ON THE CARDINALITY AND WEIGHT SPECTRA OF COMPACT SPACES, II

I. Juhász* and S. Shelah**

Abstract. Let $B(\kappa, \lambda)$ be the subalgebra of $\mathcal{P}(\kappa)$ generated by $[\kappa]^{\leq\lambda}$. It is shown that if B is any homomorphic image of $B(\kappa, \lambda)$ then either $|B| < 2^\lambda$ or $|B| = |B|^\lambda$, moreover if X is the Stone space of B then either $|X| \leq 2^{2^\lambda}$ or $|X| = |B| = |B|^\lambda$.

This implies the existence of 0-dimensional compact T_2 spaces whose cardinality and weight spectra omit lots of singular cardinals of "small" cofinality.

1. Introduction

It was shown in [J] that for every uncountable regular cardinal κ, if X is any compact T_2 space with $w(X) > \kappa$ ($|X| > \kappa$) then X has a closed subspace F such that $\kappa \leq w(F) \leq 2^{<\kappa}$ $\left(\text{resp. } \kappa \leq |F| \leq \sum\{2^{2^\lambda} : \lambda < \kappa\}\right)$. In particular, the weight or cardinality spectrum of a compact space may never omit an inaccessile cardinal, moreover under GCH the weight spectrum cannot omit any uncountable regular cardinal at all.

In the present note we prove a theorem which implies that for singular κ on the other hand there is always a 0-dimensional compact T_2 space whose cardinality and weight spectra both omit κ.

We formulate our main result in a boolean algebraic framework. The topological consequences easily follow by passing to the Stone spaces of the boolean algebras that we construct.

2. The Main Result

We start with a general combinatorial lemma on binary relations. In order to formulate it, however, we need the following definitions.

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*Research supported by the Hungarian National Foundation for Scientific Research grant no. 16391.

**Research supported by "The Israel Science Foundation", administered by The Israel Academy of Sciences and Humanities, Publication no. 612

Definition 1. Let $\prec$ be an arbitrary binary relation on a set X and τ, μ be cardinal numbers. We say that $\prec$ is τ-full if for every subset $a \subset X$ with $|a| = \tau$ there is some $x \in X$ such that $|\{y \in a: y \prec x\}| = \tau$. Moreover, $\prec$ is said to be μ-local if for every $x \in X$ we have $|\text{pred}\,(x, \prec)| \leqq \mu$, where $\text{pred}(x, \prec) = \{y \in X: y \prec x\}$.

Now, our lemma is as follows.

Lemma 2. *Let $\prec$ be a binary relation on the cardinal ϱ that is both τ-full and μ-local. Then for every almost disjoint family $\mathcal{A} \subset [\varrho]^\tau$ we have*

$$|\mathcal{A}| \leq \varrho \cdot \mu^\tau.$$

Proof. For every set $a \in \mathcal{A}$ there is a $\xi_a \in \varrho$ such that $g(a) = a \cap \text{pred}(\xi_a, \prec)$ has cardinality τ because $\prec$ is τ-full. This map g is clearly one-to-one for $\mathcal{A}$ is almost disjoint. But the range of g is a subset of $\cup\{[\text{pred}(\xi, \prec)]^\tau : \xi \in \varrho\}$ whose cardinality does not exceed $\varrho \cdot [\mu]^\tau$, and this completes the proof.

Before we formulate our main result we need some notation. Given the cardinals κ and λ (we may assume $\lambda \leq \kappa$) we denote by $B(\kappa, \lambda)$ the boolean subalgebra of the power set algebra $\mathcal{P}(\kappa)$ generated by all subsets of κ os size $\leq \lambda$. In other words

$$B(\kappa, \lambda) = [\kappa]^{\leq\lambda} \cup \{x \subset \kappa: \kappa \setminus x \in [\kappa]^{\leq\lambda}\}.$$

What we can show is that the size of a homomorphic image of $B(\kappa, \lambda)$ (as well as the size of its Stone space) has to satisfy certain restrictions, namely it is either "small" or cannot have "very small" cofinality.

Theorem 3. *Let $h: B(\kappa, \lambda) \to B$ be a homomorphism of $B(\kappa, \lambda)$ onto the boolean algebra B. Then (i) either $|B| < 2^\lambda$ or $|B|^\lambda = |B|$; (ii) if $X = St(B)$ is the Stone space of B then either $|X| \leq 2^{2^\lambda}$ or $|X| = |B| = |B|^\lambda$.*

Proof. Set $|B| = \varrho$ and assume that $\varrho \geq 2^\lambda$. Since $[\kappa]^{\leq\lambda}$ generates $B(\kappa, \lambda)$ therefore $A = h''[\kappa]^{\leq\lambda}$ generates B and thus we have $|A| = \varrho$ as well. We claim that the relation $\leq_B$ is

(a) τ-full on A for each $\tau \leq \lambda$;

(b) 2^λ-local on A.

Indeed, if $a \in [A]^\tau$ where $\tau \leq \lambda$ then there is a set $x \in [[\kappa]^{\leq\lambda}]^\tau$ such that $a = h''x$. But then $b = \cup x \in [\kappa]^{\leq\lambda}$ as well, hence $h(b) \in A$ and clearly $a \subset \text{pred}(h(b), \leq_B)$ because h is a homomorphism. This, of course, is much more than what we need for (a).

To see (b), let us first note that if $b, c \in [\kappa]^{\leq\lambda}$ and $h(b) \leq h(c)$ then $b \cap c \in [\kappa]^{\leq\lambda}$ as well and $h(b \cap c) = h(b) \wedge h(c) = h(b)$ using that h is a homomorphism again. But this implies $\text{pred}(h(c), \leq_B) = h''\mathcal{P}(c)$ for any $c \in [\kappa]^{\leq\lambda}$, consequently $|\text{pred}(h(c), \leq_B)| \leq |\mathcal{P}(c)| \leq 2^\lambda$ and this completes the proof of (b).

Applying Lemma 2 we may now conclude that for every cardinal $\tau \leq \lambda$ and for every almost disjoint family $\mathcal{A} \subset [\varrho]^\tau$ we have

$$|\mathcal{A}| \leq \varrho \cdot (2^\lambda)^\tau = \varrho.$$

This, in turn, implies $\varrho^\lambda = \varrho$. Indeed, assume that $\varrho^\lambda > \varrho$ and τ be the smallest cardinal with $\varrho^\tau > \varrho$. Then $\tau \leq \lambda$ and $\varrho^{<\tau} = \varrho$, and as is well-known, there is an almost disjoint family $\mathcal{A} \subset [{}^{<\tau}\varrho]^\tau$ of size $\varrho^\tau > \varrho$, namely $\mathcal{A} = \{A_f : f \in {}^\tau\varrho\}$ where $A_f = \{f \restriction \xi : \xi < \tau\}$ for any $f \in {}^\tau\varrho$.

Now, to prove (ii) first note that if $|B| \leq 2^\lambda$ then trivially $|X| \leq 2^{2^\lambda}$. So assume $|B| > 2^\lambda$ and in this case we prove that actually

$$|X| = 2^{2^\lambda} \cdot |B|.$$

We first show that $|X| \geqq 2^{2^\lambda} \cdot |B|$, which, as $|X| \geq |B|$ is always valid, boils down to showing that $|X| \geq 2^{2^\lambda}$.

Using that $|B| = |h''[\kappa]^{\leq\lambda}| = \varrho > 2^\lambda$ we may select a collection $\{a_\alpha : \alpha \in (2^\lambda)^+\} \subset [\kappa]^{\leq\lambda}$ such that $\alpha \neq \beta$ implies $h(a_\alpha) \neq h(a_\beta)$ and by a straight forward Δ-system argument we may also assume that $\{a_\alpha : \alpha \in (2^\lambda)^+\}$ is a Δ-system with root a. Then, as h is a homomorphism, we also have $h(a_\alpha) \wedge h(a_\beta) = h(a)$ for distinct α and β and so $\{h(a_\alpha) - h(a) : \alpha \in (2^\lambda)^+\}$ are pairwise disjoint and disdinct elements B, all but at most one of which is non-zero. However the existence of 2^λ many pairwise disjoint non-zero elements in a boolean algebra clearly implies the existence of 2^{2^λ} ultrafilters in it, hence we are done with showing $|X| \geq 2^{2^\lambda}$.

Next, to see $|X| \leq 2^{2^\lambda} \cdot |B|$ note that, again as h is a homomorphism, $h''[\kappa]^{\leq\lambda}$ is a (not necessarily proper) ideal in B, hence there is no more than one ultrafilter u on B such that $u \cap h''[\kappa]^{\leq\lambda} = \emptyset$. If, on the other hand, $u \in X$ is such that $b \in u \cap h^{cc[\kappa]^{\leq\lambda}}$ then u is generated by its subset $u \cap \text{pred}(b, \leq_B)$. However $\leq_B$ is clearly 2^λ- local on $h''[\kappa]^{\leq\lambda}$, and so we conclude that

$$\begin{aligned} |X| \leq & 1 + |\cup\{\mathcal{P}(\text{pred}(b, \leq_B)) : b \in h''[\kappa]^{\leq\lambda}\}| \leq \\ & \leq 1 + 2^{2^\lambda} \cdot |B| = 2^{2^\lambda} \cdot |B|. \end{aligned}$$

This completes the proof of our theorem.

Now let $X(\kappa, \lambda)$ be the Stone space of the boolean algebra $B(\kappa, \lambda)$. Using Stone duality and the notation of [J] the above result has the following reformulation about the weight and cardinality spectra of the 0-dimensional compact T_2 space $X(\kappa, \lambda)$.

Corollary 4.

(i) *For every* $\mu \in Sp(w, X(\kappa, \lambda))$ *we have either* $\mu < 2^\lambda$ *or* $\mu^\lambda = \mu$, *hence* $cf(\mu) > \lambda$;

(ii) *if* $\mu \in Sp(|\ |, X(\kappa, \lambda))$ *then either* $\mu < 2^{2^\lambda}$ *or* $\mu^\lambda = \mu$.

In fact, for every closed subspace Y *of* $X(\kappa, \lambda)$ *we have either* $w(Y) \leq 2^\lambda$ *or* $w(Y)^\lambda = w(Y)$ *and* $|Y| = 2^{2^\lambda} \cdot w(Y)$.

It follows from this immediately that if $2^{2^\lambda} < \kappa$ then the cardinality and weight spectra of the space $X(\kappa, \lambda)$ omit every cardinal $\mu \in (2^{2^\lambda}, \kappa]$ with $cf(\mu) \leq \lambda$. In particular, if GCH holds then $\lambda < \kappa$ implies that both $Sp(|\ |, X(\kappa, \lambda))$ and $Sp(w, X(\kappa, \lambda))$ omit all cardinals $\mu \in (\lambda, \kappa]$ with $cf(\mu) \leq \lambda$.

Note that similar omission results were obtained by van Douwen in [vD] for the case $\lambda = \omega$ and κ strong limit.

An interesting open problem arises here that we could not settle: Can one find for every cardinal κ a compact T_2 space X such that the cardinality and/or weight spectra of X omit every singular cardinal below κ?

References

[vD] E. van Douwen, *Cardinal functions on compact F-spaces and on weakly countably compact boolean algebras,* Fund. Math. **114** (1981), 236-256.

[J] I. Juhász, *On the weight spectrum of a compact spaces,* Israel J. Math. **81** (1993), 369-379.