FORCING MANY POSITIVE POLARIZED PARTITION RELATIONS BETWEEN A CARDINAL AND ITS POWERSET

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Abstract. A fairly quotable special, but still representative, case of our main result is that for $2 \leq n < \omega$, there is a natural number $m(n)$ such that, the following holds. Assume GCH: If $\lambda < \mu$ are regular, there is a cofinality preserving forcing extension in which $2^\lambda = \mu$ and, for all $\sigma < \lambda \leq \kappa < \eta$ such that $\eta^{(+m(n)-1)} \leq \mu$,

$$((\eta^{(+m(n)-1)})_\sigma) \to ((\kappa)_\sigma)^{(1)n}_\eta.$$

This generalizes results of [3], Section 1, and the forcing is a "many cardinals" version of the forcing there.

§0. INTRODUCTION.

In [3], the first author proved (with, in what follows, μ in the place of our λ, and λ in the place of our η) the consistency of:

$$\lambda < \kappa < \eta \text{ are all regular, } 2^\lambda = \eta,\ \eta \to (\eta, [\kappa; \kappa])$$

The forcing can be thought of as a "filtering through" κ of adding η many Cohen subsets of λ. Then, $\{\lambda,\ \kappa,\ \eta\}$ can be thought of as a three element set K of regular cardinals used for defining the forcing; the elements of K are taken, in the ground model, to be sufficiently far apart. An important technical notion, related to the idea of "filtering through" is the possibility of viewing $p \leq q$ as split up, in various ways, into "pure" and "apure" extensions. Schematically, but fairly accurately, the pure extensions have completeness properties, while the apure extensions have chain condition properties: see (1.7) for the former and (1.8), (1.9) for the latter.

It is natural to attempt to allow the set K of regular cardinals to be larger, and to simultaneously obtain many such, and stronger, partition relations, for example, by increasing the "dimension" (from 2 to n) and the

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number of blocks (from 2 to σ). These will all be aspects of our treatment here, see (B), below, and (c) of our main Theorem.

More specifically, we start, in V, from

$$\text{(A) } cf\ \lambda = \lambda = \lambda^{<\lambda} < \mu = \mu^{\lambda} = cf\ \mu.$$

and we fix:

(B) $K \subseteq [\lambda,\ \mu]$, a set of regular cardinals, with $\lambda,\ \mu \in K$.

In §1, we define a forcing $\mathbf{Q} = \mathbf{Q}_K$ which generalizes the forcing of [3], §1, and we prove its important properties, culminating in (1.11) and (1.12), whose statements are incorporated into our main Theorem, below (everything except item (c)).

In order to prove that $\mathbf{Q}$ preserves cofinalities we need to assume that for all cardinals, $\theta \in [\lambda,\ \mu],\ 2^{\theta} = \theta^{+}$, so the reader who is so inclined may simply assume GCH holds (in V) and make the typical GCH simplifications. Very frequently, this involves direct substitutions, such as substituting θ_κ for $(2^{<\theta_\kappa})$. However, as is usually the case, the assumption of GCH is mainly for notational convenience and to be able to state results in a simple compact form. The technical lemmas of §§1, 2 are stated in a form which makes no assumptions about cardinal exponentiation, and which indicates how the statement of the Theorem could be modified so as not to appeal to GCH at the price of allowing cardinal collapse between certain cardinals and their weak powers.

The definition of $\mathbf{Q}$ and the proofs of its properties do not require any further assumptions on K, but the proof that the polarized partition relations hold in the extension does require that we take the elements of K to be sufficiently far apart. In particular, for each $2 \le n < \omega$, there will be a natural number $m(n)$ such that (among other things), if $\kappa_1 < \kappa_2$ are successive elements of K, then $\kappa_2 \ge \kappa_1^{(+m(n)-1)}$. Thus, for given n, the "densest" possible set K consists of every $(m(n)-1)-$th regular cardinal, starting from λ, and, once again, the reader who is so inclined is invited to think only about this specific K. The statement of the result in the Abstract, above, adopts this convention on K, but the main Theorem will be stated in somewhat more general terms.

The reason for this requirement on K is that, combined with our hypotheses on cardinal exponentiation, it will guarantee that if $\kappa_1 < \kappa_2$ are successive elements of K, $\sigma < \lambda$,, then we can find $\chi,\ \tau$ with $\kappa_1 \le \chi < \tau \le \kappa_2$ such that whenever $\kappa < \kappa_1,\ \tau \to (\kappa)^{m(n)}_{\chi}$. Indeed, this will hold if we take $\kappa_1 = \chi$ and $\tau = \kappa_2$, and the statement of the result in the Abstract reflects these choices of χ and τ. See (c) of the main Theorem, below, where these ideas are precisely formulated, in somewhat more general form.

Before stating the main theorem, it remains to define the partition symbol which figures therein. Assume that τ is a cardinal, and that $(X_i : i < \sigma)$ is a pairwise disjoint family of sets each of cardinality (at least) τ. Let $X = \bigcup\{X_i : i < \sigma\}$, and let D be the set of $n-$ element subsets of X which meet each X_i in at most one element. For $a \in D$, let $a' = \{i < \sigma : a \cap X_i \neq \emptyset\}$. Then,

$$((\tau)_\sigma) \to ((\kappa)_\sigma)_\chi^{((1)_n)}$$

holds iff for all such $(X_i : i < \sigma)$, whenever F is a function from D to χ, for $i < \sigma$, there is $Y_i \subseteq X_i$, of cardinality (at least) κ such that, letting $Y = \bigcup\{Y_i : i < \sigma\}$, if $a,\ b \in D \cap [Y]^n$, and $a' = b'$, then $F(a) = F(b)$ (i.e., $(Y_i : i < \sigma)$ is canonical for F).

In addition to the above considerations, the dependence of $m(n)$ on n is related to the results of [5]. The precise formulation of these results is deferred until (2.1), where we begin to apply them. For now, we merely formulate:

(C) Given $2 \le n < \omega$, there is $m = m(n) < \omega$, sufficiently large that there is a system as in (2.1), below.

Theorem. *If, in V, λ, μ, K are as in (A), (B), above, then there is $\mathbf{Q} = \mathbf{Q}_K = (Q, \le)$ such that the empty condition of $\mathbf{Q} \Vdash 2^\lambda \ge \mu$ and forcing with $\mathbf{Q}$ adds no sequences of length $< \lambda$. Further, assuming that in V, $2^\theta = \theta^+$ for all cardinals $\theta \in [\lambda,\ \mu]$:*

(a) *card $Q = \mu$.*
(b) *Forcing with $\mathbf{Q}$ preserves cofinalities, and therefore cardinals.*
(c) *Suppose that the cardinals $\sigma,\ \kappa,\ \kappa_1,\ \chi,\ \tau,\ \kappa_2$ satisfy the following: $\kappa_1,\ \kappa_2$ are successive elements of K, and $\sigma < \lambda,\ \kappa < \kappa_1 \le \chi = \chi^\sigma < \tau \le \kappa_2$. Let $2 \le n < \omega$ and let $m = m(n)$ be as in (C), above. If, in V, $\tau \to (\kappa)_\chi^m$, then, in $V^{\mathbf{Q}}$, $((\tau)_\sigma) \to ((\kappa)_\sigma)_\chi^{((1)_n)}$.*

Remarks.

(1) *Regarding κ, clearly the most interesting case is when $\kappa = \kappa_1$; unfortunately at this point, it is unclear whether our methods, or a small variant thereof will suffice to handle this case. We are continuing to investigate this question and also the question of whether we can allow $\sigma = \lambda$, at least under the additional assumption that λ is not strongly inaccessible.*
(2) *In order to handle all $n < \omega$ simultaneously, it is natural to use measurable cardinals and and the obvious attempt to do so works*

in a straightforward way. Some significant use of large cardinals is necessary.

(3) *We treat only the extremely dispersed case, where, in the n-tuples in the domain, each coordinate comes from a different one of the σ many blocks (the superscript $((1)_n)$). It would be very desirable to allow pairs, or more, from the same block. This paper does not address this question, but for one pair, see* [1], [2], [3], §*2,* [4], [6] *and* [7].

(4) *We began work on this paper in 1986, using essentially the same approach as presented here; this work has been subject to various interruptions which has made us decide to finally present it in its present form rather than attempt to polish off various of the small questions indicated above and to optimize the results.*

(5) *When forcing, we take $p \leq q$ to mean that q gives more information. Therefore, strictly, we should speak of cofinal sets instead of dense sets, but we will stick to the more usual terminology, modulo a reversal of the partial ordering. In particular, a predense set is one whose upward closure is a final segment of the partial ordering.*

§1. THE FORCING.

We present the forcing $\mathbf{Q}$ and develop its basic properties. As mentioned above, $\mathbf{Q}$ is a "many cardinals" generalization of the forcing of [3], §1.

(1.1) Context and Preliminaries.

Let $\lambda = \lambda^{<\lambda}$, $\mu = \mu^{\lambda}$, λ, μ both be regular. Let $K \subseteq [\lambda,\ \mu]$ be a set of regular cardinals with λ, $\mu \in K$. For the remainder of this paper, λ, μ, K are fixed.

For $\kappa \in K$, let E_κ be the equivalence relation on μ defined by $i\ E_\kappa\ j$ iff $i\ +\ \kappa = j\ +\ \kappa$. For $\lambda \leq \kappa \leq \mu$, define $E_{<\kappa}$ as $id_\mu \cup \bigcup\{E_\theta :\ \theta \in K \cap \kappa\}$. For such κ, if $\kappa \notin K$, let $E_\kappa = E_{<\kappa}$. For $i < \mu$, $\lambda \leq \kappa \leq \mu$, let $[i]_\kappa =$ the E_κ−equivalence class of i, and for $A \subseteq \mu$, let $A/E_\kappa = \{[i]_\kappa :\ i \in A\}$. For such i, A, $[i]_\kappa$ is **represented in A** iff $A \cap [i]_\kappa \neq \emptyset$. If $A \subseteq B \subseteq \mu$, the $[i]_\kappa$ **grows from A to B** iff $\emptyset \neq A \cap [i]_\kappa \neq B \cap [i]_\kappa$

(1.2) Remarks.

(1) *If $\theta < \kappa$, both in K, then E_θ refines E_κ and, in fact, each E_κ class is the union of κ many E_θ classes.*

(2) *For all i, $j < \mu$, $i\ E_\mu\ j$. Thus, the following definition makes sense:*

$$\textit{if } i < j < \mu,\ \kappa(i,\ j) = \textit{ the least } \kappa \in K \textit{ such that } i\ E_\kappa\ j.$$

(1.3) Definition and Remark.

Suppose $\kappa \in K$. We define θ_κ to be the least regular cardinal which is $\geq \bigcup[(K \cap \kappa) \cup \{\lambda\}]$. Thus, in particular, $\theta_\lambda = \lambda$, if $\theta < \kappa$ are successive elements of K then $\theta_\kappa = \theta$, if $\bigcup(K \cap \kappa)$ is singular, then $\theta_\kappa = (\bigcup(K \cap \kappa))^+$, while if $\bigcup(K \cap \kappa)$ is inaccessible, then $\theta_\kappa = \bigcup(K \cap \kappa)$.

(1.4) Definition. *$q \in Q = Q_K$ iff $q :$ dom $q \rightarrow \{0,1\}$, dom $q \subseteq \mu$ and:*

(a) *for $i < \mu$, $\kappa \in K$, card$([i]_\kappa \cap$ dom $q) < \theta_\kappa$ (note: taking $\kappa = \mu$, we have card dom $q < \theta_\mu$).*

If $p,\ q \in Q$, we set $p \leq q$ iff

(b) *$p \subseteq q$,*

(c) *For all $\kappa \in K$, $\{A \in \mu/E_\kappa : A$ grows from dom p to dom $q\}$ has power $< \theta_\kappa$.*

$\mathbf{Q} = (Q,\ \leq)$.

(1.5) Definition. *For $\kappa \in K$ and $p,\ q \in Q$, let: $p \leq^{pr}_\kappa q$ iff $p \leq q$ and:*

(d) *no $E_\kappa-$class represented in dom p grows from dom p to dom q,*

and let: $p \leq^{apr}_\kappa q$ iff $p \leq q$ and:

(e) *(dom $q)/E_\kappa =$ (dom $p)/E_\kappa$.*

(1.6) Proposition.

(a) *For all $\kappa \in K$, $\leq^{pr}_\kappa$, $\leq^{apr}_\kappa$ are partial orderings of Q.*

(b) *If $p_1,\ p_2 \in Q$ and they are compatible as functions, then $p_1 \cup p_2 \in Q$; further, letting $q = p_1 \cup p_2$, if (c) of (1.4) holds between p_i and q, for $i = 1,\ 2$, then q is the join, in $\mathbf{Q}$, of p_1 and p_2.*

(c) *If $p \leq q$, $\kappa \in K$, then there are $r,\ s \in Q$ such that:*
  - *(1) $p \leq^{pr}_\kappa r \leq^{apr}_\kappa q$,*
  - *(2) $p \leq^{apr}_\kappa s \leq^{pr}_\kappa q$ and*
  - *(3) $q = r \cup s$.*

(d) *$\leq = \leq^{apr}_\mu$ (except that if $\emptyset \neq q \in Q$, then $\emptyset \leq q$, but $\emptyset \not\leq^{apr}_\kappa q$ for any $\kappa \in K$).*

(e) *If $\kappa_0 \leq \kappa_1 \leq \kappa_2$, all $\in K$, then:*
$\leq^{pr}_{\kappa_1} \subseteq \leq^{pr}_{\kappa_0}$, $\leq^{apr}_{\kappa_1} \subseteq \leq^{apr}_{\kappa_2}$.

(f) *If ($\kappa \in K$ & $s \leq^{apr}_\kappa t$ & $s \leq^{pr}_\kappa v$), then $t \cup v \in Q$ and: $s \leq (t \cup v)$, $t \leq^{pr}_\kappa (t \cup v)$, $v \leq^{apr}_\kappa (t \cup v)$.*

(g) *If $\kappa \in K$, $p \leq^*_\kappa q_i$ $(i = 1,\ 2)$, where $* \in \{pr,\ apr\}$ and $q_1,\ q_2$ are compatible in $\mathbf{Q}$, then $p \leq^*_\kappa (q_1 \cup q_2)$.*

(h) *If $p \leq^{apr}_\kappa q_1,\ q_2$ and if*

(*) if ($i \in$ dom $q_1 \backslash$dom p & $j \in$ dom $q_2\backslash$dom p) then ($[i]_{<\kappa} \neq [j]_{<\kappa}$ or $[i]_{<\kappa} \cap$dom $q_1 = [j]_{<\kappa} \cap$dom q_2))

then also $q_k \leq^*_\kappa q_1 \cup q_2$, $k = 1,\ 2$.

(i) *If $p \leq^{apr}_\kappa q_i \leq r$ for $i = 1,\ 2$, then, for such i, $q_i \leq^{apr}_\kappa q_1 \cup q_2$.*

Proof. (a) and (b) are clear. For (c), let $r = q|x$, where $\xi \in x$ iff $\xi \in dom\ q$ and ($\xi \in dom\ p$ or $[\xi]_\kappa \cap dom\ p = \emptyset$). Also, let $s = q|y$, where $\xi \in y$ iff $\xi \in dom\ q$ and $[\xi]_\kappa \cap dom\ p \neq \emptyset$. Clearly $p \ \leq^{pr}_\kappa \ r$, $p \ \leq^{apr}_\kappa \ s$; clearly $q = r \cup s$. We verify that $r \ \leq^{apr}_\kappa \ q$ and $s \ \leq^{pr}_\kappa \ q$. For the first, suppose that $\xi \in dom\ q \setminus x$. Then, $\xi \notin dom\ p$ and $[\xi]_\kappa \cap dom\ p \neq \emptyset$. Then certainly $[\xi]_\kappa \cap x \neq \emptyset$, i.e. $\xi \in \bigcup dom\ r/E_\kappa$. For the second, suppose $\xi \in dom\ q \setminus y$, but $[\xi]_\kappa \cap y \neq \emptyset$. Then, $\xi \in x \setminus dom\ p$, so $[\xi]_\kappa \cap dom\ p = \emptyset$. If $\zeta \in [\xi]_\kappa \cap y$, then $[\zeta]_\kappa \cap dom\ p \neq \emptyset$, but $[\zeta]_\kappa = [\xi]_\kappa$, contradiction.

For (d), recall that $\mu = [i]_\mu$, for all $i < \mu$. For (e), if $p \ \leq^{pr}_{\kappa_1} \ q$ and x is an $E_{\kappa_0}-$class represented in p, let x^* be the $E_{\kappa_1}-$class such that $x \subseteq x^*$. Then, x^* is represented in p and since x^* does not grow from $dom\ p$ to $dom\ q$, neither can x. Similarly, if $p \ \leq^{apr}_{\kappa_1} r$ and $\xi \in dom\ r$, there is $\zeta\ E_{\kappa_1}\ \xi$ such that $\zeta \in dom\ p$. But then, $\zeta\ E_{\kappa_2}\ \xi$, so $\xi \in [\zeta]_{\kappa_2}$.

For (f), we first show that $(dom\ t \setminus dom\ s) \cap (dom\ v \setminus dom\ s) = \emptyset$; then, by (b), $t \cup v \in Q$. It will then be clear that $s \leq (t \cup v)$. So, if $\xi \in dom\ v \setminus dom\ s$, then $[\xi]_\kappa \cap dom\ s = \emptyset$, so $\xi \notin \bigcup dom\ s/E_\kappa$ and $dom\ t \subseteq \bigcup dom\ s/E_\kappa$.

Next, we show that $t,\ v \leq t \cup v$; by (b), it will suffice to show that (c) of (1.4) holds between t and $t \cup v$ and between v and $t \cup v$. We prove the former first. So, suppose that $\tau \in K$ and first suppose that $\emptyset \neq dom\ t \cap [i]_\tau$ and $j \in (dom\ v \cap [i]_\tau) \setminus dom\ t$. Then, certainly $j \notin dom\ s$, so since $s \leq^{pr}_\kappa v$, we clearly must have that $\tau > \kappa$. Now, let $l \in [i]_\tau \cap dom\ t$. Since $s \leq^{apr}_\kappa t$, there is $a \in dom\ s$ such that $a\ E_\kappa\ l$. But then, since $\tau > \kappa$ (actually, $\geq$ would suffice here), $a\ E_\tau\ i$, so $\emptyset \neq dom\ s \cap [i]_\tau \neq dom\ s \cap [i]_\tau$. And, since $s \leq v$, there are fewer than θ_τ many such $[i]_\tau$, and we have proved that (c) of (1.4) holds between t and $t \cup v$.

To show that (c) of (1.4) holds between v and $t \cup v$, let τ be as above, and, this time, suppose that $\emptyset \neq dom\ v \cap [i]_\tau$ and that $j \in (dom\ t \cap [i]_\tau) \setminus dom\ v$. Then, certainly $j \notin dom\ s$, and so, since $s \leq^{apr}_\kappa$, $[j]_\kappa \cap dom\ s \neq \emptyset$. Thus, $[j]_\kappa$ grows from $dom\ s$ to $dom\ t$, and, since $s \leq t$, there are at most θ_κ many such $[j]_\kappa$. We consider separately the cases $\tau \geq \kappa$ and $\tau < \kappa$. In the first case, $\theta_\kappa \leq \theta_\tau$ and we have found one of at most θ_κ many $[j]_\kappa$ inside every $[i]_\tau$ which grows from $dom\ v$ to $dom\ t \cup v$, so clearly there are at most θ_τ many such $[i]_\tau$, as required. Thus, without loss of generality, we may assume that $\tau < \kappa$. In this case, we shall argue that $\emptyset \neq dom\ s \cap [i]_\tau$. Clearly this will suffice since then $[i]_\tau$ grows from $dom\ s$ to $dom\ t$, and again, since $s \leq t$, there are at most θ_τ such $[i]_\tau$, as required.

So, suppose, towards a contradiction, that $\emptyset = dom\ s \cap [i]_\tau$. Let $\xi \in [i]_\tau \cap dom\ v$, so $[\xi]_\kappa = [i]_\kappa$. But $j \in [i]_\tau \cap dom\ t$, so $j \in [i]_\kappa \cap dom\ t$. Since $s \leq^{apr}_\kappa t$, this means that $\emptyset \neq [i]_\kappa \cap dom\ s$. But then $\xi \in [i]_\kappa \cap (dom\ v \setminus dom\ s)$. This, however, is impossible, since $s \leq^{pr}_\kappa v$, which completes the proof.

We proceed, now to show that $t \leq^{pr}_\kappa t \cup v$ and that $v \leq^{apr}_\kappa t \cup v$. For the former, suppose that $\xi \in dom\ v \setminus dom\ t$. Then, $\xi \in dom\ v \setminus dom\ s$, so $[\xi]_\kappa \cap dom\ s = \emptyset$. We claim that $[\xi]_\kappa \cap dom\ t = \emptyset$. If not, and $\zeta \in [\xi]_\kappa \cap dom\ t$, then $[\zeta]_\kappa \cap dom\ s \neq \emptyset$, but, once again, $[\zeta]_\kappa = [\xi]_\kappa$, contradiction. Thus, $t \leq^{pr}_\kappa (t \cup v)$.

To see that $v \leq^{apr}_\kappa t \cup v$, suppose that $\xi \in dom\ t \setminus dom\ v$. We need to show that $[\xi]_\kappa \cap dom\ v \neq \emptyset$. This, however, is clear, because, since $\xi \in dom\ t$, $[\xi]_\kappa \cap dom\ s \neq \emptyset$, so certainly $[\xi]_\kappa \cap dom\ v \neq \emptyset$, and we have finished proving (f).

For (g), first note that if $q_i \leq r$ for $i = 1,\ 2$, then, letting $s = q_1 \cup q_2$, for such i, $q_i \leq s \leq r$. This is clear, because if $\tau \in K$ and $[j]_\tau$ grows from $dom\ q_i$ to $dom\ s$, then certainly $[j]_\tau$ grows from $dom\ q_i$ to $dom\ r$, and there are at most θ_τ such $[j]_\tau$, since $q_i \leq s$. Further, if $[j]_\tau$ grows from $dom\ s$ to $dom\ r$, then either $[j]_\tau$ grows from $dom\ q_1$ to $dom\ r$ or $[j]_\tau$ grows from $dom\ q_2$ to $dom\ r$, and again, since $q_1,\ q_2 \leq r$, there are at most θ_τ such $[j]_\tau$ for each case.

Now suppose that $*$ is *apr*. Thus, if $\xi \in dom\ s$, then, for an $i \in \{1,\ 2\}$, $\xi \in dom\ q_i$, so $[\xi]_\kappa \cap dom\ p \neq \emptyset$. It is then clear that $p \leq^*_\kappa s$, as required.

If $*$ is *pr* and $\xi \in dom\ s \setminus dom\ p$, then, letting $i \in \{1,\ 2\}$ be such that $\xi \in dom\ q_i \setminus dom\ p$, then, since $p \leq^{pr}_\kappa q_i$, clearly $[\xi]_\kappa \cap dom\ p = \emptyset$, as required.

We prove (i), before proving (h). As in (g), let $s = q_1 \cup q_2$. For $i = 1,\ 2$, we must show that $q_i \leq^{apr}_\kappa s$. We already know, from the proof of (g), that for such i, $q_i \leq s$. So, let $j = 1+(2-i)$, and suppose that $\alpha \in dom\ s \setminus dom\ q_i$. We need to show that $\emptyset \neq [\alpha]_\kappa \cap dom\ q_i$. But $\alpha \in dom\ q_j \setminus dom\ q_i$, so $\alpha \in dom\ q_j \setminus dom\ p$, so $\emptyset \neq [\alpha]_\kappa \cap dom\ p$, and the conclusion is clear.

We conclude by proving (h). For this, let $s = q_1 \cup q_2$. If we prove that q_1 and q_2 are compatible in $\mathbf{Q}$, then, by (g) and (i), we are finished. In fact, we will show directly that $q_1,\ q_2 \leq s$. By symmetry, it will suffice to prove that $q_1 \leq s$, and clearly, only (c) of (1.4) is at issue. So, let $\tau \in K$. First note that, without loss of generality, we may assume that $\tau < \kappa$. This is because, since $dom\ q_1 \setminus dom\ p$ and $dom\ q_2 \setminus dom\ p$ both have cardinality less than θ_κ, therefore so do $dom\ q_1 \setminus dom\ q_2$ and $dom\ q_2 \setminus dom\ q_1$. Then, if $\tau \geq \kappa$, in particular, fewer than θ_κ many E_τ classes grow from $dom\ q_1$ to $dom\ s$.

So, suppose $\tau > \kappa$. By hypothesis, if $[i]_\tau$ grows from $dom\ q_1$ to $dom\ s$, then $[i]_\tau \cap dom\ q_1 = [i]_\tau \cap dom\ p \neq \emptyset$, and so $[i]_\tau$ grows from $dom\ p$ to $dom\ q_2$. However, since $p \leq q_2$, there are fewer than θ_τ such $[i]_\tau$. This concludes the proof of (h) and of the Proposition.

(1.7) Proposition.

(a) *For all* $\kappa \in K$, $(Q,\ \leq^{pr}_\kappa)$ *is* κ*-complete.*
(b) $\mathbf{Q}$ *is* λ*-complete.*

Proof. The proof is routine and left to the reader.

In Proposition (1.8), which follows, we will have $\kappa \in K$ and $p \in Q$, and we introduce $\mathbf{Q}^{\mathbf{apr}}_{\kappa,\ \mathbf{p}} = (Q^{apr}_{\kappa,\ p},\ \leq^{apr}_{\kappa})$, and $Q^{apr}_{\kappa,\ p} = \{q : p\ \leq^{apr}_{\kappa}\ q\}$.

(1.8) Proposition. *If $\kappa \in K,\ p \in Q$, then $\mathbf{Q}^{\mathbf{apr}}_{\kappa,\ \mathbf{p}}$ has the $(2^{<\theta_\kappa})^+ -$ c.c..*

Proof. We should note, here, immediately, that in virtue of (1.6), (i), for $q_1,\ q_2 \geq p$, compatibility in $\mathbf{Q}^{\mathbf{apr}}_{\kappa,\ \mathbf{p}}$ is the same as compatibility in $\mathbf{Q}$, so it is the latter that we shall establish, when our statement calls for the former.

Suppose, now, that $q_i \in Q^{apr}_{\kappa,\ p}$, for $i < (2^{<\theta_\kappa})^+$. We show there is $I \subseteq (2^{<\theta_\kappa})^+$ with *card* $Y = (2^{<\theta_\kappa})^+$, such that for $i,\ j \in I,\ q_i$ and q_j are compatible in $\mathbf{Q}$. In virtue of the preceding paragraph, clearly this suffices.

For $i < (2^{<\theta_\kappa})^+$, let $d_i = dom\ q_i \setminus dom\ p$. We first show that *card* $d_i < \theta_\kappa$. Note that by (e) of (1.5), if $\alpha \in d_i$, then $[\alpha]_\kappa$ grows from *dom* p to *dom* q, and so $d_i/E_\kappa \subseteq \{A \in \mu(\kappa) :\ A$ grows from *dom* p to *dom* $q\}$. By (1.4), (c), this last set has power $< \theta_\kappa$. Finally, by (1.4), (a), for all $A \in d_i/E_\kappa$, *card* $(A \cap dom\ q_i) < \theta_\kappa$. Then, since θ_κ is regular, the conclusion that *card* $d_i < \theta_\kappa$ is clear.

Now, taking $Y_i := d_i/E_{<\kappa}$. Since each Y_i has power $< \theta_\kappa$, it is quite straightforward to conclude, combining typical Δ-system arguments with appeals to (1.6) (b) and (h).

We need a slightly more refined version of this.

(1.9) Proposition. *Suppose $\kappa \in K,\ (2^{<\theta_\kappa})^+ \leq \kappa,\ (s_i : i < i^*)$ is a $\leq^{pr}_{\kappa}$ $-$increasing sequence from Q, and suppose that for $i < i^*,\ s_i\ \leq^{apr}_{\kappa}\ t_i$, and that for $j < i < i^*,\ t_j,\ t_i$ are incompatible in $\mathbf{Q}$. Then, $i^* < (2^{<\theta_\kappa})^+$.*

Proof. If $i^* < \kappa$, we can take $s = \bigcup\{s_i : i < i^*\}$. Noting that for $j < i^*,\ s\ \leq^{apr}_{\kappa}\ (s \cup t_j)$, we can then apply (1.8). Even if $\kappa \leq i^*$, we can **essentially** argue in this fashion, by redoing the proof of (1.8). So, let $i^* = (2^{<\theta_\kappa})^+ \leq \kappa$. Let $d_i = dom\ t_i \setminus \bigcup\{dom\ s_i : i < i^*\}$. We obtain a contradiction. Then, $d_i \subseteq dom\ t_i \setminus dom\ s_i$, and, arguing as in (1.8), *card* $d_i \subseteq$ *card* $(dom\ t_i \setminus dom\ s_i) < \theta_\kappa$.

As in (1.8), for $i < i^*$, let $Y_i = d_i(< \kappa)$. Once again, we can find $I \subseteq i^*,\ Y,\ d$, and f such that *card* $I = i^*$ and for $i,\ j \in I,\ Y_i \cap Y_j = Y,\ d_i \cap \bigcup Y = d$ and $t_i \restriction d = f$. The conclusion is then as in (1.8) that for $i,\ j \in I,\ t_i$ and t_j are compatible in $\mathbf{Q}$ and therefore in $\mathbf{Q}^{\mathbf{apr}}_{\kappa,\ \mathbf{p}}$. This contradiction completes the proof of the Proposition.

(1.10) Lemma. *If $\kappa \in K,\ 2^{<\theta_\kappa} < \kappa,\ p \in Q$ and $p \mid \vdash_{\mathbf{Q}}$ "$\dot\alpha$ is an ordinal", THEN, there are q and $(r_i : i < i^*)$, all from Q, such that:*

(a) $i^* < (2^{<\theta_\kappa})^+$,
(b) $p\ \leq^{pr}_{\kappa}\ q$,
(c) $q\ \leq^{apr}_{\kappa}\ r_i$, *for all* $i < i^*$,
(d) *for some* $\alpha_i,\ r_i \mid \vdash_{\mathbf{Q}}$ "$\dot\alpha = \alpha_i$",
(e) $\{r_i : i < i^*\}$ *is predense above* q.

Proof. We shall obtain q as $q_{i^*} = \bigcup\{q_i : i < i^*\}$, where $(q_i : i < i^*)$ is $\leq^{pr}_\kappa$ −increasing, with $q_0 = p$. We work by recursion on i. Having obtained $(q_j : j \leq i)$ and $(r'_j : j < i)$ such that $(q_j : j \leq i)$ is $\leq^{pr}_\kappa$ −increasing, the $(r'_j : j < i)$ are pairwise incompatible in $\mathbf{Q}$, $q_j \ \leq^{apr}_\kappa \ r'_j$ and there is α_j such that $r'_j| \vdash_{\mathbf{Q}}$ "$\dot\alpha = \alpha_j$", note that we have the following properties:

(1) for all $j < i$, $q_i \ \leq^{apr}_\kappa \ (q_i \cup r'_j)$ (this is by (f) of (1.6) with $s = q_i,\ t = r'_j,\ v = q_i$),
(2) so, letting $r''_j = q_i \cup r'_j$, $\{r''_j : j < i\} \subseteq Q^{apr}_{\kappa,\ q_i}$.

If $\{r''_j : j < i\}$ is predense in $\mathbf{Q^{apr}_{\kappa,\ q_i}}$, then we take $i^* = i,\ q = q_i,\ r_j = r''_j$, for $j < i$, and we stop. Otherwise, there is $q' \in Q^{apr}_{\kappa,\ q_i}$ such that q' is imcompatible with each r''_j. Note that, in this case, we must have that q' is incompatible in $\mathbf{Q}$ with each r'_j, by (g) of (1.6). In this case, we shall have $i < i^*$, and we continue, so fix such q' and let $q' \leq r'$ be such that for some α, $r'| \vdash_{\mathbf{Q}}$ "$\dot\alpha = \alpha$". Applying (c) of (1.6), we get $q_i \ \leq^{pr}_\kappa \ q^* \ \leq^{apr}_\kappa \ r'$. We let $q_{i+1} = q^*,\ r'_i = r'$. By (g) of (1.6), the r'_j $(j \leq i)$ are pairwise incompatible in $\mathbf{Q}$.

If i is a limit ordinal, $i < \kappa$ and the $(q_j : j < i)$, $(r'_j : j < i)$ are definied satisfying the induction hypotheses, we let $q_i = \bigcup\{q_j : j < i\}$ (so, by (1.7), $q_i \in Q$ and is the $\leq^{pr}_\kappa$ −lub of the q_j). We must now see that the process terminates at some $i^* < (2^{<\theta_\kappa})^+$. If not, and if $(2^{<\theta_\kappa})^+ < \kappa$, let $q = \bigcup\{q_j : j < (2^{<\theta_\kappa})^+\}$, and (using the above observations), for $j < (2^{<\theta_\kappa})^+$, let $r_j = r'_j \cup q$. Then, the r_j are a pairwise incompatible family in $\mathbf{Q^{apr}_{\kappa,\ q}}$, contradicting (1.8). If $(2^{<\theta_\kappa})^+ \leq \kappa$, it is straigtforward to see that we must have $i^* < (2^{<\theta_\kappa})^+$, contradiction. This means, in particular, that $i^* < \kappa$ and then we conclude by defining q and the r_j as in the case where $(2^{<\theta_\kappa})^+ < \kappa$, but everywhere replacing $(2^{<\theta_\kappa})^+$ by i^*. This completes the proof of the Lemma.

(1.11) Proposition. *The empty condition of* $\mathbf{Q}$ *forces* $2^\lambda \geq \mu$.

Proof. For $i < \mu$, let $\mathbf{r}_i$ be the following $\mathbf{Q}$-name: $\{((\gamma,\ k),\ p) : \gamma < \lambda,\ k < 2,\ p \in Q\ \&\ p(\lambda i + \gamma) = k\}$. Since $\theta_\lambda = \lambda$, and for $i < \mu$, $[\lambda i,\ \lambda i + \lambda) = [\lambda i]_\lambda$, we clearly have that for $p \in \mathbf{Q}$, if $i_0 < i_1 < \mu, card\ A_j < \lambda$, for $j = 0,\ 1$, where, for such j, $A_j = \{\gamma < \lambda :\ \lambda i_j + \gamma \in dom\ p\}$. So, for such $p,\ i_0,\ i_1$, choosing $\gamma \in \lambda \setminus (A_0 \cup A_1)$, and letting $q = p \cup \{(\lambda i_j + \gamma,\ j) :\ j < 2\}$, we have $p \leq q$ and $q \Vdash \mathbf{r}_{i_0} \neq \mathbf{r}_{i_1}$, and the conclusion is then clear. This completes the proof of the Proposition.

(1.12) Proposition. *(Assuming that for cardinals* θ*, with* $\lambda \leq \theta \leq \mu$, $2^\theta = \theta^+$*):*

(a) *card* $Q = \mu$.
(b) *Forcing with* $\mathbf{Q}$ *adds no sequences of length* $< \lambda$.
(c) *Forcing with* $\mathbf{Q}$ *preserves cofinalities, and therefore cardinals.*

Proof. (a) is clear, and (b) follows easily, from (1.7), (b). For (c), assume, towards a contradiction, that $\tau < \sigma$, where both are regular, but that for some $q \in Q$, $q \Vdash cf\ \sigma = \tau$. By (b), we may assume that $\tau \geq \lambda$. Note that by (1.6), (d) and (1.8), with $p = \emptyset$, $\mathbf{Q}$ has the $(2^{<\theta_\mu})^+$-c.c. Further, under our additional hypotheses on cardinal exponentiation, $(2^{<\theta_\mu}) \leq \mu$, so, clearly we cannot have $\sigma > \mu$. But then there must be $\kappa \in K$ such that $\theta_\kappa \leq \tau < \kappa$. Suppose, now, that the $\mathbf{Q}$-name $\mathbf{f}$ is such that $q \Vdash \mathbf{f}$ is monotone-increasing, maps τ to σ and has range cofinal in σ. By (1.7), (a) and (1.10), applying (1.10) repeatedly to each of the names $\mathbf{f}(\alpha)$, for $\alpha < \tau$, we reach a contradiction, also using that $(2^{<\theta_\kappa}) \leq \tau$. This completes the proof of the Proposition.

§2. THE PARTITION RELATIONS.

In this section, we address item (c) of the main theorem of the Introduction. We work, first, under the simplifying assumption that $\tau < \kappa_2$. For the convenience of the reader, we will recall the context, and restate (c) as a Lemma, with this additional assumption. After the proof of the Lemma is given, we will briefly indicate the small changes necessary to accomodate the case $\tau = \kappa_2$.

So, let $\kappa_1,\ \kappa_2$ be successive members of K, let $\kappa < \kappa_1 \leq \chi = \chi^\sigma < \tau < \kappa_2$, let $\sigma < \lambda$. Assume that $2^{<\kappa_1} \leq \tau$ (in the context of (c) of the main Theorem, this will follow from the Theorem's hypotheses on cardinal exponentiation). As stated in (C) of the Introduction, for all $2 \leq n < \omega$, by examination of the methods of [5], there is sufficiently large $m(n) < \omega$ such that, assuming that, in V, $\tau \to (\kappa_1)^{m(n)}_\chi$, then, also in V, there is a system as in (2.1) below.

Lemma. *For $2 \leq n < \omega$, if, in V, $\tau \to (\kappa)^{m(n)}_\chi$, then, in $V^{\mathbf{Q}}$, $((\tau)_\sigma) \to ((\kappa)_\sigma)^{((1)_n)}_\chi$.*

Proof. Let $(A_i : i < \sigma)$ be a sequence of sets of ordinals, each of order-type τ, such that for $i < j < \sigma$, $A_i < A_j$. Let $A := \bigcup\{A_i : i < \sigma\}$. Let $D := \{a \in [A]^n :\ card(a \cap A_i) \leq 1$, for all $i < \sigma\}$. We often view the elements of D as n-tuples, enumerated in their increasing order. Let $\mathbf{c}$ be a $\mathbf{Q}$-name for a function from D to χ.

Let $p \in Q$. Using the methods of §1, we can find a $\leq^{pr}_{\kappa_2}$ -increasing sequence from Q, $\vec{p} = (p_j : j < \eta)$, with the following properties:

(1) $\eta \leq \tau$, and $p_0 = p$,
(2) for each $\vec{\alpha} = (\alpha_1,\ \cdots,\ \alpha_n) \in D$, there is $j = j_{\vec{\alpha}}$ such that in $Q^{apr}_{\kappa_2,\ p_{j+1}}$, there is a predense set, $I_{\vec{\alpha}}$ of conditions deciding $\mathbf{c}(\vec{\alpha})$.

(2.1) The system of [5]**.**

Now, let ν^* be a sufficiently large regular cardinal. Fix $<^*$, a well-ordering of H_{ν^*}. For sequences $(X_t : t \in I)$, let $u \in J(X_t : t \in I)$ iff $u \subseteq \bigcup\{X_t : t \in I\}$, *card* $u \leq n$ and for all $t \in I$, *card* $(X_t \cap u) \leq 1$. If $u, v \in J(X_t : t \in I)$, we set $u \sim v$ iff for all $t \in I$, *card* $(X_t \cap u) =$ *card* $(X_t \cap v)$. By [5] (and our choice of $m(n)$), we have the following.

Proposition. *There are* $B_i \in [A_i]^\kappa$ $(i < \sigma)$, $\mathcal{N}_u$ $(u \in J(B_i : i < \sigma)$, *and* $h_{u,\, v}$ $(u, v \in J(B_i : i < \sigma),\ u \sim v)$ *satisfying:*

(3) *each* $\mathcal{N}_u \prec (H_{\nu^*}, \in, <^*, \mathbf{c}, \vec{p}, \chi, \kappa_1, \kappa_2, K, \mathbf{Q}, (I_{\vec{\alpha}} : \alpha \in D), \cdots)$,
(4) *letting* $N_u = |\mathcal{N}_u|$, $N_u \cap (\bigcup\{B_i : i < \sigma\}) = u$, $\chi \subseteq N_u$, *card* $N_u = \chi$, $N_u^{<\chi} \subseteq N_u$,
(5) $N_u \cap N_v \subseteq N_{u\cap v}$ *(large cardinals are required for* $=$ *in place of* $\subseteq$*),*
(6) $(h_{u,\, v} :\ u, v \in J(B_i : i < \sigma),\ u \sim v)$ *is a commutative system of isomorphisms,* $h_{u,\, v} : \mathcal{N}_u \to \mathcal{N}_v$,
(7) *If* $u_k \sim v_k$, *for* $k = 1, 2$, *then* $h_{u_1,\, v_1}$ *and* $h_{u_2,\, v_2}$ *are compatible functions, when both are defined.*

(2.2) Completing the Proof.
In this subsection, we complete the proof of the Lemma. Note that our hypothesis that $\tau < \kappa_2$ guarantees that $\eta < \kappa_2$. This is the only use we make of the hypothesis that $\tau < \kappa_2$.

So, let $p^* = \bigcup\{p_j : j < \eta\}$. Then, since here, we have that $\eta < \kappa_2$, $p^* \in Q$, and is the $\leq^{pr}_{\kappa_2}$ least upper bound of the p_j, by (1.7), (a). Also, let $\gamma_i = min\ B_i$ $(i < \sigma)$, and let $J := J(\{\gamma_i\} :\ i < \sigma)$, $\tilde{J} := J(B_i :\ i < \sigma)$.

Claim 1. *If* $q \in Q^{apr}_{\kappa_2,\, p^*}$, $u \in J$, *and* $q \in N_u$, *then* $(dom\ q) \setminus (dom\ p^*) \subseteq N_u$.

Proof of Claim 1. $(dom\ q) \setminus (dom\ p^*) \in N_u$ and it has power $< \theta_{\kappa_2} = \kappa_1 \leq \chi \subseteq N_u$, so the conclusion is clear.

Claim 2. *There is* $r \in Q$, $p^* \leq^{apr}_{\kappa_2} r$ *such that:*

(1) $dom\ r \setminus dom\ p^* \subseteq \bigcup\{N_u :\ u \in J\}$
(2) *for all* $u \in J$, $p^* \cup (r|N_u) \in N_u$*; if further, card* $u = n$, *then* $p^* \cup r \restriction N_u$ *decides the value of* $\mathbf{c}(u)$.

Proof of Claim 2. Note that for the first part of (2), it suffices to have $r \restriction N_u \in N_u$, since $p^* \in N_\emptyset$. Note, also, that J has power σ, and so we enumerate J as $(u_j :\ j < \sigma)$. We shall define by recursion on $j \leq \sigma$ a sequence $(r_j :\ j \leq \sigma)$ with $r_0 := p^*$, and all $r_j \in Q^{apr}_{\kappa_2,\, p^*}$. We shall have $r := r_\sigma$. The following induction hypotheses will be in vigor, for $j \leq \sigma$.

The parallel with items (1) and (2) in the statement of the Claim should be clear.

(a) if $k+1 \leq j$ then $dom\ r_{k+1} \setminus dom\ r_k \subseteq N_{u_k}$,
(b) for all $u \in J$ and all $k \leq j$, $r_k \restriction N_u \in N_u$; if, further, $k+1 \leq j$ and $card\ u_k = n$, then $p^* \cup (r_{k+1} \restriction N_{u_k})$ decides the value of $\mathbf{c}(u_k)$
(c) $(r_k :\ k \leq j)$ is $\leq^{apr}_{\kappa_2}$ increasing.

Clearly (a) - (c) hold for $j = 0$ with $r_0 = p^*$. At limit ordinals, $\delta \leq \sigma$, we shall take $r_\delta :\ = \ \bigcup\{r_j :\ j < \delta\}$. If $\delta < \sigma$, then $\delta < \lambda \leq \kappa_1 = \theta_{\kappa_2}$. Thus, if $\delta < \sigma$, by (1.7), (b), $r_\delta \in Q$ and is the $\leq$ least upper bound of the r_j. Then, clearly also it is the $\leq^{apr}_{\kappa_2}$ least upper bound of the r_j.

If $\delta = \sigma$, then, since we are assuming $\sigma < \lambda$, we also have $\delta < \lambda$, and so, the same arguments yield the same conclusions, in this case as well.

Clearly this preserves (a), (c) and the second part of (b). We argue that it also preserves the first part of (b). So, let $u \in J$. We must see that $r_\delta \restriction N_u \in N_u$. But $r_\delta \restriction N_u = \bigcup\{r_k \restriction N_u :\ k < \delta\}$, and for all $k < \delta$, by (the first part of) (b) for (k), $r_k \restriction N_u \in N_u$. Finally, $\delta < \sigma$, and $N^\sigma_u \subseteq N_u$ and so the conclusion is clear.

So, suppose we have defined $(r_k :\ k \leq j)$ satisfying (a) - (c). We define r_{j+1} and show that (a) - (c) are preserved. Since (b) clearly corresponds to (2), and since we take $r = r_\sigma$, this will complete the proof, once we show how (1) follows from (a). This, however, is easy, since $(dom\ r) \setminus (dom\ p^*) = \bigcup\{(dom\ r_{k+1}) \setminus (dom\ r_k) :\ k < \sigma\}$, and by (a), this last is indeed included in $\bigcup\{N_u :\ u \in J\}$.

For (c) it will suffice to have $r_j \leq^{apr}_{\kappa_2} r_{j+1}$, which will be clear from construction, as will the second part of (b). Thus, we must show that there is q satisfying:

(α) $r_j \leq^{apr}_{\kappa_2} q$,
(β) $(dom\ q) \setminus (dom\ r_j) \subseteq N_{u_j}$,
(γ) if $card\ u_j = n$, then q decides the value of $\mathbf{c}(u_j)$,
(δ) for all $u \in J$, $q \restriction N_u \in N_u$.

We first argue that it will suffice to find q satisfying (α) - (γ), since any such q will automatically satisfy (δ). For this, note that if q satisfies (α), then $(dom\ q) \setminus (dom\ r_j)$ has power $< \theta_{\kappa_2} = \kappa_1 \leq \chi$. Thus, for $u \in J$, $(q \setminus r_j) \restriction N_u$ is a subset of N_u of power $< \chi$ and therefore, $(q \setminus r_j) \in N_u$. But $q \restriction N_u = (q \setminus r_j) \restriction N_u \cup r_j \restriction N_u$, and by induction hypothesis, (b), for j, $r_j \restriction N_u \in N_u$. The conclusion is then clear.

To find q satisfying (α) - (γ) is trivial if $card\ u_j < n$, so assume $card\ u_j = n$. Applying induction hypothesis (b), with $k = j$ and $u = u_j$, we have $r_j \restriction N_{u_j} \in N_{u_j}$. Since the maximal antichain in $\mathbf{Q}^{apr}_{\kappa_2,\ p^*}$ deciding $\mathbf{c}(u_j)$ is a member of N_{u_j}, and since $p^* \in N_{u_j}$, we easily find $q' \in N_{u_j}$ such that $p^* \cup r_j \restriction N_{u_j} \leq^{apr}_{\kappa_2} q'$ and such that q' decides the value of $\mathbf{c}(u_j)$. Note that, again, since $(dom\ q')\setminus(p^*\cup(dom\ r_j\restriction N_{u_j}))$ has small cardinality, compared to the closure of N_{u_j}, we will also have $(dom\ q')\setminus(dom\ p^*)\subseteq N_{u_j}$. But this makes it clear that if we take $q\ :=\ q'\cup r_j$, then q is as required. This completes the proof of Claim 2.

Now, let:

$$r^*\ :=\ p^*\cup\bigcup\{h_{u,\ v}((r\setminus p^*)\restriction N_u):\ u\in J,\ v\in\tilde{J},\ u\sim v\}.$$

We will show that $r^*\in Q$ and that whenever $u\in J,\ v\in\tilde{J}$ and $u\sim v,\ p^*\cup h_{u,\ v}(r\restriction N_u)\le r^*$. We first note that this suffices for the proof of the Lemma in our special case, since then clearly r^* forces that $(B_i:\ i<\sigma)$ is as required.

The following is the heart of the matter, and is an easy consequence of (7) of (2.1), and the arguments for the first part of (2) of Claim 2, above.

Proposition. *Suppose that for $k=1,\ 2,\ u_k\in J, v_k\in\tilde{J}$ and $u_k\sim v_k$. Let $N_k\ :=\ N_{u_k},\ N\ :=\ N_1\cap N_2$ and let $\tilde{N}=N_{u_1\cap u_2}$ (so that, by (5) of (2.1), $N\subseteq\tilde{N}$). Let $h_k\ :=\ h_{u_k,\ v_k}$. Then, $(r\setminus p^*)\restriction N\in N$ & $h_1((r\setminus p^*)\restriction N)=h_2((r\setminus p^*)\restriction N)$.*

Proof. To see that $(r\setminus p^*)\restriction N\in N_k$, we argue as in the proof of Claim 2: $(r\setminus p^*)\restriction N$ is a subset of N_k of small cardinality compared to the closure of N_k. But then, since h_1 and h_2 are compatible functions, by (7) of (2.1), the conclusion is clear.

Corollary. *$r^*\in Q$ and whenever $u\in J,\ v\in\tilde{J}$ and $u\sim v,\ p^*\cup h_{u,\ v}(r\restriction N_u)\le r^*$.*

Proof. It is immediate from the Proposition, that the $p^*\cup h_{u,\ v}((r\setminus p^*))$ are pairwise compatible as functions. To complete the proof that $r^*\in Q$, we must verify (a) of (1.4). So, suppose $i<\mu,\ \nu\in K$.

We consider separately the cases $\nu>\kappa_2,\ \nu<\kappa_2$, and the hardest case, $\nu=\kappa_2$. If $\nu>\kappa_2$, then $\kappa_2\le\theta_\nu$ and we taking the union of fewer than θ_ν conditions, so there is no problem. If $\nu<\kappa_2$, then $\theta_\nu<\kappa_1\le\chi$, so for all $v\in\tilde{J}$, either $[i]_\nu\subseteq N_v$ or $[i]_\nu\cap N_v=\emptyset$, and then the conclusion is also easy. So, suppose that $\nu=\kappa_2$, i.e., $\theta_\nu=\kappa_1$. It is here that we use that $\kappa<\kappa_1$; this permits us to argue as in the case where $\nu>\kappa_2$: we are taking the union of fewer than θ_ν conditions, and there is no problem.

To complete the proof of the Corollary, we must see that (c) of (1.4) holds (since (b) is clear). So, once again, assume $\nu\in K$. We must see that for all $u\in J,\ v\in\tilde{J}$ such that $u\sim v$, there are fewer than θ_ν many $A\in\mu/E_\nu$ such that A grows from $dom(p^*\cup h_{u,\ v}(r\restriction N_u))$ to $dom\ r^*$. Once again, is the proof that (1.4) (a), we consider separately the cases $\nu>\kappa_2,\ \nu<\kappa_2$ and $\nu=\kappa_2$. Once again, the hypothesis that $\kappa<\kappa_1$ allows us to assimilate the case $\nu=\kappa_2$ to the case $\nu>\kappa_2$, since what is really at issue is that we are taking the union of fewer than θ_ν conditions, and as before, when $\nu = \kappa_2,\ \theta_\nu = \kappa_1$. In the remaining case, where $\nu < \kappa_2$, once again we have that for all $i < \mu$ and all $w \in \tilde{J}$, either $[i]_\nu \subseteq \mathcal{N}_w$ or $[i]_\nu \cap N_w = \emptyset$, with the former holding if $[i]_\nu \in N_w$.

So, suppose that $\nu < \kappa_2$ and fix such $u,\ v$, suppose $i < \mu$ and $[i]_\nu$ grows from $dom(p^* \cup h_{u,\ v}(r \restriction N_u))$ to $dom\ r^*$. But then there are $t \in J,\ w \in \tilde{J}$ such that $t \sim w$ and $[i]_\nu \cap dom(p^* \cup h_{t,\ w}(r \restriction N_t)) \not\subseteq dom(p^* \cup h_{u,\ v}(r \restriction N_u))$. Then, $[i]_\nu \in N_v \cap N_w$. But then $[i]_\subseteq N_v \cap N_w$. Therefore, letting $b \in N_u,\ c \in N_t$ be such that $[i]_\nu = h_{u,\ v}(b) = h_{t,\ w}(c)$, we clearly have $b = c$ and $b \subseteq N_u \cap N_t$. But then, letting $x\ :=\ (dom\ r \setminus dom\ p^*) \cap b,\ x \in N_u \cap N_t$, since, once again, x is a subset of each, small in cardinality compared to the closure of each. So $h_{u,\ v}(x) = h_{t,\ w}(x)$, but this is a contradiction, since then, $[i]_\nu \cap dom(p^* \cup h_{u,\ v}(r \restriction N_u)) = (([i]_\nu \cap dom\ p^*) \cup h_{u,\ v}(x)) = (([i]_\nu \cap dom\ p^*) \cup h_{t,\ w}(x)) = [i]_\nu \cap dom(p^* \cup h_{t,\ w}(r \restriction N_t))$. This completes the proof of the Corollary, and therefore of the Lemma.

To handle the case $\tau = \kappa_2$, we take $B\ :=\ \bigcup\{B_i :\ i < \sigma\}$, we replace D, above, by $D'\ :=\ \{a \in [B]^n :\ card(a \cap B_i) \leq 1$, for all $i < \sigma\}$, and we take our $\leq^{pr}_{\kappa_2}$ -increasing sequence from Q, $\overrightarrow{p} = (p_j : j < \eta)$, to satisfy:

(1*) $\eta \leq \kappa$, and $p_0 = p$,
(2*) for each $\overrightarrow{\alpha} = (\alpha_1,\ \cdots,\ \alpha_n) \in D'$, there is $j = j_{\overrightarrow{\alpha}}$ such that in $Q^{apr}_{\kappa_2,\ p_{j+1}}$, there is a predense set, $I_{\overrightarrow{\alpha}}$ of conditions deciding $\mathbf{c}(\overrightarrow{\alpha})$.

Now $\eta \leq \kappa \leq \kappa_1 < \tau$, and then the rest of the proof goes through easily, as above.

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