations, (see below), also it implies $(\lambda, \mu) \to'_{\leq \mu'} (\lambda', \mu')$ see 0.4(1) below.

We state a definition from [Sh 8] that will be used here too. We do not consider the full generality of [Sh 8], there problems like considering K with several λ_ℓ-like $(P^2_\ell, <_\ell)$ and $|P^1_\ell| = \mu_\ell$ were addressed.
(We can use below only ordered a and increase h, it does not matter much.)

0.3 Definition. 1) An identity[1] is a pair (a, e) where a is a finite set and e is an equivalence relation on the finite subsets of a, having the property

$$b\,e\,c \Rightarrow |b| = |c|.$$

The equivalence class of b with respect to e will be denoted b/e.
2) We say that $\lambda \to (a, e)_\mu$, if for every $f : [\lambda]^{<\aleph_0} \to \mu$, there is $h : a \xrightarrow{1-1} \lambda$ such that

$$b\,e\,c \Rightarrow f(h''(b)) = f(h''(c)).$$

where

$$h''(b) = \{h(\alpha) : \alpha \in b\}.$$

3) We define

$$\mathrm{ID}(\lambda, \mu) =: \{(n, e) : n < \omega \ \&\ (n, e) \text{ is an identity and } \lambda \to (n, e)_\mu\}.$$

4) For $f : [\lambda]^{<\aleph_0} \to X$ we let

$$\begin{aligned} \mathrm{ID}(f) =: \{(n, e) :&(n, e) \text{ is an identity such that for some one-to-one function} \\ &h \text{ from } n = \{0, \dots, n-1\} \text{ to } \lambda \text{ we have} \\ &(\forall b, c \subseteq n)(b\,e\,c \Rightarrow f(h''(b)) = f(h''(c)))\} \end{aligned}$$

[1] identification in the terminology of [Sh 8]

0.4 Claim. *(Shelah [Sh 8] and the abstract [Sh:E17])* $(\lambda,\mu) \rightarrow'_{\aleph_1} (\lambda',\mu')$ *is equivalent to the existence of a function* $f : [\lambda']^{<\aleph_0} \rightarrow \mu'$ *such that*

$$\mathrm{ID}(f) \subseteq \mathrm{ID}(\lambda,\mu)$$

(more on this see [Sh 74, Th.3] statement there on $\rightarrow'_{\aleph_1}$*, see details in [Sh:E28]).*

0.5 Remark. The identities of $(\beth_\omega, \aleph_0)$ are clearly characterized by Morley's proof of Vaught's theorem (see [Mo68]). The identities of $(\aleph_\omega, \aleph_0)$ are stated explicitly in [Sh 37] and [Sh 49], when $\aleph_\omega \le 2^{\aleph_0}$ where it is also shown that $(\aleph_\omega, \aleph_0) \rightarrow' (2^{\aleph_0}, \aleph_0)$. For $(\aleph_1, \aleph_0)$, the identities are characterized in [Sh 74] (for some details see [Sh:E28]). The identities for λ-like models, λ strongly ω-Mahlo are clear, see Schmerl and Shelah [ScSh 20] (for strongly n-Mahlo this gives positive results, subsequently sharpened (replacing $n+2$ by n) and the negative results proved by Schmerl, see [Sch85]).

We generally neglect here three cardinal theorems and λ-like model (and combinations, see [Sh 8], [Sh 18], the positive results like 0.4 are similar). Recently Shelah and Vaananen [ShVa 790] deal with recursiveness, completeness and identities, see also [ShVa:E47].

In Gilschrist, Shelah [GcSh 491] and [GcSh 583], we dealt with 2-identities.

0.6 Definition. 1) A two-identity or k-identity[2] is a pair (a,e) where a is a finite set and e is an equivalence relation on $[a]^k$. Let $\lambda \rightarrow (a,e)_\mu$ mean $\lambda \rightarrow (a,e^+)_\mu$ where $be^+c \Leftrightarrow (bec) \vee (b = c \subseteq a)$ for any $b,c \subseteq a$.
1A) A $(\le k)$-identity is defined similarly using $[a]^{\le k}$.
2) We define

$$\mathrm{ID}_k(\lambda,\mu) =: \{(n,e) : (n,e) \text{ is a } k\text{-identity and } \lambda \rightarrow (n,e)_\mu\}.$$

2A) We define $\mathrm{ID}_2(f)$ when $f : [\lambda]^2 \rightarrow X$ as

$$\mathrm{ID}_2(f) = \Big\{ (n,e) : (n,e) \text{ is a two-identity such that for some } h, \\ \text{a one-to-one function from } \{0,\dots,n-1\} \text{ into } \lambda \\ \text{we have } \{\ell_1,\ell_2\}e\{k_1,k_2\} \text{ implies that } \ell_1 \ne \ell_2 \in \{0,\dots,n-1\}, \\ k_1 \ne k_2 \in \{0,\dots,n-1\} \text{ and } f(\{h(\ell_1),h(\ell_2)\}) = f(\{h(k_1),h(k_2)\}) \Big\}.$$

[2] it is not an identity as e is an equivalence relation on too small set

2B) We define $\mathrm{ID}_k(f)$ when $f : [\lambda]^k \to X$ as $\{(n,e) : (n,e)$ is a k-identity such that if uev so $u,v \subseteq \{0,\dots,n-1\}$ satisfies $|u| = |v| \le k$ then $f(\{h(\ell) : \ell \in u\}), f(\{h(\ell) : \ell \in v\})\}$.
2C) We define $\mathrm{ID}_{\le k}(f)$ when $f : [\lambda]^{\le k} \to X$ similarly.
3) Let us define

$$\begin{aligned}\mathrm{ID}_2^{\circledast} =: \{({}^n 2, e) : \ & ({}^n 2, e) \text{ is a two-identity and if} \\ & \{\eta_1,\eta_2\},\{\nu_1,\nu_2\} \text{ are } \subseteq {}^n 2, \text{ then} \\ & \{\eta_1,\eta_2\} e \{\nu_1,\nu_2\} \Rightarrow \eta_1 \cap \eta_2 = \nu_1 \cap \nu_2\}.\end{aligned}$$

By [Sh 49], under the assumption $\aleph_\omega < 2^{\aleph_0}$, the families $\mathrm{ID}_2(\aleph_\omega,\aleph_0)$ and $\mathrm{ID}_2^{\circledast}$ coincide (up to an isomorphism of identities). In Gilchrist and Shelah [GcSh 491] and [GcSh 583] we considered the question of the equality between these $\mathrm{ID}_2(2^{\aleph_0},\aleph_0)$ and $\mathrm{ID}_2^{\circledast}$ under the assumption $2^{\aleph_0} = \aleph_2$. We showed that consistently the answer may be "yes" and may be "no".

Note that $(\aleph_n,\aleph_0) \nrightarrow (\aleph_\omega,\aleph_0)$ so $\mathrm{ID}(\aleph_n,\aleph_0) \ne \mathrm{ID}(\aleph_\omega,\aleph_0)$, but for identities for pairs (i.e. ID_2) the question is meaningful.

The history of the problem suggested to me that there should be a model where $K_{(\lambda,\mu)}$ is not $\aleph_0$-compact for some λ,μ; I do not know about the opinion of others and it was not easy for me as I thought a priori. As mathematicians do not feel that a strong expectation makes a proof, I was quite happy to be able to prove the existence of such a model. This was part of my lectures in a 1995 seminar in Jerusalem and notes of the lecture were taken by Mirna Dzamonja and I thank her for this, but because the proof was not complete, its publications were delayed.

I thank the referee for various corrections and Peter Komjath for detecting a problem in the previous proof of 5.13 and Alon Siton for some corrections.

The following is the main result of this paper (proved in 3.4):

0.7 Main Theorem. Con*(the pair* $(\aleph_n,\aleph_0)$ *is not* $\aleph_0$*-compact* $+2^{\aleph_0} \ge \aleph_n$*) for* $n \ge 4$.

Later in the paper we deal with the case $n = 2$ which is somewhat more involved. This is the simplest case by a reasonable measure: if you do not like to use large cardinals then assuming that there is no inaccessible in $\mathbf{L}$ (or much less), all pairs (μ^+,μ) are known to be $\aleph_0$-compact and if in addition $\mathbf{V} = \mathbf{L}$ also all logic $L(\exists^{\ge\lambda}), \lambda > \aleph_0$ are (by putting together already known results; $\mathbf{V} = \mathbf{L}$ is used just to imply that there is no limit, uncountable not strong limit cardinal; so adding G.C.H. suffice).

How much this consistency result will mean to a model theorist, let us not elaborate, but instead say an anecdote about Jensen. He is reputed to have said: " When I started working on the two-cardinal problem, I was told it was the heart of model theory. Once I succeeded to prove something, they told me what I did was pure set theory, and were not very interested"; also, mathematics is not immune to fashion changes.

My feeling is that there are probably more positive theorems in this subject waiting to be discovered. Anyway, let us state the following

Thesis Independence results help us clear away the waste, so the possible treasures can stand out.

Of course, I have to admit that, having spent quite some time on the independence results, I sometimes look for the negative of the picture given by this thesis.

The strategy of our proof is as follows. It seems natural to consider the simplest case, i.e., that of two-place functions, and try to get the incompactness by constructing a sequence $\langle f_k : k < \omega \rangle$ of functions from $[\aleph_n]^2$ into $\aleph_0$ such that for all k we have $\mathrm{ID}_2(f_k) \supseteq \mathrm{ID}_2(f_{k+1})$, yet for no $f : [\aleph_n]^2 \to \aleph_0$ do we have $\mathrm{ID}_2(f) \subseteq \bigcap\limits_{k<\omega} \mathrm{ID}_2(f_k)$. This suffices. Related proofs to our main results were [Sh 522].

Note that another interpretation of 0.7 is that if we add to first order logic the cardinality quantifiers $(\exists^{\geq \lambda} x)$ for $\lambda = \aleph_1, \aleph_2, \aleph_3, \aleph_4$ we get a noncompact logic.

We thank the referee for many helpful comments and the reader should thank him also for urging the inclusion of several proofs.

This work is continued in [ShVa 790] and [Sh 824].

§1 Relevant Identities

We commence by several definitions. For simplicity, for us all identities, colorings etc. will be 2-place.

1.1 Definition. 1) For $m, \ell < \omega$ let

$$\mathrm{dom}_{\ell,m} = \{\eta \in {}^{\ell+1}\omega : \eta \restriction \ell \in {}^{\ell}2 \text{ and } \eta(\ell) < m\}$$

$$\begin{aligned}\mathrm{ID}^1_{\ell,m} = \{(\mathrm{dom}_{\ell,m}, e) :\, & e \text{ is an equivalence relation on } [\mathrm{dom}_{\ell,m}]^2 \\ & \text{such that } \{\eta_1,\eta_2\} e \{\nu_1,\nu_2\} \\ & \Rightarrow \eta_1 \cap \eta_2 = \nu_1 \cap \nu_2 \wedge \ell g(\eta_1 \cap \eta_2) < \ell\}.\end{aligned}$$

2) Let

$$\mathrm{ID}^1_\ell = \cup\{\mathrm{ID}^1_{\ell,m} : m < \omega\}$$

$$\mathrm{ID}^1 = \cup\{\mathrm{ID}^1_\ell : \ell < \omega\}.$$

3) For $\mathbf{s} = (\mathrm{dom}_{\ell,m}, e) \in \mathrm{ID}^1_{\ell,m}$ and $\nu \in {}^{\ell \geq}2$ let

$$\mathrm{dom}^{[\nu]}_{\ell,m} = \{\rho \in \ \mathrm{dom}_{\ell,m} : \nu \trianglelefteq \rho\}$$

and if $\nu \in {}^{\ell >}2$ we let

$$e_{<\nu>}(\mathbf{s}) = e \restriction \{\{\eta_0,\eta_1\} : \nu\char`^ < i > \triangleleft \eta_i \text{ for } i = 0,1\}.$$

We use $\mathbf{s}$ to denote identities so $\mathbf{s} = (\mathrm{dom}_{\mathbf{s}}, e(\mathbf{s}))$; and if $\mathbf{s} \in \mathrm{ID}^1$ then let $\mathbf{s} = (\mathrm{dom}_{\ell(\mathbf{s}),m(\mathbf{s})}, e(\mathbf{s}))$.
4) An equivalence class is non-trivial <u>if</u> it is not a singleton.

Note that it follows that every e-equivalence class is an $e_{<\nu>}$-equivalence class for some ν. We restrict ourselves to

1.2 Definition. 1) Let $\mathrm{ID}^2_{\ell,m}$ be the set of $\mathbf{s} \in \mathrm{ID}^1_{\ell,m}$ such that for every $\nu \in {}^{\ell>}2$ the equivalence relation $e_{<\nu>}(\mathbf{s})$ has at most one non-singleton equivalence class, which we call $e_{[\nu]} = e_{[\nu]}(\mathbf{s})$.

So we also allow $e_{<\nu>}(\mathbf{s}) =$ empty, in which case we choose a representative equivalence class $e_{[\nu]}$ as the first one under, say, lexicographical ordering.
2) $\mathrm{ID}^2_\ell = \cup\{\mathrm{ID}^2_{\ell,m} : m < \omega\}$.

1.3 Definition. 0) We say $\mathscr{C}$ is a closure operation on a set W when: $\mathscr{C}$ is a function from the family of subsets of W to itself and $\mathscr{U}_1 \subseteq \mathscr{U}_2 \subseteq W \Rightarrow \mathscr{C}(\mathscr{U}_1) \subseteq \mathscr{C}(\mathscr{U}_2) = \mathscr{C}(\mathscr{C}(\mathscr{U}_2)) \subseteq W$ and $\mathscr{C}(\mathscr{U}) = \cup\{\mathscr{C}(u) : u \subseteq \mathscr{U}$ finite$\}$.
1) We define for $k, \ell, m < \omega$; $\mathscr{C}$ a closure operation on $\mathrm{dom}_{\ell,m}$ when $\mathbf{s} = (\mathrm{dom}_{\ell,m}, e)$ is $(k, \mathscr{C})$-nice: the demands are

- (a) $\mathbf{s} \in \mathrm{ID}^1_{\ell,m}$
- (b) if $\nu \in {}^\ell 2$ and $(\nu \restriction i)\hat{}\langle 1 - \nu(i)\rangle \triangleleft \rho_i \in \mathrm{dom}_{\ell,m}$ for each $i < \ell$ then $\{\eta : \nu \triangleleft \eta \in \mathrm{dom}_{\ell,m}$ and for each $i < \ell$ the set $\{\rho_i, \eta\}/e$ is not a singleton$\}$ has at least two members
- (c) if $u \subseteq \mathrm{dom}_{\ell,m}$ and $|u| \le k$ then we can find $a(\mathbf{s}, \mathscr{C})$-decomposition of (u_1, u_2) of u which means that:
  - (α) $u = u_1 \cup u_2$
  - (β) $\mathscr{C}(u_1 \cap u_2) \cap (u_1 \cup u_2) \subseteq u_1$
  - (γ) if $\alpha \in u_1 \backslash u_2$ and $\beta \in u_2 \backslash u_1$ then $\{\alpha, \beta\}$ is not an edge of the graph $H[\mathbf{s}]$, see below.
- (d) for each $\nu \in {}^{\ell>}2$ the graphs $(\{\rho \in \mathrm{dom}_{\ell,m} : \nu \triangleleft \rho\}, e_\nu)$ has a cycle but no cycle with $\le k$ nodes

2) We can interpret $\mathbf{s} = (\mathrm{dom}_{\ell,m}, e)$ as the graph $H[\mathbf{s}]$ with set of nodes $\mathrm{dom}_{\ell,m}$ and set of edges $\{\{\eta, \nu\} : \{\eta, \nu\}/e$ not a singleton (and of course $\eta \ne \nu$ are from $\mathrm{dom}_{\ell,m})\}$.
3) We may write $e(\mathbf{s})$ instead of $\mathbf{s}$ if $\mathrm{dom}_{\ell,m}$ can be reconstructed from e (e.g. if the graph has no isolated point (e.g. if it is 0-nice, see clause (b) of part (1)). Saying nice we mean $[\log_2(m)]$-nice.

1.4 Claim. *1) If (λ, μ) is $\aleph_0$-compact and $c_n : [\lambda]^{<\aleph_0} \to \mu$ and $\Gamma_n = \mathrm{ID}(c_n)$ for $n < \omega$, <u>then</u> for some $c : [\lambda]^{<\aleph_0} \to \mu$ we have $ID(c) \subseteq \bigcap\limits_{n<\omega} \Gamma_n$ (in fact equality holds).*
2) Similarly using ID_2.

Remark. By the same proof, if we just assume $(\lambda_1,\mu_1) \to'_{\aleph_1} (\lambda_2,\mu_2)$ and $c_n : [\lambda_1]^{<\aleph_0} \to \mu_1$, <u>then</u> we can deduce that there is $c : [\lambda_2]^{<\aleph_0} \to \mu_2$ satisfying $\mathrm{ID}(c) \subseteq \bigcap_{n<\omega} \mathrm{ID}(c_n)$.

Proof. Straightforward.
1) In details, let F_m be an m-place function symbol and P the distinguished unary predicate and let $T = \{\psi_n : n < \omega\} \cup \{\neg\psi_{\mathbf{s}} : \mathbf{s}$ is an identity of the form (n,e) not from $\bigcap_{n<\omega} \mathrm{ID}(c_n)\}$ where

(a) $\psi_n = (\forall x_0)(\forall x_1)\ldots(\forall x_{n-1})(P(F_n(x_0,\ldots,x_{n-1}))$ & $\wedge\{(\forall x_0)\ldots(\forall x_{n-1})$ $F_n(x_0,\ldots,x_{n-1}) = F_n(x_{\pi(0)},\ldots,x_{\pi(n-1)}) : \pi$ is a permutation of $\{0,\ldots,n-1\}\}$

(b) if $\mathbf{s} = (n,e)$ is an identity then $\psi_{\mathbf{s}} = (\exists x_0)\ldots(\exists x_{n-1})[\bigwedge_{\ell<m<n} x_\ell \neq x_m$ & $\bigwedge_{b_1,b_2\subseteq n, b_1 e b_2} F_{|b_1|}(\ldots,x_\ell,\ldots)_{\ell\in b_1} = F_{|b_2|}(\ldots,x_\ell,\ldots,)_{\ell\in b_2}]$.

Clearly T is a (first order) countable theory so as by the assumption the pair (λ,μ) is $\aleph_0$-compact it suffices to prove the following two statements $\boxtimes_1,\boxtimes_2$.

$\boxtimes_1$ if $M \in K_{(\lambda,\mu)}$ is a model of T, <u>then</u> there is $c : [\lambda]^{<\aleph_0} \to \mu$ such that $\mathrm{ID}(c) \subseteq \bigcap_{n<\omega} \Gamma_n$.
[Why does $\boxtimes_1$ hold? There is $N \cong M$ such that N has universe $|N| = \lambda$ and $P^N = \mu$. Now we define c: if $u \in [\lambda]^{<\aleph_0}$, let $\{\alpha^u_\ell : \ell < |u|\}$ enumerate u in increasing order and let $c(u) = F^N_{|u|}(\alpha^u_0,\alpha^u_1,\ldots,\alpha^u_{|u|-1})$. Note that because $N \models \psi_n$ for $n < \omega$ clearly c is a function from $[\lambda]^{<\aleph_0}$ <u>into</u> μ. Also because $N \models \psi_n$, if $n < \omega$ and $\alpha_0,\ldots,\alpha_{n-1} < \lambda$ are with no repetitions then $F^N_n(\alpha_0,\ldots,\alpha_{n-1}) = c\{\alpha_0,\ldots,\alpha_{n-1}\}$. Now if $\mathbf{s} \in \mathrm{ID}(c)$ let $\mathbf{s} = (n,e)$ and let $u = \{\alpha_0,\ldots,\alpha_{n-1}\} \in [\lambda]^n \subseteq [\lambda]^{<\aleph_0}$ exemplify that $\mathbf{s} \in \mathrm{ID}(c)$, hence easily $N \models \psi_{\mathbf{s}}$ so necessarily $\neg\psi_{\mathbf{s}} \notin T$ hence $\mathbf{s} \in \bigcap_{n<\omega} \Gamma_n$. This implies that $\mathrm{ID}(c) \subseteq \bigcap_{n<\omega} \Gamma_n$ is as required.]

$\boxtimes_2$ if $T' \subseteq T$ is finite then T' has a model in $K_{(\lambda,\mu)}$.

[Why? So T' is included in $\{\psi_m : m < m^*\} \cup \{\neg\psi_{\mathbf{s}_k} : k < k^*\}$ for some $m^* < \omega, k^* < \omega, \mathbf{s}_k = (n_k,e_k)$ an identity not from $\bigcap_{\ell<\omega} \mathrm{ID}(c_\ell)$, so we can find $\ell(k) < \omega$

such that $\mathbf{s}_k \notin \mathrm{ID}(c_{\ell(k)})$. Let H be a one-to-one function from ${}^{k^*}\mu$ into μ. We define a model M:

- (a) its universe $|M|$ is λ
- (b) $P^M = \mu$
- (c) if $n < \omega, \{\alpha_0, \dots, \alpha_{n-1}\} \in [\lambda]^n$ then
$F_n^M(\alpha_0, \dots, \alpha_{n-1}) = H(c_{\ell(0)}\{\alpha_0, \dots, \alpha_{n-1}\},$
$c_{\ell(1)}\{\alpha_0, \dots, \alpha_{n-1}\}, \dots, c_{\ell(k^*-1)}\{\alpha_0, \dots, \alpha_{n-1}\})$.

If $n < \omega$ and $\alpha_0, \dots, \alpha_{n-1} < \lambda$ are with repetitions we let $F_n^M(\alpha_0, \dots, \alpha_{n-1}) = 0$. Clearly M is a model from $K_{(\lambda,\mu)}$ of the vocabulary of T. Also M satisfies each sentence ψ_m by the way we have defined F_m^M. Lastly, for $k < k^*, M \models \neg\psi_{\mathbf{s}_k}$ because $(n_k, e_k) \notin \mathrm{ID}(c_{\ell(k)})$ by the choice of the F_n's as H is a one-to-one function.] $\square_{1.4}$

Of course

1.5 Observation. 1) For every $\ell < \omega, k < \omega$ for some m there is a k-nice $\mathbf{s} = (\mathrm{dom}_{\ell,m}, e)$.
2) If $\mathbf{s}$ is k-nice and $m \leq k$, <u>then</u> $\mathbf{s}$ is m-nice.

Proof. 1) Choose m large enough and choose random enough appropriate graphs.
2) Easy. $\square_{1.5}$

1.6 Definition. We say that $\langle v_{s,\iota} : s \in Y^+$ and $\iota < 2\rangle$ is a special I-system (and we let $v_s = \cup\{v_{s,\iota} : \iota < 2\}, Y_j = Y^+ \cap [I]^j$ for $j \in \{0, 1, 2\}$

- (a) I is a linear order, $Y^+ \subseteq [I]^{\leq 2}$ and $\emptyset, \{a\} \in Y^+$ when $a \in I$
- (b) $v_{s,\iota}$ is a set of ordinals and $v_{s,0} = v_{s,1}$ if $s \in Y_0 \cup Y_2$
- (c) $\mathrm{otp}(v_s)$ depends just on $|s|$ and $\mathrm{otp}(v_{s,\iota})$ depend just on $|s|$ and ι
- (d) if $I \models$ "$a < b < c$" then
  - (α) $v_{\{a\},0} = v_{\{a,b\}} \cap v_{\{0,c\}}$
  - (β) $v_{\{c\},1} = v_{\{a,c\}} \cap v_{\{b,c\}}$
- (e) if $a, b, c, d \in I$ with no repetition then $v_\emptyset = v_{\{a,b\}} \cap v_{\{c,d\}}$
- (f) if $I \models$ "$a < b, c < d$" then $\mathrm{OP}_{v_{\{c,d\},0,v_{\{a,b\}}}}$
  - (α) maps $v_{\{a\},0}$ onto $v_{\{c\},0}$ and
  - (β) maps $v_{\{b\},1}$ onto $v_{\{d\},1}$
  - (γ) is the identity on $v_\emptyset$

(g) if $a, b \in I$ then $\mathrm{OP}_{v_{\{a\}}, v_{\{b\}}}$ maps $v_{\{b\},\iota}$ onto $v_{\{a\},\iota}$ for $\iota = 0, 1$ and is the identity on $v_\emptyset$.

§2 Definition of the Forcing

We have outlined the intended end of the proof at the end of the introductory section. It is to construct a sequence of functions $\langle f_n : n < \omega \rangle$ with certain properties. As we have adopted the decision of dealing only with 2-identities from ID_ℓ, all our functions will be colorings of pairs, and we shall generally use the letter c for them.

Our present theorem 0.7 deals with $\aleph_4$, but we may as well be talking about some $\aleph_{n(*)}$ for a fixed natural number $n(*) \geq 2$. Of course, the set of identities will depend on $n(*)$. We shall henceforth work with $n(*)$, keeping in mind that the relevant case for Theorem 0.7 is $n(*) = 4$. Also we fix $\ell(*) = n(*) + 1$ on which the identities depend (but vary m). Another observation about the proof is that we can replace $\aleph_0$ with an uncountable cardinal κ such that $\kappa = \kappa^{<\kappa}$ replacing $\aleph_n$ by κ^{+n}. Of course, the pair (κ^{+n}, κ) is compact because $[\kappa = \kappa^{\aleph_0} < \lambda \Rightarrow (\lambda, \kappa)$ is $\leq \kappa$-compact], however, much of the analysis holds.

We may replace $(\aleph_n, \aleph_0)$ by $(\kappa^{+n(*)}, \kappa)$ if $\kappa^{+n(*)} \leq 2^{\aleph_0}$; we hope to return to this elsewhere.

To consider (κ^+, κ) we need large cardinals; even more so for considering $(\mu^+, \mu), \mu$ strong limit singular of cofinality $\aleph_0$, and even $(\kappa^{+n}, \kappa), \mu \leq \kappa < \kappa^{+n} \leq \mu^{\aleph_0}$.

We now describe the idea behind the definition of the forcing notion we shall be concerned with. Each "component" of the forcing notion is supposed to add a coloring

$$c : [\lambda]^2 \to \mu$$

preserving some of the possible 2-identities, while "killing" all those which were not preserved, in other words it is concerned with adding f_n; specifically we concentrate on the case $\lambda = \aleph_{n(*)}, \mu = \aleph_0$. Hence, at first glance the forcing will be defined so that to preserve an identity we have to work hard proving some kind of amalgamation for the forcing notion, while killing an identity is a consequence of adding a colouring exemplifying it. By preserving a set Γ of identities, we mean that $\Gamma \subseteq \mathrm{ID}(c)$, and more seriously $\Gamma \subseteq \mathrm{ID}_2(\lambda, \mu)$; we restrict ourselves to some ID^*, an infinite set of 2-identities.

We shall choose $\mathrm{ID}^* \subseteq \mathrm{ID}_2^{\circledast}$ below small enough such that we can handle the identities in it.

We define the forcing by putting in its definition, for each identity that we want to preserve, a clause specifically assuring this. Naturally this implies that not only the desired identities are preserved, but also some others so making an identity be not in $\mathrm{ID}_2(\lambda, \mu)$ becomes now the hard part. So, we lower our sights and simply hope that, if $\Gamma \subseteq \mathrm{ID}^*$ is the set of identities that we want to preserve, than no identity $(a, e) \in \mathrm{ID}^* \setminus \Gamma$ is preserved; this may depend on Γ.

How does this control over the set of identities help to obtain the non-compactness? We shall choose sets $\Gamma_n \subseteq \mathrm{ID}^*$ of possible identities for $n < \omega$. The forcing we referred to above, let us call it $\mathbb{P}^{\Gamma_n}$, add a colouring $\underset{\sim}{c}_n : [\lambda]^2 \to \omega$ such that $\mathrm{ID}_2(\underset{\sim}{c}_n)$ includes Γ_n and is disjoint to $\mathrm{ID}^* \backslash \Gamma_n$; also it will turn out to have a strong form of the *ccc*. We shall force with $\mathbb{P} =: \prod\limits_{n \in \omega} \mathbb{P}^{\Gamma_n}$, where the product is taken with finite support. Because of the strong version of *ccc* possessed by each $\mathbb{P}^{\Gamma_n}$, also $\mathbb{P}$ will have *ccc*. Now, in $\mathbf{V}^{\mathbb{P}}$ we have for every n a colouring $c_n : [\lambda]^2 \to \omega$ which preserves the identities in Γ_n, moreover $\mathbf{V}^{\mathbb{P}} \models \Gamma_n \subseteq \mathrm{ID}(c_n) \cap \mathrm{ID}^*$.

We shall in fact obtain that

$$\mathrm{ID}^* = \Gamma_0 \supseteq \Gamma_1 \ \ \& \ \ \Gamma_1 \supseteq \Gamma_2 \ \ \& \ \ \ldots \ \ \& \ \ \bigcap_{n<\omega} \Gamma_n = \emptyset \ \ \& \ \ \mathrm{ID}(c_n) \cap \Gamma_0 = \Gamma_n.$$

If we have $\aleph_0$-compactness for $(\lambda, \aleph_0)$, then by 1.4(2) there must be a colouring $c : [\lambda]^2 \to \omega$ in $\mathbf{V}^{\mathbb{P}}$ such that

$$\mathrm{ID}_2(c) \cap \Gamma_0 \subseteq \bigcap_{n<\omega} \Gamma_n = \emptyset.$$

We can find a name $\underset{\sim}{c}$ in $\mathbf{V}$ for such c, so by *ccc*, for every $\{\alpha, \beta\} \in [\lambda]^2$, the name $\underset{\sim}{c}(\{\alpha, \beta\})$ depends only on $\aleph_0$ "coordinates". At this point a first approximation to what we do is to apply a relative of Erdös-Rado theorem to prove that there are an n, a large enough $W \subseteq \lambda$ and for every $\{\alpha, \beta\} \in [W]^2$ a condition $p_{\{\alpha,\beta\}} \in \prod\limits_{\ell<n} \mathbb{P}^{\Gamma_\ell}$, such that $p_{\{\alpha,\beta\}}$ forces a value to $\underset{\sim}{c}(\{\alpha, \beta\})$ in a "uniform" enough way. We shall be able to extend enough of the conditions $p_{\{\alpha,\beta\}}$ by a single condition p^* in $\prod\limits_{\ell<n} \mathbb{P}^{\Gamma_\ell}$, which gives an identity in $\mathrm{ID}_2(\underset{\sim}{c})$ which belongs to $\bigcap\limits_{\ell<n} \Gamma_\ell \setminus \Gamma_n$, contradiction.

Before we give the definition of the forcing, we need to introduce a notion of closure. The properties of the closure operation are the ones possible to obtain for $(\lambda, \aleph_0)$, but not for $(\aleph_\omega, \aleph_0)$. We of course need to use somewhere such a property, as we know in *ZFC* that $(\aleph_\omega, \aleph_0)$ has all those identities, i.e. $\mathrm{ID}_2^{\circledast} = \mathrm{ID}_2(\aleph_\omega, \aleph_0)$. On a similar proof see [Sh 424] (for ω-place functions) and also (2-place functions), [Sh 522]. The definition of the closure in [GcSh 491] is close to ours, but note that the hard clause from [GcSh 491] is not needed here.

2.1 Definition. 1) Let $\mathrm{ID}^*_{\ell(*)} =: \{\mathbf{s} \in \mathrm{ID}^2_{\ell(*)} : \mathbf{s}$ is 0-nice$\}$.
2) We say that $\mathbf{s}_1, \mathbf{s}_2 \in \mathrm{ID}^*_{\ell(*)}$ are far when: $\mathbf{s}_2$ is $|\mathrm{dom}_{\mathbf{s}_1}|$-nice or $\mathbf{s}_1$ is $|\mathrm{dom}_{\mathbf{s}_2}|$-nice.

Remark. We can consider $\{\mathbf{s}_n : n < \omega\}$, which hopefully will be independent, i.e. for every $X \subseteq \omega$ for some c.c.c. forcing notion $\mathbb{P}$, in $\mathbf{V}^{\mathbb{P}}$ we have $\lambda \rightarrow (\mathbf{s}_n)_\mu$ iff $n \in X$. It is natural to try $\{\mathbf{s}_n : n < \omega\}$ where $\mathbf{s}_n = (\mathrm{dom}_{\ell(*),m_n}, e_n)$ where $m_n = n$ (or 2^{2^n} may be more convenient) and e_n is $[\log\log(n)]$-nice.

2.2 Definition. [λ is our fixed cardinal.]
1) Let M^* (or M^*_λ) be a model with universe λ, countable vocabulary, and its relations and functions are exactly those defined in $(\mathscr{H}(\chi), \in, <^*_\chi)$ for $\chi = \lambda^+$ (and some choice of $<^*_\chi$, a well ordering of $\mathscr{H}(\chi)$).
2) For $\bar\alpha \in {}^{\omega>}(M^*_\lambda)$ let $c\ell_\ell(\bar\alpha) = \{\beta < \lambda :$ for some first order $\varphi(y, \bar x)$ we have $M^*_\lambda \models \varphi[\beta, \bar\alpha]$ & $(\exists^{\le \aleph_\ell} x)\varphi(x, \bar\alpha)\}$ and $c\ell(\bar\alpha) = \{\beta < \lambda :$ for some first order $\varphi(y, \bar x)$ we have $M^*_\lambda \models \varphi[\beta, \bar\alpha]$ & $(\exists^{<\aleph_0} x)\varphi(x, \bar\alpha)\}$.
3) For a model M and $A \subseteq M$ let $c\ell_M(A)$ be the smallest set of elements of M including A and closed under the functions of M (so including the individual constants).

Note that
<u>2.3 Fact</u>: If $\beta_0, \beta_1 \in c\ell_{\ell+1}(\bar\alpha)$ <u>then</u> for some $i \in \{0,1\}$ we have $\beta_i \in c\ell_\ell(\bar\alpha\,\hat{}\,\langle \beta_{1-i}\rangle)$.

Proof. Easy.

The idea of our forcing notion is to do historical forcing (see [RoSh 733] for more on historical forcing and its history). That is, we put in only those conditions which we have to put in order to meet our demands, so every condition in the forcing has a definite rule of creation. In particular, (see below), in the definition of our partial colourings, we avoid giving the same color to any pairs for which we can afford this, if the rule of creation is to be respected. We note that the situation here is not as involved as the one of [RoSh 733], and we do not in fact need the actual history of every condition.

We proceed to the formal definition of our forcing.

Clearly case 0 for $k \geq 0$ is not necessary from a historical point of view but it simplifies our treatment later; also case 1 is used in clause (η) of case 3.

Note that in case 2 below we do not require that the conditions are isomorphic over their common part (which is natural for historic forcing) as the present choice simplifies clause $(\zeta)(iv)$ in case 3.

Remark. Saharon, check if you want to add the definition of weakly far here.

2.4 Main Definition. Let $n(*) \geq 2, n(*) \leq \ell(*) < \omega, \lambda = \aleph_{n(*)}, \mu = \aleph_0$ be fixed. All closure operations we shall use are understood to refer to $M^*_{\aleph_{n(*)}}$ from 2.2(1). Let $\Gamma \subseteq \mathrm{ID}^*_{\ell(*)}$ be given. For two sets u and v of ordinals with $\mathrm{otp}(u) = \mathrm{otp}(v)$, we let $OP_{v,u}$ stand for the unique order preserving 1-1 function from u to v. For finite $u \subseteq M^*_{\aleph_{n(*)}}$ let $\mathscr{C}_u$, a closure operation on u be defined by $\mathscr{C}_u(v) = u \cap c\ell_{M^*_{\aleph_{n(*)}}}(v)$ for every $v \subseteq u$.

We shall define $\mathbb{P} =: \mathbb{P}_\Gamma = \mathbb{P}^\lambda_\Gamma$, it is $\subseteq \mathbb{P}^*_\lambda$.

Members of $\mathbb{P}^*_\lambda$ are the pairs of the form $p = (u, c) =: (u^p, c^p)$ with

$$u \in [\lambda]^{<\aleph_0} \text{ and } c : [u]^2 \to \omega.$$

The order in $\mathbb{P}^*_\lambda$ is defined by

$$(u_1, c_1) \leq (u_2, c_2) \Leftrightarrow (u_1 \subseteq u_2 \ \ \& \ \ c_1 = c_2 \restriction [u_1]^2)$$

For $p \in \mathbb{P}^*_\lambda$ let $n(p) = \sup(\mathrm{Rang}(c^p)) + 1$; this is $< \omega$.
We now say which pairs (u, c) of the above form (i.e. $(u, c) \in \mathbb{P}^*_\lambda$) will enter $\mathbb{P}$. We shall have $\mathbb{P} = \bigcup\limits_{k<\omega} \mathbb{P}_k$ where $\mathbb{P}_k =: \mathbb{P}^{\lambda,\Gamma}_k$ are defined by induction on $k < \omega$, as follows.

<u>Case 0</u>: <u>$k = 4\ell$</u>. If $k = 0$ let $\mathbb{P}_0 =: \{(\emptyset, \emptyset)\}$.

If <u>$k = 4\ell > 0$</u>, a pair $(u, c) \in \mathbb{P}_k$ <u>iff</u> for some $(u', c') \in \bigcup\limits_{m<k} \mathbb{P}_m$ we have $u \subseteq u'$ and $c = c' \restriction [u]^2$; we write $(u, c) = (u', c') \restriction u$.

<u>Case 1</u>: <u>$k = 4\ell + 1$</u>. (This rule of creation is needed for density arguments.)

A pair (u, c) is in $\mathbb{P}_k$ <u>iff</u> (it belong to $\mathbb{P}^*_\lambda$ and) there is a $p_1 = (u_1, c_1) \in \bigcup\limits_{m<k} \mathbb{P}_m$ and $\alpha < \lambda$ satisfying $\alpha \notin u_1$ such that:

- (a) $u = u_1 \cup \{\alpha\}$,
- (b) $c \restriction [u_1]^2 = c_1$ and
- (c) For every $\{\beta, \gamma\}$ and $\{\beta', \gamma'\}$ in $[u]^2$ which are not equal, if $c(\{\beta, \gamma\})$ and $c(\{\beta', \gamma'\})$ are equal, <u>then</u> $\{\beta, \gamma\}, \{\beta', \gamma'\} \in [u_1]^2$. (Hence, c does not add any new equalities except for those already given by c_1.)

<u>Case 2</u>: <u>$k = 4\ell + 2$</u>. (This rule of creation is needed for free amalgamation, used in the Δ-system arguments for the proof of the *c.c.c.*.)

A pair (u, c) is in $\mathbb{P}_k$ iff (it belongs to $\mathbb{P}^*_\lambda$ and) there are $(u_1, c_1), (u_2, c_2) \in \bigcup\limits_{m<k} \mathbb{P}_m$ for which we have

- (a) $u = u_1 \cup u_2$
- (b) $c \restriction [u_1]^2 = c_1$ and $c \restriction [u_2]^2 = c_2$
- (c) c does not add any unnecessary equalities, i.e., if $\{\beta, \gamma\}$ and $\{\beta', \gamma'\}$ are distinct and in $[u]^2$ and $c(\{\beta, \gamma\}) = c(\{\beta', \gamma'\})$, then $\{\{\beta, \gamma\}, \{\beta', \gamma'\}\} \subseteq [u_1]^2 \cup [u_2]^2$.
  Note that $[u_1]^2 \cap [u_2]^2 = [u_1 \cap u_2]^2$
- (d) $c\ell_0(u_1 \cap u_2) \cap (u_1 \cup u_2) \subseteq u_1$ (usually $c\ell_0(u_1 \cap u_2) \cap (u_1 \cup u_2) \subseteq u_1 \cap u_2$) is O.K. too for present §2, §3 but not, it seems, in 4.6).

Main rule:

Case 3: $k = 4\ell + 3$. (This rule[3] is like the previous one, but the amalgamation is taken over $(\mathbf{s}, \mathscr{C})$ where $\mathbf{s} = (\mathrm{dom}_{\ell(*),m}, e) \in \Gamma$).

A pair $p = (u, c) \in \mathbb{P}_k$ iff there are $(\mathbf{s}, \mathscr{C}) \in \Gamma, \mathbf{s} = (\mathrm{dom}_{\ell(*),m(*)}, e)$ and objects $I, Y^+, Y_0, Y_1, Y_2, \bar{v}^\ell$ for $\ell < \ell(*), \bar{w}, \bar{p}$ (actually I, Y^+, Y_0, Y_1, Y_2 depends on $\mathbf{s}$ only)

$\boxtimes(A)$
- (a) I is $\mathrm{dom}_{\ell(*),m(*)}$ linearly ordered by $<_{\mathrm{lex}}$
- (b) $Y^+ = Y_0 \cup Y_1 \cup Y_2$ where $Y_0 = \{\emptyset\}, Y_1 = [I]^1, Y_2 = \{t \in [I]^2 : t/e$ not a singleton$\}$
- (c) $\bar{v}^\ell = \langle v_{s,\iota} : s \in Y^+$ and $\iota < 2$ is a special system of sets and recall that $\ell = |s| \Rightarrow v^\ell_s$ is
  $v_{s,0} = v_{s,1}$ if $s \in Y_0 \cup Y_2$ and $v_{s,0} \cup v_{s,1}$ if $s \in Y_1$
- (d) $\bar{w} = \langle w_\ell : \ell \leq \ell(*) \rangle$

$\boxtimes(B)$
- (a) w_ℓ, v^ℓ_s are finite sets of ordinals $< \lambda$
- (b) $w_\ell \subseteq w_{\ell+1}$ and $v^\ell_s \subseteq v^{\ell+1}_s$ for $\ell < \ell(*), s \in Y^+, \iota < 2$
- (c) $w_\ell = \cup\{v_{t,\iota} : t \in Y^+, \iota < 2\}$ for $\ell \leq \ell(*)$
- (d) $v^\ell_s \cap v^\ell_t \subseteq v^\ell_{s \cap t}$ for $s, t \in Y^+$
- (e) $v^{\ell+1}_{s,\iota} \cap w_\ell = v^\ell_s$
- (f) if $s, t \in Y_\iota, |s| = |t|$ then $\mathrm{otp}(v^\ell_s) = \mathrm{otp}(v^\ell_t)$ and $\mathrm{otp}(v^\ell_{s,\iota}) = \mathrm{otp}(v^\ell_{t,\iota})$
- (g) $c\ell(v^\ell_{t,\iota}) \cap w_\ell = v^\ell_{t,\iota}$.

$\boxtimes(C)$
- (a) $p^\ell_{s,\iota} \in P^*_\lambda$ when $s \in Y_1 \leq 2$ or $s \in Y_0 \cup Y_2, \iota = 0$ when $s \in Y_0 \cup Y_2$

[3]you may understand it better seeing how it is used in the proof of 3.2

(b) $u^{p^\ell_{s,\iota}} = v^\ell_{s,\iota}$ and if $s \in Y_1$ then $p^\ell_{s,0} \restriction (v^\ell_{s,0} \cap v^\ell_{s,1}) = p^\ell_{s,1} \restriction (v^\ell_{s,0} \cap v^\ell_{s,1})$ and call it v^ℓ_s

(c) if $t = \{\eta, \nu\} \in Y$ and $\eta <_I \nu$ then $p^\ell_t \restriction v^\ell_{\{\eta\},0} = p^\ell_{\{\eta\}}, p^\ell_t \restriction v^\ell_{\{\nu\},1} = p^\ell_{\{\nu\}}$

(d) if $\eta \in \mathrm{dom}_{\ell(*),m(*)}$ then $p^\ell_{\{\eta\}} \restriction v^\ell_\emptyset = p^\ell_\emptyset$

(e) $p^\ell_{s,\iota} \le p^{\ell+1}_{s,\iota}$

(f) if $\eta, \nu \in \mathrm{dom}_{\ell(*),m(*)}$ and $\eta \restriction \ell = \nu \restriction \ell$ and $\iota < 2$ then $\mathrm{OP}_{v_{\{\nu\},\iota},\{v_\eta\},\iota}$ maps $p^\ell_{\{\eta\},\iota}$ to $p^\ell_{\{\nu\},\iota}$

(g) if $\{\eta_0, \nu_0\}, \{\eta_1, \nu_1\} \in Y_2$ and $\eta_0 \restriction \ell = \nu_0 \restriction \ell = \eta_1 \restriction \ell = \nu_1 \restriction \ell$ and $\eta_0(\ell) = 0 = \eta_1(\ell) \neq \nu_0(\ell) = 1 = \nu_1(\ell)$ then $\mathrm{OP}_{v_{\{\eta_1,\nu_1\}},v_{\{\eta_0,\nu_0\}}}$ maps $p^\ell_{\{\eta_0,\nu_0\}}$ to $p^\ell_{\{\eta_1,\nu_1\}}$

(h) if $\nu \in {}^{\ell(*)>}2, \nu\hat{\ } < 0 > \triangleleft \eta_0, \nu_0$ and $\nu\hat{\ } < 1 > \triangleleft \eta_1, \nu_1$ and $\ell \le \ell(*)$ then $\mathrm{OP}_{\{\eta_0,\eta_1\},\{\nu_0,\nu_1\}}$

(α) maps $p^\ell_{\{\nu_0,\nu_1\}}$ onto $p^\ell_{\{\eta_0,\eta_1\}}$

(β) maps $p^\ell_{\{\nu_0\},0}$ onto $p^\ell_{\{\eta_0\},0}$

(γ) maps $p^\ell_{\{\nu_1\},1}$ onto $p^\ell_{\{\eta_1\},1}$

(δ) maps $p^\ell_\emptyset$ onto itself (actually follows)

$\boxtimes(D)$ (a) $u = w_{\ell(*)}$

(b) $p \restriction v^{\ell(*)}_t = p^{\ell(*)}_t$ when $t \in Y_2$ (hence $p \restriction v^\ell_{t,\iota} = p^\ell_{t,\iota}$ when $t \in Y^+$, $\iota = 0$ or $t \in Y_1, \iota = 1$)

(c) if $c^p\{\alpha_1, \beta_1\} = c^p\{\alpha_2, \beta_2\}$ where $\alpha_1 < \alpha_2$ are from $u, \beta_1 < \beta_2$ are from u then $\{\alpha_1, \beta_2\} \in \cup\{[v^{\ell(*)}_t]^2 : t \in Y_2\}$

(d) $p^\ell_{t,\iota} \in \mathbb{P}_{<k}$ when defined

(e) if $w \subseteq W, \gamma \in W \backslash w$ and $|w \backslash \gamma| \le 1$ then $v_{\{\gamma\}} \backslash v_\emptyset$ is disjoint to $\cup\{v_t : t \in [w]^{\le 2}\}$

(f) if $w \subseteq W, \beta < \alpha$ are from $W, \{\beta, \gamma\} \nsubseteq w$ and $|w \backslash \gamma| \le 1$ then $v_{\{\beta,\gamma\}} \backslash (v_{\{\beta\}} \cup v_{\{\gamma\}})$ is disjoint to $\cup\{v_t : t \in [w]^{\le 2}\}$.

2.5 Claim. *1) $\mathbb{P}^\lambda_\Gamma$ satisfies the c.c.c. and even the Knaster condition.*
2) For each $\alpha < \lambda$ the set $\mathscr{I}_\alpha = \{p \in \mathbb{P}^\lambda_\Gamma : \alpha \in u^p\}$ is dense open.
3) $\Vdash_{\mathbb{P}^\lambda_\Gamma}$ "$\underset{\sim}{c} = \cup\{c^p : p \in \underset{\sim}{G}\}$ is a function from $[\lambda]^2$ to ω".

Proof. 1) By Case 2.

In detail, assume that $p_\varepsilon \in \mathbb{P}^\lambda_\Gamma$ for $\varepsilon < \omega_1$ and let $p_\varepsilon = (u_\varepsilon, c_\varepsilon)$. As each u_ε is a finite subset of λ, by the Δ-system lemma without loss of generality for some finite $u^* \subseteq \lambda$ we have: if $\varepsilon < \zeta < \omega_1$ then $u_\varepsilon \cap u_\zeta = u^*$. By further shrinking, without loss of generality $\alpha \in u^* \Rightarrow \langle |u_\varepsilon \cap \alpha| : \varepsilon < \omega_1 \rangle$ is constant and $\varepsilon < \zeta < \omega_1 \Rightarrow |u_\varepsilon| = |u_\zeta|$. Also without loss of generality the set $\{(\ell, m, k)$: for some $\alpha \in u_\varepsilon$ and $\beta \in u_\varepsilon$ we have $\ell = |\alpha \cap u_\varepsilon|, m = |\beta \cap u_\varepsilon|$ and $k = c_\varepsilon\{\alpha, \beta\}\}$ does not depend on ε. We can conclude that $\varepsilon < \zeta < \omega_1 \Rightarrow \mathrm{OP}_{u_\zeta, u_\varepsilon}$ maps p_ε to p_ζ over u^*. Clearly for $\varepsilon < \omega_1$, the set $c\ell(u_\varepsilon)$ is countable hence for every $\zeta < \omega_1$ large enough we have $u_\zeta \cap c\ell_0(u_\varepsilon) = u^*$ so restricting $\langle p_\varepsilon : \varepsilon < \omega_1 \rangle$ to a club we get that $\varepsilon < \zeta < \omega_1 \Rightarrow c\ell_0(u_\varepsilon) \cap u_\zeta = u^*$ (this is much more than needed). Now for any $\varepsilon < \zeta < \omega_1$ we can define $q_{\varepsilon,\zeta} = (u_{\varepsilon,\zeta}, c_{\varepsilon,\zeta})$ with $u_{\varepsilon,\zeta} = u_\varepsilon \cup u_\zeta$ and $c_{\varepsilon,\zeta} : [u_{\varepsilon,\zeta}]^2 \rightarrow \omega$ is defined as follows: for $\alpha < \beta$ in $u_{\varepsilon,\zeta}$ let $c_{\varepsilon,\zeta}\{\alpha, \beta\}$ be $c_\varepsilon\{\alpha, \beta\}$ if defined, $c_\zeta\{\alpha, \beta\}$ if defined, and otherwise $\sup(\mathrm{Rang}(c_\varepsilon)) + 1 + (|u_{\varepsilon,\zeta} \cap \alpha| + |u_{\varepsilon,\zeta} \cap \beta|)^2 + |u_{\varepsilon,\zeta} \cap \alpha|$. Now $q_{\varepsilon,\zeta} \in \mathbb{P}^\lambda_\Gamma$ by case 2, and $p_\varepsilon \leq q_{\varepsilon,\zeta}, p_\zeta \leq q_{\varepsilon,\zeta}$ by the definition of order.
2) By Case 1.

In detail, let $p \in \mathbb{P}^\lambda_\Gamma$ and $\alpha < \lambda$ and we shall find q such that $p \leq q \in \mathscr{I}_\alpha$. If $\alpha \in u^p$ let $q = p$, otherwise define $q = (u^q, c^q)$ as follows $u^q = u^p \cup \{\alpha\}$ and for $\beta < \gamma \in u^q$ we let $c^q\{\beta, \gamma\}$ be: $c^p\{\beta, \gamma\}$ when it is well defined and $\sup(\mathrm{Rang}(c^p)) + 1 + (|\beta \cap u^q| + |\gamma \cap u^q|)^2 + |\beta \cap u^q|$ when otherwise. Now $q \in \mathbb{P}^\lambda_\Gamma$ by case 1 of Definition 2.4, $p \leq q$ by the order's definition and $q \in \mathscr{I}_\alpha$ trivially.
3) Follows from part (2). $\square_{2.5}$

§3 Why does the forcing work

We shall use the following claim for $\mu = \aleph_0$

3.1 Claim. *1) If $f : [\lambda]^2 \to \mu$ and M is an algebra with universe $\lambda, |\tau_M| \le \mu$ and $w_t \subseteq \lambda, |w_t| < \aleph_0$ for $t \in [\lambda]^2$ and $\lambda \ge \beth_2(\mu^+)^+$, <u>then</u> for some special system $\langle v_{t,\iota} : t \in [W]^{\le 2}, \iota < 2\rangle$ of sets we have:*

- (a) *$W \subseteq \lambda$ is infinite in fact $|W| = \mu^{++}$*
- (b) *$f \restriction [W]^2$ is constant*
- (c) *$t \cup w_t \subseteq v_t \in [\lambda]^{<\aleph_0}$ for $t \in [W]^2$*
- (d) *$OP_{v_{\{\beta\},\iota},v_{\{\alpha\},\iota}}$ maps α to β when $\alpha, \beta \in W, \iota < 2$*
- (e) *if $s,t \in [W]^{\le 2}$ then $v_s \cap c\ell(v_t) \subseteq v_t$ except possibly when for some $\alpha < \beta < \gamma$ we have $\{s,t\} = \{\{\alpha,\beta\},\{\beta,\gamma\}\}$.*

2) If $u \in [\lambda]^{<\aleph_0} \Rightarrow c\ell_M(u) \in [M]^{<\mu}$, <u>then</u> $\lambda = (\beth_2(\mu))^+$ is enough.

Remark. 1) See more in [Sh 289]; this is done for completeness.
2) We can use $\langle v_\eta : \eta \in \dim_{\ell(*),m(*)}\rangle$ being as required just when necessary.

Proof. 1) Let $w_t \cup t = \{\zeta_{t,\ell} : \ell < n_t\}$ with no repetitions and we define the function c, c_0, c_1 with domain $[\lambda]^3$ as follows: if $\alpha < \beta < \gamma < \lambda$ then

$$c_0\{\alpha,\beta,\gamma\} = \{(\ell_1,\ell_2) : \ell_1 < n_{\{\alpha,\beta\}}, \ell_2 < n_{\{\alpha,\gamma\}} \text{ and } \zeta_{\{\alpha,\beta\},\ell_1} = \zeta_{\{\alpha,\gamma\},\ell_2}\}$$

$$c_1\{\alpha,\beta,\gamma\} = \{(\ell_1,\ell_2) : \ell_1 < n_{\{\alpha,\gamma\}}, \ell_2 < n_{\{\beta,\gamma\}} \text{ and } \zeta_{\{\alpha,\gamma\},\ell_1} = \zeta_{\{\beta,\gamma\},\ell_2}\}$$

$$c\{\alpha,\beta,\gamma\} = (c_0\{\alpha,\beta,\gamma\}, c_1\{\alpha,\beta,\gamma\}, f\{\alpha,\beta\}).$$

By Erdös-Rado theorem for some $W_1 \subseteq \lambda$ of cardinality and even order type μ^{++} for part (1), μ^+ for part (2) such that $c \restriction [W_1]^3$ is constant. Let $\{\alpha_\varepsilon : \varepsilon < \mu^{++}\}$ list W_1 in increasing order. If $2 < i < \mu^{++}$, let

$$\begin{aligned} v_{\{\alpha_i\}} =: \{\zeta_{\{\alpha_i,\alpha_{i+1}\},\ell_1} : &\text{ for some } \ell_2 \text{ we have} \\ &(\ell_1,\ell_2) \in c_0\{\alpha_i,\alpha_{i+1},\alpha_{i+2}\}\}\cup \\ &\{\zeta_{\{\alpha_0,\alpha_i\},\ell_1} : \text{ for some } \ell_2 \text{ we have} \\ &(\ell_1,\ell_2) \in c_1\{\alpha_0,\alpha_1,\alpha_i\}\} \end{aligned}$$

(clearly $\alpha_i \in v_{\{\alpha_i\}}$).

For $i < j$ in $(2, \mu^{++})$ let $v_{\{\alpha_i,\alpha_j\}} = v_{\{\alpha_i\}} \cup v_{\{\alpha_j\}} \cup w_{\{\alpha_i,\alpha_j\}}$. Now for some unbounded $W_2 \subseteq W_1 \backslash \{\alpha_0, \alpha_1\}$ and $Y \in [\lambda]^{\le \mu}$ we have:

if $\alpha \ne \beta \in W_2$ then $c\ell_M(v_{\{\alpha\}}) \cap c\ell_M(v_{\{\beta\}}) \subseteq Y$.

Now by induction on $\varepsilon < \mu^{++}$ we can choose $\gamma_\varepsilon \in W_2$ strictly increasing with $\varepsilon, \gamma_\varepsilon$ large enough. It is easy to check that $W = \{\gamma_\varepsilon : \varepsilon < \mu^{++}\}$ is as required.

2) The same proof. $\square_{3.1}$

* * *

3.2 The preservation Claim. *Let $n(*), \ell(*), \lambda, \mu = \aleph_0$ be as in Definition 2.4 and assume $\lambda > \beth_2(\mu^+)$.*

1) If $\mathbb{P} = \mathbb{P}^\lambda_\Gamma$ and $(\mathrm{dom}_{\ell(*),m}, e) \in \Gamma \subseteq \mathrm{ID}^*_{\ell(*)}$ <u>then</u> in $\mathbf{V}^{\mathbb{P}}$ we have $(\mathrm{dom}_{\ell(*),m}, e) \in \mathrm{ID}_2(\lambda, \aleph_0)$.

2) Assume that $\mathbb{P} = \prod\limits_{n<\gamma} \mathbb{P}^\lambda_{\Gamma_n}$, the product with finite support where $\Gamma_n \subseteq \mathrm{ID}^*_{\ell(*)}$ and $\gamma \le \omega$ and $p^* \in \mathbb{P}$ forces that $\underset{\sim}{c}$ is a function from $[\lambda]^2$ to ω. <u>Then</u> for some finite $d \subseteq \gamma$ for any $\mathbf{s} \in \bigcap\limits_{n \in d} \Gamma_n$ we have $p^* \nVdash_{\mathbb{P}}$ "$\mathbf{s} \notin \mathrm{ID}_2(\underset{\sim}{c})$".

Proof. 1) Follows from (2), letting $\gamma = 1, \Gamma_0 = \Gamma$.

2) Assume $p^* \in \mathbb{P}$ and $p^* \Vdash_{\mathbb{P}}$ "$\underset{\sim}{c}$ is a function from $[\lambda]^2$ to ω". Assume toward contradiction that $\exists \mathbf{s}, \mathbf{s} \in \bigcap\limits_{n\in d} \Gamma_n$, and $p^* \Vdash_{\mathbb{P}}$ "$\mathbf{s} \notin \mathrm{ID}_2(\underset{\sim}{c})$". Let $k(*) = 2^{\ell(*)} - 1$ and let $k(\nu) = |\{\rho \in {}^{\ell(*)>}2 : \rho <_{\mathrm{lex}} \nu\}|$ for $\nu \in {}^{\ell(*)>}2$. For $p \in \mathbb{P}$ let $u[p] = \cup\{u^{p(n)} : n \in \mathrm{Dom}(p)\}$, so $u[p] \in [\lambda]^{<\aleph_0}$ and for any $q \in \mathbb{P}$ we let $n[q] = \sup(\cup\{\mathrm{Rang}(c^{q(n)}) : n \in \mathrm{Dom}(q)\})$. For any $\alpha < \beta < \lambda$ letting $t = \{\alpha, \beta\}$ we define, by induction on $k \le k(*)$ the triple $(n_{t,k}, w_{t,k}, d_{t,k})$ such that:

(*) $n_{t,k} < \omega, w_{t,k} \in [\lambda]^{<\aleph_0}$ and $d_{t,k} \subseteq \gamma$ is finite.

Case 1: $k=0: n_{t,k} = n[p^*]+2$ and $w_{t,k} = \{\alpha,\beta\} \cup u^{p^*}$ and $d_{t,k} =$ Dom(p^*).

Case 2: $k+1$:

Let $\mathscr{P}_{t,k} = \{q \in \mathbb{P} : p^* \le q, u[q] \subseteq w_{t,k}$ and $n[q] \le n_{t,k}$ and $\mathrm{Dom}(q) \subseteq d_{t,k}\}$; clearly it is a finite set, and for every $q \in \mathscr{P}_{t,k}$ we choose $p_{t,q}$ such that $q \le p_{t,q} \in \mathbb{P}$ and $p_{t,q}$ forces a value, say $\zeta_{t,q}$ to $\underset{\sim}{c}(t)$. Now we let

$$w_{t,k+1} = \bigcup \{u[p_{t,q}] : q \in \mathscr{P}_{t,k}\} \cup w_{t,k}.$$

$$d_{t,k+1} = \cup\{\mathrm{Dom}(p_{t,q}) : q \in \mathscr{P}_{t,k}\} \cup d_{t,k}$$

$$n_{t,k+1} = \mathrm{Max}\{|w_{t,k+1}|^2, n_{t,k}+1, n[p_{t,q}]+1 : q \in \mathscr{P}_{t,k}\}$$

We next define an equivalence relation E on $[\lambda]^2 : t_1 E t_2$ iff letting $t_1 = \{\alpha_1,\beta_1\}, t_2 = \{\alpha_2,\beta_2\}, \alpha_1 < \beta_1, \alpha_2 < \beta_2$ and letting $h = OP_{w_{\{\alpha_2,\beta_2\},k(*)}, w_{\{\alpha_1,\beta_1\},k(*)}}$, we have

- (i) $w_{t_1,k(*)}, w_{t_2,k(*)}$ has the same number of elements
- (ii) h maps α_1 to α_2 and β_1 to β_2 and $w_{t_1,k}$ onto $w_{t_2,k}$ for $k \le k(*)$ (so h is onto)
- (iii) $d_{t_1,k} = d_{t_2,k}$ for $k \le k(*)$.
  We define also $\hat{h}$, in the next way: $\hat{h}(q_1) = q_2$ if $\mathrm{Dom}(q_1) = \mathrm{Dom}(q_2)$ and $\hat{h}$ maps $u[q_1]$ onto $u[q_2]$, and for every α,β in $u[q_1]$ we have $\mathbf{c}^{q_1}(\{\alpha,\beta\}) = \mathbf{c}^{q_2}(\{h(\alpha),h(\beta)\})$, so $\hat{h}$ maps $\mathscr{P}_{t,k}$ onto $\mathscr{P}_{t_2,k}$
- (iv) if $q_1 \in \mathscr{P}_{t_1,k}$ and $k < k(*)$ then $\hat{h}$ maps q_1 to some $q_2 \in \mathscr{P}_{t_2,k}$ and it maps p_{t_1,q_1} to p_{t_2,q_2} and we have $\zeta_{t_1,q_1} = \zeta_{t_2,q_2}$.

Clearly E has $\le \aleph_0$ equivalence classes. So let $c : [\lambda]^2 \to \aleph_0$ be such that $c(t_1) = c(t_2) \Leftrightarrow t_1 E t_2$ and let $w_t = w_{t,k(*)}$.

By Claim 3.1, recalling that we have assumed $\lambda > \beth_2(\aleph_1)$ we can find $W \subseteq \lambda$ of cardinality $\aleph_2$ and $\bar{v} = \langle v_{t,\iota} : t \in [W]^{\le 2}, \iota < 2\rangle$ as there; i.e., we apply it to an expansion of M^*_λ such that $c\ell_0(-) = c\ell_M(-)$.

Let $d^*_k = d_{t,k} \subseteq \omega$ for $t \in [W]^2$ and $k \le k(*)$, now we choose $d = d^*_{k(*)} \subseteq \gamma$, and we shall show that it is as required in the claim. Let $\mathbf{s} = (\mathrm{dom}_{\ell(*),m(*)}, e) \in \bigcap\limits_{\ell \in d} \Gamma_\ell$.

Let I be $\mathrm{dom}_{\ell(*),m(*)} = \mathrm{dom}_{\mathbf{s}}$ ordered lexicographically, $Y_0 = \emptyset, Y_1 = [I]^1, Y_2 = \{t \in [I]^2 : t/e_{\mathbf{s}}$ is not a singleton$\}$. We choose $\alpha_\eta \in W$ for $\eta \in I$ increasing with η.

Let $v^\ell_{t,\iota} =: v_{\{\alpha_\eta:\eta\in t\},\iota}$ and $w_\ell =: \cup\{v^\ell_{t,\iota} : t \in Y^+, \iota < 2\}$. Now we choose $p^\ell_{t,\iota}$ for $t \in Y^+, \iota < 2$ by induction on $\ell \le k(*)$ such that

$\circledast_\ell$ (a) $p^\ell_{t,\iota} \in \mathbb{P}^\lambda_\Gamma$, $\mathrm{Dom}(p^\ell_{t,\iota}) \subseteq d_{\{\alpha_\eta:\eta\in t\},\iota}$ for $t \in Y^+, \iota < 2$ (and $p^\ell_{t,\iota} = p^\ell_t$ if

$t \in Y_0 \cup Y_2$

(b) $u^{p^\ell_{t,\iota}(\beta)} = v^\ell_{t,\iota}$ for $t \in Y^+, \iota < 2, \beta \in d_{\{\alpha_\eta:\eta\in t\},\iota}$

(c) $p^\ell_{t,\iota} \in \mathscr{P}_{\{\alpha_\eta:\eta\in t\},\ell}$ implies (a)+(b)

(d) if $k < \ell$ then $p^k_{t,\iota} = p^\ell_{t,\iota} \restriction v^k_{t,\iota}$
(recall $q = p \restriction u$ mean $\mathrm{Dom}(q) = \mathrm{Dom}(p), q(\beta)$
$= (p(\beta) \restriction (u \cap \mathrm{Dom}(p(\beta)))$

(e) if $t = \{\eta, \nu\}, \eta <_{\mathrm{lex}} \nu$ then

(α) $p^\ell_{\{\eta\},0} = p^\ell_{\{\eta,\nu\}} \restriction v^\ell_{\{\eta\},0}$

(β) $p^\ell_{\{\nu\},1} = p^\ell_{\{\eta,\nu\}} \restriction v^\ell_{\{\nu\},1}$

(f) if $\{\eta, \nu\} \in Y_2$ and $\ell = \ell g(\eta \cap \nu) + 1$ then $p^\ell_{\{\eta,\nu\}}$ forces a value to $\underset{\sim}{c}\{\alpha_\eta, \alpha_\nu\}$ (which in fact is $\zeta_{\{\alpha_\eta,\alpha_\nu\},q}, q = p^{\ell-1}_{\{\eta,\nu\}}$)

(g) the demand on commuting with OP from Definition 2.4, Case 3 holds.

There is no problem to carry the induction.
[For $\ell = 0$ we already know only p^* and $u^{p^*} \subseteq v_\emptyset$ and the demand we have to satisfy are from clauses (a),(b),(c),(e),(g). This is straight.

For $\ell = k+1$, we choose $\eta^* <_{\mathrm{lex}} \nu^*$ such that $\{\eta^*, \nu^*\}$ is as in clause (f) and then continue as before.]

Lastly, let p^+ be such that $\mathrm{Dom}(p^+) = d^*_{k(*)}$ and for each $\beta \in d^*_{k(*)}$

$$u^{p^+(\beta)} = \cup\{u^{p^{k(*)}_y(\beta)} : y \in Y_2\};$$

$$c^{p^+(\beta)} \text{ extend each } c^{p^{k(*)}_y(\beta)} \text{ otherwise is } 1-to-1 \text{ with new values.}$$

So $p^+ \geq p^*$ forces that $\{\alpha_\eta : \eta \in \mathrm{dom}_{\ell(*),m(*)}\}$ exemplify $\mathbf{s} = (\mathrm{dom}_{\ell(*),m(*)}, e) \in \mathrm{ID}_2(\underset{\sim}{c})$, a contradiction. $\square_{3.2}$

3.3 The example Claim. *Let $n(*) \geq 4, \ell(*) > n(*), \lambda = \aleph_{n(*)}, \mu = \aleph_0$. Assume*

(a) $\mathbf{s}^* = (\mathrm{dom}_{\ell(*),m(*)}, e^*) \in \mathrm{ID}^*_{\ell(*)}$, *(see Definition 2.1)*

(b) $\Gamma \subseteq \mathrm{ID}^*_{\ell(*)}$

(c) *if $\mathbf{s} \in \Gamma$ then $\mathbf{s}$ and $\mathbf{s}^*$ are far (see Definition 2.1)*

(d) $\mathbb{P} = \mathbb{P}^\lambda_\Gamma$

(e) *$\underset{\sim}{c}$ is the $\mathbb{P}$-name $\cup\{c^p : p \in \underset{\sim}{G}_{\mathbb{P}}\}$.*

Then $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{c}$ *is a function from* $[\lambda]^2$ *to* μ *exemplifying* $(\mathrm{dom}_{\ell(*),m(*)}, e^*)$ *does not belong to* $\mathrm{ID}_2(\lambda, \aleph_0)$".

Proof. So assume toward contradiction that $p \in \mathbb{P}$ and $\alpha_\eta < \lambda$ for $\eta \in \mathrm{dom}_{\ell(*),m(*)}$ are such that p forces that $\eta \mapsto \alpha_\eta$ is a counterexample, i.e. $\langle \alpha_\eta : \eta \in \mathrm{dom}_{\ell(*),m(*)} \rangle$ is with no repetitions and p forces that $t_1 e^* t_2 \Rightarrow \underset{\sim}{c}(\{\alpha_\eta : \eta \in t_1\}) = \underset{\sim}{c}(\{\alpha_\eta : \eta \in t_2\})$. By 2.5(2) without loss of generality $\{\alpha_\eta : \eta \in \mathrm{dom}_{\ell(*),m(*)}\} \subseteq u^p$.

Let $Y = Y_{e^*} = \{y : y \in \mathrm{Dom}(e^*)$ and y/e^* is not a singleton$\}$ and for $\nu \in {}^{\ell(*)>}2$ let $Y_\nu = Y_{\nu,e^*} = \{\{\eta_0, \eta_1\} \in Y_{e^*} : \nu\hat{\ } < i > \trianglelefteq \eta_i$ for $i = 0, 1\}$ as in the previous proof. We now choose by induction on $\ell \leq n(*)$ the objects $\eta_\ell, \nu_\ell, Z_\ell$ and first order formulas $\varphi_\ell(x, y_0, \ldots, y_{\ell-1})$ and $<^\ell_{y_0,\ldots,y_{\ell-1}} (x, \bar{y})$ in the vocabulary of M^*_λ such that:

$\boxtimes(a)$ $\nu_\ell \in {}^\ell 2, \eta_\ell \in \mathrm{dom}_{\ell(*),m(*)}$ and $M^*_\lambda \models (\exists^{\leq \aleph_{n(*)-\ell}} x)\varphi_\ell(x, \alpha_{\eta_0}, \ldots, \alpha_{\eta_{\ell-1}})$

(b) $<^\ell_{\alpha_{\eta_0},\ldots,\alpha_{\eta_{\ell-1}}}$ is a well ordering of $\{x : M^*_\lambda \models \varphi_\ell[x, \alpha_{\eta_0}, \ldots, \alpha_{\eta_{\ell-1}}]\}$ of order type a cardinal $\leq \aleph_{n(*)-\ell}$

(c) $\nu_0 = <>, \varphi_0 = [x = x]$

(d) $\nu_{\ell+1} = (\eta_\ell \restriction \ell)\hat{\ }\langle 1 - \eta_\ell(\ell) \rangle$ and $\nu_\ell \triangleleft \eta_\ell$

(e) $Z_\ell = \{\eta : \nu_\ell \triangleleft \eta \in \mathrm{dom}_{\ell(*),m(*)}$ and $\{\eta_s, \eta\} \in e_{[v \restriction s]}$ for $s = 0, 1, \ldots, \ell - 1\}$

(f) $\eta \in Z_\ell \Rightarrow \alpha_\eta \in \{\beta : M^*_\lambda \models \varphi_\ell[\beta, \alpha_{\eta_0}, \ldots, \alpha_{\eta_{\ell-1}}]\}$

(g) η_ℓ is such that:

(α) $\nu_\ell \triangleleft \eta_\ell \in Z_\ell$

(β) if $\nu_\ell \trianglelefteq \eta \in Z_\ell$ then $\alpha_\eta \leq^\ell_{\alpha_{\eta_0},\ldots,\alpha_{\eta_{\ell-1}}} \alpha_{\eta_\ell}$.

(See similar proof with more details in 4.3).

Let $\nu^* = \nu_{n(*)}, Z = Z_{n(*)}, Z^+ = \{\eta_\ell : \ell < n(*)\} \cup Z$; note that by Definition 1.3(1), clause (b) and Definition 2.1 we have $|Z| \geq 2$, i.e., this is part of $(\mathrm{dom}_{\ell(*),m(*)}, e^*)$ being 0-nice. For $\nu \in \{\nu_\ell : \ell < n(*)\}$ let s_ν be such that: $\rho_1 \cap \rho_2 = \nu$ & $\rho_1, \rho_2 \in Z^+ \Rightarrow s_\nu = c\{\alpha_{\rho_1}, \alpha_{\rho_2}\}$ (clearly exists). By case 0 in Definition 2.4, without loss of generality

$$u^p = \{\alpha_\eta : \eta \in Z^+\},$$

that is, we may forget the other $\alpha \in u^p$; by claim 3.2 we have $p \in \mathbb{P}^\lambda_\emptyset$ so for some k we have $p \in \mathbb{P}^{\lambda,\emptyset}_k$.

So we have

$\boxplus$ $\langle \eta_\ell : \ell \le n(*) \rangle, Z, Z^+, \langle \nu_\ell : \ell \le n(*) \rangle, \langle s_{\eta_\ell \restriction \ell} : \ell < n(*) \rangle$ and p are as above, that is

- (i) (α) $\eta_\ell \in \mathrm{dom}_{\ell(*),m(*)}$
  - (β) $\nu_0 = <>, \nu_{\ell+1} = (\eta_\ell \restriction \ell)\hat{\ }\langle (1 - \eta_\ell(\ell)) \rangle$,
  - (γ) $\nu_\ell \triangleleft \eta_\ell$
  - (δ) $Z = \{\rho \in \mathrm{dom}_{\ell(*),m(*)} : \nu_{n(*)} \triangleleft \rho$ and $\{\eta_\ell, \rho\}/e$ is not a singleton for each $\ell < n(*)\}$ and let $Z^+ = Z \cup \{\eta_\ell : \ell < n(*)\}$
- (ii) $p \in \mathbb{P}_k^{\lambda,\emptyset}$
- (iii) $\alpha_\eta \in u^p$ for $\eta \in Z^+$
- (iv) $\langle \alpha_\eta : \eta \in Z^+ \rangle$ is with no repetitions
- (v) $c^p \restriction \{\alpha_\eta : \eta \in Z^+\}$ satisfies: if $\ell < n(*)$ and $\nu \in Z \cup \{\eta_t : \ell < t < n(*)\}$ so $\eta_\ell \cap \nu = \eta_\ell \restriction \ell$ <u>then</u> ($\alpha_\nu \ne \alpha_{\eta_\ell}$ and) $c\{\alpha_\nu, \alpha_{\eta_\ell}\} = s_{\eta_\ell \restriction \ell}$
- (vi) $\{\alpha_\eta : \eta \in Z\} \subseteq c\ell_0\{\alpha_{\eta_\ell} : \ell \le n(*)\}$
- (vii) Z has at least two members (actually follows)
- $(viii)$ $Z = Z_0 \cup Z_1$ where $Z_i = \bigcup\limits_{\ell < n(*)} \{\rho \in Z_\ell : \rho(n(*)) = i\}$ for $i = 0, 1$.

For $t = \{\rho_1, \rho_2\} \in \mathrm{dom}_{\ell(*),m(*)}$ let $\ell = \ell g(\rho_1 \cap \rho_2)$ [necessary]

$$e_i^* =: \{\{\rho_1, \rho_2\} : \rho_1, \rho_2 \in \mathrm{dom}_{\ell(*),m(*)} \text{ and } \ell g(\rho_1 \cap \rho_2) = i \le \ell(*) \text{ and } \{\rho_1, \rho_2\}/e_{\rho_1 \cap \rho_2} \text{ is not a singleton}\}$$

$$e_i^{*^+} = e_i^* \cup \{\{\eta\} : \eta \in \mathrm{dom}_{\ell(*),m(*)}\} \cup \{\emptyset\}$$

Among all such examples choose one with $k < \omega$ minimal. The proof now splits according to the cases in Definition 2.4.

<u>Case 0</u>: $k = 0$.
Trivial.

<u>Case 1</u>: $k = 4\ell + 1$.
Let p_1, α be as there, so recall that $\{\alpha, \beta\} e^{p_1} \{\alpha', \beta'\} \Rightarrow \{\alpha, \beta\} = \{\alpha', \beta'\}$. Hence obviously, by clauses (v) and (vii) above, $\eta \in Z^+ \Rightarrow \alpha_\eta \ne \alpha$, so $\{\alpha_\eta : \eta \in Z^+\} \subseteq u^{p_1}$, contradicting the minimality of k.

Case 2: $k = 4\ell + 2$.

Let $p_i = (u_i, c_i) \in \bigcup_{\ell < k} \mathbb{P}_\ell^{\lambda,\emptyset}$ for $i = 1, 2$ be as there. We now prove by induction on $\ell < n(*)$ that $\alpha_{\eta_\ell} \in u_1 \cap u_2$. If $\ell < n(*)$ and it is true for every $\ell' < \ell$, but (for some $i \in \{1, 2\}$), $\alpha_{\eta_\ell} \in u_i \backslash u_{3-i}$, it follows by clause (v) of $\boxplus$ that the sequence $\langle \mathbf{c}(\{\alpha_{\eta_\ell}, \alpha_n u\}) : \nu \in Z_\ell^*\rangle$ is constant where we let $Z_\ell^* = \{\eta_{\ell+1}, \eta_{\ell+2}, \dots, \eta_{n(*)-1}\} \cup Z$, hence $\{\alpha_\nu : \nu \in Z_\ell^*\}$ is disjoint to $u_{3-i} \backslash u_i$, so $\{\alpha_\nu : \nu \in Z^+\} \subseteq u_i$, so we get contradiction to the minimality of k.

As $\{\alpha_{\eta_\ell} : \ell < n(*)\} \subseteq u_2 \cap u_1$ necessarily (by clause (vi) of $\boxplus$) we have $\{\alpha_\nu : \nu \in Z_{n(*)}^*\} = \{\alpha_\nu, \nu \in Z\} \subseteq c\ell_0\{\alpha_{\eta_\ell} : \ell < n(*)\} \subseteq c\ell_0(u_2 \cap u_1)$. But $\{\alpha_\nu : \nu \in Z_{n(*)}^*\} \subseteq u_2 \cup u_1$ by $\boxplus(iii)$, and we know that $c\ell_0(u_2 \cap u_1) \cap (u_2 \cup u_1) \subseteq u_1$ by clause (d) of Definition 2.4, Case 2 hence $\{\alpha_\nu : \nu \in Z_{n(*)}^*\} \subseteq u_1$ contradiction to "k minimal".

Case 3: $k = 4\ell + 3$.

So let $\mathbf{s} \in \Gamma, p, \langle w_\ell : \ell \le \ell(*)\rangle, \langle v_t^\ell \in Y^+, \ell \le \ell(*)\rangle, \langle q_\ell : \ell \le \ell(*)\rangle, \langle f_{t,\iota}^\ell : t \in Y^+, \iota \le 2\rangle$ be as in definition 2.4. Let $j \le \ell(*)$ be minimal such that $\{\alpha_\eta : \eta \in Z^+\} \subseteq w_j$.

Subcase 3A: $j = 0$ and $\{\alpha_{\eta_\ell} : \ell \le n(*)\}$ is included in $v_\emptyset^j$ then (recalling $t \in Y^+ \Rightarrow c\ell(v_t^j) \cap w_\ell = v_t^j$ by clause (B)(g) of definition 2.4 we have $\{\alpha_\rho : \rho \in Z^+\} \subseteq v_\emptyset^j$ but $p \restriction v_\emptyset^j = p_\emptyset^j$ (by (D)(b)) and $p_\emptyset^j \in \cup\{\mathbb{P}_{k'} : k' < k\}$ and this is impossible by the induction hypothesis.

Case 3B: $j = 0$ but not Case 3A.

So for some $\ell \le n(*), \alpha_{\eta_\ell} \notin v_\emptyset^j$ hence by (B(x)) we have $\langle c^{q_0}\{\alpha_{\eta_\ell}, \alpha_\rho\} : \rho \in Z\rangle$ is constant, and $p \in Z \Rightarrow \alpha_\rho \notin v_\emptyset^j$.

Now we use $\{\{\eta_0, \eta_1\} \in e_{\nu_{n(*)}}^{\mathbf{s}} : \eta_0 \in Z_0, \eta_1 \in Z_1\}$. It is included in some equivalence class of e^* and by 2.1 we get contradiction to the clause (c) of the assumptions on $\mathbf{s}$ and Γ.

Subcase 3C: $j = i + 1 \le n(*)$ and $t \in e_i^* \Rightarrow \{\alpha_{\eta_\ell} : \ell \le n(*)\} \nsubseteq w_i \cup v_t^{i+1}$ (but is $\subseteq \cup\{v_t^j : t \in y_i\} \cup w_i$ [necessary?]) not really or we demand equality).

For $\ell \le n(*)$ such that $\alpha_{\eta_\ell} \notin w_i$, let $t_\ell \in e_i^*$ be minimal such that $\alpha_{\eta_\ell} \in v_{t_\ell}^j$. By the assumption of this subcase, for some $\ell(1) < \ell(2) \le n(*)$ we have $t \in e_i^* \Rightarrow \{\alpha_{\eta_{\ell(1)}}, \alpha_{\eta_{\ell(2)}}\} \nsubseteq w_i \cup v_t^{i+1}$. But this implies that $\mathbf{c}\{\alpha_{\eta_{\ell(1)}}, \alpha_{\eta_{\ell(2)}}\}$ appears only one in c^{q_i}, easy contradiction.

Subcase 3D: $j = i + 1, t \in e_i^*, \{\alpha_{\eta_\ell} : \ell \le n(*)\} \subseteq w_i \cup v_t^{i+1}$ but for every $s \in e_i^*, \{\alpha_{\eta_\ell} : \ell \le n(*)\} \nsubseteq v_s^{i+1}$.

This implies that for some $\ell(1) < \ell(2) \le n(*), \{\alpha_{\eta_{\ell(1)}}, \alpha_{\eta_{\ell(2)}}\}$ is not in $\cup\{[v^{i+1}_s]^2 : s \in e^*_i\} \cup [w_i]^2$.

By clause (B)(x) it follows that $c^{q_i}\{\alpha_{\eta_{\ell(1)}}, \alpha_{\eta_{\ell(2)}}\}$ is not in $\cup\{\mathrm{Rang}(c^{p^{i+1}_s}) : s \in e^*_i\}$.

By clause (B)(y) this implies that $\rho_1 \ne \rho_2 \in Z \Rightarrow \{\rho_1, \rho_2\} \notin \cup\{[v^{i+1}_s]^2 : s \in e^*_i\} \cup [w_i]^2$. We are done by $\mathbf{s}$ being Γ and $\mathbf{s}^+$ being far, i.e., clause (c) of the assumption.

Subcase 3E: $j = i+1, t \in e^*_i$ and $\{\alpha_{\eta_\ell} : \ell \le n(*)\} \subseteq v^{i+1}_t$.

Recall that $\{\alpha_\rho : \rho \in Z^+\} \subseteq c\ell\{\alpha_{\eta_\ell} : \ell \le n(*)\}$ by $\boxplus$(vi), so together $\{\alpha_\rho : \rho \in Z^+\} \subseteq v^{i+1}_t$ but $q_{i+1} \restriction v^{i+1}_t = p^{i+1}_t \in \mathbb{P}_{<k}$.

Together we have covered all the cases. $\square_{3.3}$

3.4 Theorem. *Let $n(*) = 4$ (or just $n(*) \ge 4$), $\lambda = \aleph_{n(*)}, \ell(*) = n(*)+1$ and $2^{\aleph_\ell} = \aleph_{\ell+1}$ for $\ell < n(*)$.*
1) For some c.c.c. forcing $\mathbb{P}$ of cardinality λ in $\mathbf{V}^{\mathbb{P}}$ the pair $(\lambda, \aleph_0)$ is not $\aleph_0$-compact.
2) If in addition $\chi = \chi^{\aleph_0} \ge \lambda$ there is a forcing notion $\mathbb{P}$ of cardinality χ such that $\mathbf{V}^{\mathbb{P}} \models$ "$2^{\aleph_0} = \chi$" and the pair $(\lambda, \aleph_0)$ is not compact.
3) There is an infinite $\Gamma^* \subseteq \mathrm{ID}^*_{\ell(*)}$ which is recursive and for every $\Gamma \subseteq \Gamma^*$ for some forcing notin $\mathbb{P}$, in fact $\mathbb{P}_\Gamma$, in $\mathbf{V}^{\mathbb{P}}$ we have $\Gamma = \mathrm{ID}_2(\lambda, \aleph_0) \cap \mathrm{ID}^*_{\ell(*)}$.

Proof. 1) Let $\Gamma_n = \{\mathbf{s} \in \mathrm{ID}^*_{\ell(*)} : \mathbf{s} \text{ is } n\text{-nice}\}$, see Definition 2.1 and 1.3, clearly $\Gamma_{n+1} \subseteq \Gamma_n$ and $\Gamma_n \ne \emptyset$ (see 1.5) for $n < \omega$ and $\emptyset = \bigcap\limits_{n<\omega} \Gamma_n$ and let $\mathbb{P}_n = \mathbb{P}^\lambda_{\Gamma_n}$ and let $\underset{\sim}{c}_n = \cup\{c^p : p \in \underset{\sim}{G}_{\mathbb{P}_n}\}$, it is a $\mathbb{P}_n$-name and $\mathbb{P}$ is the product $\prod\limits_{n<\omega} \mathbb{P}_n$ with finite support. Now the forcing notion $\mathbb{P}$ satisfies the c.c.c. as $\mathbb{P}_n$ satisfies the Knaster condition (by 2.5(1)). By 3.3 we know that $\Vdash$ "$\mathrm{ID}_2(\underset{\sim}{c}_n) \cap \mathrm{ID}^*_{\ell(*)} \subseteq \Gamma_n$" for $\mathbb{P}_n$ hence for $\mathbb{P}$, in fact it is not hard to check that equality holds. If $\aleph_0$-compactness holds then in $\mathbf{V}^{\mathbb{P}}$ for some $c : [\lambda]^2 \to \omega$ we have $\mathrm{ID}_2(c) \cap \mathrm{ID}^*_{\ell(*)} \subseteq \bigcap\limits_n \Gamma_n = \emptyset$ by claim 1.4. But in $\mathbf{V}^{\mathbb{P}}$, if $c : [\lambda]^2 \to \omega$ then by 3.2(2) it realizes some $\mathbf{s} \in \cup\{\Gamma_n : n < \omega\} \subseteq \mathrm{ID}^*_{\ell(*)}$ (even k-nice one for every $k < \omega$).

Together we get that the pair $(\lambda, \aleph_0)$ is not $\aleph_0$-compact.
2) We let $\mathbb{Q}$ be adding χ Cohen reals, i.e. $\{h : h \text{ a finite function from } \chi \text{ to } \{0,1\}\}$ ordered by inclusion. Let $\mathbb{P}$ be as above and force with $\mathbb{P}^+ = \mathbb{P} * \mathbb{Q}$, now it is easy to check that $\mathbb{P}^+$ is as required.

3) Choose $\mathbf{s}_k \in \mathrm{ID}^*_{\ell(*)}$ by induction on $k < \omega$ such that $\mathbf{s}_k$ is $\sup\{|\mathrm{dom}_{\mathbf{s}_\ell}| : \ell < k\}$-nice let $\Gamma = \{\mathbf{s}_k : k < \omega\}$ and use $\mathbb{P} = \mathbb{P}_\Gamma$. $\square_{3.4}$

§4 Improvements and additions

Though our original intention was to deal with the possible incompactness of the pair $(\aleph_2, \aleph_0)$, we have so far dealt with $(\lambda, \aleph_0)$ where $2^{\aleph_0} \geq \lambda = \aleph_{n(*)}$ & $n(*) \geq 4$. For dealing with $(\aleph_3, \aleph_0), (\aleph_2, \aleph_0)$, that is $n(*) = 3, 2$ we need to choose M^*_λ more carefully.

What is the problem in §3 concerning $n(*) = 2$?
On the one hand in the proof of 3.3 we need that there are many dependencies among ordinals $< \lambda$ by M^*_λ; so if λ is smaller this is easier, but so far the gain was only enabling us to use smaller $\ell(*)$ which really just make us use larger $\ell(*)$ help.

On the other hand, in the proof of 3.2 we use 3.1, a partition theorem, so here if λ is bigger it is easier. But instead we can use demands specifically on M^*_λ. Along those lines we may succeed for $n(*) = 3$ using 3.1(1) rather than 3.1(2) but we still have problems for the pair $(\aleph_2, \aleph_0)$; here we change the main definition 2.4, in case 3 changes $\langle v_y : y \in Y^+ \rangle$, i.e. for $\eta \in \text{dom}_{\mathbf{s}}$ we have $v^+_{\{\eta\}}, v^-_{\{\eta\}}$ instead $v_{\{\eta\}}$. For this we have to carefully reconsider 3.2, but the parallel of 3.1 is easier. Note that in §2, §3 we could have used a nontransitive version of $c\ell_M(-)$.

4.1 Definition. We say that M^* is $(\lambda, < \mu, n(*), \ell(*))$-suitable <u>if</u>:

- (a) M^* is a model of cardinality λ
- (b) $\mu < \lambda \leq \mu^{+n(*)}$ and $n(*) < \ell(*) < \omega$
- (c) τ_{M^*}, the vocabulary of M^*, is of cardinality $\leq \mu$
- (d) for every subset A of M^* of cardinality $< \mu$, the set $c\ell_{M^*}(A)$ has cardinality $< \mu$.
- (e) for some $m^* < \omega$ we have: if $\mathbf{s} = (\text{dom}_{\ell(*),m}, e) \in \text{ID}^*_{\ell(*)}$ and $a_\eta \in M^*$ for $\eta \in \text{dom}_{\ell(*),m}$ and $\mathbf{s}$ is m^*-nice, $m > m^*$, <u>then</u> we can find $\langle \eta_\ell : \ell < n(*) \rangle$ and $\langle \nu_\ell : \ell \leq n(*) \rangle$ such that
  - (α) $\eta_\ell \in \text{dom}_{\ell(*),m}$
  - (β) $\nu_0 = <>, \nu_{\ell+1} = (\eta_\ell \restriction \ell)\hat{\ }\langle 1 - \eta_\ell(\ell) \rangle$
  - (γ) $\nu_\ell \triangleleft \eta_\ell$
  - (δ) $Z = \{\rho \in \text{dom}_{\ell(*),m} : \nu_{n(*)} \triangleleft \rho$ and in the graph $H[e], \rho$ is connected to η_ℓ for $\ell = 0, \ldots, n(*) - 1\}$
  - (ε) if $\ell(1) < \ell(2) < n(*)$ then $\{\eta_{\ell(1)}, \eta_{\ell(2)}\}$ is an edge of the graph $H[e]$
  - (ζ) $\{\alpha_\rho : \rho \in Z\} \subseteq c\ell_{M^*}\{\alpha_{\eta_\ell} : \ell < n(*)\}$.

4.2 Definition. 1) We say that M^* is explicitly[1] $(\lambda, <\mu, n(*))$-suitable when:

(a) M^* is a model of cardinality λ

(b) $\lambda = \mu^{+n(*)}$

(c) τ_{M^*}, the vocabulary of M^*, is of cardinality $\le \mu$

(d) for $A \subseteq M^*$ of cardinality $< \mu$, the set $c\ell_{M^*}(A)$ has cardinality $< \mu$ and $A \ne \emptyset \wedge \mu > \aleph_0 \Rightarrow \omega \subseteq c\ell_{M^*}(A)$

(e) for some $\langle R_\ell : \ell \le n(*)\rangle$ we have

(α) R_ℓ is an $(\ell+2)$-place predicate in τ_{M^*}; we may write $R_\ell(x, y, z_0, \dots, z_{\ell-1})$ as $x <_{z_0,\dots,z_{\ell-1}} y$ or $x <_{\langle z_0,\dots,z_{\ell-1}\rangle} y$

(β) for any $c_0, \dots, c_{\ell-1} \in M^*$, the two place relation $<_{c_0,\dots,c_{\ell-1}}$ (i.e. $\{(a,b) : \langle a, b, c_0, \dots, c_{\ell-1}\rangle \in R_\ell^{M^*}\}$) is a well ordering of $A_{c_0,\dots,c_{\ell-1}} =: A_{\langle c_0,\dots,c_{\ell-1}\rangle} =: \{b : (\exists x)(x <_{c_0,\dots,c_{\ell-1}} b \vee b <_{c_0,\dots,c_{\ell-1}} x)\}$ of order-type a cardinal

(γ) $R_0^{M^*}$ is a well ordering of M^* of order type λ

(δ) if $\bar c = \langle c_\ell : \ell < k\rangle$ and $<_{\bar c}$ is a well ordering of $A_{\bar c}$ of order type μ^{+m} then for every $c_k \in M^*$ we have $A_{\bar c \char`\^ \langle c_k\rangle} = \{a \in A_{\bar c} : a <_{\bar c} c_k\}$ so is empty if $c_k \notin A_{\bar c}$ so if $\ell g(\bar c) = n(*)$ this is a definition of $A_{\bar c \frown <c_k>}$ as it is not covered by clause (β)

(ε) if $\bar c = \langle c_\ell : \ell < k\rangle \in {}^k(M^*)$ and $|A_{\bar c}| < \mu$ then $A_{\bar c} \subseteq c\ell_{M^*}(\bar c)$.

2) We say that M^* is explicitly[2] $(\lambda, <\mu, n(*))$-suitable when:

$(a) - (d)$ as in part (1)

(e) for some $\langle R_\ell : \ell \le n(*)\rangle$ we have (like (e) but we each time add z's and see clause (δ))

(α) R_ℓ is a $(2\ell+2)$-place predicate in τ_{M^*}; we may write $R_\ell(x, y, z_0, \dots, z_{2\ell-1})$ or $x <_{z_0,\dots,z_{2\ell-1}} y$ or $x <_{\langle z_0,\dots,z_{2\ell-1}\rangle} y$

(β) for any $c_0, \dots, c_{2\ell-1} \in M^*$ the two-place relation $<_{c_0,\dots,c_{2\ell-1}}$ (i.e., $\{(a,b) : \langle a, b, c_0, \dots, c_{2\ell-1}\rangle \in R_\ell^{M^*}\}$) is a well ordering of $A_{c_0,\dots,c_{2\ell-1}} = A_{(c_0,\dots,c_{2\ell-1})} = \{b :$ for some $a, \langle a, b, c_0, \dots, c_{2\ell-1}\rangle \in R_\ell^{M^*}$ or $\langle b, a, c_0, \dots, c_{2\ell-1}\rangle \in R_\ell^{M^*}\}$ of order type a cardinal

(γ) $R_0^{M^*}$ is a well ordering of M^* of order type λ; for simplicity $R_0^{M^*} = c \restriction \lambda$

(δ) if $\bar c = \langle c_\ell : \ell < 2k\rangle$ and $<_{\bar c}$ is a well ordering of $A_{\bar c}$ of order type μ^{+m} <u>then</u> for any $c_{2k}, c_{2k+1} \in M^*$ we have $A_{\bar c \char`\^ \langle c_{2k}, c_{2k+1}\rangle}$ is empty if $\{c_{2k}, c_{2k+1}\} \nsubseteq A_{\bar c}$ and otherwise is $\{a \in A_{\bar c} : a <_{\bar c} c_{2k}$ and $a < c_{2k+1}\}$; if $k = n(*)$ this is a definition of $A_{\bar c \frown <c_{2k}, c_{2k+1}>}$.

4.3 Observation. 1) If M is an explicitly[1] $(\lambda, < \mu, n(*))$-suitable model, then M is a $(\lambda, < \mu, n(*) + 1, \ell(*))$-suitable model if $\ell(*) > n(*) + 1$.
2) If M is an explicitly[2] $(\lambda, < \mu, n(*))$-suitable model, then M is a $(\lambda, < \mu, 2n(*) + 2, 2n(*) + 3)$-suitable model.

Proof. 1) Straightforward, similar to inside the proof of 3.3 and as we shall use part (2) only and the proof of (1) is similar but simpler, we do not elaborate.
2) Clearly clauses (a) - (d) of Definition 4.1 holds, so we deal with clause (e). So assume $\ell(*) \geq 2n(*)$ and $\mathbf{s} = (\mathrm{dom}_{\ell(*),m}, e) \in \mathrm{ID}^*_{\ell(*)}$ and $\alpha_\eta \in M$ for $\eta \in \mathrm{dom}_{\ell(*),m}$ are pairwise distinct. We choose by induction on $\ell \leq n(*)$ the objects $\eta_{2\ell}, \nu_{2\ell+1}, Z_{2\ell}, \eta_{2\ell+1}, \nu_{2\ell+2}, Z_{2\ell+1}$ such that node $\nu_0 =<>$ and $\nu_{2\ell+2}$ is chosen in stage ℓ

$\boxtimes$(a) $\nu_\ell \in {}^\ell 2, \eta_\ell \in \mathrm{dom}_{\ell(*),m}$ and $M \models (\exists^{\leq \aleph_{n(*)-\ell}} x)\varphi_\ell(x, \alpha_{\eta_0}, \ldots, \alpha_{\eta_{2\ell-1}})$

(b) $<^\ell_{\alpha_{\eta_0},\ldots,\alpha_{\eta_{2\ell-1}}}$ is a well ordering of $A_{\langle \alpha_{\eta_0},\ldots,\alpha_{2\ell-1}\rangle} =: \{x : M \models \varphi_\ell[x, \alpha_{\eta_0}, \ldots, \alpha_{\eta_{2\ell-1}}]\}$ of order type a cardinal $\leq \aleph_{n(*)-\ell}$

(c) $\nu_0 =<>, \varphi_0 = [x = x]$

(d) $\nu_{\ell+1} = (\eta_\ell \restriction \ell)\char`^\langle 1 - \eta_\ell(\ell)\rangle$

(e) $Z_\ell = \{\eta : \nu_{2\ell} \triangleleft \eta \in \mathrm{dom}_{\ell(*),m}$ and $\{\eta_s, \eta\} \in e_{\nu \restriction s}$ for $s = 0, 1, \ldots, \ell - 1\}$

(f) $\eta \in Z_\ell \Rightarrow \alpha_\eta \in A_{\langle \alpha_{\eta_k} : k < 2\ell\rangle}$

(g) η_ℓ is such that:

(α) $\nu_\ell \triangleleft \eta_\ell \in Z_\ell$

(β) if $\nu_\ell \trianglelefteq \eta \in Z_\ell$ then $[\ell$ even $\Rightarrow \alpha_\eta \leq_{\alpha_{\eta_0},\ldots,\alpha_{\eta_{\ell-1}}} \alpha_{\eta_\ell}]$ and $[\ell$ odd $\Rightarrow \alpha_\eta \leq \alpha_{\eta_\ell}]$.

How do we do the induction step? Arriving to ℓ we have already defined $\langle \nu_k : k \leq 2\ell\rangle, \langle \eta_k : k < 2\ell\rangle$ and $\langle Z_k : k < 2\ell\rangle$, recalling $\nu_0 =<>$. So by the clause (e) (= definition of Z_k) also $Z_{2\ell}$ is well defined and $\{\alpha_\eta : \eta \in Z_{2\ell}\}$ is included in $A_{\langle \alpha_{\eta_k} : k<2\ell\rangle}$ and let $\eta_{2\ell} \in Z_{2\ell}$ be such that $\eta \in Z_{2\ell} \Rightarrow \alpha_\eta \leq_{\langle a_{\eta_k} : k<2\ell\rangle} \alpha_{\eta_{2\ell}}$ and $\nu_{2\ell+1} = \nu_{2\ell}\char`^\langle 1 - \eta_{2\ell}(2\ell)\rangle = (\eta_{2\ell} \restriction (2\ell))\char`^\langle 1 - \eta_{2\ell}(2\ell)\rangle$ so $Z_{2\ell+1}$ is well defined. Let $\eta_{2\ell+1} \in Z_{2\ell+1}$ be such that $\eta \in Z_{2\ell} \Rightarrow a_\eta \leq \alpha_{\eta_{2\ell+1}}$ and $\nu_{2\ell+2} = \nu_{2\ell+1}\char`^\langle 1 - \eta_{2\ell+1}(2\ell + 1)\rangle$ and we have carried the induction. $\square_{4.3}$

Are there such models? We shall use 4.4(2), the others are for completeness (i.e. part (3) is needed for $\lambda = \aleph_3$ and part (4) says concerning $\lambda = \aleph_2$ it suffices to use ID_3^*):

4.4 Observation. 1) For μ regular uncountable, there is an explicitly[1] $(\mu^{+2}, < \mu, 2)$-suitable model.
2) If $\mu = \aleph_0$, then there is an explictly[2] $(\mu^{+2}, < \mu, 2)$-suitable model.
3) If μ is regular uncountable, $t = 1$ or $\mu = \aleph_0$ & $t = 2$ and $n \in [3, \omega)$, <u>then</u> there is an explicitly^t $(\mu^{+n}, < \mu, n)$-suitable model.
4) If $2^{\aleph_0} = \aleph_1, \mu = \aleph_0$ <u>then</u> for some $\aleph_2$-c.c., $\aleph_1$-complete forcing notion $\mathbb{Q}$ of cardinality $\aleph_2$ in $\mathbf{V}^{\mathbb{Q}}$ there is an explicitly $(\aleph_2, < \aleph_0, 2)$-suitable model.

4.5 Remark. It should be clear that if $\mathbf{V} = \mathbf{L}$ (or just $\neg \exists 0^{\#}$), then this works also for singular μ <u>but</u> more reasonable is to use non-transitive closure.

Proof of 4.4. 1), 2) Let $t = 1$ for part (1) and $t = 2$ for part (2). Let $n(*) = 2$ and $\lambda = \mu^{+2}$. We choose M_α by induction on $\alpha \le \lambda$ such that:

- (α) M_α is a τ^--model where $\tau^- = \{R_0, R_1, R_2\}$ with R_ℓ is $(t\ell + 2)$-predicate and $x <_{\bar z} y$ means $R_\ell(x, y, \bar z)$
- (β) M_α is increasing with α and has universe $1 + \alpha$
- (γ) $R_0^{M_\alpha}$ is $< \restriction \alpha$ (and $A^{M_\alpha}_{<>} = \alpha$)
- (δ) for $\bar c \in {}^{tk}(M_\alpha), k = 0, 1, 2$ we have $<_{\bar c}$ is a well ordering of $A^{M_\alpha}_{\bar c} =: \{a : M_\alpha \models (\exists x)(a <_{\bar c} x \vee x <_{\bar c} a)\}$ of order type a cardinal $< \mu^{+(n(*)+1-k)}$
- $(\varepsilon)(i)$ if $t = 1, \bar c \in {}^k(M_\alpha), k = 0, 1, 2$ and $d \in A^{M_\alpha}_{\bar c}$ then $A^{M_\alpha}_{\bar c \char`\^ <d>} = \{a \in A^{M_\alpha}_{\bar c} : M_\alpha \models a <_{\bar c} d\}$
- (ii) if $t = 2, \bar c \in {}^{2k}(M_\alpha), k = 0, 1, 2$ and $d_0, d_1 \in A^{M_\alpha}_{\bar c}$ then $A^{M_\alpha}_{\bar c \char`\^ \langle d_0, d_1 \rangle} = \{a \in A^{M_\alpha}_{\bar c} : M_\alpha \models$ "$a <_{\bar c} d_0$ & $a < d_1$"$\}$
- (ζ) if A is a subset of M_α of cardinality $< \mu$ then $c\ell^*_{M_\alpha}(A)$ is of cardinality $< \mu$ and $c\ell^*_{M_\alpha}(c\ell^*_{M_\alpha}(A)) = c\ell^*_{M_\alpha}(A)$ where
  - $\boxtimes$ for $A \subseteq M_\alpha, c\ell^*_{M_\alpha}(A)$ is the minimal set B such that: $A \subseteq B$ and $(\forall \bar c \in {}^{3t}B)(|A^{M_\alpha}_{\bar c}| < \mu \Rightarrow A_{\bar c} \subseteq B)$; clearly B exists and $c\ell^*_{M_\alpha}(\emptyset) = \emptyset$
- (η) for every $\beta < \alpha, k = 1, 2$ and $\bar c \in {}^k(M_\beta)$ we have $A^{M_\alpha}_{\bar c} = A^{M_\beta}_{\bar c}$
- (θ) if $A \subseteq \beta < \alpha$ then $c\ell^*_{M_\beta}(A) = c\ell^*_{M_\alpha}(A)$
- (ι) if $t = 2$ and $\mu = \aleph_0$ and $A \subseteq \alpha$ is finite, β is the last element in A, <u>then</u> for some finite $B \subseteq \beta$ we have $c\ell^*_{M_\alpha}(A) = \{\beta\} \cup c\ell^*_{M_\beta}(B)$.

We leave the cases $\alpha < \mu$ and α a limit ordinal to the reader (for (ζ) we use (θ)) and assume $\alpha = \beta + 1$ and M_γ for $\gamma \le \beta$ are defined. We can choose $\langle B_{\beta,i} : i < \mu^+ \rangle$, a (not necessarily strictly) increasing sequence of subsets of β, each of cardinality $\le \mu, B_{\beta,0} = \emptyset$ and $\cup\{B_{\beta,i} : i < \mu^+\} = \beta$ and $c\ell^*_{M_\beta}(B_{\beta,i}) = B_{\alpha,i}$.

For each $i < \mu^+$ let $\langle B_{\beta,i,\varepsilon} : \varepsilon < \mu \rangle$ be (not necessarily strictly) increasing sequence of subsets of $B_{\beta,i}$ with union $B_{\beta,i}$ such that $c\ell^*_{M_\beta}(B_{\beta,i,\varepsilon}) = B_{\beta,i,\varepsilon}, B_{\beta,i,0} = \emptyset$. Let $<^*_\beta$ be a well ordering of $\{\gamma : \gamma < \beta\}$ such that each $B_{\beta,i}$ is an initial segment so it has order type μ^+. For $\gamma \in B_{\beta,i+1} \backslash B_{\beta,i}$ let $<^*_{\beta,\gamma}$ be a well ordering of $A^*_{(\beta,\gamma)} = \{\xi : \xi <^*_\beta \gamma\}$ of order type $\le \mu$ such that $(\forall \varepsilon < \mu)(B_{\beta,i+1,\varepsilon} \cap A^*_{(\beta,\gamma)})$ is an initial segment of $A^*_{(\beta,\gamma)}$ by $<^*_{\beta,\gamma}$.

Now we define M_α:
universe is α
$R_0^{M_\alpha} = < \restriction \alpha$

<u>Case 1</u>: $t = 1$.

$R_1^{M_\alpha} = R_1^{M_\beta} \cup \{(a, b, \beta) : a <^*_\beta b\}$

$R_2^{M_\alpha} = R_2^{M_\beta} \cup \{(a, b, \beta, \gamma) : \gamma < \beta$, and $a <^*_{\beta,\gamma} b$ hence $a <^*_\beta \gamma$ & $b <^*_\beta \gamma$ and $a, b \in B_{\beta,i+1}$ for the unique i such that $\gamma \in B_{\beta,i+1} \backslash B_{\beta,i}\}$.

<u>Case 2</u>: $t = 2$.

$$R_1^{M_\alpha} = R_1^{M_\beta} \cup \{(a, b, \beta, \gamma) : a <^*_\beta b \text{ and } a < \gamma, b < \gamma \text{ and, of course, } a, b, \beta \in \alpha\}.$$

$$R_2^{M_\alpha} = R_2^{M_\beta} \cup \{(a, b, \beta, \gamma_0, \beta_1, \gamma_1) : a, b, \gamma_0, \beta_1 \in \alpha \text{ and } a < \beta, b < \beta, a < \gamma_0, b < \gamma_0, a, b, \beta_1, \gamma_1 \in A_{<\beta,\gamma_0>} \text{ and } a <^*_{\beta,\gamma_0} b \text{ and } a < \gamma_1, b < \gamma_1\}.$$

To check for clause (ζ) is easy if $\mu = \text{cf}(\mu) > \aleph_0$ and follows by clause (ι) if $\mu = \aleph_0$. Having carried the induction we define M: it is M_λ expanded by $\langle F_i^M : i < \mu \rangle$ such that: if $\bar{c} \in {}^{3t}\lambda = {}^{3t}(M_\lambda)$ and $A_{\bar{c}}$ is a non empty well defined and of cardinality $< \mu$ (which follows) then $\{F_i^M(\bar{c}) : i < \mu\}$ list $A_{<c_0,c_1,c_2>} \cup \{0\}$ otherwise $\{F_i^M(\bar{c}) : i < \mu\}$ is $\{0\}$.
3) Similar and used only for $(\aleph_3, \aleph_0)$ so we do not elaborate.
4) Let $\mathbb{Q}$ be defined as follows:
$p \in \mathbb{Q}$ <u>iff</u>

- (α) p is a τ^--model, as in (α) of the proof of part (1)
- (β) the universe univ(p) of p is a countable subset of λ, we let $A^p_{<>} =$ univ(p)
- (γ) $R_0^p = < \restriction$ univ(p) and $<_{<>} = R_0^p$

(δ) if $\bar{c} \in {}^{tk}(\text{univ}(p)), k = 1, 2$ then $<_{\bar{c}} = <^p_{\bar{c}}$ is a well ordering of $A^p_{\bar{c}} = \{a \in p : p \models (\exists x)(a <_{\bar{c}} x \vee x <_{\bar{c}} a)\}$ and for $d \in A^p_{\bar{c}}$ we let $A^p_{\bar{c}\char`\^<d>} = \{a \in A^p_{\bar{c}} : a <^p_{\bar{c}} d\}$

(ε) $(A^p_{\bar{c}}, <_{\bar{c}})$ has order type ω if $k = 2$

(ζ) if $A \subseteq$ univ(p) is finite, then $c\ell^*_p(A)$ is finite (is defined as in(2)).

<u>the order</u>:
$\mathbb{Q} \models p \leq q$ <u>iff</u>

(i) p is a submodel of q

(ii) if $\bar{c} \in {}^2(\text{univ}(p))$ then $A^p_{\bar{c}} = A^q_{\bar{c}}$

(iii) if $\bar{c} \in {}^1(\text{univ}(p))$ then $A^p_{\bar{c}}$ is an initial segment of $A^q_{\bar{c}}$ by $<_{\bar{c}}$.

The rest should be clear. $\square_{4.4}$

4.6 Claim. *Assume (main case is $n(*) = 2$)*

($*$) $2 \leq n(*) < \omega, \lambda = \aleph_{n(*)}, 2^{\aleph_0} < \lambda, \ell(*) = 2n(*) + 3$ *and* $\lambda \leq \chi = \chi^{\aleph_0}$.

1) For some forcing notion $\mathbb{P}^*$ we have

(a) *$\mathbb{P}^*$ is a forcing notion of cardinality χ*

(b) *$\mathbb{P}^*$ satisfies the c.c.c.*

(c) *in $\mathbf{V}^{\mathbb{P}^*}$ the pair $(\aleph_{n(*)}, \aleph_0)$ is not compact*

(d) *in $\mathbf{V}^{\mathbb{P}^*}$ we have $2^{\aleph_0} = \chi$.*

2) There is an infinite $\Gamma^* \subseteq \text{ID}^*_{\ell(*)}$ which is recursive and for every $\Gamma \subseteq \Gamma^*$ for some forcing notin $\mathbb{P}$, in fact $\mathbb{P}_\Gamma$, in $\mathbf{V}^{\mathbb{P}}$ we have $\Gamma = \text{ID}_2(\lambda, \aleph_0) \cap \text{ID}^*_{\ell(*)}$.

Proof. We repeat §2, §3 with the following changes.

If $n(*) \geq 3$, we need change (A) below and using 3.1(2) instead of 3.1(1). For $n(*) = 2$ we need all the changes below

(A) we replace M^*_λ by any model as in 4.4(2) if $n(*) = 2$, 4.4(3) if $n(*) \geq 3$

(B) in 3.1

  (a) we assume $\lambda \geq (2^\mu)^+, \mu = \aleph_0, (\forall A \in [M]^{<\mu})(|c\ell_M(A)| < \mu)$

  (b) the conclusion: weaken $|W| = \mu^{++}$ to W infinite

  (c) proof:

Let $g : [\lambda]^2 \to \omega$ be $g(t) = |c\ell_M(t \cup w_t)| < \omega$.
Let $W_1 \in [\lambda]^{\mu^+}$ be such that $g \restriction [W_1]^2$ is constant say $k(*)$ and $f \restriction [W_1]^2$ is constantly γ. Let $c\ell_M(t) = \{\zeta_{t,\ell} : \ell < g(t)\}$. By Ramsey theorem, there is an infinite $W \subseteq W_1$ such that:

- $\circledast$ the truth value on $\zeta_{\{\alpha_1,\beta_1\},\ell_1} = \zeta_{\{\alpha_2,\beta_2\},\ell_2}$ depend just on ℓ_1, ℓ_2, T.V.(α_i, β_j), T.V.$(\beta_j < \alpha_i)$ for $i, j \in \{1, 2\}$.

The conclusion should be clear. $\square_{4.6}$

§5 Open Problems and concluding remarks

We finish the paper by listing some problems (some are old, see [\CK]).

5.1 Question: Suppose that λ is strongly inaccessible, $\mu > \aleph_0$ is regular not Mahlo and $\square_\mu$. Then $\lambda \rightarrow \mu$ in the λ-like model sense, i.e. if a first order ψ has a λ-like model then it has a μ-like model.
If λ is ω-Mahlo, the answer is yes, see [ScSh 20] by appropriate partition theorems. The assumption that μ is not Mahlo is necessary by Schmerl, see [Sch85].

5.2 Question: (Maybe under $\mathbf{V} = \mathbf{L}$.) Suppose that $\lambda^{\beth_\omega(\kappa)} = \lambda$ and $\lambda_1^{<\lambda_1} = \lambda_1 > \kappa_1$. Then $(\lambda^+, \lambda, \kappa) \rightarrow (\lambda_1^+, \lambda_1, \kappa_1)$.

5.3 Question: (GCH) If λ and μ are strong limit singulars and λ is a limit of supercompacts, then $(\lambda^+, \lambda) \rightarrow (\mu^+, \mu)$.

5.4 Question: Find a universe with $(\beth_2(\aleph_0), \aleph_0) \rightarrow (2^{2^\lambda}, \lambda)$ for every λ.
(The author has a written sketch of a result which is close to this one. He starts with $\aleph_0 = \kappa_0 < \kappa_1 < \ldots < \kappa_m$ which are supercompacts and let $\mathbb{P}_n$ be the forcing which adds κ_{n+1} Cohen subsets to κ_n in $V^{\mathbb{P}_0 * \mathbb{P}_1 \ldots \mathbb{P}_{n-1}}$ for $n < m$. The idea is using the partition on trees from [Sh 288, §4]).

5.5 Question: Are all pairs in the set

$$\{(\lambda, \mu) : 2^\mu = \mu^+ \ \ \& \ \ \mu = \mu^{<\mu} \ \ \& \ \ \mu^{+\omega} \leq \lambda \leq 2^{\mu^+}\}$$

such that there is μ^+-tree with $\geq \lambda, \mu^+$-branches, equivalent for the two cardinal problem?

More related to this particular work are
5.6 Question:

1) Can we find $n < \omega$ and an infinite set Γ^* of identities (or 2-identities) such that for any $\Gamma \subseteq \Gamma^*$ for some forcing notion $\mathbb{P}$ in $\mathbf{V}^{\mathbb{P}}$ we have $\Gamma = \Gamma^* \cap \mathrm{ID}(\aleph_n, \aleph_0)$.
2) In (1) we can consider (λ, μ) with $\mu = \mu^{\aleph_0}, \lambda = \mu^{+n}$, so we ask: can we find a forcing notion $\mathbb{P}$ not adding reals such that for every $\Gamma \subseteq \Gamma^*$ for some $\mu = \mu^{<\mu}$ we have $\Gamma = \Gamma^* \cap \mathrm{ID}(\mu^{+n}, \mu)$.

5.7 Question: 1) Can we get results parallel to 3.4 for $(\aleph_2, \aleph_1) + 2^{\aleph_0} \geq \aleph_2$ (so we should start with a large cardinal, at least a Mahlo).
2) The parallel to 5.6(1),(2).

<u>5.8 Question</u>: 1) Can we get results parallel to 3.4 for $(\aleph_{\omega+1}, \aleph_\omega)+$ G.C.H. or $(\mu^+, \mu), \mu$ strong limit singular + G.C.H.
2) The parallel to 5.6(1),(2).

<u>5.9 Question</u>: How does assuming MA $+2^{\aleph_0} > \aleph_n$ influence ID$(\aleph_n, \aleph_0)$? (see below).

<u>We end with some comments</u>:

5.10 Definition. 1) For $k \le \aleph_0$, we say (λ, μ) has k-simple identities <u>when</u> $(a, e) \in \mathrm{ID}(\lambda, \mu) \Rightarrow (a, e') \in \mathrm{ID}(\lambda, \mu)$ whenever:

$(*)_k$ $a \subseteq \omega, (a, e)$ is an identity of (λ, μ) and e' is defined by

$$be'c \text{ iff } |b| = |c| \ \& \ (\forall b', c')[b' \subseteq b \ \& \ |b'| \le k \ \& \ c' = OP_{c,b}(b') \Rightarrow b'ec'].$$

$$\text{recalling } OP_{B,A}(\alpha) = \beta \text{ iff } \alpha \in A \ \& \ \beta \in B \ \& \ \ \mathrm{otp}(\alpha \cap A) = \ \mathrm{otp}(\beta \cap B).$$

5.11 Claim. *1) If (λ_1, μ_1) has k-simple identities and there is $f : [\lambda_2]^{\le k} \to \mu_2$ such that $\mathrm{ID}_{\le k}(f) \subseteq \mathrm{ID}_{\le k}(\lambda_1, \mu_1)$, <u>then</u> $(\lambda_1, \mu_1) \to (\lambda_2, \mu_2)$.*
2) If $\mathrm{cf}(\lambda_1) > \mu_1$, then we can use f with domain $[\lambda_2]^{\le k} \backslash [\lambda_2]^{\le 1}$.

Proof. Should be easy.

5.12 Claim. *1) [MA $+2^{\aleph_0} > \aleph_n$]. The[4] pair $(\aleph_n, \aleph_0)$ has 3-simple identities.*
2) If $\mu = \mu^{<\mu}$ and $\chi = \chi^{<\chi} > \mu$ <u>then</u> for some μ^+-c.c., $(< \mu)$-complete forcing notion $\mathbb{P}$ of cardinality χ, in $\mathbf{V}^{\mathbb{P}}$ we have $2^\mu = \chi$ and $\mu^{+n} < \chi \Rightarrow (\mu^{+n}, \mu)$ has 3-simple identities; moreover, if $\lambda < \chi$ and $\mathbf{c} : [\lambda]^{<\aleph_0} \to \mu$ then for some $\mathbf{c}' : [\lambda]^{\le 3} \to \mu$ we have: if $n < \omega, \alpha_0, \dots, \alpha_{n-1} < \lambda, \beta_0, \dots, \beta_{n-1}, < \lambda$ and $u \subseteq n \wedge |u| \le 3 \Rightarrow \mathbf{c}'\{\alpha_\ell : \ell \in u\} = \mathbf{c}'\{\beta_\ell : \ell \in u\}$ <u>then</u> $\mathbf{c}\{\alpha_\ell : \ell < n\} = \mathbf{c}\{\beta_\ell : \ell < n\}$.
3) If $m < n < \omega, \mu = \mu^{<\mu}$, <u>then</u> (μ^{+n}, μ^{+m}) has $(m+3)$-simple identities in $\mathbf{V}^{\mathbb{P}}$ for appropriate μ^+-c.c. $(< \mu)$-complete forcing notion.

Proof. 1) For any $c : [\aleph_n]^{<\aleph_0} \to \omega$ we define a forcing notion $\mathbb{P} = \mathbb{P}_c$ as follows:
$p \in \mathbb{P}$ <u>iff</u>:

(a) $p = (u, f) = (u^p, f^p)$

[4] Of course the needed version of MA is quite weak; going more deeply in [Sh 522]. There original version say 2-simplicity and Peter Komjath note that its proof was wrong.

(b) u is a finite subset of $\aleph_n$

(c) f is a function from $[u]^{\leq 3}$ to ω

(d) if $k < \omega, k \geq 3$ and $\alpha_0, \ldots, \alpha_{k-1}$ are from u with no repetitions $\beta_0, \ldots, \beta_{k-1}$ are from u with no repetitions and $[\ell(0) < \ell(1) < \ell(2) < k \Rightarrow f(\{\alpha_{\ell(0)}, \alpha_{\ell(1)}, \alpha_{\ell(2)}\}) = f(\{\beta_{\ell(0)}, \beta_{\ell(1)}, \beta_{\ell(2)}\})]$, then $c(\{\alpha_0, \ldots, \alpha_{k-1}\}) = c(\{\beta_0, \ldots, \beta_{k-1}\})$.

The rest should be clear.
2), 3) Similar (use e.g. [Sh 546]). $\square_{5.12}$

We can give an alternative proof of [Sh 49], note that by absoluteness the assumption MA is not a real one; it can be eliminated and $(\mu^{+\omega}, \mu) \rightarrow' (2^{\aleph_0}, \aleph_0)$ can be deduced.

We further can ask:

5.13 Question: Assume $\Gamma_i \subseteq \mathrm{ID}^*$ for $i < i^*, \mathbb{P}$ is $\Pi\{\mathbb{P}^\lambda_{\Gamma_i} : i < i^*\}$ with finite support, $c : [\aleph_{n(*)}]^2 \rightarrow \omega$ in $\mathbf{V}^{\mathbb{P}}$ then $\mathrm{ID}(c)$ is not too far from some $\bigcup\limits_{i \in w} \Gamma_i, w \subseteq i^*$ finite.

5.14 Discussion: We can look more at ordered identities (recall)

$(*)_1$ for $\mathbf{c}_i : [\lambda]^{<\aleph_0} \rightarrow \mu$ let $\mathrm{OID}(c) = \{(a, e) : a$ a set of ordinals and there is an ordered preserving $f : a \rightarrow \lambda$ such that $b_1 e b_2 \Rightarrow \mathbf{c}(f''(b_1)) = \mathbf{c}(f''(b_2))\}$ and $\mathrm{OID}(\lambda, \mu) = \{(n, e) : (n, e) \in \mathrm{OID}(\mathbf{c})$ for every $\mathbf{c} : [\lambda]^{<\aleph_0} \rightarrow \mu\}$, and similarly OID_2, OID_k.

Of course,

$(*)_2$ $\mathrm{ID}(\lambda, \mu)$ can be computed from $\mathrm{OID}(\lambda, \mu)$.

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