SPECIAL SUBSETS OF ${}^{\mathrm{CF}(\mu)}\mu$, BOOLEAN ALGEBRAS AND MAHARAM MEASURE ALGEBRAS
SH620

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Abstract. The original theme of the paper is the existence proof of "there is $\bar\eta = \langle \eta_\alpha : \alpha < \lambda\rangle$ which is a (λ, J)-sequence for $\bar I = \langle I_i : i < \delta\rangle$, a sequence of ideals". This can be thought of as a generalization to Luzin sets and to Sierpinski sets, but for the product $\prod\limits_{i<\delta} \mathrm{Dom}(I_i)$, the existence proofs are related to pcf.

The second theme is when does a Boolean algebra $\mathbf{B}$ have a free caliber λ (i.e. if $X \subseteq \mathbf{B}$ and $|X| = \lambda$, then for some $Y \subseteq X$ with $|Y| = \lambda$ and Y is independent). We consider it for $\mathbf{B}$ being a Maharam measure algebra, or $\mathbf{B}$ a (small) product of free Boolean algebras, and κ-cc Boolean algebras. A central case is $\lambda = (\beth_\omega)^+$ or more generally, $\lambda = \mu^+$ for μ strong limit singular of "small" cofinality. Second case is $\mu = \mu^{<\kappa} < \lambda < 2^\mu$; the main case is λ regular but we also have things to say on the singular case. Lastly, we deal with ultraproducts of Boolean algebras in relation to irr(-) and s(-) etc.

Key words and phrases. Set theory, Boolean algebras, Maharam algebras, pcf, caliber.

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Annotated Content

§0 Introduction

§1. The framework and an illustration

We define when "$\bar\eta = \langle \eta_\alpha : \alpha < \lambda\rangle$ is a (λ, I, J)-sequence for $\bar I = \langle I_i : i < \delta\rangle$", which means ($I = J^{\mathrm{bd}}_\lambda$ for simplicity) that each η_α belongs to $\prod\limits_{i<\delta} \mathrm{Dom}(I_i)$ and that for any $\bar A = \langle A_i : i < \delta\rangle \in \prod\limits_{i<\delta} I_i$ for every large enough $\alpha < \lambda$, the sequence η_α "run away" from $\bar A$ i.e. for the J-majority of $i < \delta$ we have $\eta_\alpha(i) \notin A_i$. We give the easy existence if I_i is κ_i-complete and $\langle \kappa_i : i < \delta\rangle$ are strictly increasing converging to a strong limit (singular) μ which satisfies $\mu^+ = 2^\mu = \lambda$ (1.9). We define normality, explain how by the existence of such $\bar\eta$, colouring properties can be lifted (1.6). As an illustration we prove (the well known result) that, e.g., if $\lambda = 2^{\beth_\omega} = \beth^+_\omega$, then $\beth^+_\omega$ is not a free caliber of the Maharam measure algebra (i.e., some set X of λ elements, is non-independent, in fact in a more specific way). For this we use ideals related to the Erdös-Rado Theorem.

§2. There are large free subsets

Why does the application in §1 involve λ "near" a strong limit singular μ of cofinality $\aleph_0$? We show that this was necessary: if $\mu^{\aleph_0} < \lambda \le 2^\lambda$ and $\mathrm{cf}(\lambda)$ is large enough ($> \beth_2$ is OK, $> 2^{\aleph_0}$ is almost OK, but involves more pcf considerations), then λ is a free caliber of the Maharam measure algebra. We use: if $\lambda > 2^\kappa, f_\alpha \in {}^\kappa\,\mathrm{Ord}$ for $\alpha < \lambda, \alpha \ne \beta \Rightarrow f_\alpha/J + f_\beta/J$, <u>then</u> (almost always) for some ideal I on *kappa* extending J and $X \in [\lambda]^\lambda, \langle f_\alpha/I : \alpha < \lambda\rangle$ are pairwise $\ne_I$.

§3. Strong independence in Maharam measure algebras

We define when "$\bar\eta$ is a super (λ, I, J)-sequence for $\bar I$". The strengthening is that we now can deal with n-tuples (any $n < \omega$) and prove the easy existence (see 3.1, 3.2). We define for a set of λ intervals in a Boolean algebra variants of independence and strong negation of it (3.4) and apply it to prove existence of strongly λ-anti-independent set in Maharam Measure algebra (3.7), which (by 3.8) suffices for having a subalgebra of dimension λ with no independent set of cardinality λ. This completes the consistency part of the solution of a problem, which was to characterize all cardinals λ which can have this property.

We prove here, e.g., if $\lambda = \beth_{\omega+1} = \beth^+_\omega$, then there is a Hausdorff compact zero dimensional topological space with measure on the family of the Borel subsets such that it has dimension λ, so as a measure space is isomorphic to the Maharam measure space $\mathscr{B}(\lambda)$, but there is no homomorphism from X onto ${}^\mu 2$ (see 3.9). We finish by some easy examples.

§4. The interesting ideals and the direct pcf application

We return to our original aim: existence of λ-sequences for $\bar{I}$. In 4.1 we consider some ideals ($J^{\rm bd}_A$, $\prod\limits_{\ell<n} J_\ell$, $J^{\rm bd}_{\langle\lambda_\ell:\ell<n\rangle} = \prod\limits_{\ell} J^{\rm bd}_{\lambda_\ell}$, each λ_ℓ regular, in the cases $\lambda_\ell < \lambda_{\ell+1}$, $\lambda_\ell > \lambda_{\ell+1}$, $\lambda_\ell > 2^{\lambda_{\ell+1}}$). We point out (4.9) that for $\bar{I} = \langle J^{\rm bd}_{\lambda_i} : i < \delta\rangle$, if $\lambda = \mathrm{tcf}(\prod\limits_{i<\delta} \lambda_i/J^{\rm bd}_\delta)$ we get existence directly from the pcf theory. We then turn to the case $I_i = \prod\limits_{\ell<n_i} J^{\rm bd}_{\lambda_{i,\ell}}$, give a sufficient pcf condition for the existence when $\langle\lambda_{i,\ell} : \ell < n\rangle$ is increasing (4.11) and then prove that this condition occurs not rarely (in 4.14), so if $\lambda = \prod\limits_{i<\delta} \lambda_i/J^{\rm bd}_\delta$, λ_i increasing, we can "group together" intervals of λ_i; and the existence of such $\langle\lambda_i : i < \delta\rangle$ is an important theme of pcf theory.

§5. λ-sequences for decreasing $\bar{\lambda}^i$ by pcf

We consider cases with $I_i = J^{\rm bd}_{\langle\lambda_{i,\ell}:\ell<n_i\rangle}$, $\langle\lambda_{i,\ell} : \ell < n_i\rangle$ a decreasing sequence of regulars. We prove the existence by using twice cases of true cofinalities, and show that if the pcf structure is not so simple then there are such cases (e.g. $\beth_{\omega_i+1} > \beth^{+\omega}_{\omega_i}$). We concentrate on the case $i < \delta \Rightarrow n_i = n$, and then indicate the changes needed in the general case.

§6. Products of Boolean Algebras

Monk asks about the free caliber of products of $\mathbf{B}_i = FBA(\chi_i)$ = the free algebra with χ_i generators, for $i < \delta$. In fact he asks whether $\lambda = \beth^+_\omega$ is a free caliber of the product of the $FBA(\beth_n)$ for $n < \omega$. But we think that the intention was to ask if $\lambda = \mathrm{cf}(\lambda) > 2^{|\delta|}$ is a free caliber of $\prod\limits_{i<\delta} \mathbf{B}_i$. Note that this product satisfies the $(2^{|\delta|})^+$-c.c. In fact it has cellularity $2^{|\delta|}$, so "tends to have free calibers". We show that if there is a normal super (λ, J)-sequence $\bar{\eta}$ for appropriate $\bar{I} = \langle I_n : n < \omega\rangle$, then λ is not a free caliber of $\prod\limits_{n<\omega} FBA(|\mathrm{Dom}\ I_n|)$ (see 6.4, 6.5), so a negative answer is possible. Now being "near a strong limit singular of cofinality $\aleph_0$" is necessary as a result parallel to that of §2 holds (see 6.6).

Though the choice of $\beth_\omega$ was probably just natural as the first case to consider, actually the product of uncountably many $FBA(\chi_i)$'s behave differently e.g. $\prod\limits_{i<\omega_1} FBA(\beth_i)$ has free caliber $(\beth_{\omega_1})^+$! (see 6.7). The proof involves pcf considerations dealt with in §7. We turn to another problem of Monk ([M2, Problem 34]), this time giving unambivalent solution. If κ is weakly inaccessible with $\langle 2^\mu : \mu < \kappa\rangle$ not eventually constant, then there is a κ-c.c. Boolean algebra of cardinality $2^{<\kappa}$ and no independent subsets of cardinality κ^+ (see 6.10, using the existence of suitable trees). We note that results similar to countable products hold for the completion of $FBA(\chi)$.

We end by deducing from Gitik Shelah [GiSh 597] complementary consistency results (so e.g. the first question is not answerable in ZFC) and phrasing the principles involved, so slightly sharpening the previous results. (See 6.14 - 6.17). So together with the earlier part of the section we have answered [M2, Problems 35,36] and [M2, Problems 32,33] in the case we are near a strongly limit singular cardinal.

§7. A nice subfamily of function exists

For completeness we deal with the following: $f_\alpha \in {}^\theta\mathrm{Ord}$ for $\alpha < \lambda$ are given, $2^\theta < \lambda = \mathrm{cf}(\lambda)$ and we would like to get approximation to "for some $X \subseteq \lambda$, $|X| = \lambda$, $\langle f_\alpha : \alpha \in X\rangle$ is a Δ-system", continuing [Sh 430, Claim 6.6D]. We phrase a special case (7.3) and deal with some variants.

§8. Consistency of "$\mathscr{P}(\omega_1)$ has a free caliber" and discussion of pcf

We deal with another of Monk's problems, [M2, Problem 37], proving the consistency of "there is no complete Boolean algebra $\mathbf{B}$ of cardinality $2^{\aleph_1}$ with empty free caliber" (in fact $\aleph_{\omega_1+1} = 2^{\aleph_1}$ is always a free caliber of $\mathbf{B}$). The universe is obtained by adding $\aleph_{\omega_1+1}$ Cohens to a model of ZFC + GCH, and the proof uses §7. We finish by discussing some pcf problems: pcf preserves being even; and we state a consequence of $\{\mu : \mu \text{ strong limit}, \mathrm{cf}(\mu) = \aleph_0, \mu^+ = 2^\mu\}$ being unbounded (here?).

§9. Having a λ-sequence for a sequence of non-stationary ideals

We return to the original theme, for a more restricted case. We assume $\lambda = \mathrm{cf}(2^\mu)$ where μ is strong limit singular, and in this section $\lambda = 2^\mu$ i.e. 2^μ is regular (for the singular case see §10). We get quite strong results: (fix $n(*) < \omega$ for simplicity) for some ideal J on $\mathrm{cf}(\mu)$ (usually $J^{\mathrm{bd}}_{\mathrm{cf}(\mu)}$, always close to it) we can find $\langle \bar\lambda^i : i < \mathrm{cf}(\mu)\rangle$, $i < j \Rightarrow \max(\bar\lambda^i) < \min(\bar\lambda^j)$, $\bar\lambda^i = \langle \lambda_{i,\ell} : \ell < n(*)\rangle$, $\lambda_{i,\ell+1} > 2^{\lambda_{i,\ell}}$ ($\lambda_{i,\ell}$ regular of course, $\mu = \sup_{i<\mathrm{cf}(\mu)} \lambda_{i,0}$), such that there is a (λ, J)-sequence for $\bar I = \langle J^{\mathrm{bd}}_{\bar\lambda^i} : i < \mathrm{cf}(\mu)\rangle$. This is nice (compare with §5) but we get much more: $\bar I$ is a sequence of nonstationary ideals and even $\langle \prod\limits_{\ell<n(*)} J^{\mathrm{nst},\sigma}_{\lambda_{i,\ell}} : i < \mathrm{cf}(\mu)\rangle$ where $J^{\mathrm{nst},\sigma}_\chi = \{A : A \cap \{\delta < \chi : \mathrm{cf}(\delta) = \sigma\} \text{ is not stationary}\}$ and $\sigma = \mathrm{cf}(\sigma) \in (\mathrm{cf}(\mu), \mu)$.

We then work more and get versions with club guessing ideals. We deal further with the version we get for the case $\mathrm{cf}(\mu) = \aleph_0$. (So it is less clear which ideals J can be used.)

§10. The power of a strong limit singular is itself singular: existence

We do the parallel of the first theorem of §9 in the case 2^μ is singular.

§11. Preliminaries to the construction of ccc Boolean algebras with no large independent sets

Here the problem is whether every κ-c.c. Boolean algebra has free caliber λ; the case of being "near a strong limit singular μ of cofinality $< \kappa$" was considered in [Sh 575], we deal with the case $\mu = \mu^{<\kappa} < \lambda < 2^\mu$. Here we make the set theoretic preparation for a proof of the consistency of a negative answer with strong violation of GCH. We use Boolean algebras generated by x_α's freely except for $x_\alpha \cap x_\beta \cap x_\gamma = 0$ for $\{\alpha, \beta, \gamma\} \in W$ for some set W of triples with intersection having at most one element. The point is that the properties of "$\bar{\eta}$ is a λ-sequence for $\bar{I}$" with such ideals I (unlike the ones associated with the Erdös-Rado theorem) are preserved by adding many Cohens to μ (where $\mu \ll |\mathrm{Dom}(I_i)|$ etc.).

§12. Constructing ccc Boolean algebras with no large independent sets

We complete the consistency results for which the ground was prepared in §11. We construct the relevant Boolean algebra using a (λ, J)-sequence for $\bar{I}$, $\bar{I}$ as there, using, as building blocks, Boolean algebras generated e.g. from the triple system. So we will give sufficient conditions for the κ-c.c. and other properties of the Boolean algebra.

§13. The singular case

We continue §11, §12 by dealing here with the case λ is singular but $(\forall \alpha < \lambda)$ $(|\alpha|^{<\kappa} < \lambda)$, note that the forcing from §12 essentially creates only such cases.

§14. Getting free caliber for regular cardinals

We continue dealing with κ-c.c. Boolean algebras, giving a sufficient condition for λ being a free caliber, hence a consistency follows (complementing §11 and §12; together this solves [M2, Problems 32,33] in the case we are not near a strong limit singular cardinal; thus together with §6 this gives a solution).

§15. On irr: The invariant of the ultraproduct, greater than the ultraproduct of invariants

We prove the consistency of $\mathrm{irr}\big(\prod\limits_{n<\omega} \mathbf{B}_n/\mathscr{D}\big) > \prod\limits_{n<\omega} \mathrm{irr}(\mathbf{B}_n)/\mathscr{D}$ where $\mathscr{D}$ is a nonprincipal ultrafilter on ω and $\mathrm{irr}(\mathbf{B}) = \mathrm{irr}_\omega(\mathbf{B})$ and $\mathrm{irr}_n(\mathbf{B}) = \sup\{|X| : X \subset \mathbf{B}$ and if $x_0, x_1, \dots, x_m$ are distinct members of X, $m < n$ then $x_0 \notin \langle x_1, \dots, x_m\rangle_{\mathbf{B}}\}$. The way is to build $\mathbf{B}_n$ with $\mathrm{irr}_n(\mathbf{B}_n) = \lambda^+$, $\mathrm{irr}_{2n+1}(\mathbf{B}_n) = \lambda$, $\lambda = \lambda^{\aleph_0}$. Our earlier tries as the approximation to $\mathbf{B}_n$ did not work. So the point is a version of n-graded independence phrased as $\langle F_\ell : \ell < n\rangle$, then solve [M2, Problem 26]. We then deal with $s(-)$, $\mathrm{hL}(-)$, $\mathrm{hd}(-)$ and $\mathrm{Length}(-)$, using the construction of §12 in ZFC, and solving [M2, Problems 22,46,51,55].

§0 Introduction

Our original aim was to construct special subsets of $\prod_{i<\delta} \lambda_i$, concentrating particularly on the case when λ_i converge to a strong limit singular.

This continues [Sh 575] (so [Sh:g], [Sh 462], Rosłanowski and Shelah [RoSh 534]), but as these are essentially notes from the author's lectures in Madison, they are self contained. (§1, §4 just represent old material, adding an illustration for Maharam algebras).

Some sections improve the general existence theorems. The main new point is the case when we use

$$I_i = \prod_{\ell<n_i} J^{\mathrm{bd}}_{\lambda_{\ell,i}} \quad \text{with the } \lambda_{\ell,i}\text{'s are regular decreasing}$$

(as well as the case of the nonstationary ideal). We shall discuss this below and give the definition after we first fix some notation.

0.1 Notation. 1) I denotes an ideal on a set $\mathrm{Dom}(I)$, which means that I is a subset of $\mathscr{P}(\mathrm{Dom}(I))$ closed under (finite) unions and subsets, $\mathrm{Dom}(I) \notin I$, and usually for simplicity, all singletons are assumed to belong to I.

I is κ-complete if it is closed under unions of $< \kappa$ elements.
2) I, J denote ideals.
3) $I^+ = \{A \subseteq \mathrm{Dom}(I) : A \notin I\}$.
4) If A is a set of ordinals with no last member we let

$$J^{\mathrm{bd}}_A = \{B \subseteq A : B \text{ a bounded subset of } A\}.$$

5) The completeness of the ideal I, $\mathrm{comp}(I)$ is the maximal κ such that I is κ-complete (it is necessarily a well-defined regular cardinal).
6) $[A]^\kappa = \{a \subseteq A : |a| = \kappa\}$, $[A]^{<\kappa} = \{a \subseteq A : |a| < \kappa\}$ etc.
7) $\mathrm{cov}(\lambda,\mu,\theta,\sigma) = \mathrm{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^{<\mu}$, and for every $a \in [\lambda]^{<\theta}$ there are $\alpha < \sigma$ and $a_i \in \mathscr{P}$ for $i < \alpha$ such that $a \subseteq \bigcup_{i<\alpha} a_i\}$.
8) For sets u, v of ordinals let $\mathrm{OP}_{u,v}$ be the function from u into v such that: $\beta = \mathrm{OP}_{u,v}(\alpha) \Leftrightarrow \alpha \in u \ \& \ \beta \in v \ \& \ \mathrm{otp}(u \cap \alpha) = \mathrm{otp}(v \cap \beta)$.
9) Terms are τ.

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0.2 Definition. We say $\bar\eta = \langle \eta_\alpha : \alpha < \lambda \rangle$ is a (λ, I, J)-sequence for $\bar I = \langle I_i : i < \delta \rangle$ <u>if</u>

(a) I is an ideal on λ (if not mentioned, we assume $I = J^{\rm bd}_\lambda$), I_i is an ideal on $\mathrm{Dom}(I_i)$,

(b) J is an ideal on δ (if not mentioned, we assume $J = J^{\rm bd}_\delta$),

(c) $\eta_\alpha \in \prod\limits_{i<\delta} \mathrm{Dom}(I_i)$,

(d) If $X \in I^+$ <u>then</u>

$$\{i < \delta : \{\eta_\alpha(i) : \alpha \in X\} \in I_i\} \in J.$$

By [Sh 575], if I_i is κ_i-complete, $\kappa_i > \sum_{j<i} \kappa_j$, $\mu = \sum_{i<\delta} \kappa_i$ strong limit, $|\mathrm{Dom}(I_i)| < \mu$ and $2^\mu = \mu^+ = \lambda$, then there is such a sequence. We recall this in §1.

As an example of the application of such $\bar\eta$, we presented the following (presented in 1.16): Suppose that $\mathscr{B}$ is a Maharam measure algebra of dimension $\geq \mu$, $\mathrm{cf}(\mu) = \aleph_0$. Then we can find $a_\alpha \in \mathscr{B}$ for $\alpha < \lambda$ such that $\mathrm{Leb}(a_\alpha) > 0$ and

$$(\forall X \in [\lambda]^\lambda)(\exists n)(\forall \alpha_0 < \cdots < \alpha_n \in X) \bigcap_{i \leq n} a_{\alpha_i} = 0.$$

A "neighborhood" of μ being strong limit of cofinality $\aleph_0$ is necessary.
Our usual case, which we call normal is: $\kappa_i > \prod\limits_{j<i} |\mathrm{Dom}(I_j)|$ (this was not used in the measure algebra application, but it is still good to have).

Main point: The main new point of this paper is to build a (λ, I, J)-sequence $\bar\eta$ for certain $\bar I$ without using $2^\mu = \mu^+$. We describe the cases of $\bar I$ which we can handle.

Case 1: The easiest case of $I_i : I_i = J^{\rm bd}_{\lambda_i}$, $\lambda = \mathrm{cf}\Big(\prod\limits_{i<\delta} \lambda_i / J\Big)$.
We only need to translate from the known pcf results.

Case 2:

$$I_i = \prod_{\ell < n_i} J^{\rm bd}_{\lambda_{\ell,i}},$$

where $\lambda_{\ell,i}$ are regular increasing with ℓ and i, and J is an ideal on $\{(i,\ell) : i < \delta, \ell < n_i\}$ such that

$$(\forall X \in J)(\exists^{(J^{\rm bd}_\delta)^+} i)(\bigwedge_{\ell < n_i} (i,\ell) \notin X),$$

and where for ideals J_m $(m < n)$

$$\prod_{m<n} J_m =: \{X \subseteq \times_{m<n} \mathrm{Dom}(J^m) : \neg\exists^{J_0^+} x_0 \exists^{J_1^+} x_1 \cdots \exists^{J_{n-1}^+} x_{n-1}(\langle x_0, \dots, x_{n-1}\rangle \in X)\}$$

Starting from reasonable pcf assumptions and working a little, we can handle this case as well.

Main Case 3:

$$I_i = \prod_{\ell<n_i} J^{\mathrm{bd}}_{\lambda_{\ell,i}},$$

$\lambda_{\ell,i}$ regular *decreasing* with ℓ.

We prove: If $\bigwedge_i n_i = n$, and $\lambda_\ell = \mathrm{tcf}\big(\prod_{i<\delta} \lambda_{\ell,i}/J'\big)$ for $\ell < n$, then we can find $\langle \eta_{\bar\alpha} : \bar\alpha \in \prod_{\ell<n} \lambda_\ell\rangle$ with $\eta_{\bar\alpha}(i) \in \prod_{\ell<n} \lambda_{\ell,i}, i < \delta$ such that

$$\text{if} \quad X \in \Big(\prod_{\ell<n} J^{\mathrm{bd}}_{\lambda_\ell}\Big)^+ =: \Big(J^{\mathrm{bd}}_{\langle\lambda_\ell:\ell<n\rangle}\Big)^+$$

$$\text{then} \quad \Big\{i < \delta : \{\eta_{\bar\alpha}(i) : \bar\alpha \in X\} \in \Big(J^{\mathrm{bd}}_{\langle\lambda_{\ell,i}:\ell<n\rangle}\Big)\Big\} \in J'.$$

Interesting instances: $\lambda_{\ell,i}$ decreasing with ℓ and $i < j \Rightarrow \lambda_{\ell,i} < \lambda_{n,j}$.

Case 4: Like Case 3, but using the nonstationary ideal, or nonstationary ideal restricted so some "large subset" of $\lambda_{\ell,i}$ instead of $J^{\mathrm{bd}}_{\lambda_{\ell,i}}$.

Case 5: Like Case 3 but using a suitable club guessing ideal $\mathrm{id}^a(\overline{C}^{\ell,i})$).

On history, background etc. and on Boolean algebras, see Monk [M1], [M2]. This works continues [Sh 575] and it evolved as follows. Getting the thesis of Carrieres, which was based on [Sh:92], we started thinking again on "free calibers", this time on measure algebras. We noted that [Sh 575] gives the answer if, e.g. $\lambda = (\beth_\omega)^+ = \beth_{\omega+1}$, and started to think of what is called here "there is a (λ, J)-sequence for $\bar I$ ". We started to lecture on it (§1, §4, then §5,§9,§10; in Madison, Fall 1996). Meanwhile Mirna Džamonja asked me doesn't this solve a problem from her thesis. This was not actually the case, but it became so in §3. Then she similarly brought me p. 256 of Monk [M2] and this influenced the most of the rest of the paper, while later I also looked at pages 255, 257 of [M2], but not so carefully. Lastly, §15 is looking back at the problems from [RoSh 534]. Some of the sections are (revisions of) notes from my lectures. So I would like to thank

Christian Carrieres, Donald Monk and the participants of the seminar in Madison for their influence, and mainly Mirna Džamonja for god-mothering this paper in many ways and to David Fremlin who lately informed me that 1.16 was well known and 3.8, 3.15 have already appeared in Plebanek [Pl1], [Pl2].

Concerning §3, the question was asked for $\lambda = \aleph_1$ by Haydon and appeared in Fremlin's book [Fre]. Haydon [Ha1], [Ha2] and Kunen [Ku81] independently proved it to be consistent for $\lambda = \aleph_1$ assuming CH. The question from [Ha1] and [Fre] was what happens with $\aleph_1$ under MA. Recently, Plebanek [Pl1], [Pl2] proved that under MA all regular cardinals $\geq \aleph_2$ fail the property, and finally Fremlin [Fre] gave the negative answer to the original question of Haydon by showing that under MA the property fails for $\aleph_1$. Džamonja and Kunen [DK1], [DK2] considered the general case (any λ) and topological variants.

§1 The framework and an illustration

We are considering a sequence $\langle I_i : i < \delta\rangle$ of ideals, and we would like to find a sequence $\bar\eta = \langle \eta_\alpha : \alpha < \lambda\rangle$ of members of $\prod\limits_{i<\delta} \mathrm{Dom}(I_i)$ which "runs away" from $\bar A = \langle A_i : i < \delta\rangle$ when $A_i \in I_i$ (see definition 1.1 below).

When I_i is κ_i-complete, $\kappa_i > \prod\limits_{j<i} |\mathrm{Dom}(I_j)|$, $\mu = \sum\limits_{i<\delta} \kappa_i$ strong limit singular, $\lambda = \mu^+ = 2^\mu$, this is easy. We present this (all from [Sh 575]) and, for illustration, an example.

1.1 Definition. 1) We say that $\bar\eta$ is a (λ, I, J)-sequence for $\bar I$ <u>if</u>:

- (a) J is an ideal on δ and I is an ideal on λ,
- (b) $\bar I = \langle I_i : i < \delta\rangle$, where I_i is an ideal on $\mathrm{Dom}(I_i)$,
- (c) $\bar\eta = \langle \eta_\alpha : \alpha < \lambda\rangle$ where $\eta_\alpha \in \prod\limits_{i<\delta} \mathrm{Dom}(I_i)$,
- (d) if $X \in I^+$ then

$$\{i < \delta : \{\eta_\alpha(i) : \alpha \in X\} \in I_i\} \in J.$$

2) We say $\bar\eta$ is a weakly (λ, I, J)-sequence for $\bar I$ if we weaken clause (d) to

(d^-) if $X \in I^+$ then

$$\big\{i < \delta : \{\eta_\alpha(i) : \alpha \in X\} \in I_i^+\big\} \in J^+.$$

3) We may omit J if $J = J^{\mathrm{bd}}_\delta$, we may omit I if $I = J^{\mathrm{bd}}_\lambda$, and then we may say "$\bar\eta$ is a λ-sequence for $\bar I$.

We can replace λ by another index set.

1.2 Definition. 1) We say $\bar\eta$ is normally a (λ, I, J)-sequence for $\bar I$ (or in short, "$\bar\eta$ is normal", when $\bar I, I, J$ are clear) <u>if</u>:

- $(*)$ for every $i < \delta$,

$$\mathrm{comp}(I_i) > |\{\eta_\alpha \restriction i : \alpha < \lambda\}|.$$

2) We say $\bar{I} = \langle I_i : i < \lambda \rangle$ is normal if

$$\mathrm{comp}(I_i) > \prod_{j<i} |\mathrm{Dom}(I_i)|.$$

1.3 Observation. If $\bar{I} = \langle I_i : i < \delta \rangle$ is normal and $\bar{\eta}$ is a (λ, I, J)-sequence for $\bar{I}$ <u>then</u> $\bar{\eta}$ is a normal (i.e., normally a (λ, I, J)-sequence for $\bar{I}$).

Proof. As for each $i < \delta$

$$|\{\eta_\alpha \restriction i : \alpha < \lambda\}| \le |\prod_{j<i} \mathrm{Dom}(I_j)| = \prod_{j<i} |\mathrm{Dom}(I_j)| < \mathrm{comp}(I_i).$$

1.4 Discussion. Why is normality (and sequences $\bar{\eta}$ in general) of interest? Think for example, of having for each $i < \delta$, a colouring $\mathbf{c}_i$, say a function with domain $[\mathrm{Dom}(I_i)]^2$ (or even $[\mathrm{Dom}(I_i)]^{<\aleph_0}$), call its range the set of colours. These colourings are assumed to satisfy "for every $X \in I_i^+$ we can find some $Y \subseteq X$ with $Y \in I^+$, such that $\mathbf{c}_i \restriction [Y]^2$ (or $c_i \restriction [Y]^{<\aleph_0}$) is of some constant pattern". Now using $\bar{\eta}$ we can define a colouring $\mathbf{c}$ on $[\lambda]^2$ (or $[\lambda]^{<\aleph_0}$) "induced by the $\langle \mathbf{c}_i : i < \delta \rangle$", e.g.,

$$\begin{aligned} \mathbf{c}(\{\alpha, \beta\}) = \mathbf{c}_{i(\alpha,\beta)}(\{\eta_\alpha(i(\alpha,\beta)), \eta_\beta(i(\alpha,\beta))\}) \\ \text{where } i(\alpha,\beta) = \mathrm{Min}\{i : \eta_\alpha(i) \neq \eta_\beta(i)\}. \end{aligned}$$

Now, normality (or weak normality) is a natural assumption, because of the following:

1.5 Claim. *If $\bar{\eta}$ is a normally (λ, I, J)-sequence for $\bar{I}$, (or weakly so) and $X \in I^+$, <u>then</u> the following set is $= \delta \bmod J$ (or $\neq \emptyset \bmod J$):*

$$\begin{aligned} Y = \Big\{ i < \delta : &\text{for some } \nu \in \prod_{j<i} \mathrm{Dom}(I_j) \text{ and } X_i \in I_i^+ \\ &\text{we have} : (\forall x \in X_i)(\exists \alpha \in X)[\nu = \eta_\alpha \restriction i \ \& \ x = \eta_\alpha(i)] \Big\}. \end{aligned}$$

Proof. Let $X_i = \{\eta_\alpha(i) : \alpha \in X\}$, by the definitions it is enough to prove

$(*)$ if $X_i \in I_i^+$ then $i \in Y$.

Let $Z_i = \{\eta_\alpha \restriction i : \alpha < \lambda\}$, so $Z_i \subseteq \prod_{j<i} \mathrm{Dom}(I_j)$ and $|Z_i| < \mathrm{comp}(I_i)$ by the normality of $\bar\eta$. Now for each $\nu \in Z_i$ let us define

$$X^i_\nu = \{\eta_\alpha(i) : \alpha \in X \quad \text{and} \quad \eta_\alpha \restriction i = \nu\}.$$

Clearly $X_i = \bigcup\{X^i_\nu : \nu \in Z_i\}$, and I_i is $|Z_i|^+$-complete (as $|Z_i| < \mathrm{comp}(I_i)$). As $X_i \in I^+_i$, necessarily for some $\nu \in Z$ we have $X^i_\nu \in I^+_i$. This exemplifies that $i \in Y$, as required. $\square_{1.5}$

1.6 Conclusion. Assume

- (a) $\bar\eta$ is a normal weak (λ, I, J)-sequence for $\bar I$
- (b) $\mathbf{c}_i$ is a function from ${}^{\omega>}(\mathrm{Dom}(I_i))$ to a set $\mathbf{C}$ of colours (or from $[\mathrm{Dom}(I_i)]^{<\aleph_0}$)
- (c) $\mathbf{d}$ is a function from ${}^{\omega>}\varepsilon(*)$ (or from $[\varepsilon(*)]^{<\aleph_0}$) to $\mathbf{C}$
- (d) $\mathbf{c}_i$ exemplifies $I_i \nrightarrow (\mathbf{d})$ which means
  - $(*)$ for every $X \in I^+_i$ we can find distinct $x_\zeta \in X$ for $\zeta < \varepsilon(*)$ such that: if $n < \omega$ and $\zeta_0 < \cdots < \zeta_{n-1} < \varepsilon(*)$ then

$$\mathbf{c}_i(\langle x_{\zeta_0}, \dots, x_{\zeta_{n-1}}\rangle) = \mathbf{d}(\langle \zeta_0, \dots, \zeta_{n-1}\rangle)$$

$$(\text{or } \mathbf{c}_i(\{x_{\zeta_0}, \dots, x_{\zeta_{n-1}}\}) = \mathbf{d}(\{\zeta_0, \dots, \zeta_{n-1}\}))$$

- (e) We define the colouring $\mathbf{c}$ such that for all $n < \omega$

$$\mathbf{c}(\langle \alpha_0, \dots, \alpha_{n-1}\rangle) = \mathbf{c}_i(\langle \eta_{\alpha_0}(i), \dots, \eta_{\alpha_{n-1}}(i)\rangle)$$

$$(\text{or } \mathbf{c}(\{\alpha_0, \dots, \alpha_{n-1}\}) = \mathbf{c}_i(\{\eta_{\alpha_0}(i), \dots, \eta_{\alpha_{n-1}}(i)\}),$$

$$\ell < m < n \Rightarrow i = \mathrm{Min}\{j < \delta : \eta_{\alpha_\ell}(j) \neq \eta_{\alpha_m}(j)\}.$$

Then $\mathbf{c}$ exemplifies $I \nrightarrow (\mathbf{d})$.

Proof. Why? If $X \in I^+$, let Y be the set as in Claim 1.5, hence $Y \in J^+$. Pick an $i \in Y$, so there is $X_i \in I^+_i$ and ν exemplifying that $i \in Y$. Let $\{x_\zeta : \zeta < \varepsilon(*)\}$ exemplify that $I_i \nrightarrow (\mathbf{d})$. For $\zeta < \varepsilon(*)$, let $\alpha_\zeta \in X$ be such that $\eta_{\alpha_\zeta} \restriction i = \nu$ and $\eta_{\alpha_\zeta}(i) = x_\zeta$. Hence for all $n < \omega$ and $\zeta_0 < \dots \zeta_{n-1} < \varepsilon(*)$ we have

$$\mathbf{c}(\langle \alpha_{\zeta_0}, \dots \alpha_{\zeta_{n-1}}\rangle) = \mathbf{c}_i(\langle x_{\zeta_0}, \dots x_{\zeta_{n-1}}\rangle) = \mathbf{d}(\langle \zeta_0, \dots \zeta_{n-1}\rangle).$$

$\square_{1.6}$

1.7 Comments: 1) Of course in 1.6 we can restrict ourselves to colouring of pairs. Note that the conclusion works for all $\mathbf{d}$'s simultaneously. Also, additional properties of the $\mathbf{c}_i$'s are automatically inherited by $\mathbf{c}$, see 1.8 below.
2) We can also be interested in colours of n-tuples, $n > 3$, where $i < \delta$ as in clause (e) of 1.6 does not exist.
3) What is the gain in the conclusion?

A reasonable gain is "catching" more cardinals, i.e. if $I_i = J^{\mathrm{bd}}_{\lambda_i}, I = J^{\mathrm{bd}}_\lambda$, then in addition to having an example for λ_i we have one for λ. A better gain is when I is simpler than the I_i's. The best situation is when we essentially can get $I = J^{\mathrm{bd}}_\lambda, J = J^{\mathrm{bd}}_\delta$ for all normal $\bar I$ with $\langle |\mathrm{Dom}(I_i)| : i < \delta\rangle$ increasing with limit μ. Assuming a case of G.C.H. this is trivially true.

* * *

Normally we can find many tuples for which there is $i < \delta$ as in clause (e) of 1.6.

1.8 Fact. In 1.6 if $\theta = (2^{|\delta|})^+$, or at least $\theta = \mathrm{cf}(\theta)$ & $(\forall \alpha < \theta)(|\alpha|^{|\delta|} < \theta)$ then:

- $(*)$ for every $X \in [\lambda]^\theta$, we can find $Y \in [X]^\theta$ and $i < \delta$ and a 1-to-1 function h from Y into $\mathrm{Dom}(I_i)$ such that
$$\mathbf{c}(\langle \alpha_0, \dots, \alpha_{n-1}\rangle) = \mathbf{c}_i(\langle h(\alpha_0), \dots, h(\alpha_{n-1})\rangle)$$
for $\alpha_0, \dots, \alpha_{n-1} \in Y$ (actually $h(\alpha) = \eta_\alpha(i)$, where for all α we have $\eta_\alpha \restriction i = \nu$ for some $\nu \in \prod\limits_{j<i} \mathrm{Dom}(I_j)$).

Proof. By the Δ-system lemma applied to $\{\{\eta_\alpha \restriction i : i < \delta\} : \alpha \in X\}$. More elaborately, let χ be large enough, and let $M \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ be such that $\{\theta, X, I, J, \bar I, \bar\eta\} \subseteq M$ and ${}^\delta M \subseteq M$, while $\|M\| = \theta$ and $M \cap \theta$ is an ordinal $< \theta$. If we choose $\alpha \in X \backslash M$, then we can choose $i < \delta$ such that $\eta_\alpha \restriction i \in M$, $\eta_\alpha \restriction (i+1) \notin M$ (exists as ${}^\delta M \subseteq M$). Now notice that for some such α and i the set $Z =: \{\eta_\beta(i) : \beta \in X, \eta_\beta \restriction i = \eta_\alpha \restriction i\}$ has cardinality θ; this holds by clause (d) of Definition 1.1. Let $h : Z \to X$ be such that $\gamma \in Z \Rightarrow \eta_{h(\gamma)} \restriction i = \eta_\alpha \restriction i$ and $\eta_{h(\gamma)}(i) = \gamma$. Lastly let $Y = \mathrm{Rang}(h)$. $\square_{1.8}$

1.9 Lemma. *Assume*

- *(a)* I_i *is a* κ_i*-complete ideal on* λ_i *for* $i < \delta$*, and* δ *is a limit ordinal,*
- *(b)* $\kappa_i = \mathrm{cf}(\kappa_i) > \sum_{j<i} \kappa_j$,

(c) $\mu = \sup_{i<\delta} \kappa_i = \sup_{i<\delta} \lambda_i$,

(d) $\mathrm{cf}(I_i, \subseteq) \le \mu^+$ *(usually in applications it is* $< \mu$ *as usually* $2^{\lambda_i} < \mu$*; the cofinality is that of a partially ordered set),*

(e) $\lambda = \mu^+ = \mu^{|\delta|}$ *(so* $\lambda = \lambda^{|\delta|}$*; note that* $\mu^{|\delta|} \ge \mu^{\mathrm{cf}(\mu)} \ge \mu^+$ *always).*

Then some $\bar\eta$ *is a* μ^+*-sequence for* $\langle I_i : i < \delta\rangle$.

1.10 Remark. 1) We shall focus on the case μ as strong limit singular, $\delta = \mathrm{cf}(\mu)$.
(2) We can weaken the requirement $\lambda = \mu^+$, but not now and here.

Proof of 1.9. Let $\mathscr{Y}_i \subseteq I_i$ be cofinal,

$$|\mathscr{Y}_i| \le \lambda.$$

So $\left|\prod\limits_{i<\delta} \mathscr{Y}_i\right| \le \lambda^{|\delta|} = \lambda$, and we can list $\prod\limits_{i<\delta} \mathscr{Y}_i$ as $\left\langle \langle A^\zeta_i : i < \delta\rangle : \zeta < \lambda \right\rangle$, where $A^\zeta_i \in \mathscr{Y}_i$.

For $\zeta < \lambda$, let $\langle \beta(\zeta,\varepsilon) : \varepsilon < \mu\rangle$ list $\{\beta : \beta < \max\{\mu, \zeta\}\}$ (or $\{\beta : \beta \le \zeta\}$).

Now by induction on $\zeta < \lambda$, we choose a function $\eta_\zeta \in \prod\limits_{i<\delta} \lambda_i$. Let $\eta_\zeta(i)$ be any member of

$$\lambda_i \backslash \bigcup \{A^{\beta(\zeta,\varepsilon)}_i : \varepsilon < \sum_{j<i} \kappa_j\}.$$

[Why can we choose such $\eta_\zeta(i)$? Because $A^{\beta(\zeta,\varepsilon)}_i \in I_i$ and I_i is κ_i-complete and $\kappa_i > \sum_{j<i} \kappa_j$].

We claim that $\bar\eta =: \langle \eta_\zeta : \zeta < \lambda\rangle$ is as required. Let X be unbounded $\subseteq \lambda$, we need to show Y is co-bounded in δ, where

$$Y =: \{i : \{\eta_\alpha(i) : \alpha \in X\} \in I^+_i\}.$$

Let $A^*_i = \{\eta_\alpha(i) : \alpha \in X\}$ for every $i \notin Y$. Let $A^*_i =: \emptyset$ for $i \in Y$. Let $A_i \in \mathscr{Y}_i$, $A_i \supseteq A^*_i$. Let $\zeta < \lambda$ be such that $\langle A_i : i < \delta\rangle = \langle A^\zeta_i : i < \delta\rangle$. So for every $\alpha \in X \backslash (\zeta + 1)$, for every $i < \delta$ large enough $\eta_\alpha(i) \notin A_i$.

[Large enough means: Just that letting $\varepsilon = \varepsilon_{\alpha,\zeta} < \mu$ be such that $\zeta = \beta(\alpha, \varepsilon)$ and letting $i^* = i^*_{\alpha,\zeta}$ be such that $\sum_{j<i^*} \kappa_j > \varepsilon$, then $i \in [i^*, \delta) \Rightarrow \eta_\alpha(i) \notin A_i$]. $\square_{1.9}$

<u>1.11 Example</u>: $\lambda = \mu^+ = 2^\mu$, μ strong limit of cofinality $\aleph_0$. Let $\mu = \sum_{n<\omega} \mu_n$. Without loss of generality $\mu_{n+1} > \beth_{n+7}(\mu_n)$. Let $D_n = [\beth_{n+3}(\mu_n)^+]^n$

$$I_n =: \{X \subseteq D_n : \text{there is } h : X \to 2^{\mu_n} \text{ such that for no infinite subset } A \text{ of } (\beth_{n+3}(\mu_n))^+ \text{ is } h \restriction (X \cap [A]^n) \text{ constant}\}.$$

1.12 Fact. I_n is an ideal.

1.13 Fact. The ideal I_n is not trivial (so $D_n \notin I_n$).
[Why? By the Erdös-Rado Theorem, see 1.17 - 1.18 for a detailed explanation.]

1.14 Fact. I_n is μ_n^+-complete.
[Why? If $h_i : D_n \to 2^{\mu_n}$ $(i < \mu_n)$, then there is $h : D_n \to 2^{\mu_n}$ such that $h(x) = h(y) \Rightarrow \bigwedge_i h_i(x) = h_i(y)$].

1.15 Conclusion. So, By Lemma 1.9, there is $\bar{\eta} = \langle \eta_i : i < \mu^+ = \lambda \rangle$ which is a λ-sequence for $\langle I_n : n < \omega \rangle$.

We apply Conclusion 1.15 to measure algebras getting a well known result: <u>1.16 Application</u>
Assume $\lambda = \mu^+$ and μ is a strong limit singular of cofinality $\aleph_0$ (i.e., as in 1.11). If $\mathscr{B}$ is a measure algebra (Maharam) of dimension $\geq \mu$, we can find $a_\alpha \in \mathscr{B}$ for $\alpha < \lambda$ with $\mathrm{Leb}(a_\alpha) > 0$ for each α, such that for every $X \in [\lambda]^\lambda$ we can find $n^* < \omega$, $\alpha_1, \dots, \alpha_{n^*} \in X$ satisfying

$$\mathscr{B} \models \bigcap_{\ell=1}^{n^*} a_{\alpha_\ell} = 0.$$

Proof. Let $\bar{\eta}$ and I_n be as in conclusion 1.15 (all in the context of Example 1.11). Let $\langle x_{n,\alpha} : n < \omega, \alpha < \beth_{n+3}(\mu_n)^+ \rangle$ be independent in the sense of measure, all elements of $\mathscr{B}$ and of measure $1/2$.

For any $\eta \in \prod_{n<\omega} D_n$, let

$$y_{\eta,n} = y_{\eta(n)} = 1 - \bigcap_{\beta \in \eta(n)} x_{n,\beta} - \bigcap_{\beta \in \eta(n)} (1_{\mathscr{B}} - x_{n,\beta}).$$

Note that $\bigcap_{\beta \in \eta(n)} x_{n,\beta}$ has measure 2^{-n} (by the choice of the $x_{n,\alpha}$'s). So $\mathrm{Leb}(y_{\eta,n}) = 1 - 2 \cdot 2^{-n}$ (hence $\mathrm{Leb}(y_{\eta(n)}) > 0$ if $n \geq 2$). Let

$$y_\eta = \bigcap_{n \geq 5} y_{\eta,n} \in \mathscr{B}.$$

So $\mathrm{Leb}(y_\eta) \geq 1 - 2 \cdot \sum_{n \geq 5} 2^{-n} = 1 - 2 \cdot 2^{-4} = 1 - 2^{-3} > 1/2$. We let $a_\alpha = y_{\eta_\alpha}$ for $\alpha < \lambda$. We check that $\langle a_\alpha : \alpha < \lambda \rangle$ is as required. Suppose $X \in [\lambda]^\lambda$. So, as $\bar{I}$ is normal, for some $n > 5$ and $\nu \in \prod_{\ell < n} D_\ell$ we have

$$Y_\nu =: \{\eta_\alpha(n) : \alpha \in X, \eta_\alpha \restriction n = \nu\} \in I_n^+.$$

(Note that ν is not really needed for the rest of the proof.)

So there is $\{\gamma_\ell : \ell < \omega\} \subseteq \beth_{n+3}(\mu_n)^+$ increasing such that

$$[\{\gamma_\ell : \ell < \omega\}]^n \subseteq Y_\nu.$$

We use just $\langle \gamma_\ell : \ell < 2n - 1 \rangle$.

For $u \in [\{\gamma_\ell : \ell < 2n - 1\}]^n$ let $\alpha(u) \in X$ be such that

$$\eta_{\alpha(u)}(n) = u.$$

It is enough to show that in $\mathscr{B}$

$$\bigcap_u y_{\eta_{\alpha(u)}} \leq \bigcap_u a_{\alpha(u)} = 0.$$

So suppose that there is $z \in \mathscr{B}$ with $\mathrm{Leb}(z) > 0$ and such that $z \leq \bigcap_u y_{\eta_{\alpha(u)}}$. Then without loss of generality

$$\ell < 2n - 1 \Rightarrow z \leq x_{n,\gamma_\ell} \vee z \leq 1 - x_{n,\gamma_\ell}.$$

<u>Case 1</u>: $|\{\ell : z \leq x_{n,\gamma_\ell}\}| \geq n$. Let $u \in [\{\gamma_\ell : \ell < 2n - 1\}]^n$ be such that

$$\bigwedge_{\gamma_\ell \in u} (z \leq x_{n,\gamma_\ell}).$$

So $zle \bigcap_{\gamma_\ell \in u} x_{n,\gamma_\ell}$. But $z \leq y_{\eta_{\alpha(u)}} \leq 1_{\mathscr{B}} - \bigcap_{\gamma \in u} x_{n,\gamma}$, a contradiction.

<u>Case 2</u>: Not Case 1. So necessarily $|\{\ell : z \leq 1_{\mathscr{B}} - x_{n,\gamma_\ell}\}| \geq n$ and continue as above using $1_{\mathscr{B}} - x_{n,\gamma_\ell}$. $\square_{1.16}$

Let us elaborate on the ideals used above.

1.17 Definition. For n, λ, ε let

$$\mathrm{ERJ}^{n,\varepsilon}_\lambda = J^{n,\varepsilon}_\lambda = \{A \subseteq [\lambda]^n : \text{ there is no } w \subseteq \lambda \text{ satisfying } \mathrm{otp}(w) = \varepsilon \text{ and } [w]^n \subseteq A\}$$

$$\mathrm{ERI}^{n,\varepsilon}_{\lambda,\mu} = I^{n,\varepsilon}_{\lambda,\mu} = \{A \subseteq [\lambda]^n : \text{there are } A_i \in J^{n,\varepsilon}_\lambda \text{ for } i < i(*) < \mu \text{ such that } A = \bigcup_{i<i(*)} A_i\}.$$

1.18 Fact. 1) $I^{n,\varepsilon}_{\lambda,\mu}$ is a cf(μ)-complete ideal on $[\lambda]^n$, not necessarily proper (see (2)); note that $J^{n,\varepsilon}_\lambda$ is not necessarily an ideal.
2) $I^{n,\varepsilon}_{\lambda,\mu}$ is a proper ideal, i.e., $[\lambda]^n \notin I^{n,\varepsilon}_{\lambda,\mu}$ iff

$$\chi < \mu \Rightarrow \lambda \to (\varepsilon)^n_\chi.$$

3) $I_n = I^{n,\omega}_{\beth_{n+7}(\mu_n)^+,(2^{\mu_n})^+}$ (where I_n and $\langle \mu_n : n < \omega\rangle$ are from 1.11).
4) In the proof of 1.16 we could have used less, for example

$$I_n = I^{n,2n+1}_{\beth_{n+1}(\mu_n)^+,\mu_n^+}$$

as $\beth_{n+1}(\mu_n)^+ \to (\mu_n^+)^n_{\mu_n}$ for $n \geq 1$.

Proof. (3) First direction.

Let $A \in I_n$, so there is $h : A \to 2^{\mu_n}$ witnessing it. Let $A_i = h^{-1}(i)$ for $i < 2^{\mu_n}$ Now $X \subseteq \lambda$, $|X| \geq \aleph_0 \;\Rightarrow\; [X]^n \nsubseteq A_i$, by the choice of A. Hence

$$A_i \in J^{n,\omega}_{\beth_{n+7}(\mu_n)^+}.$$

Hence

$$A \in I^{n,\omega}_{\beth_{n+7}(\mu_n)^+,(2^{\mu_n})^+}.$$

Second direction: Let $A \in I^{n,\omega}_{\beth_{n+7}(\mu_n)^+,(2^{\mu_n})^+}$, so there are A_i (for $i < i(*) < (2^{\mu_n})^+$) such that $A_i \in J^{n,\omega}_{\beth_{n+7}(\mu_n)^+}$ and $A = \bigcup\limits_{i<i(*)} A_i$.

Renaming, without loss of generality $i(*) \leq 2^{\mu_n}$, and let

$$A'_i = \begin{cases} A_i \setminus \bigcup_{j<i} A_i & \text{if } i < i(*) \\ \emptyset & \text{otherwise, i.e., if } i \in [i(*), 2^{\lambda_n}). \end{cases}$$

So $\langle A'_i : i < i(*) \rangle$ is a partition of A. As $A_i \in J^{n,\omega}_{\beth_{n+7}(\mu_n)^+}$, we know that $\neg(\exists X \subseteq \beth_{n+7}(\mu_n)^+$ infinite) $([X]^n \subseteq A_i)$. Hence, letting $\kappa = \beth_{n+7}(\mu_n)^+$

$$\neg(\exists X \subseteq \kappa \ \text{ infinite}) \ \ ([X]^n \subseteq A_i).$$

Define $h : A \to 2^{\mu_n}$ by

$$h(\bar\alpha) = i \quad \text{ iff } \bar\alpha \in A'_i,$$

so h witnesses $A \in I_n$. $\square_{1.18}$

1.19 Definition. 1) A set $W \subseteq [\lambda]^{<\aleph_0}$ is called a ccc base <u>if</u>:

$(*)$ for $u \neq v$ in W, $|u \cap v| < |u|/2$.

2) For $W \subseteq [\lambda]^{<\aleph_0}$ let

$$I_\lambda[W] = \{A \subseteq \lambda : W \cap [A]^{<\aleph_0} = \emptyset\},$$

$$I_{\lambda,\kappa}[W] = \big\{A \subseteq \lambda : A \text{ is the union of } < \kappa \text{ members of } I_\lambda[W]\big\}.$$

3) For a Boolean algebra $\mathbf{B}$ we define $I_{\mathbf{B},\kappa}$ by letting : $X \in I_{\mathbf{B},\kappa}$ iff $X \subseteq \mathbf{B}\setminus\{1\}$ is the union of $< \kappa$ ideals of $\mathbf{B}$.

1.20 Claim. *1) Assume*

(a) *$\bar\eta$ is a (λ, J)-sequence for $\bar I = \langle I_i : i < \delta\rangle$, and $\mathrm{cf}(\lambda) > \delta$*

(b) *for $i < \delta$, the function $h_i : \mathrm{Dom}(I_i) \to \lambda_i$ satisfies*

$$\alpha < \lambda_i \Rightarrow \{x \in \mathrm{Dom}(I_i) : h_i(x) < \alpha\} \in I_i.$$

Let $\bar h = \langle h_i : i < \delta\rangle$ and let $f_\alpha = \bar h \circ \eta_\alpha =: \langle h_i(\eta_\alpha(i)) : i < \delta\rangle$ $\big(\in \prod\limits_{i<\delta} \lambda_i\big)$.
<u>Then</u>

(c) $(\forall f \in \prod\limits_{i<\delta} \lambda_i)(\forall^{J^{bd}_\lambda} \gamma < \lambda)(f <_J f_\gamma)$

(d) *for some club E of λ, we have*

$(d)_E$ *if $\alpha < \varepsilon \leq \beta < \lambda$ and $\varepsilon \in E$ then $f_\alpha <_J f_\beta$.*

(So if $X \in [\lambda]^\lambda$, $(\forall \delta \in E)$ $|X \cap (\delta, \min(E \backslash (\delta+1))]| \leq 1$ then $\langle f_\alpha : \alpha \in X \rangle$ is $<_J$-increasing cofinal in $\prod\limits_{i<\delta} \lambda_i$.)

2) If $\bar f = \langle f_\alpha : \alpha < \lambda \rangle$, E satisfies $(d)_E$ (and of course $\sup_{i<\delta} \lambda_i < \lambda$) and $\mu < \lambda$ <u>then</u> without loss of generality for X as in $(d)_E$ the sequence $\bar f \restriction X$ is μ-free (see Definition 1.21(1) below), moreover $\bar f$ is (μ, E)-free (see below clause (1) of 1.21), provided that $(*)$ or just the weaker $(*)'$ or just $(*)''$ below holds where:

$\boxtimes$ *for $\lambda > \mu$ and $\bar\lambda = \langle \lambda_i : i < \delta \rangle$ we consider the conditions*

- $(*)$ $\lambda = \chi^+, \chi = \mathrm{cf}(\chi) \geq \mu = \lim_J \langle \lambda_i : i < \delta \rangle$ *for some χ,*
- $(*)'$ $\mu = \lim_J \lambda_i$ *and* $\{\delta < \lambda : \mathrm{cf}(f(\delta)) < \mu\} \in I[\lambda]$,
- $(*)''$ *there is $\bar f' = \langle f'_\alpha : \alpha < \lambda \rangle$ which is $<_J$-increasing cofinal in $(\prod\limits_{i<\delta} \lambda_i, <_J)$ and is μ-free.*

1.21 Definition. Let J be an ideal and $\bar f = \langle f_\alpha : \alpha < \alpha^* \rangle$ a sequence of functions from $\mathrm{Dom}(J)$ into the ordinals.
1) $\bar f$ is μ-free <u>if</u> for $X \in [\alpha^*]^{<\mu}$ we can find $\bar s = \langle s_\alpha : \alpha \in X \rangle$, $s_\alpha \in J$ such that

$$[\alpha < \beta \ \& \ \alpha \in X \ \& \ \beta \in X \ \& \ i \in J \backslash s_\alpha \backslash s_\beta] \Rightarrow f_\alpha(i) < f_\beta(i).$$

2) $\bar f$ is (μ, E)-free for J <u>if</u> for $X \in [\alpha^*]^{<\mu}$ we can find $\bar s = \langle s_\alpha : \alpha \in X \rangle$, $s_\alpha \in J$ such that

$$[\alpha \leq \delta < \beta \ \& \ \alpha \in X \ \& \ \delta \in E \ \& \ \beta \in X \ \& \ i \in \delta \backslash s_\alpha \backslash s_\beta] \Rightarrow f_\alpha(i) < f_\beta(i).$$

Proof of 1.20. 1) <u>Clause (c)</u>: Let $f \in \prod\limits_i \lambda_i$, so $A_{f,i} =: \{x \in \mathrm{Dom}(I_1) : h_i(x) \leq f(\alpha)\} \in I_i$ hence by clause (a) which we are assuming, for some $\gamma^* < \lambda$ we have $\gamma \in [\gamma^*, \lambda) \Rightarrow \{i < \delta : f_\gamma(i) \notin A_{f,i}\} \in J$. By the definition of f_γ this means $\gamma \in [\gamma^*, \lambda) \Rightarrow \{i < \delta : \neg(f(i) < f_\gamma(i))\} \in J$, so we are done.

<u>Clause (d)</u>: By Clause (c) for each $\beta < \lambda$ there is $\gamma_\beta < \lambda$ such that $\gamma \in [\gamma_\beta, \lambda) \Rightarrow f_\beta <_J f_\gamma$. Let $E = \{\delta < \lambda : \delta$ is a limit ordinal such that $(\forall \beta < \delta)(\gamma_\beta < \delta)$. Now E is as required.
2) As in [Sh:g, II,§1,I]. $\square_{1.20}$

1.22 Remark. This applies to the construction in §4, §5, etc., (e.g., construction from $\lambda = \prod\limits_{i<\delta} \lambda_i / J^{bd}_\delta$).

§2 There are large free subsets

The reader may wonder if really something like $\lambda = \mathrm{cf}(\lambda) \in (\mu, 2^\mu]$ for μ strong limit singular, is necessary for 1.16. As in [Sh 575], the answer is yes, though not for the same reason.

Of course, in what follows, Maharam measure algebra can be replaced by any measure algebra. The interesting case is $(\exists \chi)(\chi < \lambda \leq \chi^{\aleph_0})$.

2.1 Fact. Let $\mathscr{B}$ be a Maharam measure algebra. If $\beth_2 \leq \mu = \mu^{\aleph_0} < \mathrm{cf}(\lambda) \leq \lambda \leq 2^\mu$ and $a_\alpha \in \mathscr{B}^+$, (so $\mathrm{Leb}(a_\alpha) > 0$) for $\alpha < \lambda$ are pairwise distinct, <u>then</u> for some $X \in [\lambda]^\lambda$ we have:

- $(*)$ any nontrivial Boolean combination of finitely many members of $\{a_\alpha : \alpha \in X\}$ has positive measure.

Proof. Let $\{x_i : i < i(*)\}$ be a basis of the Maharam measure algebra (so each x_i has measure $1/2$ and x_i's are measure-theoretically independent). So for each $\alpha < \lambda$ we can find ordinals $i(\alpha, n) < i(*)$ for $n < \omega$, and a Boolean term τ_α such that $a_\alpha = \tau_\alpha(x_{i(\alpha,0)}, x_{i(\alpha,1)}, \ldots)$. Note that this equality is only modulo the ideal of null sets. Remember

- $(*)_0$ we can replace $\langle x_\alpha : \alpha < \lambda \rangle$ by $\langle x_\alpha : \alpha \in X \rangle$ for any $X \in [\lambda]^*$.

Without loss of generality, each τ_α is a countable intersection of a countable union of finite Boolean combinations of the x_i's. Again without loss of generality, $\langle i(\alpha, n) : n < \omega \rangle$ is with no repetition. Note that without loss of generality

$$i(*) = \{i(\alpha, n) : \alpha < \lambda \text{ and } n < \omega\}.$$

Hence without loss of generality $i(*) \leq \lambda$, hence without loss of generality $i(*) = \lambda$. By Engelking Karlowicz Theorem [EK], clearly we can divide λ to μ sets $\langle X_\zeta : \zeta < \mu \rangle$ such that

- $(*)_1$ the sets $A_{\zeta,n} =: \{i(\alpha, n) : \alpha \in X_\zeta\}$ for each ζ satisfy: $\langle A_{\zeta,n} : n < \omega \rangle$ are pairwise disjoint.

As the number of possible terms τ_α is $\leq 2^{\aleph_0} \leq \mu$ by $(*)_0$ without loss of generality

- $(*)_2$ if $\alpha, \beta \in X_\zeta$ then $\tau_\alpha = \tau_\beta$, call it τ^ζ.

Note also

$(*)_3$ if $Y \subseteq X_\zeta$ then

$$\begin{aligned}\mathrm{ind}(Y) =: \Big\{\alpha \in Y : & \text{for no } m < \omega \text{ and } \beta_0, \dots, \beta_{m-1} \in Y \cap \alpha \text{ do we have :} \\ & a_\alpha \in \text{ the complete subalgebra generated by} \\ & \{x_{i(\beta_\ell, n)} : \ell < m, n < \omega\}\Big\}\end{aligned}$$

satisfies $|\mathrm{ind}(Y)| + 2^{\aleph_0} \geq |Y|$.

[Why? We can prove by induction on $\alpha \notin \mathrm{ind}(Y)$ that for some $m < \omega$ and $\beta_0, \beta_1, \dots, \beta_{m-1} \in \mathrm{ind}(Y) \cap \alpha$ we have $a_\alpha \in$ the complete subalgebra of $\mathscr{B}$ generated by $\{x_{i(\beta_\ell, n)} : \ell < m, n < \omega\}$, using the transitive character of this property. Now for each $m < \omega$ and $\beta_0, \dots, \beta_{m-1} \in \mathrm{ind}(Y)$, the number of a_α such that $a_\alpha \in$ (the subalgebra generated by $\{x_{i(\beta_\ell, n)} : \ell < m, n < \omega\}$) is at most continuum.]

As $\mathrm{cf}(\lambda) > \mu$, for at least one $\zeta < \mu$, $|X_\zeta| = \lambda$, hence by $(*)_3$ we have $|\mathrm{ind}(X_\zeta)| = \lambda$. So, without loss of generality

$(*)_4(a)$ the sets $A_n = \{i(\alpha, n) : \alpha < \lambda\}$ are pairwise disjoint,

(b) $\tau_\alpha = \tau$ for $\alpha < \lambda$,

(c) for no $m < \omega$ and $\beta_0 < \dots < \beta_m < \lambda$ do we have $a_{\beta_m} \in$ the complete subalgebra generated by $\{x_{i(\beta_\ell, n)} : \ell < m, n < \omega\}$.

Now for each $\alpha < \lambda$ we define an ideal I'_α on ω (thought apriori $I_\alpha = \mathscr{P}(\omega)$ is allowed): it is the ideal generated by the sets

$$Z_{\alpha,\beta} =: \{n < \omega : i(\beta, n) = i(\alpha, n)\} \quad \text{for } \beta < \alpha.$$

and (where $\mathrm{ch}_A(n)$ is 1 if $n \in A$ and 0 if $n \notin A$ for any $A \subseteq \omega$)

$$\begin{aligned}J =: \big\{A \subseteq \omega : & \tau(x_0, x_2, \dots, x_{2n}, \dots) \\ & = \tau(x_{0+\mathrm{ch}_A(0)}, x_{2+\mathrm{ch}_A(1)}, \dots, x_{2n+\mathrm{ch}_A(n)}, \dots)\big\}.\end{aligned}$$

As $\{x_i : i < i(*)\}$ is free (in the measure theoretic sense),

$(*)_5$ if

(a) for $A \in J$, and $\{\alpha_m : n < \omega\}$ and $\{\beta_n : n < \omega\}$ such that $\alpha_n < i(*)$ are with no repetition and $\beta_n < i(*)$ with no repetition, and

(b) $(\forall m, n < \omega)[\alpha_n = \beta_m \Leftrightarrow n = m \ \&\ n \notin A]$
<u>then</u> $\tau(x_{\alpha_0}, \dots) = \tau(x_{\beta_0}, \dots)$ in $\mathscr{B}$, of course.

(Just apply the definition of J to $\langle x_{\alpha_0}, x_{\beta_0}, x_{\alpha_1}, \dots\rangle$). By transitivity of equality (i.e., using $\langle \gamma_n : n < \omega\rangle$ such that $n \notin A \Rightarrow \gamma_n = \alpha_n = \beta_n$ and $n \in A \Rightarrow \gamma_n \notin \{\alpha_m : m < \omega\} \cup \{\beta_m : m < \omega\} \cup \{\gamma_m : m \notin n\}$) we get

$(*)_6$ if (a) of $(*)_5$ then

$$(\forall n < \omega)[n \notin A \;\Rightarrow\; \alpha_n = \beta_n] \quad \Rightarrow \quad \tau(x_{\alpha_0}, \dots) = \tau(x_{\beta_0}, \dots).$$

Hence J is closed under subsets and (finite) unions, that is J is an ideal on ω. Let I_α be the ideal on ω generated by $I'_\alpha \cup J$. By clause (c) of $(*)_4$ and $(*)_6$ we know that $\omega \notin I'_\alpha$; recall that I'_α is an ideal on ω though it is possible that singletons are not in I_α (a violation of a convention in §0). [In fact we could have eliminated this violation, but there is no reason to put extra work for it.] Also $J \subseteq I_\alpha$.

Now, the number of possible ideals on ω is at most $\beth_2 \le \mu <$ cf(λ), so it suffices to prove

$(*)_7$ if $Y \subseteq \lambda$, $\alpha \in Y \Rightarrow I_\alpha \subseteq I$, where I is an ideal on ω (so $\omega \notin I$ but singletons may or may not belong to I) extending J, then any finite Boolean combination of $\{a_\alpha : \alpha \in Y\}$ has positive measure.

Proof of $(*)_7$. Let $\beta_0 < \dots < \beta_{m-1}$ be from Y. Let

$$A = \{n < \omega : \text{ for some } \ell < k < m \text{ we have } i(\beta_\ell, n) = i(\beta_k, n)\}.$$

By the definition of $Z_{\alpha,\beta}$, clearly $A \in I$. For $Z \subseteq i(*)$ let $\mathscr{B}^*[Z]$ be the complete subalgebra of $\mathscr{B}$ generated by $\{x_\beta : \beta \in Z\}$. We let $\mathscr{B}^* =: \mathscr{B}^*[Z_*]$ where $Z_* = \{i(\beta_\ell, n) : \ell < m, n \in A\}$. Let $\mathscr{B}^*_\ell =: \mathscr{B}^*[\{i(\beta_\ell, n) : n \in A\}]$.

As $\mathscr{B}^*_\ell$ is complete, for each $\ell < m$ we can find $b^-_\ell, b^+_\ell \in \mathscr{B}^*_\ell$ such that

(i) $b^-_\ell \le a_{\beta_\ell} \le b^+_\ell$,
(ii) if $c \in \mathscr{B}^*_\ell$ then $c \le a_{\beta_\ell} \Rightarrow c \le b^-_\ell$ and $c \ge a_{\beta_\ell} \Rightarrow c \ge b^+_\ell$.

By the definition of $\mathscr{B}^*$ and the assumptions on $\langle x_i : i < i(*)\rangle$ and on $\langle a_\alpha : \alpha < \lambda\rangle$ clearly

$(*)_8$ if $\{i(\beta_\ell, n) : n \in A\} \subseteq Z$ and $\{i(\beta_\ell, n) : n \in \omega \backslash A\} \cap Z = \emptyset$ and $Z \subseteq i(*)$ then

$(ii)_Z$ if $c \in \mathscr{B}^*[Z]$, then $c \le a_{\beta_\ell} \;\Rightarrow\; c \le b^-_\ell$ and $c \ge a_{\beta_\ell} \;\Rightarrow\; c \ge b^+_\ell$.

Obviously, for some Boolean terms τ^-_ℓ, τ^+_ℓ we have
$b^-_\ell = \tau^-_\ell(\dots, x_{i(\beta_\ell,n)},\dots)_{n\in A}$
$b^+_\ell = \tau^+_\ell(\dots, x_{i(\beta_\ell,n)},\dots)_{n\in A}$.
Now, as $\tau_\alpha = \tau$ for $\alpha \in Y$, clearly $\tau^-_\ell = \tau^-$ and $\tau^+_\ell = \tau^+$ for some fixed τ^- and τ^+. Also $b^-_\ell < b^+_\ell$ as otherwise $\omega\backslash A \in J$. Let $b_\ell = b^+_\ell - b^-_\ell$ so $\mathrm{Leb}(b_\ell) > 0$, and for some term τ^*, $b_\ell = \tau^*(\dots, x_{i(\beta_l,n)},\dots)_{n\in A}$, and let $b = \bigcap\limits_{\ell<m} b_\ell \in \mathscr{B}^*$.

Clearly

$(*)_9$ $\mathrm{Leb}(b) > 0 \Rightarrow$ any Boolean combination of the $a_{\beta_\ell}(\ell < m)$ has positive measure.

[Why? prove it on $\{a_{\beta_\ell} : \ell < m'\}$ by induction on $m' \le m$ using $(*)_8$.]

For proving $\mathrm{Leb}\Big(\bigcap\limits_{\ell<m} b_\ell\Big) > 0$, we define an equivalence relation E on ω:

$$\begin{aligned} n_1 E n_2 \text{ iff for every } \ell < k < m \text{ we have}\\ i(\beta_\ell, n_1) = i(\beta_k, n_1) \Leftrightarrow\ i(\beta_\ell,n_2) = i(\beta_k,n_2). \end{aligned}$$

Clearly E has finitely many equivalence classes, say $A_0, A_1, \dots, A_{k(*)-1}$. For $k_1 \le k(*)$ and $\bar\ell = \langle \ell_k : k_1 \le k < k(*)\rangle$ satisfying $\ell_k < m$ let

$$\begin{aligned} Z_{k_1,\bar\ell} = \big\{\tau^*(\dots, x_{i(\gamma_n,n)},\dots) :\ & \text{for every } k < k(*),\ \text{for some } \ell < m \text{ we have}\\ & \langle \gamma_n : n \in A_k\rangle = \langle i(\beta_n,\ell) : n \in A_k\rangle,\\ & \text{but if } k \ge k_1 \text{ then } \ell = \ell_k\big\}. \end{aligned}$$

We prove by induction on $k_1 \le k(*)$ that for any appropriate $\bar\ell$

$$c_{\bar\ell} =: \mathrm{Leb}(\bigcap\{b : b \in Z_{k_1,\bar\ell}\}) > 0.$$

(In fact the measure does not depend on $\bar\ell$.)

For $k_1 = k(*)$ we have $\{b_\ell : \ell < m\} \subseteq Z_{k_1,\langle\,\rangle}$ so this gives the desired conclusion.

<u>The case $k_1 = 0$</u>:

It is trivial: $Z_{0,\bar\ell}$ is a singleton $\{\tau^*(\dots, x_{i(\gamma_n,n)},\dots)\}$, where $\gamma_n \in A_n$ so obviously it has positive measure.

<u>The case $k_1 + 1$</u>:

So let $\bar{\ell} = \langle \ell_k : k_1 + 1 \le k < k(*) \rangle$, and we know that for each $n < \omega$ the element $d_\ell = c_{\langle n \rangle \hat{}\, \bar{\ell}}$ is > 0. For $t < m$ let f_t be a function from $Y = \big\{ i(\beta_\eta, \ell) : \ell < \omega$ such that if $\ell \in A_k$ then $k \in [k_1 + 1, k(*)] \Rightarrow n = \ell_k$ and $k = k_1 \Rightarrow n = 0$ and $k < k_1 \Rightarrow n < m \big\}$ into λ, f_t is one to one, f_t is the identity on $Y^* = \{ i(\beta_n, \ell) \in Y : \ell \notin A_{k_1} \}$ and $\langle \mathrm{Rang}(f_t \restriction (Y \backslash Y^*)) : t < m \rangle$ are pairwise disjoint and

$$\ell \in A_{k_1} \Rightarrow f_t(i(\beta_0, \ell)) = i(\beta_t, \ell).$$

Now we can imitate the beginning of the proof of $(*)_5$ and get $\bigcap_{n<m} d_n > 0$. Let $Y_t =$ Rang(f_t), and note that f_0 is the identity and $Y_0 = Y$. Clearly f_t induces an isomorphism from $\mathscr{B}[Y_0]$ onto $\mathscr{B}[Y_t]$. Call it $\hat{f}_t$ and easily $d_t =: \hat{f}_t(d_0)$. So we can imitate the beginning of the proof of $(*)_5$ and get $\bigcap\limits_{n<m} d_n > 0$. But

$$c_i = \bigcap_{n<m} c_{\langle n \rangle \hat{}\, \bar{\ell}} = \bigcap_{n<m} d_n > 0$$

as required. $\square_{2.1}$

2.2 Discussion. 1) The proof of 2.1 gives more, almost a division to $\le \mu$ subfamilies of independent elements (in the Boolean algebra sense), see **?** below.
$\longrightarrow$ scite{2.16} undefined
2) We may wonder if "$\mu \ge \beth_2$" is necessary. Actually it almost is not (see 2.5 below) but cf$(\lambda) > 2^{\aleph_0}$ is essential (see 3.11 below).

We shall see below (in 2.5) what we can get from the proof of 2.1.

2.3 Definition. For a Boolean algebra $\mathbf{B}$ we say $\langle \langle a_\alpha, b_\alpha \rangle : \alpha < \alpha^* \rangle$ is an explicitly independent sequence of intervals in $\mathbf{B}$ if:

(a) $\mathbf{B} \models a_\alpha < b_\alpha$,

(b) if $u_0, u_1 \subseteq \alpha^*$ are finite and disjoint then

$$\mathbf{B} \models \bigcap_{\alpha \in u_0} b_\alpha \cap \bigcap_{\alpha \in u_1} (-a_\alpha) > 0.$$

2.4 Claim. *Assume*

$(*)_Y[X](a)$ *$|X| = \chi$ and $\mathscr{B}(X)$ is a Maharam measure algebra with free basis $\{x_i : i \in X\}$. For $Z \subseteq X$ we let $\mathscr{B}(Z)$ be the complete subalgebra of $\mathscr{B}(X)$ generated by $\{x_i : i \in Z\}$*

$(b)_Y$ *$a_\alpha \in \mathscr{B}^+$ (i.e., Leb$(a_\alpha) > 0$) for $\alpha \in Y$ and $\beta < \alpha \ \Rightarrow\ a_\beta \neq a_\alpha$, while $|Y| = \lambda, Y$ a set of ordinals for simplicity.*

1) If $\lambda = \mathrm{cf}(\lambda) > \aleph_1$ <u>then</u> for some $Y' \in [Y]^\lambda$, $Z \in [X]^{<\lambda}$ and $a^-_\alpha \le a^+_\alpha$ from $\mathscr{B}(Z)$ we have:

- *(i) for $c \in \mathscr{B}(Z)$ we have $c \le a_\alpha \ \Rightarrow\ c \le a^-_\alpha$ and $a_\alpha \le c \ \Rightarrow\ a^+_\alpha \le c$,*
- *(ii) if $u \in [Y']^{<\aleph_0}$, $\eta \in {}^u2$ and*

$$\bigcap\{a^+_\alpha : \alpha \in u, \eta(\alpha) = 1\} \cap \bigcap\{1 - a^-_\alpha : \alpha \in u, \eta(\alpha) = 0\} \neq 0,$$

then $\bigcap\limits_{\alpha \in u} a^{[\eta(\alpha)]}_\alpha \neq 0$, where $c^{[0]} = -c$, $c^{[1]} = c$.

2) Assume $\inf\{Leb(a_\alpha \triangle b) : b \in \langle a_\beta : \beta < \alpha\rangle\}_{\mathscr{B}} > 0$ for $\alpha \in Y$. <u>Then</u> in part (1) we can demand $a^-_\alpha < a^+_\alpha$. Hence

- *($*$) there is $Y'' \in [Y']^\lambda$ such that $\langle a_\alpha : \alpha \in Y''\rangle$ is independent iff there is $Y'' \in [Y']^\lambda$ such that $\langle (a^-_\alpha, a^+_\alpha) : \alpha \in Y''\rangle$ is explicitly independent. (See Definition 2.3 above.)*

3) If $|Y| = \lambda > |X| = \chi$ and $\chi_1 < \chi, \sigma = \mathrm{cov}(\chi, \chi_1^+, \aleph_1, 2) < \lambda$ <u>then</u> Y can be represented as the union of $\le \sigma$ subsets Y' such that for each there is $Z \in [\chi]^{\le \chi_1}$ satisfying $\{a_\alpha : \alpha \in Y'\} \subseteq \mathscr{B}(Z)$.
4) If the clause (α) below holds then we can represent Y as the union of $\le \mu$ subsets Y' each satisfying (c) below (and $(b)_{Y'}$),

- *$(c)_{Y'}$ $a_\alpha = \tau(\dots, x_{i(\alpha,n)}, \dots)_{n<\omega}$, $n \neq m \ \Rightarrow\ i(\alpha, n) \neq i(\alpha, m)$ and the sets $A_n(Y') = \{i(\alpha, n) : n < \omega\}$ are pairwise disjoint, where*
- *$(\alpha)(i)$ $2^{\aleph_0} \le \mu = \mu^{\aleph_0}$ and $2^\mu \ge \lambda$ or at least*
- *(ii) $2^{\aleph_0} \le \mu$ and the density of the ($< \aleph_1$)-base product ${}^\omega\chi$ is $\le \mu$.*

5) If Y' is as in (4), i.e., satisfies clause (c), <u>then</u> any finite intersection of a_α's (for $\alpha \in Y'$) is not zero.
6) If Y' is as in (4), i.e., satisfies clause (c) <u>then</u> Y' is the union of $\le \beth_2$ subsets Y'', such that

- *$(*)_{Y''}$ there is an algebra M with universe Y'' and $\le \beth_1$ functions (with finite arity, of course) such that: if $[u \subseteq Y'', \alpha \in u \ \Rightarrow \alpha \notin cl_M\{u \cap \alpha\}]$, then $\langle a_\alpha : \alpha \in u\rangle$ is independent.*

Proof. Straight and/or included in the proof of 2.1. $\square_{2.4}$

2.5 Claim. *In 2.1 we can weaken "$\mu \geq \beth_2$" to "$\mu \geq 2^{\aleph_0}$" or even "$\mathrm{cf}(\lambda) > 2^{\aleph_0}$" except possibly when λ is singular but $\boxtimes$ below fails:*

> $\boxtimes$ *for any countable set $\mathfrak{a}$ of regulars, $|\mathrm{pcf}(\mathfrak{a})| \leq \aleph_0$ or $(*)$ from 2.6.*

Proof. Without loss of generality we assume $(*)_4$ from the proof of 2.1 (as the proof of 2.1 up to that point works here too). Let J be as there, so J is an ideal on ω, so

(+) J is an ideal on ω and $\langle i(\alpha, n) : n < \omega \rangle / J$ for $\alpha < \lambda$ are pairwise distinct;

by the following observation 2.6 for some ideal I on ω extending J and $X \in [\lambda]^\lambda$, we have

$$\alpha \in X \ \& \ \beta \in X \ \& \ \alpha \neq \beta \Rightarrow \ \{n : i(\alpha, n) = i(\beta, n)\} \in I.$$

This is enough for continuing with the old proof of 2.1. $\square_{2.5}$

2.6 Fact. 1) If J is an ideal on κ, $\langle f_\alpha / J : \alpha < \lambda \rangle$ are pairwise distinct functions in ${}^\kappa\mathrm{Ord}$ and $\theta = \mathrm{cf}(\lambda) > 2^\kappa$ <u>then</u> for some ideal I on κ extending J and $X \in [\lambda]^\lambda$ we have:

$$\alpha \in X \ \& \ \beta \in X \ \& \ \alpha \neq \beta \Rightarrow f_\alpha \neq_I f_\beta$$

except possibly when

$(*)$ λ is singular and $\neg \boxtimes_\kappa$, where

$\boxtimes_\kappa$ for any set $\mathfrak{a}$ of regular cardinals $> \kappa$ we have $|\mathfrak{a}| \leq \kappa \ \Rightarrow \ |\mathrm{pcf}(\mathfrak{a})| \leq \kappa$.

2) We can replace $(*)$ by

$(*)'$ λ is singular and $\neg \boxtimes^+_{\kappa,\lambda}$ or $\neg \boxtimes^-_{\kappa,\lambda}$, where

$\boxtimes^+_{\kappa,\lambda}$ for no set $\mathfrak{a}$ of regular cardinals $> \kappa$, do we have $|\mathfrak{a}| \leq \kappa$ and $\lambda = \sup(\lambda \cap \mathrm{pcf}(\mathfrak{a}))$

$\boxtimes^-_{\kappa,\lambda}$ there are no χ, $\mathrm{cf}(\lambda) = \theta < \chi < \lambda$ and increasing sequences $\bar{\lambda}^\zeta = \langle \lambda^\zeta_i : i < \kappa \rangle$ of regular cardinals $\in (2^\kappa, \chi)$ such that $\langle \max \ \mathrm{pcf}\{\lambda^\zeta_i : i < \kappa\} : \zeta < \theta \rangle$ is increasing with limit λ but for every ultrafilter $\mathscr{D}$ on κ we have

$$\sup \Big\{ \mathrm{tcf}\Big(\prod_{i<\kappa} \lambda^\zeta_i / \mathscr{D} \Big) : \zeta < \theta \Big\} < \lambda.$$

Proof. 1) Follows by (2).
2) The proof is split to cases.

Case 1: λ is regular. We apply 7.5 here which is [Sh 430, 6.6D] (or in more details [Sh 513, 6.2]).

Case 2: λ singular. First note Subfact: $\neg\boxtimes^+_{\lambda,\kappa} \Rightarrow \boxtimes^-_{\lambda,\kappa}$.

[Why? Let $\mathfrak{a}$ exemplify $\neg\boxtimes^+_{\lambda,\kappa}$, let $\theta_\varepsilon \in \text{ pcf}(\mathfrak{a}) \setminus \{\lambda\}$ be increasing for $\varepsilon < \theta$ with limit λ. Let $\mathfrak{b}_\varepsilon \subseteq \mathfrak{a}$ be such that $\theta_\varepsilon = \max \text{pcf}(\mathfrak{b}_\varepsilon)$ and let $\langle \lambda_\zeta : \zeta < \kappa\rangle$ list $\mathfrak{a}$ and let $\lambda^\varepsilon_\zeta$ be: λ_ζ if $\lambda_\zeta \in \mathfrak{b}_\varepsilon$ and $(2^\kappa)^+$ if $\lambda_\zeta \notin \mathfrak{b}_\varepsilon$. Now $\bar\lambda^\varepsilon = \langle \lambda^\varepsilon_\zeta : \zeta < \kappa\rangle$ exemplifies $\neg\boxtimes_{\lambda,\kappa}$. First max pcf $\{\lambda^\varepsilon_\zeta : \zeta < \kappa\} = \theta_\varepsilon < \lambda$ and θ_ε is increasing with limit $\sup(\lambda \cap \text{ pcf}(\mathfrak{a}))$. Secondly, for every ultrafilter $\mathscr{D}$ on κ for each ε we have $\text{tcf}\Big(\prod\limits_{\zeta<\kappa} \lambda^\varepsilon_\zeta/\mathscr{D}\Big)$ is $(2^\kappa)^+$ or is $\text{tcf}\Big(\prod\limits_{\zeta<\kappa} \lambda_\zeta/\mathscr{D}\Big)$. (Simplify the first case if $\{\zeta < \kappa : \lambda_\zeta \notin \mathfrak{b}_\varepsilon\} \in \mathscr{D}$ and the second case if $\{\zeta < \kappa : \lambda_\zeta \in \mathfrak{b}_\varepsilon\} \in \mathscr{D}$.) So now if $\text{tcf}\Big(\prod\limits_{\zeta<\kappa} \lambda_\zeta/\mathscr{D}\Big) \geq \lambda$ implies $\text{tcf}\Big(\prod\limits_{\zeta<\kappa} \lambda^\varepsilon_\zeta/\mathscr{D}\Big) = (2^\kappa)^+$ as the later is $\leq \theta_\varepsilon < \lambda$, so really there is no ultrafilter $\mathscr{D}$ on κ for which $\sup\Big\{\text{tcf}\Big(\prod\limits_{\zeta<\kappa} \lambda^\varepsilon_\zeta/\mathscr{D}\Big) : \varepsilon < \theta\Big\} < \lambda$, so the second demand in $\boxtimes^-_{\lambda,\kappa}$ holds. $\square_?$

$\longrightarrow$ scite{2.6A} undefined

Continuation of the proof of 2.6. Now we assume $\boxtimes^-_{\lambda,\kappa}$. For every regular $\sigma \in (2^\kappa, \lambda)$ we apply 7.5 to $\langle f_\alpha : \alpha < \sigma\rangle$, so we can find $A_\sigma \subseteq \kappa$ and $\langle \gamma_{\sigma,i} : i < \kappa\rangle$ such that

$(*)_0$ for every sequence $\langle \beta_i : i \in A_\sigma\rangle$ satisfying $\beta_i < \gamma_{\sigma,i}$ there are σ ordinals $\alpha < \sigma$ for which

$$i \in A_\sigma \Rightarrow \beta_i < f_\alpha(i) < \gamma_{\sigma,i}, i \in \kappa\backslash A_\sigma \Rightarrow f_\alpha(i) = \gamma_{\sigma,i},$$

$(**)_1$ $B \in J \quad\Rightarrow\quad \sigma \in \text{ pcf}\{\text{cf}(\gamma_{\sigma,i}) : i < \kappa,\ i \in A_\sigma,\ i \notin B\}$.

Let $J_\sigma = \{B \subseteq \kappa : \max \text{pcf}\{\text{cf}(\gamma_{\sigma,i}) : i \in \kappa\backslash A_\sigma;\text{ and } i \in B\} < \sigma\}$, so clearly $\sigma = \text{ tcf}\Big(\prod\limits_{i<\kappa} \text{cf}(\gamma_{\sigma,i})/J_\sigma\Big)$ and $J \subseteq J_\sigma$. Let A'_σ be such that $A'_\sigma \subseteq A_\sigma$, and $\sigma = \max \text{pcf}\{\text{cf}(\gamma_{\sigma,i}) : i \in A'_\sigma\}$. Also, as $\theta = \text{cf}(\lambda) > 2^\kappa$, for some $A' \subseteq \kappa$ (infinite) the set $\Theta = \{\sigma : 2^\kappa < \sigma = \text{ cf}(\sigma) < \theta;\text{ and } A'_\sigma = A'\}$ is unbounded in λ. Let $\langle \sigma_\varepsilon : \varepsilon < \theta\rangle$ be an increasing unbounded sequence of members of Θ, such that its limit is λ. Apply 7.5 (see Case 1) to $\langle g_\varepsilon \restriction A' : \varepsilon < \theta\rangle$, where $g_\varepsilon(i) = \gamma_{\sigma_\varepsilon,i}$, and get $\langle \beta^*_i : i \in A'\rangle$ and $B' \subseteq A'$ such that

(*) if $\langle \beta_i : i \in A' \rangle$ satisfies $i \in A' \Rightarrow \beta_i < \beta^*_i$ <u>then</u> for unboundedly many ordinals $\varepsilon < \theta$

$$i \in B' \Rightarrow \beta_i < \gamma_{\sigma_\varepsilon, i} < \beta^*_i,$$

$$i \in A' \backslash B' \Rightarrow \gamma_{\sigma_\varepsilon, i} = \beta^*_i.$$

Can $B' = \emptyset$? This would mean that for some unbounded $X \subseteq \theta$ we have

$$\varepsilon \in X \quad \Rightarrow \quad (\forall i \in A')[\gamma_{\sigma_\varepsilon, i} = \beta^*_i]$$

hence $\{\sigma_\varepsilon : \varepsilon \in X\} \subseteq \text{pcf}\{\text{cf}(\beta^*_i) : i \in A'\}$, so $\{\text{cf}(\beta^*_i) : i \in A'\}$ has pcf of cardinality $\geq \theta > 2^\kappa$ whereas $|A'| \leq \kappa$, contradiction, so really $B' \neq \emptyset$.

As we are assuming $\neg \boxtimes^-_{\kappa,\lambda}$, there is an ultrafilter $\mathscr{D}$ on A' such that

$$\lambda \leq \sup \Big\{ \text{tcf}\Big(\prod_{i \in A'} \gamma_{\sigma_\varepsilon, i} / \mathscr{D} \Big) : \varepsilon < \theta \Big\}.$$

Clearly

$$\text{tcf}\Big(\prod_{i \in A'} \gamma_{\sigma_\varepsilon, i} / \mathscr{D} \Big) \leq \sigma_\varepsilon < \lambda$$

(by the choice of $A'_{\sigma_\varepsilon} = A'$). Without loss of generality $\sigma_\varepsilon > \theta$ for each $\varepsilon < \theta$. So we can choose, for each ε, a function $h_\varepsilon \in \prod\limits_{i \in A'} \gamma_{\sigma_\varepsilon, i}$ such that

(*) if $\zeta < \theta$ and $\zeta \neq \varepsilon$, while $\langle \gamma_{\sigma_\zeta, i} : i \in A' \rangle \leq_{\mathscr{D}} \langle \gamma_{\sigma_\varepsilon, i} : i \in A' \rangle$ <u>then</u>

$$\langle \gamma_{\sigma_\zeta, i} : i \in A' \rangle <_{\mathscr{D}} h_\varepsilon.$$

(Note that $\langle \gamma_{\sigma_\zeta, i} : i \in A' \rangle \neq_{\mathscr{D}} \langle \gamma_{\sigma_\varepsilon, i} : i \in A' \rangle$ because of the cofinalities of the respective ultraproducts.) So, considering $\mathscr{D}$ as an ultrafilter on κ:

$$X_\varepsilon = \big\{ \alpha < \sigma_\varepsilon : h_\varepsilon <_{\mathscr{D}} f_\alpha <_{\mathscr{D}} \langle \gamma_{\sigma_\varepsilon, i} : i < \kappa \rangle, \text{ but } \\ \beta < \alpha \;\Rightarrow\; \neg (f_\alpha \leq_{\mathscr{D}} f_\beta <_{\mathscr{D}} \langle \gamma_{\sigma_\varepsilon, i} : i < \kappa \rangle) \big\}$$

has cardinality σ_ε. So $X = \bigcup\limits_{\varepsilon < \sigma} X_\varepsilon$ is as required.

$\square_{2.6}$

We may wonder whether we can remove or at least weaken the assumption (*); the answer is:

2.7 Claim. *1) If $\kappa \le \lambda$ and $\theta = \mathrm{cf}(\lambda) < \lambda$, and $\boxtimes^-_{\kappa,\lambda}$ (from 2.6) <u>then</u> for some $f_\alpha \in {}^\kappa\lambda$ (for $\alpha < \lambda$) the conclusion of 2.6(1) fails.*

Proof. 1) Let χ, λ^ζ_i $(i < \kappa, \zeta < \theta)$ be as in $\boxtimes^-_{\kappa,\lambda}$.

Let $\mathfrak{a}_\zeta =: \{\lambda^\zeta_i : i < \kappa\}$, and $\sigma_\zeta = \max \mathrm{pcf}(\mathfrak{a}_\zeta)$. Without loss of generality $\langle \sigma_\zeta : \zeta < \theta \rangle$ is increasing with limit λ. By [\Sh:g<II , §3] for each $\zeta < \theta$ we can find $\langle f^\zeta_\alpha : \alpha < \sigma_\zeta \rangle$ be such that:

$$\mathfrak{b} \subseteq \mathfrak{a}_\zeta \Rightarrow |\{f^\zeta_\alpha \restriction \mathfrak{b} : \alpha < \max \mathrm{pcf}(\mathfrak{a}_\zeta)\}| = \max \mathrm{pcf}(\mathfrak{b}).$$

Define $\langle f_\alpha : \alpha < \lambda \rangle$ by: $f_\alpha(i) = f^{\zeta(\alpha)}_\alpha(\lambda^\zeta_i)$ where $\zeta(\alpha) = \min\{\zeta : \sigma_\zeta > \alpha\}$. Now check. $\square_{2.7}$

2.8 Discussion. 1) So if $2^\kappa < \lambda$, $\theta = \mathrm{cf}(\lambda)$ then 2.7 shows that 2.6 is the best possible. (Of course, we still do not know if $\boxtimes^-_{\lambda,\kappa}$ is possible). See more in 3.13.
2) Note: If $\mathrm{cf}(\lambda) > 2^\kappa$, and

$$(\forall \mathfrak{a})(\mathfrak{a} \subseteq \mathrm{Reg} \ \& \ |\mathfrak{a}| \le \kappa < \min(\mathfrak{a}) \ \Rightarrow \ |\mathrm{pcf}(\mathfrak{a})| \le |\mathfrak{a}|),$$

<u>then</u> $\square^-_{\lambda,\kappa}$ cannot occur as without loss of generality

$$J_\zeta = \{A \subseteq \kappa : \max \mathrm{pcf}\{\lambda^\zeta_i : i \in A\} < \max \mathrm{pcf}\{\lambda^\zeta_i : i < \kappa\}\}$$

does not depend on ζ.

§3 Strong independence in Maharam measure algebras

3.1 Claim. *Assume*

(a) I_i *is a* κ_i*-complete ideal on* λ_i *for* $i < \delta$,

(b) $\kappa_i > \sum_{j<i} \kappa_j$,

(c) $\mu = \sup_{i<\delta} \kappa_i$ *is strong limit singular,*

(d) $\lambda_i < \mu$,

(e) $\lambda = \mu^+ = 2^\mu$.

Then there is $\bar\eta$ a super λ-sequence for $\langle I_i : i < \delta\rangle$, where

3.2 Definition. We say $\bar\eta$ is a super (λ, J)-sequence for $\langle I_i : i < \delta\rangle$ if, in addition (to the demands in 1.9)

(∗) for every $n < \omega$ and $\beta_{\alpha,\ell} < \lambda$ (for $\alpha < \lambda$, $\ell < n$) increasing with ℓ, pairwise distinct (i.e. $\beta_{\alpha_1,\ell_1} = \beta_{\alpha_2,\ell_2} \Rightarrow \alpha_1 = \alpha_2 \ \& \ \ell_1 = \ell_2$) we have

$$\Big\{ i < \delta : \{\langle \eta_{\beta_{\alpha,\ell}}(i) : \ell < n\rangle : \alpha < \lambda\} \in \prod_{\ell<n} I_i \Big\} \in J.$$

Moreover

(∗) if $n < \omega$, $\beta_{\alpha,\ell} < \lambda$ (for $\alpha < \lambda$, $\ell < n$), $\beta_{\alpha,\ell} < \beta_{\alpha,\ell+1}$, and the $\beta_{\alpha,\ell}$ are pairwise distinct then for some $A \in J$ we have:
if $m < \omega$, $i_0 < i_1 < \cdots i_{m-1}$ belong to $\delta \backslash A$, then

$$\{\langle\langle \eta_{\beta_{\alpha,\ell}}(i_t) : \ell < n\rangle : t < m\rangle : \alpha < \lambda\} \in \Big(\prod_{t<m}\Big(\prod_{\ell<n} I_{i_t}\Big)\Big)^+.$$

Proof. Like the proof of 1.9. $\square_{3.1}$

3.3 Example: $\lambda = \mu^+ = 2^\mu$, $\mu = \sum_{i<\kappa} \lambda_i$, $i < j \ \Rightarrow \ \delta = \kappa < \lambda_i < \lambda_j < \mu$ and each λ_i is measurable with a $\big(\aleph_0 + \sum_{j<i} \lambda_j\big)^+$-complete normal (or just Ramsey for n_i) ultrafilter $\mathscr{D}_i$ on λ_i.

Let $\bar n = \langle n_i : i < \kappa\rangle$, $i < n_i < \omega$, (if $\kappa = \aleph_0$, $n_i = i$ we may omit it)

$$I_i = \{A \subseteq [\lambda_i]^{n_i} : \text{ for some } B \in \mathscr{D}_i \text{ we have } [B]^{n_i} \cap A = \emptyset\}.$$

Then

$(*)_1$ Claim 3.1 applies,

$(*)_2$ for every $m < \omega$ and $X \in \prod\limits_{\ell < m} I_i$ we can find $A \in \mathscr{D}_i$ such that:

$$\{\bar{s} : \bar{s} = \langle s_\ell : \ell < m \rangle, s_\ell \in [A]^{n_i}, s_\ell < s_{\ell+1}\} \cap X = \emptyset.$$

3.4 Definition. 1) For a Boolean algebra $\mathbf{B}$ we say $\langle (a_\alpha, b_\alpha) : \alpha < \alpha^* \rangle$ is a strongly independent sequence of intervals if

(a) $\mathbf{B} \models a_\alpha < b_\alpha$,

(b) if $\mathbf{B}'$ is a Boolean algebra extending $\mathbf{B}$ and $n < \omega$, $\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \alpha^*$ and $\mathbf{B}' \models$ "$a_{\alpha_\ell} \leq x_\ell \leq b_{\alpha_\ell}$" for $\ell < n$, <u>then</u> any non-trivial Boolean combination of $\langle x_\ell : \ell < n \rangle$ is non-zero (in $\mathbf{B}'$).

2) We say, for a Boolean algebra $\mathbf{B}$ that $\langle (a_\alpha, b_\alpha) : \alpha < \alpha^* \rangle$ is a λ-anti independent sequence of intervals <u>if</u>:

(a) $\mathbf{B} \models a_\alpha \leq b_\alpha$,

(b) if $\mathbf{B}'$ is a Boolean algebra extending $\mathbf{B}$ and $X \in [\alpha^*]^\lambda$ and $\mathbf{B}' \models$ "$a_\alpha \leq x_\alpha \leq b_\alpha$" for $\alpha \in X$, then there are $n < \omega$ and $\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1}$ from X such that some non-trivial Boolean combination of $\langle x_{\alpha_\ell} : \ell < n \rangle$ is zero.

3) We say $\langle (a_\alpha, b_\alpha) : \alpha < \alpha^* \rangle$ is an independent sequence of intervals in the Boolean algebra $\mathbf{B}$ <u>if</u> letting $\mathbf{B}', x_\alpha$ be as in 3.5 below, we have $\langle x_\alpha : \alpha < \alpha^* \rangle$ is independent (in $\mathbf{B}'$).
4) We say $\langle (a_\alpha, b_\alpha) : \alpha < \alpha^* \rangle$ is a strongly λ-anti-independent sequence of intervals for the Boolean algebra $\mathbf{B}$ <u>if</u>:

(a) $\mathbf{B} \models a_\alpha \leq b_\alpha$,

(b) if $\mathbf{B}', X, x_\alpha (\alpha \in X)$ are as in 3.4(2)(b) above, <u>then</u> the Boolean subalgebra of $\mathbf{B}'$ generated by $\{x_\alpha : \alpha \in X\}$ contains no free subset of cardinality λ.

5) We say $\langle (a_\alpha, b_\alpha) : \alpha < \alpha^* \rangle$ is mediumly λ-anti independent (sequence of intervals of the Boolean algebra $\mathbf{B}$) <u>if</u>

(a) $\mathbf{B} \models a_\alpha \leq b_\alpha$,

(b) if $\mathbf{B}'$ is the free extension of $\mathbf{B}$ for $\langle (a_\alpha, b_\alpha) : \alpha < \alpha^* \rangle$ (see 3.5 below), then the Boolean subalgebra of $\mathbf{B}'$ generated by $\{x_\alpha : \alpha < \alpha^*\}$ contains no free subalgebra of cardinality λ.

3.5 Definition. We say that $\mathbf{B}' = \mathbf{B}'(\mathbf{B}, \langle (a_\alpha, b_\alpha) : \alpha < \alpha^* \rangle)$, or $\mathbf{B}'$ is the free extension of $\mathbf{B}$ for $\langle (a_\alpha, b_\alpha) : \alpha < \alpha^* \rangle$, <u>if</u>:

- $(*)$ $\mathbf{B}'$ is the algebra freely generated by $\mathbf{B} \cup \{x_\alpha : \alpha < \alpha^*\}$ except for the equations:
  - (a) the equations which $\mathbf{B}$ satisfies,
  - (b) $a_\alpha \leq x_\alpha \leq b_\alpha$, for $\alpha < \alpha^*$.

3.6 Observation. 1) In 3.4(3), if $\mathbf{B} \subseteq \mathscr{B}(\alpha_0)$, $\alpha_0 + \omega + \alpha^* \leq \alpha_1$ <u>then</u> we can embed $\mathbf{B}'$ into $\mathscr{B}(\alpha_1)$ over $\mathbf{B}$.

2) There are obvious implications among the notion from Definition 3.4 and some equivalences: independent (3.4(3)) with explicitly independent; and stronger independent with "(a) of 3.4(1) and if $\alpha_1, \dots, \alpha_2$, $\beta_1, \dots, \beta_m < \alpha^*$ with no repetition,

$$\mathbf{B} \models \text{“} \bigcap_{\ell=1}^{n} a_{\alpha_\ell} \cap \bigcap_{\ell=1}^{m} (-b_{\beta_\ell}) > 0 \text{ ”}.$$

3.7 Lemma. *Assume μ is strong limit singular of countable cofinality and $\lambda = \mu^+ = 2^\mu$.*
<u>Then</u> in $\mathscr{B}(\mu)$, (the Maharam measure algebra of dimension μ) we can find a sequence $\langle (a_\alpha, b_\alpha) : \alpha < \mu \rangle$ such that:

- *(a) $\mathscr{B}(\mu) \models a_\alpha < b_\alpha$,*
- *(b) $\langle (a_\alpha, b_\alpha) : \alpha < \lambda \rangle$ is strongly λ-anti independent.*

Remark. What is the difference with 1.16 Note that 3.4(ii)(b) speaks on "no free subset of the Boolean algebra", not just of the set.

Proof. 1) Let $\mu = \sum_{n<\omega} \lambda^0_n$, (we may demand $\beth_{n+8}(\lambda^0_n) < \lambda^0_{n+1} < \mu$) and let I_n be $\mathrm{ERI}^{n,h}_{\beth_{n-1}(\lambda^0_n)^+, (\lambda^0_n)^+}$ (see Definition 1.17, they were used in the proof of 1.13). Let $\bar\eta = \langle \eta_\alpha : \alpha < \lambda \rangle$ be as guaranteed by 3.1 (so $\lg(\eta_\alpha) = \omega$, $\eta_\alpha(n) \in [\lambda_n]^n$, where $\lambda_n = \beth_{n-1}(\lambda^0_n)^+$. So I_{n+1} is $|\mathrm{Dom}(I_n)|^+$-complete, (we could also have $\langle I_n : n < \omega \rangle$ is normal). Renaming, let x^n_α (for $n < \omega$, $\alpha < \lambda_n$) be the free generators of the Maharam algebra.

Define for $\alpha < \lambda$ and $n < \omega$

$$a^*_{\alpha,n} = \bigcap \{x^m_\beta : \beta \text{ appears in } \eta_\alpha(m)\}$$

$$b^*_{\alpha,n} = \bigcup\{(1 - x^m_\beta : \beta \text{ appears in } \eta_\alpha(m)\}.$$

We define by induction on n, the elements $a_{\alpha,n}$, $b_{\alpha,n}$ as follows: for $n < 5$ let $a_{\alpha,n} = 0$, $b_{\alpha,n} = 1$. For $n \geq 5$ we let $a_{\alpha,n} = a_{\alpha,n-1} \cup (a^*_{\alpha,n} \cap b_{\alpha,n})$ and $b_{\alpha,n} = b_{\alpha,n+1} \cap (b^*_{\alpha,n} \cup a_{\alpha,n})$. We can prove by induction on $n < \omega$ that $a_{\alpha,n-1} \leq a_{\alpha,n} \leq b_{\alpha,n} \leq b_{\alpha,n-1}$. We can compute the measure, e.g., let $(b_{\alpha,n} - a_{\alpha,n}) = \prod\{1 - 2^{-(\ell-1)} : 5 \leq \ell \leq n\}$.

Let $a_\alpha = \bigcup\limits_{n<\omega} a_{\alpha,n} \in \mathscr{B}(\mu)$, $b_\alpha = \bigcap\limits_{n<\omega} b_{\alpha,n} \in \mathscr{B}(\mu)$.

So clearly $\mathscr{B}(\mu) \vDash a_\alpha \leq b_\alpha$, and by the measure computations above, $\mathscr{B}(\mu) \vDash a_\alpha < b_\alpha$. So $\langle (a_\alpha, b_\alpha) : \alpha < \lambda\rangle$ is a sequence of intervals. Suppose $\mathbf{B}$, c_α (for $\alpha < \lambda$), is a counterexample to the conclusion so there is an independent subset $\{d_\alpha : \alpha < \lambda\}$ of $\langle c_\alpha : \alpha < \lambda\rangle_{\mathbf{B}} \subseteq \mathbf{B}$. Thus, for each $\alpha < \lambda$ for some $k_\alpha < \omega$ and a Boolean term $\tau = \tau_\alpha(x_0, \dots, x_{k_\alpha - 1})$ and some $\beta_{\alpha,0} < \beta_{\alpha,1} < \dots < \beta_{\alpha,k_\alpha-1}$ we have $d_\alpha = \tau_\alpha(c_{\beta_{\alpha,0}}, c_{\beta_{\alpha,1}}, \dots, c_{\beta_{\alpha,k_\alpha-1}})$.

As we can replace $\{d_\alpha : \alpha < \lambda\}$ by any subset of the same cardinality without loss of generality $\tau_\alpha = \tau$, so let $k_\alpha = k(*)$.

Similarly, by the Δ-system argument without loss of generality for some $k < k(*)$ we have

$$\ell < k \Rightarrow \beta_{\alpha,\ell} = \beta_\ell \text{ and } \alpha(1) < \alpha(2) \Rightarrow \beta_{\alpha(1),k(*)-1} < \beta_{\alpha(2),k}.$$

Let $X_n = \{\langle \eta_{\beta_{\alpha,\ell}}(n) : k \leq \ell < k(*)\rangle : \alpha < \lambda\} \subseteq {}^{(k(*)-k)}([\lambda_n]^n)$. So we know that $B = \{n < \omega : n \geq k(*) - k \text{ and } X_n \in (\prod\limits_{\ell=k}^{k(*)-1} I_n)^+\} \in J^+$. Let $n \in B$. We can find a function $h : X_n \to \lambda$ such that

$$\bar{t} \in X_n \ \ \& \ \ h(\bar{t}) = \alpha \Rightarrow \bar{t} = \langle \eta_{\beta_{\alpha,\ell}}(n) : k \leq \ell < k(*)\rangle.$$

Let $m(*) < \omega$ be large enough, a power of 2 for simplicity.

As $X_n \in (\prod\limits_{\ell=k}^{k(*)-1} I_n)^+$, we can find $\langle S_\ell : \ell \in [k, k(*)]\rangle$ and $\langle u_{\bar{s}} : \bar{s} \in S_\ell\rangle$ for $\ell \in [k, k(*))$ such that

(a) $S_k = \{\langle\ \ \rangle\}$,
(b) $u_{\bar{s}} \in [\lambda_n]^{m(*)}$,
(c) the $u_{\bar{s}}$'s are pairwise disjoint,
(d) $S_{\ell+1} = \{\bar{s}^\frown\langle w\rangle : \bar{s} \in S_\ell, w \in [u_{\bar{s}}]^n\}$,
(e) $S_{k(*)} \subseteq X_n$.

(We just do it by induction on ℓ using the definition of $\prod\limits_{\ell=k}^{k(*)-1} I_n$ and the definition of I_ℓ.) So it suffices to show that $\langle d_{h(\bar t)} : \bar t \in S_{k(*)}\rangle$ is not independent. For this just note:

$\otimes$ for every $\epsilon \in \mathbb{R}^{>0}$ if n is large enough compared to $k(*), 1/\epsilon$, and $m(*)$ is large enough compared to n <u>then</u> for every ultrafilter $\mathscr{D}$ on $\mathscr{B}(\mu)$ we can by downward induction on $\ell = k, \dots, k(*)-1$ find $u^-_{\bar s} \in [u_{\bar s}]^{m(*)/2^{k(*)-\ell}}$ and $\eta_{\bar s} \in {}^{\{k,\dots,k(*)-1\}}2$ for $\bar s \in S_\ell$ such that: $\bar s \trianglelefteq \bar t \in S_{\ell_1}$ and $\ell \le \ell_1 < k(*)$ and $\alpha \in u^-_{\bar t} \Rightarrow [x^n_\alpha \in \mathscr{D} \equiv \eta_{\bar s}(\ell_1) = 1]$.

Now let $\eta^* = \eta_{\langle\,\rangle}$ (i.e., $\eta_{\bar s}$ for the unique $\bar s \in S_0$) and for $m < k(*)$ letting $S'_m = \left\{\bar s \in S_m : \text{if } \ell < m \text{ then } \bar s(\ell) \in [u^-_{\bar s \restriction \ell}]^n\right\}$, we have $\bar s \in S'_{k(*)} \Rightarrow d_{h(\bar s)} \in \mathscr{D}$ or $\bar s \in S'_{k(*)} \Rightarrow d_{h(\bar s)} \notin \mathscr{D}$.

So to prove that $\langle d_\alpha : \alpha < \lambda\rangle$ is not independent it suffices to find $S \subseteq S_{k(*)}$ such that

$$\otimes_S \qquad \bigcap_{\alpha\in S} d_\alpha \cap \bigcap_{\alpha\in S_{k(*)}\setminus S} d_\alpha = 0,$$

or equivalently

$\otimes'_S$ for no ultrafilter $\mathscr{D}$ on $\mathscr{B}(\mu)$ do we have

$$\alpha \in S_{k(*)} \Rightarrow [d_\alpha \in \mathscr{D} \equiv \alpha \in S].$$

By the argument above it will suffice to have

$\otimes''_S$ <u>if</u> $\langle u^-_{\bar s} : \bar s \in \cup\{S'_\ell : \ell < k(*)\rangle$ satisfies: $S'_0 = S_0$, $S'_\ell \subseteq S_\ell$,

$$\bar s \in S'_\ell \Rightarrow u^-_{\bar s} \in [u_{\bar s}]^{m(*)/2^{2k(*)-\ell}}$$

and
$S'_{\ell+1} = \{\bar s{}^\frown\langle w\rangle : \bar s \in S'_\ell \text{ and } w \in [u^-_{\bar s}]^n\}$ <u>then</u> $S \cap S'_{k(*)} \notin \{\emptyset, S\}$.

Now, not only that this is trivial by the probabilistic existence proof á la Erdös but the proof gives much more than enough. $\square_{3.7}$

3.8 Claim. *: Assume*

(∗) *λ is regular $> \aleph_0$ and $\langle (a_\alpha, b_\alpha) : \alpha < \lambda\rangle$ is a strongly (or just mediumly) λ-anti-independent sequence of pairs from $\mathscr{B}(\lambda)$ satisfying $a_\alpha < b_\alpha$.*

Then:

(a) *There is* $\mathbf{B}'$*, such that:*
 (α) $\mathbf{B}'$ *is a subalgebra of* $\mathscr{B}(\lambda)$*,*
 (β) $\mathbf{B}'$ *has cardinality* λ *and even dimension* λ*,*
 (γ) *there is no subset of* $\mathbf{B}'$ *of cardinality* λ *which is independent.*

(b) *Let* $\mathbf{B}'$, $x_\alpha(\alpha < \lambda)$ *be as in 3.5, then the Boolean algebra in clause (a) can be chosen isomorphic to* $\langle x_\alpha : \alpha < \lambda\rangle_{\mathbf{B}'}$.

Proof. Straight. Clause (a) follows from clause (b). For clause (b) apply Definition 3.4(5) and 3.6. (Note: we can use $\mathscr{B}' \subseteq \mathscr{B}(\lambda + \lambda)$). It has already been done by Plebanek [Pl1]. $\square_{3.8}$

3.9 Conclusion. For λ as in 3.7 (i.e., $\lambda = \mu^+ = 2^\mu$, μ strong limit $>$ $\mathrm{cf}(\mu) = \aleph_0$) or just as in $(*)$ of 3.8, we have

$(*)$ there is a topological space X which is Hausdorff, compact zero dimensional, with a measure Leb on the Borel sets such that it has dimension λ, so as a measure space is isomorphic to $\mathscr{B}(\lambda)$ but there is no homomorphism from X onto ${}^\lambda 2$.

Proof. By 3.7(1) $(*)$ of 3.8 holds so we can restrict ourselves to this case. So by 3.8 we know that clause (a) of 3.8 holds. Now it follows that $(*)$ holds, more specifically, that the Čech–Stone compactification of $\mathbf{B}'$ (i.e., the set of ultrafilters of $\mathbf{B}'$ with the natural topology) and the measure of $\mathbf{B}'$ (which is just the restriction of the one on $\mathbf{B}(\lambda)$) satisfies $(*)$ of 3.9. $\square_{3.9}$

3.10 Example: Assume $\mathscr{B}$ is a Maharam measure algebra of dimension μ and free basis $\langle x_\alpha : \alpha < \mu\rangle$, $\mu \geq \lambda >$ $\mathrm{cf}(\lambda) = \aleph_0$. Then $(*)_{2,\lambda}$ below holds, where

$(*)_{2,\lambda}$ there are positive pairwise distinct members a_α of $\mathscr{B}(\mu)$ for $\alpha < \mu$, such that for every $X \in [\lambda]^\lambda$ for some $\alpha \neq \beta$ from X, $a_\alpha \cap a_\beta = 0$.

Proof. Trivial: let $\lambda = \sum_{n<\omega} \lambda_n$, $\lambda_n < \lambda_{n+1}$ and for $\alpha \in \big(\bigcup\limits_{\ell<n} \lambda_\ell, \lambda_n\big)$ we let $a_\alpha = x_{\omega+\alpha} \cap (x_n - \bigcup\limits_{m<n} x_m)$. $\square_{3.10}$

3.11 Fact. Suppose $\aleph_0 <$ $\mathrm{cf}(\lambda) < \lambda$ and there are positive $b_\alpha \in \mathscr{B}(\mathrm{cf}(\lambda))$ for $\alpha <$ $\mathrm{cf}(\lambda)$ such that for every $X \in [\mathrm{cf}(\lambda)]^{\mathrm{cf}}(\lambda)$ for some $m < \omega$ and $\beta_0, \dots, \beta_m \in X$ we have $\mathrm{Leb}\Big(\bigcap\limits_{\ell\le m} b_{\beta_\ell}\Big) = 0$ and $\mu \ge \lambda$. <u>Then</u> we can find pairwise distinct $a_\alpha \in \mathscr{B}(\lambda)$ for $\alpha < \lambda$ such that for every $X \in [\lambda]^\lambda$ for some $m < \omega$, $\beta_0, \dots, \beta_m \in X$ we have $\mathrm{Leb}\Big(\bigcap_{\ell\le m} a_{\beta_\ell}\Big) = 0$, i.e., $\mathscr{B}(\lambda) \vDash \bigcap\limits_{\ell\le m} a_{\beta_\ell} = 0$.

Proof. Like the proof of 3.10 replacing $x_n - \bigcup_{m<n} x_m$ (for $n < \omega$) by b_α (for $\alpha <$ cf(λ)). (Just say that if cf(λ) is a precaliber of $\mathscr{B}$ then so is λ.)

3.12 Remark. 1) By 2.1 we have in 3.11 that necessarily cf$(\lambda) \le \beth_2$ is normally cf$(\lambda) \le \beth_1$.
2) Note that 3.13 elaborates 2.7 above and 3.15 is complementary to §2.

<u>3.13 Example</u>: Assume $\aleph_0 \le \sigma \le \theta =$ cf$(\lambda) \le 2^\sigma \le \mu < \lambda$,

$$\lambda = \sup\big\{\max \mathrm{pcf}(\mathfrak{a}) : \mathfrak{a} \subseteq \mathrm{Reg} \cap \mu \backslash 2^\sigma,\ |\mathfrak{a}| = \sigma,\ [\mathfrak{a}]^{<\sigma} \subseteq J_{<\max\ \mathrm{pcf}}(\mathfrak{a})[\mathfrak{a}], \\ \text{and } \sup(\mathrm{pcf}(\mathfrak{a})\backslash\{\max\ \mathrm{pcf}(\mathfrak{a})\}) \le \mu\big\}$$

and there is $\mathscr{A} \subseteq [\sigma]^\sigma$ such that $|\mathscr{A}| \ge \theta$ and

$$A \ne B\ \ \&\ \ A \in \mathscr{A}\ \ \&\ \ \mathbf{B} \in \mathscr{A} \Rightarrow |A \cap B| < \sigma.$$

Or just for no uniform ultrafilter $\mathscr{D}$ on σ do we have $|\mathscr{D} \cap \mathscr{A}| \ge \sigma$.
<u>Then</u> we can find ordinals $i(\alpha, \varepsilon)$ for $\alpha < \lambda, \varepsilon < \sigma$ such that

- (a) for $\alpha \ne \beta$, $\{\varepsilon : i(\alpha,\varepsilon) \ne i(\beta,\varepsilon)\}$ is infinite. Moreover
- $(a)^+$ for any $\lambda' < \lambda$ for some ultrafilter $\mathscr{D}$ on σ, $\{\langle i(\alpha,\varepsilon) : \varepsilon < \sigma\rangle/\mathscr{D} : \alpha < \lambda\}$ has cardinality $\ge \lambda'$,
- (b) for no ultrafilter $\mathscr{D}$ on σ do we have $\{\langle i(\alpha,\varepsilon) : \varepsilon < \sigma\rangle/\mathscr{D} : \alpha < \lambda\}$ have cardinality λ.

[Why? Let

$$\lambda = \sum_{\zeta<\theta} \lambda_\zeta, \quad \lambda_\zeta < \lambda, \quad \lambda_\zeta = \max\ \mathrm{pcf}(\mathfrak{a}_\zeta),$$

$$|\mathfrak{a}_\zeta| = \sigma, \quad [\mathfrak{a}_\zeta]^{<\sigma} \subseteq J_{<\lambda_i}[\mathfrak{a}_\zeta], \quad \mu \ge \sup(\mathrm{pcf}(\mathfrak{a}_\zeta)\backslash\{\lambda_\zeta\}).$$

Let $f^\zeta_\alpha \in \prod \mathfrak{a}_\zeta$ for $\zeta < \theta$, $\alpha < \lambda_\zeta$ be such that $\langle f^\zeta_\alpha : \alpha < \lambda_\zeta\rangle$ is $<_{J_{<\lambda_\zeta}[\mathfrak{a}_\zeta]}$–increasing cofinal and $\mathfrak{b} \in J_{<\lambda_\zeta}(\mathfrak{a}_\zeta) \ \Rightarrow\ \mu \ge |\{f^\zeta_\alpha \restriction \mathfrak{b} : \alpha < \lambda_\zeta\}|$. Let $\mathscr{A} = \{A_\zeta : \zeta < \theta\}$, let $\mathfrak{a}_\zeta = \{\tau^\zeta_\varepsilon : \varepsilon \in A_\zeta\}$. Lastly $i(\alpha,\varepsilon)$ is

$$f^\zeta_\alpha(\varepsilon) \text{ if } \bigcup_{\xi \le \zeta} \lambda_\xi \le \alpha < \lambda_\zeta \ \ \& \ \ \varepsilon \in A_\zeta,$$

$$\zeta \text{ if } \bigcup_{\xi < \zeta} \lambda_\xi \le \alpha < \lambda_\zeta \ \ \& \ \ \varepsilon \notin A_\zeta.$$

Now check.

3.14 Remark. There are easy sufficient conditions: if $2^\sigma < \mu^1 \le \mu$, $\mathrm{cf}(\mu^1) = \sigma$, $\mathrm{pp}(\mu^1) \ge \lambda$, $(\forall \chi < \mu^1)(\mathrm{cf}(\chi) \le \sigma \to \ \mathrm{pp}(\chi) < \mu^1)$ and $\lambda < \mu^{+\omega}$ or at least $\lambda = \sup\big\{\chi : \mu < \chi = \ \mathrm{cf}(\chi) < \lambda$ and $\neg(\exists \mathfrak{a})(\mathfrak{a} \subset \ \mathrm{Reg} \cap \chi \backslash \mu \ \ \& \ \ |\mathfrak{a}| \le \sigma \ \ \& \ \ \chi \in \mathrm{pcf}(\mathfrak{a}))\big\}$.

3.15 Example: Assume

- (a) $\aleph_0 < \theta = \ \mathrm{cf}(\lambda) \le 2^{\aleph_0} < \mu < \lambda$,
- (b) there is a θ–Luzin subset of ${}^\omega 2$.

Then

- (α) there are pairwise disjoint $a_\alpha \in \mathscr{B}(\mu)$ for $\alpha < \lambda$ such that for no $X \in [\lambda]^\lambda$ is $\langle a_\alpha : \alpha \in X\rangle$ free
- (β) moreover, for $X \in [\lambda]^\lambda$ for some $n < \omega$ and $\beta_0 < \beta_1 < \cdots < \beta_n$ from X we have $\mathscr{B}(\lambda) \models \bigcap\limits_{\ell \le n} a_{\beta_\ell} = 0$.

Proof. (Has already appeared in Plebanek [Pl1].) By 3.11 it suffices to prove its assumption. Let for $n < \omega$, $\langle c_{n,\ell} : \ell < (n+1)^2\rangle$ be a sequence of pairwise disjoint members of $\mathscr{B}(\omega)$ with union 1, each with each with measure $1/n^2$. For $\eta \in \prod\limits_{n<\omega} (n+1)^2$ let $b_\eta = \bigcap\limits_{n<\omega} (1 - c_{\eta,\eta(\ell)})$. Now suppose

- $(*)$ $X \subseteq {}^\omega 2$, $|X| = \theta$, and if $Y \in [X]^\theta$ then for some $n < \omega$ and $\nu \in \prod\limits_{\ell<n} (\ell+1)^2$ we have
$$\{\ell : \ell < (n+1)^2\} = \{\eta(n) : \eta \restriction n = \mu, \ \eta \in Y\}.$$

So $\{b_\eta : \eta \in X\}$ is as required. Lastly from clause (c) of the assumption there is X as required in $(*)$ so, we are done. $\square_{3.15}$

3.16 Remark. 1) So we can weaken clause (c) of the assumption to $(*)$ from the proof, or variants of it.
2) Note that strong negation of (c) of 3.15 which is consistent, implies the inverse situation.

§4 The interesting ideals and the direct pcf application

Our problem, the existence of (λ, I, J)–sequences for $\bar{I}$, depends much on the ideals I_i we use. Under strong set theoretic assumptions, there are λ-sequences $\bar{\eta}$ by 1.9 (and 3.1); but we would like to prove their existence (i.e., in ZFC). For some ideals, by [Sh:g] we will have many cases of existence, e.g., when I_i is $J^{bd}_{\lambda_i}$, λ_i regular. But we are more interested in the existence for more complicated ideals. The first step up are $J^{bd}_{\bar{\lambda}}$ with $\bar{\lambda}$ a (finite) strictly increasing sequence of cardinals. The proof for them is not much harder than with the J^{bd}_{λ}'s. We then consider the central ideal here: $J^{bd}_{\bar{\lambda}}$ for $\bar{\lambda}$ a (strictly) decreasing sequence of regular cardinals, and explain why the existence of $\bar{\eta}$ for these ideals is more useful. We also consider their strong relative which comes from the nonstationary ideal. We would of course love to have even stronger ideals but there are indications that for those which we considered and failed, the failure is not completely due to incompetence, i.e., there are related independence results (see later). We commence this section by reviewing some general definitions, some of them used earlier in the paper.

4.1 Definition. 1) For a set A of ordinals with no last element (mainly $A = \lambda = \mathrm{cf}(\lambda)$)

$$J^{\mathrm{bd}}_A = \{B : B \subseteq A \text{ is bounded}\}.$$

2) If $A \subseteq$ Ord is such that cf otp$(A)) > \aleph_0$ and A stationary in $\sup(A)$, we let

$$J^{\mathrm{nst}}_A = \{B \subseteq A : \ B \text{ is not a stationary subset of } \sup(A)\}.$$

3) If $A \subseteq$ Ord, $\theta = \mathrm{cf}(\theta) < \mathrm{cf}(\mathrm{otp}(A))$ and

$$\{\delta < \sup(A) : \delta \in A, \ \mathrm{cf}(\delta) = \theta\}$$

is a stationary subset of $\sup(A)$, then let

$$J^{\mathrm{nst},\theta}_A = \{B \subseteq A : \{\delta \in B : \ \mathrm{cf}(\delta) = \theta\} \text{ is a nonstationary subset of } \sup(A)\}.$$

4.2 Definition. 1) For an ideal J let $(\exists^{J^+} x)\varphi(x)$ mean that

$$\{x \in \mathrm{Dom}(J) : \varphi(x)\} \in J^+.$$

2) For an ideal J let $(\forall^J x)\varphi(x)$ mean

$$\{x \in \mathrm{Dom}(J) : \neg\varphi(x)\} \in J.$$

4.3 Definition. 1) $J = \prod\limits_{\ell<n} J_\ell$ is the following ideal on $\prod\limits_{\ell<n} \mathrm{Dom}(J_\ell)$:
for $X \subseteq \prod\limits_{\ell<n} \mathrm{Dom}(J_\ell)$ we have

$$X \in J^+ \text{ iff } (\exists^{J^+_0} x_0)(\exists^{J^+_1} x_1)\cdots(\exists^{J^+_{n-1}} x_{n-1})[\langle x_0,\dots,x_{n-1}\rangle \in X].$$

2) If $\bar\lambda = \langle \lambda_\ell : \ell < n\rangle$ we let:

(a) $J^{\mathrm{bd}}_{\bar\lambda} = \prod\limits_{\ell<n} J^{\mathrm{bd}}_{\lambda_\ell}$

(b) if $\mathrm{cf}(\lambda_\ell) > \aleph_0$ for $\ell < n$ then we let

$$J^{\mathrm{nst}}_{\bar\lambda} = \prod_{\ell<n} J^{\mathrm{nst}}_{\lambda_\ell}.$$

(c) if $\mathrm{cf}(\lambda_\ell) > \theta = \mathrm{cf}(\theta)$ for $\ell < n$ then we let

$$J^{\mathrm{nst},\theta}_{\bar\lambda} = \prod_{\ell<n} J^{\mathrm{nst},\theta}_{\lambda_\ell}$$

(d) if $\bar\theta = \langle \theta_\ell : \ell < n\rangle$ and $\mathrm{cf}(\lambda_\ell) > \theta_\ell = \mathrm{cf}(\theta_\ell)$ for $\ell < n$ then we let

$$J^{\mathrm{nst},\bar\theta}_{\bar\lambda} = \prod_{\ell<n} J^{\mathrm{nst},\theta_\ell}_{\lambda_\ell}.$$

4.4 Claim. *If $\bar\lambda = \langle \lambda_\ell : \ell < n\rangle$ is a strictly increasing sequence of regular cardinals then the following conditions (a)–(d) on $X \subseteq \prod\limits_{\ell<n} \lambda_\ell$ $\mathrm{Dom}(J^{\mathrm{bd}}_{\bar\lambda})$ are equivalent:*

(a) $X \in (J^{\mathrm{bd}}_{\bar\lambda})^+$;

(b) *for no $\bar\alpha \in \prod\limits_{\ell<n} \lambda_\ell$ do we have*

$$(\forall \bar\beta \in X)(\neg(\bar\alpha < \bar\beta)), \qquad \text{where } \bar\beta < \bar\alpha =: \bigwedge_{\ell<n} \beta_\ell < \alpha_\ell;$$

(c) *we can find $\langle \alpha_\eta : \eta \in \bigcup\limits_{m\le n} \prod\limits_{\ell<m} \lambda_\ell\rangle$ such that:*

(i) $\alpha_\eta < \lambda_{\lg(\eta)}$,

(ii) $\alpha_{\eta\frown\langle i\rangle} < \alpha_{\eta\frown\langle j\rangle}$ *for* $i < j < \lambda_{\lg(\eta)+1}$,

(iii) $\eta \in \prod\limits_{\ell<n} \lambda_\ell \Rightarrow \langle \alpha_{\eta\restriction\ell} : \ell \le n\rangle \in X$;

(d) *Like (c), adding*

(iv) $\alpha_\eta = \alpha_\nu \Rightarrow \eta = \nu$.

Proof. Straight. For (b) $\Rightarrow$ (c) use induction on $n = \ell g(\bar\lambda)$, see the proof at the end of the proof of 4.11, of $(*)$ there. $\square_{4.4}$

4.5 Discussion. From 4.4, we see that for for $X \in (J^{bd}_{\bar\lambda})^+$ there are patterns which necessarily occur as subsets of X. These are essentially like the branches (= maximal nodes) of a tree with n levels, with a linear order on each level and with no dependencies between the different levels. These patterns were explored in [Sh 462], [RoSh 534], [Sh 575]. The patterns considered there can be represented as a set $\Delta \subseteq \prod\limits_{\ell<n} B_\ell$, $B_\ell \subseteq$ Ord such that $\eta(i) = \nu(i) \Rightarrow \eta \restriction i = \nu \restriction i$ (i.e., treeness). Now look at $J^{bd}_{\bar\lambda}$, where the gain is that Δ does not have a tree, that is, we have any $\Delta \subseteq \prod\limits_{\ell<n} B_\ell$, $B_\ell \subseteq$ Ord, so that $\eta,\nu \in \Delta$ can have $\{\ell < n : \eta(\ell) = \nu(\ell)\}$ being arbitrary (rather than being an initial segment), of course this depends on the ideal.

4.6 Claim. *Assume $\bar J = \langle J_\ell : \ell < n\rangle$ and J_ℓ is a κ_ℓ-complete ideal on λ_ℓ. We also demand $\kappa_\ell > \lambda_k$ when $\ell > k$. Let $J = \prod\limits_{\ell<n} J_\ell$.*

1) The following conditions on $X \subseteq \prod\limits_{\ell<n} \lambda_\ell$ are equivalent:

(a) $X \in J^+$;

(b) *for no $\bar A = \langle A_\ell : \ell < n\rangle$, $A_\ell \in J_\ell$ do we have*

$$\bar\beta \in X \Rightarrow \bigvee_\ell \beta_\ell \in A_\ell;$$

(c) *we can find $\langle \alpha_\eta : \eta \in \bigcup\limits_{m\le n} \prod\limits_{\ell<m} \lambda_\ell\rangle$ such that $\alpha_\eta < \lambda_{\lg}(\eta)$ and*

$(*)$ *for each $\nu \in \prod\limits_{\ell<n} \lambda_\ell$ we have*

$$\langle \alpha_{\nu\restriction(\ell+1)} : \ell < n\rangle \in X.$$

2) If $[A \subseteq \lambda_\ell \ \& \ |A| < \lambda_\ell] \ \Rightarrow \ A \in J_\ell$ *then we can add*

(d) *like (c), but adding*

(iii) $\alpha_{\nu\frown\langle i\rangle} < \alpha_{\nu\frown\langle j\rangle}$ *if* $i < j < \lambda_{\lg(\nu)+1}$.

Proof. Similar to 4.4.

4.7 Claim. *Let* $\bar\lambda = \langle \lambda_\ell : \ell < n\rangle$ *be a decreasing sequence of regular cardinals.*

(1) *If* $\lambda_\ell > 2^{\lambda_{\ell+1}}$ *for* $\ell < n$, *then:*
   (∗) *for every* $A \in (J^{\rm bd}_{\bar\lambda})^+$, *there are* $A_\ell \in (J^{\rm bd}_{\lambda_\ell})^+$ *such that* $\prod\limits_{\ell<n} A_\ell \subseteq A$.

(2) *If* $J = \prod\limits_{\ell<n} J_\ell$ *and* J_ℓ *is a* $(2^{\lambda_{\ell+1}})^+$*–complete ideal on* λ_ℓ, *<u>then</u>* (∗) *holds, with* J *in place of* $J^{\rm bd}_{\bar\lambda}$ *and* J_ℓ *in place of* $J^{\rm bd}_{\lambda_\ell}$.

(3) *For every* $A \in (J^{\rm bd}_{\bar\lambda})^+$ *and* $k < \omega$ *we can find* $B_\ell \in [\lambda_\ell]^k$ *such that* $\prod\limits_{\ell<n} B_\ell \subseteq A$.

(4) *In (3), instead of* k *and* $J^{\rm bd}_{\lambda_\ell}$ *(for* $\ell < n$*) we can use any* κ *and* $((\lambda_{\ell+1})^\kappa)^+$*-complete ideal* J_ℓ *on* λ_ℓ *for* $\ell < n$.

Proof. E.g., (3). We prove it by induction on n.

<u>$n = 1$</u>. Trivial, as singletons are in the ideal.

<u>$n + 1$</u>. Let $X_0 = \big\{\alpha < \lambda_0 : \{\bar\alpha \in \prod\limits_{\ell=1}^{n-1} \lambda_\ell : \langle\alpha\rangle\frown\bar\alpha \in A\} \in (\prod\limits_{\ell=1}^{n-1} J^{bd}_{\lambda_\ell})^+\big\}$.

Clearly, $X_0 \in (J^{bd}_{\lambda_0})^+$.

By the induction hypothesis, for each $\alpha \in X_0$, there is $\langle B^\alpha_\ell : \ell = 1, \dots, n-1\rangle$, such that

$$B^\alpha_\ell \in [\lambda_\ell]^k \quad \text{and} \quad \prod_{\ell=1}^{n-1} B^\alpha_\ell \subseteq \{\bar\alpha \in \prod_{\ell=1}^{n-1} \lambda_\ell : \langle\alpha\rangle\hat{\ }\bar\alpha \in A\} =: \bar B^\alpha.$$

So X_0 is the union of $\prod\limits_{\ell=1}^{n-1} \lambda^k_\ell = \lambda_1$ sets $X_0[\bar B] = \{\alpha \in X_0 : \bar B^\alpha = \bar B\}$, so for some $\bar B$, $|X_0[\bar B]| \ge k$ and let $B_0 =$ first k members of $X_{0,\bar B}$. $\square_{4.7}$

4.8 Definition. For a partial order P let $\mathrm{tcf}(P) = \lambda$ iff there is an increasing cofinal sequence of length λ in P (tcf – stands for true cofinality); so e.g., $(\omega, <) \times (\omega_1, <)$ has no true cofinality, but $\mathrm{tcf}\prod(\aleph_n, <)/\mathscr{D}$ is well defined if $\mathscr{D}$ is an ultrafilter on ω.

4.9 Fact. 1) If $J \supseteq J^{\mathrm{bd}}_\delta$ is an ideal, $\lambda_i = \mathrm{cf}(\lambda_i) > \delta$, for $i < \delta$ and $\lambda = \mathrm{tcf}(\prod\limits_{i<\delta} \lambda_i/J)$, then there is a (λ, J)-sequence $\bar\eta = \langle \eta_\alpha : \alpha < \lambda\rangle$ for $\langle J^{bd}_{\lambda_i} : i < \delta\rangle$.
2) If λ_i is increasing in i then $\langle J^{bd}_{\lambda_i} : i < \delta\rangle$ is normal (hence $\bar\eta$ is normal) provided that $\delta = \omega$ or at least

$(*)_1$ $\lambda > \prod\limits_{j<i} \lambda_j$ for $i < \delta$.

3) If we just ask $\bar\eta$ to be normal it suffices to demand

$(*)_2$ $\lambda_i >$ max $\mathrm{pcf}\{\lambda_j : j < i\}$ for $i < \delta$.

Proof. In $\prod\limits_{i<\delta} \lambda_i/J$, there is a cofinal increasing sequence $\langle f_\alpha : \alpha < \lambda\rangle$. It is as required, as we now show. Let $X \in [\lambda]^\lambda$, let $X_i = \{f_\alpha(i) : \alpha \in X\}$ for $i < \delta$. Define $f \in \prod\limits_{i<\delta} \lambda_i$:

$$f(i) = \begin{cases} \sup(X_i) + 1 & \text{if } \sup(X_i) < \lambda_i \\ 0 & \text{otherwise.} \end{cases}$$

But $\langle f_\alpha : \alpha < \lambda\rangle$ is cofinal, so for some $\alpha_0 < \lambda$, $f <_J f_{\alpha_0}$. Now $X \in [\lambda]^\lambda$, so for some α_1, we have $\alpha_0 < \alpha_1 \in X$. As $\langle f_\alpha : \alpha < \lambda\rangle$ is increasing, $f_{\alpha_0} <_J f_{\alpha_1}$, hence $f <_J f_{\alpha_1}$. So $A = \{i : f(i) \geq f_{\alpha_1}(i)\} \in J$. But $f_{\alpha_1}(i) \in X_i$, so $i \in \delta \backslash A \Rightarrow \lambda_i = \sup(X_i)$.
2) Easy.
3) By [Sh:g, II,3.5]. $\square_{4.9}$

4.10 Comment: 1) This is good e.g. to lift a colouring of the λ_i's to one of λ. But we would like to have an upgrade as well.
2) The kind of assumptions of 4.9 is the central interest in [Sh:g].

4.11 Claim. *Assume $\bar\lambda^i = \langle \lambda_{i,\ell} : \ell < n_i\rangle$ is an increasing sequence of regulars $> \delta$ for $i < \delta$. Also assume that J is an ideal on $\{(i,\ell) : i < \delta, \ell < n_i\}$ and*

$$\lambda = \mathrm{tcf}(\prod_{i,\ell} \lambda_{i,\ell}/J),$$

and for some ideal J' on δ, we have $J' \supseteq J^{\rm bd}_\delta$ and J is generated by

$$\big\{\{(i,n) : n < n_i, i \in A\} : A \in J'\big\}.$$

Then there is a (λ, J')-sequence $\bar\eta$ for $\langle J^{\rm bd}_{\bar\lambda^i} : i < \delta\rangle$.
2) $\langle J^{\rm bd}_{\bar\lambda^i} : i < \delta\rangle$ is normal (hence $\bar\eta$ above is normal) if

$(*)_1$ $\delta = \omega$ *and* $i < j < \delta \Rightarrow \lambda_{i,n_i-1} < \lambda_{j,0}$ *or*
$(*)_2$ $\prod\{\lambda_{i,\ell} : i < j, \ell < n_j\} < \lambda_{j,0}$.

3) If we ask just $\bar\eta$ to be normal it suffices to demand

$(*)_3$ $\max \operatorname{pcf}\{\lambda_{i,\ell} : i < j, \ell < n_j\} < \lambda_{j,0}$.

Proof. Again, let $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ be $<_J$–increasing cofinal. Let $\eta_\alpha(i) = \langle f_\alpha(i,\ell) : \ell < n_i\rangle \in \prod \bar\lambda^i$. Let $X \in [\lambda]^\lambda$. Let $X_i = \{\eta_\alpha(i) : \alpha \in X\}$. If $X_i \in J^{bd}_{\prod \bar\lambda^i}$, then there is $\bar\alpha^i \in \prod \bar\lambda^i = \prod\limits_{\ell<n_i} \lambda_{i,\ell}$ such that

$$(*) \qquad \bar\beta \in X_i \Rightarrow \bigvee_{\ell<n_i} \beta_\ell < \alpha^i_\ell.$$

(We return to this at the end of the proof.)

So let $f \in \prod\limits_{i,\ell} \lambda_{i,\ell}$ be given by $f(i,\ell) = \alpha^i_\ell$. So, as before, for some $\alpha \in X$, $f <_J f_\alpha$. So

$$A = \Big\{i : \bigwedge_{\ell<n_i} f((i,\ell)) \geq f_\alpha((i,\ell))\Big\} \in J'.$$

Now for $i \in \delta\backslash A$ we have $X_i \notin J^{bd}_{\prod \bar\lambda^i}$.

[Why $(*)$? Prove the existence of $\bar\alpha^i$, for notational convenience denoted here by $\bar\beta$, by induction on n_i. Here we use "increasing $\bar\lambda^i$".

$n_i = 1$. Clear

$n_i = k+1$. For $\alpha < \lambda_{i,0}$ define

$$X_{i,\alpha} = \{\bar\beta \restriction [1, n_i) : \bar\beta \in X_i\}.$$

So we know that for some $\gamma_0 < \lambda_{i,0}$

$$\alpha \in [\gamma_0, \lambda_{i,0}] \Rightarrow X_{i,\alpha} \in J^{\mathrm{bd}}_{\prod_{\ell=1}^{n-1} \lambda_{i,\ell}}.$$

So for each such α we have $\bar{\beta}^\alpha \in \prod\limits_{\ell=1}^{n} \lambda_{i,\ell}$ as given by the induction hypothesis. Let

$$\beta_\ell = \begin{cases} \gamma_0 + 1 & \text{if } \ell = 1, \\ \bigcup\{\beta^\alpha_\ell : \alpha \in [\gamma_0, \lambda_{i,0})\} & \text{otherwise.} \end{cases}$$

Why is the latter $< \lambda_{i,\ell}$? As $\lambda_{i,0} < \mathrm{cf}(\lambda_{i,\ell})$.] $\square_{4.11}$

4.12 Question: Are there many cases fitting the framework of 4.11?

4.13 Answer: Not so few. E.g., for any κ, for many $\lambda = \mathrm{cf}(\lambda)$ we have that $\lambda = \mathrm{tcf}\Big(\prod\limits_{i<\kappa} \lambda_i / J^{\mathrm{bd}}_\kappa\Big)$ for some sequence $\langle \lambda_i : i < \kappa \rangle$. E.g., if $\aleph_0 < \mathrm{cf}(\delta) = \kappa$ and $\kappa < \mu = \beth_\delta < \lambda = \mathrm{cf}(\lambda) \le \beth_{\delta+1}$ or just $\aleph_0 < \kappa = \mathrm{cf}(\mu) < \lambda = \mathrm{cf}(\lambda) \le \mu^\kappa$ and $(\forall \chi < \mu)[\chi^\kappa < \mu]$ then there is an increasing sequence of regulars $\langle \lambda_i : i < \kappa \rangle$ with limit $\beth_\delta$ or μ respectively as above. [Why? see [Sh:g, VIII,§1,2.6].] Even if $\kappa = \aleph_0$ this holds for many λ's, e.g., if $\mu < \lambda < \mu^{+\omega_1}$ or just $|\{\chi : \mu < \chi < \lambda;$ and $\chi = \aleph_\chi\}| < \mu$ see [Sh:g, IX] and use 4.14 below.

Note that by the pcf–theorem (see [Sh:g, VIII,2.6])

4.14 Claim. *Assume I to be an ideal on δ, and $\lambda_{i,\ell} = \mathrm{cf}(\lambda_{i,\ell}) > |\delta|$ for $i < \delta$ and $\ell < n_i$ and $0 < n_i < \omega$. Then the following are equivalent*

(a) *for every $\langle k_i : i < \delta \rangle \in \prod\limits_{i<\delta} n_i$ we have*

$$\lambda = \mathrm{tcf}\Big(\prod_{i<\delta} \lambda_{i,k_i} / I\Big).$$

(b) *Letting*

$$I' = \Big\{A \subseteq \bigcup_{i<\delta} \{i\} \times n_i : \text{for some} B \in I \text{ we have} A \subseteq \bigcup_{i \in B} \{i\} \times n_i\Big\},$$

we have $\prod \lambda_{i,n}/I'$ has true cofinality λ.

Proof. Let A^*, B^* be a partition of $\bigcup\limits_{i<\theta} \{i\} \times n$ such that

$$\lambda = \max \mathrm{pcf}\{\lambda_{i,n} : (i,n) \in A^*\} \quad \text{and} \quad \lambda \notin \mathrm{pcf}\{\lambda_{i,n} : (i,n) \in B^*\}$$

(exists by the pcf theorem). Now:

(a) ⇒ (b)

If $\prod_{i,n} \lambda_{i,n}/I'$ does not have true cofinality λ, then for some $A \in (I')^+$ we have that $\prod_{(i,n)\in A} \lambda_{i,n}/I'$ has true cofinality $\lambda' \neq \lambda$ (here we use the pcf theorem) and without loss of generality $A \subseteq A^* \ \vee \ A \subseteq B^*$, hence $\lambda \notin \text{pcf}\{\lambda_{i,n} : (i,n) \in A\}$. Let $B = \{i < \delta : (\exists n < n_i)[(i,n) \in A]\}$, so by the definition of I' we know $B \in I^+$. So, for $i \in B$ we can choose $k_i \in \{0,\dots,n_i - 1\}$ such that $(i,k_i) \in A$. So $\{(i,k_i) : i \in B\} \subseteq A$ hence $\text{pcf}\{\lambda_{i,k_i} : i \in B\} \subseteq \text{pcf}\{\lambda_{i,k} : (i,k) \in A\}$, but λ does not belong to the later, hence not to the former, contradicting (a).

¬(a) ⇒ ¬(b)

So there is $\langle k_i : i < \delta\rangle \in \prod_{i<\delta} n_i$ such that $\neg[\text{tcf}(\prod \lambda_{i,k_i}/I) = \lambda]$ hence by the pcf theorem, for some $A \in (I)^+$, we have $\max \text{pcf}\{\lambda_{i,k_i} : i \in A\} < \lambda$. Let $B = \{(i,k_i) : i \in A\}$, so clearly $\max \text{pcf}\{\lambda_{i,k_i} : (i,k_i) \in B\} < \lambda$. But by the definition of I', we have $B \in (I')^+$ so we get contradiction to (b). $\square_{4.14}$

4.15 Remark. See more on related topics in [Sh 589].

§5 λ-sequences for decreasing $\bar\lambda^i$ by pcf

5.1 Discussion. Our aim here is to get "decreasing $\bar\lambda$" from "increasing $\bar\lambda$" (for $J^{bd}_{\bar\lambda}$), in some sense, to "make gold from lead". We do this by using pcf assumptions, then proving that these assumptions are very reasonable.

(Note: when we cannot materialize the pcf assumptions the situation is close to SCH, and then we have other avenues for construction of λ-sequences for some I, e.g., (1.9, 3.1).)

* * *

In the following claim the interesting case is when λ_ℓ are increasing, $\bar\lambda^i = \langle \lambda_{\ell,i} : \ell < n\rangle$ decreasing sequence of regular cardinals, $\lambda_{\ell,i} > \prod\limits_{\substack{j<i\\ m<n}} \lambda_{m,j}$, or at least $\lambda_{\ell,i} > \max\ \mathrm{pcf}\{\lambda_{m,j} : m < n, j < i\}$.

5.2 Claim. *Assume*

(a) $\bar\lambda = \langle \lambda_\ell : \ell < n\rangle$, $\bar\lambda^i = \langle \lambda_{\ell,i} : \ell < n\rangle$ *for* $i < \delta$,
(b) I *is an ideal on* δ,
(c) $\lambda_\ell = \mathrm{tcf}\Big(\prod\limits_{i<\delta} \lambda_{\ell,i}/I\Big)$ *for* $\ell < n$,
(d) $\bar f^\ell = \langle f_{\ell,\alpha} : \alpha < \lambda_\ell\rangle$ *is* $<_I$*–increasing and cofinal in* $\prod\limits_{i<\delta} \lambda_{\ell,i}$,
(e) $\delta < \lambda_{\ell,i} = \mathrm{cf}(\lambda_{\ell,i})$,
(f) *for* $\bar\alpha \in \prod\limits_{\ell<n} \lambda_\ell$ *let* $f_{\bar\alpha}$ *be defined by* $f_{\bar\alpha}(i) = \langle f_{\ell,\alpha_\ell}(i) : \ell < n\rangle \in \prod\limits_{\ell<n} \lambda_{\ell,i}$.

Then for any $X \in (J^{\mathrm{bd}}_{\bar\lambda})^+$ *we have*

$$\big\{i : \{f_{\bar\alpha}(i) : \bar\alpha \in X\} \in J^{\mathrm{bd}}_{\bar\lambda^i}\big\} \in I.$$

Proof. Let $X_i = \{f_{\bar\alpha}(i) : \bar\alpha \in X\}$ and let $B = \{i < \delta : X_i \in J^{\mathrm{bd}}_{\bar\lambda^i}\}$.

Assume $B \in I^+$ and we shall get a contradiction. For each $i \in B$, $m < n$ and $\bar\alpha \in \prod\limits_{\ell<m} \lambda_{\ell,i}$, let

$$X^i_{\bar\alpha} = \Big\{\bar\beta \in \prod_{\ell=m}^{n-1} \lambda_{\ell,i} : \bar\alpha\,\hat{}\,\bar\beta \in X_i\Big\}$$

and let

$$g_i(\bar{\alpha}) = \min\Big\{\gamma \leq \lambda_{m,i} : \text{ if } \beta \in [\gamma, \lambda_{m,i}) \text{ then } X^i_{\bar{\alpha}\frown\langle\beta\rangle} \in J^{\rm bd}_{\prod_{\ell=m+1}^{n-1}\lambda_{\ell,i}}\Big\}.$$

This definition just unravels the definition of $J^{\rm bd}_{\bar{\lambda}^i}$; note

$(*)$ $X^i_{\langle\,\rangle} = X_i \in J^{\rm bd}_{\bar{\lambda}^i}$,

$(*)'$ if $X^i_{\bar{\alpha}} \in J^{\rm bd}_{\prod_{\ell\geq\,\lg(\bar{\alpha})}}\lambda_{\ell,i}$ then $g_i(\bar{\alpha}) < \lambda_{\lg}(\bar{\alpha})$.

Now we choose by induction on $m < n$ ordinals $\alpha_m < \lambda_m$ such that for $m \leq n$ we have

$(*)_m$ $B_m =: \big\{i \in B : X^i_{\langle f_{\ell,\alpha_\ell}(i):\ell<m\rangle} \in J^{\rm bd}_{\prod_{\ell\geq m}\lambda_{\ell,i}}\big\} = B \bmod I.$

So, stipulating $J^{\rm bd}_{\prod_{\ell\geq n}\lambda_{\ell,i}} = \{\emptyset\}$, the ideal on $\{\langle\,\rangle\}$, we have that $(*)_0$ holds with $B = B_0$.

If $(*)_m$ is true, clearly

$$\langle g_i(\langle f_{\ell,\alpha_\ell}(i) : \ell < m\rangle) : i \in B_m\rangle$$

is in $\prod\limits_{i<\delta}\lambda_{m,i}$. But $B_m \in I^+$ and $\langle f_{m,\alpha} : \alpha < \lambda_m\rangle$ is $<_I$–increasing cofinal in $\prod\limits_{i<\delta}\lambda_{m,i}$. So for some α_m

$$B'_m = \{i \in B_m : g_i(\langle f_\ell, \alpha_\ell(i) : \ell < m\rangle) \geq \alpha_m\} \in I.$$

Defining B_{m+1} using this α_m, we easily obtain

$$B_{m+1} \supseteq B_m \backslash B'_m \text{ so we see that } (*)_{m+1} \text{ holds.}$$

So

$$\bar{\alpha} = \langle\alpha_\ell : \ell < n\rangle \in \prod_{\ell<n}\lambda_\ell$$

is well defined.

In the inductive definition of α_m, any larger α'_m would serve in place of α_m (of course it would influence the future choices). So, in addition to $(*)_m$, we can demand

$$(**)_m \qquad \Big\{\bar\beta \in \prod_{\ell=m}^{n-1} \lambda_\ell : \langle \alpha_\ell : \ell < m\rangle^\frown \bar\beta \in X\Big\} \in (J^{\rm bd}_{\bar\lambda \restriction [m,n)})^+.$$

So from $(**)_n$ we get $\langle \alpha_\ell : \ell < n\rangle \in X$ hence for all i we have $\langle\rangle \in X^i_{\langle f_{\ell,\alpha_\ell}(i):\ell<n\rangle}$, by the definition. But

$$B_n = \{i \in B : X^i_{\langle f_{\ell,\alpha_\ell}(i):\ell<n\rangle} \in J^{\rm bd}_{\prod_{\ell \ge n} \lambda_{\ell,i}}\} = B \bmod I,$$

so $B_n \neq \emptyset$, and if $i \in B_n$ this means $X^i_{\langle f_{\ell,\alpha_\ell}(i):\ell<n\rangle} \in J^{\rm bd}_{\prod_{\ell\ge n}\lambda_{\ell,i}} = \{\emptyset\}$ so $X^i_{\langle f_{i,\alpha_\ell}(i):\ell<\rangle} = \emptyset$, contradicting the previous sentence. $\square_{5.2}$

In fact, more generally,

5.3 Claim. . *Assume*

(a) $\bar\eta^\ell = \langle \eta^\ell_\alpha : \alpha < \lambda\rangle$ *in an* (I,J,λ)*-sequence for* $\langle I_{i,\ell} : i < \delta\rangle$ *for each* $\ell < n$,
(b) $I_i = \prod\limits_{\ell<n} I_{i,\ell}$,
(c) $\bar\eta = \langle \eta_\alpha : \alpha < \lambda\rangle$, *where* $\eta_\alpha \in \prod\limits_{i<\delta} Dom(I_i)$ *and*

$$\eta_\alpha(i) = \langle \eta^\ell_\alpha(i) : \ell < n\rangle.$$

Then $\bar\eta$ *is an* (I,J,λ)*-sequence for* $\langle I_i : i < \delta\rangle$.

Proof. Like the proof of 5.2. $\square_{5.3}$

5.4 Claim. . *Assume*

(a) $\lambda = \operatorname{tcf}\Big(\prod\limits_{i<\delta} \theta_{\ell,i}/J\Big)$ *for* $\ell < n$ *and* $\theta_{\ell,i}$ *are increasing with* ℓ;
(b) $\theta_{\ell,i} = \operatorname{tcf}\Big(\prod\limits_{\varepsilon<\varepsilon_i} \tau_{\ell,i,\varepsilon}/J_i\Big)$ *and* $\tau_{\ell,i,\varepsilon}$ *are regular decreasing with* ℓ, *i.e.,* $\tau_{\ell,i,\varepsilon} \ge \tau_{\ell+1,i,\varepsilon}$ *(the interesting case is* $>$*).*

Let

$$J^* = \Big\{A : A \subseteq \{(\ell,i,\varepsilon) : \ell < n, i < \delta, \varepsilon < \varepsilon_i\} \textit{ and;}$$
$$\bigwedge_\ell (\forall^J i)(\forall^{J_i}\varepsilon)[(\ell,i,\varepsilon) \notin A]\Big\},$$

and let

$$I_{i,\varepsilon} = \prod_{\ell<n} J^{\rm bd}_{\tau_{\ell,i,\varepsilon}}.$$

Then

$$\lambda = \operatorname{tcf}\Big(\prod_{i,\varepsilon} \tau_{\ell,i,\varepsilon}/J^*\Big)$$

and we can find $\bar\eta_\alpha \in \prod\limits_{i,\varepsilon} I_{i,\varepsilon}$ *for* $\alpha < \lambda$ *such* $\langle \eta_\alpha : \alpha < \lambda\rangle$ *is a* (λ, J^*)*-sequence for* $\langle I_{i,\varepsilon} : i, \varepsilon\rangle$.

Proof. Straight. (Using 5.3 and [Sh:g, I,2.10]). $\square_{5.4}$

5.5 Example: Assume

(∗) $\langle \lambda_i : i < \delta\rangle$ is a strictly increasing sequence of regulars, $\delta < \lambda_0$, $\lambda = \operatorname{tcf}\big(\prod\limits_{i<\delta} \lambda_i/J^{\rm bd}_\delta\big)$.

5.6 Discussion. This may seem a strong assumption, but getting such representations is central in [Sh:g]. If μ is strong limit singular

$\otimes$ $\aleph_0 < \kappa = \operatorname{cf}(\mu) < \mu < \lambda = \operatorname{cf}(\lambda) \le 2^\mu$,

then there is such $\langle \lambda_i : i < \operatorname{cf}(\mu)\rangle$, $\lambda_i < \mu = \sup(\lambda_i)$. So without loss of generality $2^{\lambda_i} < \lambda_{i+1}$ (see 4.13).
Now fix n for simplicity. Let

$$\lambda_{\ell,i} = \lambda_{n\times i+n-\ell}.$$

So

$$\bar\lambda^i = \langle \lambda_{\ell,i} : \ell < n\rangle \text{ is strictly decreasing.}$$

In 4.14 an example is given for 5.2.

For 5.4 we have e.g.

5.7 Claim. *Assume*

- (a) *μ is strong limit,*
- (b) $\aleph_0 = \mathrm{cf}(\mu) < \mu$,
- (c) $2^\mu \geq \mu^{+\omega+1} = \lambda$ *[also* $\mu^{+\omega_4+\omega+1} = \lambda$ *is OK, or just* $\lambda = \mu^{+\delta+1} < \mathrm{pp}^+(\mu)$ *and* $\mathrm{cf}(\mu^{+\delta}) < \mu$*].*

Then:
1) We can find $\lambda_{\ell,i}$, k_ℓ *such that* $(\ell \leq i < \omega)$*:*

- (A) $\lambda_{i,\ell} < \mu = \sum_{m,j} \lambda_{m,j}$,
- (B) $2^{\lambda_{\ell+1,i}} < \lambda_{\ell,i}$ *and* $2^{\lambda_{0,i}} < 2^{\lambda_{i+1,i+1}}$,
- (C) $\mathrm{tcf}\big(\prod\limits_{i<\omega} \lambda_{\ell,i}/J^{\mathrm{bd}}_\omega\big) = \mu^{+k_\ell}$,
- (D) $0 < k_m < k_{m+1} < \omega$,
- (E) $\lambda = \mathrm{tcf}\Big(\prod\limits_{m<\omega} \mu^{+k_m}/J^{\mathrm{bd}}_\omega\Big)$.

2) For every $n < \omega$*, we can find* $J, \lambda'_{\ell,i}$ $(\ell < n, i < \omega)$ *such that:*

- (i) *there is* $\bar\eta$ *a* λ*-sequence for* $\langle J^{\mathrm{bd}}_{\langle \lambda'_{\ell,i}:\ell<n\rangle} : i < \omega\rangle$,
- (ii) $2^{\lambda'_{\ell+1,i}} < \lambda'_{\ell,i}$,
- (iii) $2^{\lambda'_{0,i}} < \lambda'_{n-1,i+1}$,
- (iv) $(\forall A \in J)(\exists^\infty i)\ [n \times \{i\} \cap A = \emptyset]$.

5.8 Remark. 1) This Claim can be used with no further reference to pcf: just for any μ as in (a)–(c), we have $\bar\eta$ for which we can construct colourings, objects, etc.
2) There are theorems with n increasing, they are somewhat cumbersome.
Of course, we can use

$$I'_m = \prod_{i=n_m}^{n_m+1} J^{\mathrm{bd}}_{\langle \lambda_{\ell,i}:\ell<n_m\rangle}.$$

3) <u>Note</u>: $2^\mu \geq \mu^{\omega+1}$ is a strong negation of $2^\mu = \mu^+$ which was very useful here. (Our general theme is: $\neg$SCH is a good hypothesis) and we shall deal with closing the gap.
4) <u>Note</u>: if $2^\mu = \mu^{+n(*)}$, we can prove nice things with $I = J^{\mathrm{bd}}_{\langle \mu^{+n(*)-\ell},\ell<n(*)\rangle}$.
5) If $\aleph_0 <\ \mathrm{cf}(\mu) < \mu$ the parallel claim is even easier, and μ being a strong limit is necessary only for (B).

Proof of 5.7. 1) We will just give a series of quotations.

First $\mathrm{cf}(\mu^{+\omega}) = \aleph_0$, so by [Sh:g, II,1.6], there is an increasing sequence $\langle \theta_i : i < \omega\rangle$ of regulars with limit $\mu^{+\omega}$ such that

$$\lambda = \mu^{+\omega+1} = \mathrm{tcf}\Big(\prod_{i<\omega} \theta_i / J^{\mathrm{bd}}_\omega\Big),$$

so for i large enough $\theta_i > \mu$. So without loss of generality $\bigwedge_i \theta_i > \mu$.

So let $\theta_i = \mu^{+k_i}$, $k_i \in (0,\omega)$ strictly increasing. By [Sh:g, 5.9,p.408], we have $\mathrm{pp}(\mu) > \mu^{+k_i}$. (We would like to have $\mathrm{pp}(\mu) = 2^\mu$, but only "almost proved".) This means by the no hole theorem [Sh:g, 2.3] that for some countable set $\mathfrak{a}_\ell$ of regulars $< \mu$, $\mu = \sup(\mathfrak{a}_\ell)$ and $\mu^{+k_\ell} \in \mathrm{pcf}(\mathfrak{a}_\ell)$. So by the pcf theorem, without loss of generality $\mu^{+k_\ell} = \max\ \mathrm{pcf}(\mathfrak{a}_\ell)$ and $\mu^+, \dots, \mu^{+(k_\ell - 1)} \notin \mathrm{pcf}(\mathfrak{a}_\ell)$ (alternatively use [Sh:g, VIII,§1]).

So necessarily

$$\mu^{+k_\ell} = \mathrm{tcf}\big(\prod \mathfrak{a}_\ell / J^{\mathrm{bd}}_{\mathfrak{a}_\ell}\big).$$

Let $\mu = \sum_{n<\omega} \mu_n$, $\mu_n < \mu_{n+1} < \mu$. We start choosing $\lambda_{\ell,i}$ by induction on i, for all i by downward induction on ℓ, so that

$$\lambda_{\ell,i} > \mu_i, \quad \lambda_{\ell,i} \in \mathfrak{a}_i,$$

and (B) holds. So, as $\lambda_{\ell,i} \in \mathfrak{a}_i$ and $\lambda_{\ell,i}$ is increasing with i, with limit μ, we have

$$\mathrm{tcf}\Big(\prod_i \lambda_{\ell,i} / J^{\mathrm{bd}}_\omega\Big) = \mu^{+k_\ell}.$$

2) Let $h : \omega \to \omega$ be such that $(\forall m)(\exists^{\aleph_0} i)\ (h(i) = m)$. Choose by induction on i, $\lambda'_{\ell,i} \in \{\lambda_{h(i),m} : m < \omega\}$ such that (b) + (c) of (2) hold. For each i we do this by downward induction on ℓ. Then apply the last theorem. $\square_{5.8}$

We may deal with all n's at once, at some price. The simplest case is:

5.9 Claim. . *Assume*

(a) $\langle A_\ell : \ell < \omega\rangle$ *is a sequence of pairwise disjoint sets,*

(b) $\lambda = \mathrm{tcf}\big(\prod\limits_{n<\omega} \theta_n / J^{\mathrm{bd}}_\omega\big)$,

(c) $\theta_n = \mathrm{tcf}\big(\prod\limits_{\ell<\omega} \tau_{n,\ell} / J^{\mathrm{bd}}_\omega\big)$, $\tau_{n,\ell}$ *regular* $> \aleph_0$,

(d) $h : \omega \to \omega$ *is such that* $|h^{-1}(\{n\})| = \aleph_0$, $J = \big\{A \subseteq \omega \times \omega : (\forall^{J^{\mathrm{bd}}_\omega} n)\ (\forall^{J^{\mathrm{bd}}_\omega} m)$ $(h(n) = \emptyset = A \cap \{m\} \times [h(n), 2h(n))\big\}$.

__Then__ *there is a* (λ, J)*-sequence for* $\langle J^{\mathrm{bd}}_{\tau_{n,\ell}} : (n,\ell) \in \omega \times \omega\rangle$.

Proof. Straight.

5.10 Remark. 1) We can replace $\langle \theta_n : n < \omega\rangle$ by $\langle \theta_i : i < \delta\rangle$.
2) Another way to get an example for 5.4 is to have $\langle \mu_i : i < \kappa\rangle$ increasing continuous, $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, $\kappa < \mu_0$, $\mu = \mu_\kappa = \sum_{i<\kappa} \mu_i$, $\mathrm{cf}(\mu_i) \leq |\delta|$, $\mathrm{pp}_{|\delta|}(\mu_i) < \mu_{i+1}$, $\chi_i = |\mathrm{Reg} \cap [\mu_i, \mathrm{pp}^+_{|\delta|}(\mu_i))|$, $S \subseteq \kappa$ stationary such that for every $S' \subseteq S$ stationary we have $\prod\limits_{i \in S'} \chi_i > \chi_\kappa$.
3) In all the cases here we can get normality as in §4.
4) See 1.19, 1.20.

§6 Products of Boolean Algebras

Monk asks [M2, Problem 35,p.15]:

6.1 Monk's Problem. Does $\prod_{n<\omega} \mathrm{FBA}(\beth_n)$ have free caliber $\beth^+_\omega$?

Here:

6.2 Notation. FBA(β) is the Boolean algebra freely generated by $\langle x_\alpha : \alpha < \beta \rangle$.

6.3 Definition. 1) We say that the cardinal λ is a free caliber of the Boolean algebra $\mathbf{B}$ if for every $X \in [\mathbf{B}]^\lambda$ there is $Y \in [\mathbf{B}]^\lambda$ such that Y is independent in $\mathbf{B}$, so if $\mathrm{cf}(\lambda) > \|\mathbf{B}\|$ this holds trivially.
2) $\mathrm{FreeCal}(\mathbf{B}) = \{\lambda \leq |\mathbf{B}| : \lambda \text{ is a free caliber of } \mathbf{B}\}$.
We show that, e.g., if $\beth^+_\omega = 2^{\beth_\omega}$ then the answer is NO.

6.4 Claim. *Assume:*

(a) *there is a normal*[1] *super (λ, J)-sequence $\bar\eta$ for $\bar I = \langle I_i : i < \delta \rangle$,*
(b) $I_i =\ ERI^2_{\lambda_i,\kappa_i} =: \big\{X \subseteq [\lambda_i]^2$: *for some* $h : X \to \kappa_i, |\mathrm{Rang}(h) < \kappa_i$, *and for no* $u \in [\lambda_i]^{\aleph_0}$ *do we have* ($h \restriction [u]^2$ *constant*) & $[u]^2 \subseteq X\big\}$,
(c) $\delta < \omega_1$.

Then λ is not a free caliber of $\prod_{i<\delta} FBA(\lambda_i)$.

6.5 Remark. By 3.1, if $\lambda = \mu^+ = 2^\mu$, μ strong limit $> \aleph_0 =\ \mathrm{cf}(\mu)$, then we can find such κ_i, $\lambda_i < \mu$ and $\bar\eta$ for $\delta = \omega$.

Proof. By renaming without loss of generality

$$(*)_1 \qquad \eta_\alpha(i) \geq \sum_{j<i} \lambda_j.$$

Let $\eta_\alpha(i) = \{f^0_\alpha(i), f^1_\alpha(i)\}$, $f^0_\alpha(i) < f^1_\alpha(i)$ $(< \lambda_i)$. First we deal with the case $\delta = \omega$, as its notation is simpler. Let $\mathbf{B}_n =\ \mathrm{FBA}(\lambda_n)$ be freely generated by $\{x^n_\alpha : \alpha < \lambda_n\}$. We define $g^*_\alpha \in \prod_{n<\omega} \mathbf{B}_n$ for $\alpha < \lambda$ by

[1] if $\bar I$ is normal, i.e. $\kappa_{i+1} > \lambda_i$, the normality of $\bar\eta$ follows.

$$g^*_\alpha(\ell) = \bigcap_{k<\ell} \left(x^\ell_{f^0_\alpha(k)} - x^\ell_{f^1_\alpha(k)} \right).$$

Note:

$\otimes_1$ for $\alpha < \beta < \lambda$, we have g^*_α, g^*_β are distinct elements of $\prod\limits_{n<\omega} \mathbf{B}_n$,

$\otimes_2$ if $f^0_n(\beta) = f^1_n(\alpha)$ and $m > n$ then $\mathbf{B}_m \models g^*_\alpha(m) \cap g^*_\beta(m) = 0$.

[Why? As $x^m_{f^0_n(\alpha)} - x^m_{f^1_n(\alpha)}$ is disjoint to $x^m_{f^0_n(\beta)} - x^m_{f^1_n(\beta)}$.]

$\otimes_3$ if $n < \omega$ and for $i = 1, 2$ we have $\alpha_i, \beta_i < \lambda$ and $f^0_n(\beta_i) = f^1_n(\alpha_i)$ and

$$\bigwedge_{k<n} f^0_k(\alpha_1) = f^0_k(\alpha_2) \quad \text{and} \quad \bigwedge_{k<n} f^1_k(\beta_1) = f^1_k(\beta_2)$$

then

$$\prod_{n<\omega} \mathbf{B}_m \models g^*_{\alpha_1} \cap g^*_{\beta_1} = g^*_{\alpha_2} \cap g^*_{\beta_2}.$$

[Why? Check each coordinate in the product, for $m > n$ use $\otimes_2$ to show that both sides are zero, and if $m \leq n$ use the last two assumptions.]

Now if $X \in [\lambda]^\lambda$ then there are such $\alpha_1, \alpha_2, \beta_1, \beta_2$ (using the choice of $\bar\eta$ and its normality).

What if $\delta > \omega$ is a limit ordinal of cofinality ω? Let $\delta = \cup\{i_n : n < \omega\}, 0 = i_0 < i, i_n < i_n < 0$ and for $i < \delta, n[i]$ is the unique $n < \omega$ such that $i_n \leq i < i_{n+1}$ and let $\delta = \bigcup\limits_{n<\omega} u_n, u_n \subseteq u_{n+1}, u_n$ finite for $n < h$. We let $g^*_\alpha \in \prod\limits_{i<\delta} B_i$ for $\alpha < \delta$, be the function

$$g^*_\alpha(i) = \cap\{x^i_{f^0_\alpha(j)} - x^i_{f^1_\alpha(j)} : j \in u_{n[i]}\}.$$

$\square_{6.4}$

6.6 Claim. *Assume:*

$(*)(a)$ $\mu = \mu^\theta < \lambda = \mathrm{cf}(\lambda) \leq 2^\mu$, *and* $\langle \chi_i : i < \theta \rangle$ *a sequence of cardinals, or*

(b) $2^\theta < \lambda = \mathrm{cf}(\lambda)$ *and in the* $(< \theta^+)$*-base product topology on* ${}^{\sup(\chi_i)}2$ *the density is* $< \lambda$, *or at least in the box product topology on* $\prod\limits_{i<\theta} ({}^{\chi_i}2)$ *(where each* ${}^{\chi_i}2$ *has Tychonov topology) has density* $< \lambda$.

<u>Then</u> $\prod_{i<\theta}$ *FBA*(χ_i) *has free caliber* λ.

Proof. As in §2. $\square_{6.4}$

Probably the choice of the product of $\langle$ FBA$(\beth_n) : n < \omega\rangle$ in the original question was chosen just as the simplest case, as is often done. But in this case the products of uncountably many free Boolean algebras behave differently.

6.7 Claim. *Assume* $\lambda =$ cf$(\lambda) > 2^\theta$, cf$(\theta) > \aleph_0$ *and* $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$ *and each* χ_i *is a cardinal. <u>Then</u>* $\prod_{i<\theta}$ *FBA*(χ_i) *has free caliber* λ.

Proof. First assume a stronger assumption

$(*)$ $\lambda = \mu^+$, cf$(\mu) = \theta > \aleph_0$ and $(\forall \alpha < \mu)(|\alpha|^\theta < \mu)$,

or alternatively

$(*)^-$ $\lambda =$ cf(λ) and $\mu > 2^\theta$ are as in 7.5 below and we assume $i < \theta \Rightarrow \chi_i \le \mu$.

(This was our first proof. It possibly covers all cases under some reasonable pcf hypothesis, and illuminates the method).

Let $g^*_\alpha \in \prod_{i<\theta}$ FBA(χ_i) for $\alpha < \lambda$ be pairwise distinct, and we should find $X \in [\lambda]^\lambda$ such that $\langle g^*_\alpha : \alpha \in X\rangle$ is independent. Let

$$g^*_\alpha(i) = \tau_{\alpha,i}(x_{\beta_{\alpha,i,0}}, x_{\beta_{\alpha,i,1}}, \dots, x_{\beta_{\alpha,i,m(\alpha,i)-1}}),$$

where $\tau_{\alpha,i}$ is a Boolean term. Without loss of generality no $x_{\beta_{\alpha,i,\ell}}$ is redundant, $\beta_{\alpha,i,m}$ increasing with m. As $2^\theta < \lambda =$ cf(λ) without loss of generality $\tau_{\alpha,i} = \tau_i$ and so $m(\alpha,i) = m(i)$ for every $\alpha < \lambda, i < \theta$. Let f_α be the function with domain θ, $f_\alpha(i) = \langle \beta_{\alpha,i,\ell} : \ell < m(i)\rangle$. Let $f^{[\ell]}_\alpha(i) = \beta_{\alpha,i,\ell}$, so Dom$(f^{[\ell]}_\alpha) = \{i < \theta : \ell < m(i)\}$.

If $(*)$ holds then by 7.3(2) and 7.4(2) (see later) we have

⊚ there are u^*, m^*, v, $\bar\beta^*$, X such that
- (a) $u^* \in [\theta]^\theta$ and $X \in [\lambda]^\lambda$,
- (b) $i \in u^* \Rightarrow m(i) = m^*$,
- (c) $v \subseteq m^*$ but $v \ne m^*$,
- (d) $\bar\beta^* = \langle \beta^*_{\ell,i} : \ell < m^*, i \in u^*\rangle$,
- (e) $\ell \in v \Rightarrow \langle f^{[\ell]}_\alpha \restriction u^* : \alpha \in X\rangle$ is $<_{J^{\mathrm{bd}}_{u^*}}$-increasing and cofinal in $\prod_{i\in u^*} \beta^*_{\ell,i}$,

(f) $\ell \in m^* \backslash v \ \Rightarrow \ f^{[\ell]}_\alpha \restriction u^* = \langle \beta^*_{\ell,i} : i \in u^* \rangle$,

(g) for every $\bar\gamma \in \prod\limits_{\substack{\ell \in v \\ i \in u^*}} \beta^*_{\ell,i}$ for λ ordinals $\alpha \in X$ we have,

$$i \in u^* \ \& \ \ell \in v \quad \Rightarrow \quad \gamma_{\ell,i} < f^{[\ell]}_\alpha(i) < \beta^*_{\ell,i},$$

(h) if $\ell \in v$, $\alpha \in X$, $i \in u^*$ then $f^{[\ell]}_\alpha(i) > \sup\big\{\beta^*_{\ell_1,i_1} : \beta^*_{\ell_1,i_1} < \beta^*_{\ell,i}$ where $\ell_1 < m^*$ and $i_1 < \theta\big\}$ and $\alpha < \beta \in X$ implies: for every $i \in u^*$ large enough we have $f^{[\ell]}_\beta(i) > \max\big\{f^{[\ell_1]}_\alpha(i_1) : \beta^*_{\ell_1,i_1} = \beta^*_{\ell,i}$ and $\ell_1 < m^*$ and $i_1 < \theta\big\}$ (the interesting case is $i_1 = i$).

Now for any $n < \omega$, and $\alpha_0 < \cdots < \alpha_{n-1}$ from X, we have

$\otimes$ for every $i \in u^*$ large enough

$$\langle f_{\alpha_0}(i), f_{\alpha_1}(i), \ldots, f_{\alpha_{n-1}}(i) \rangle = \big\langle \langle \beta_{\alpha_0,i,\ell} : \ell < m^* \rangle, \langle \beta_{\alpha_1,i,\ell} : \ell < m^* \rangle, \ldots, \langle \beta_{\alpha_{n-1},i,\ell} : \ell < m^* \rangle \big\rangle$$

is as in a Δ–system, in fact

$$\beta_{\alpha_{k(1)},i,\ell(1)} = \beta_{\alpha_{k(2)},i,\ell(2)} \Rightarrow (k(1),\ell(1)) = (k(2),\ell(2)) \vee (\ell(1) = \ell(2) \in v).$$

As $v \neq \{0,1,\ldots,m^*-1\}$ and in τ no variable is redundant clearly

$\otimes'$ for every $i \in u^*$ large enough, $\langle \tau(x_{\beta_{\alpha_0,i,0}},\ldots), \tau(x_{\beta_{\alpha_1,i,0}},\ldots),\ldots \rangle$ is independent.

This implies that $\langle g^*_{\alpha_\ell} : \ell < n \rangle$ is independent (in $\prod\limits_{i<\theta} \mathrm{FBA}(\chi_i)$) as required.

If we do not have $(*)$ or $(*)^-$, by $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$ and $2^\theta < \lambda = \mathrm{cf}(\lambda)$ without loss of generality for some $\tau = \tau(x_1,\ldots,x_{n-1})$ and infinite $u \subseteq \theta$, and some $X \in [\lambda]^\lambda$ we have: $\langle f_\alpha \restriction u : \alpha \in X \rangle$ is with no repetition, $\tau_{\alpha,i} = \tau$ for $\alpha \in X$, $i \in u$. So without loss of generality $u = \theta$. Then we can find an ultrafilter $\mathscr{D}$ on θ as in 7.7 below and then the proof above works. $\square_{6.7}$

6.8 Comment. Before we use 7.7, we wonder if "$\chi_i \le \mu$" is necessary in $(*)^-$ of 6.7. This is quite straight. We can omit it if

$$\mathfrak{a} \subseteq \text{ Reg } \cap \lambda \backslash \mu, |\mathfrak{a}| \leq \theta \Rightarrow \max \text{ pcf}(\mathfrak{a}) < \lambda.$$

6.9 Problem. 1) Which of the following statements is consistent with ZFC:

- (a) μ is strong limit, $\text{cf}(\mu) = \aleph_0$, and for every $\lambda \in \text{ Reg} \cap (\mu, 2^\mu]$ and cardinals χ_n such that $\mu = \sum\limits_{n<\omega} \chi_n$, λ is a free caliber of $\prod\limits_{n<\omega} \text{FBA}(\chi_n)$.
  (What about "some such λ"? See 6.14 below.)
- (b) the same for all such μ.

2) Can you prove in ZFC that for some strong limit μ, $\theta = \text{ cf}(\mu) < \mu$ and for some set $\langle \mathfrak{a}_i : i < \sigma \rangle$ where $\sigma = \theta^+$ or $\sigma = (2^\theta)^+$, pairwise disjoint there is $\lambda \in (\mu, 2^\mu] \cap \bigcap\limits_{i<\sigma} \text{ pcf}(\mathfrak{a}_i)$.

Now we turn to another of Monk's problems.

6.10 Claim. *Assume*

- $(*)$ $\kappa > \aleph_0$ *is weakly inaccessible and* $\langle 2^\mu : \mu < \kappa \rangle$ *is not eventually constant.*

<u>*Then*</u>

- (a) *there is a* κ*-c.c. Boolean algebra of cardinality* $2^{<\kappa}$*, with no independent subset of cardinality* κ^+.

Proof. There are sequences $\langle (\mathscr{I}_i, \mathscr{J}_i) : i < \kappa \rangle$, $\langle (\kappa_i, \lambda_i) : i < \kappa \rangle$ such that $\mathscr{J}_i$ is a dense linear order of cardinality λ_i and $\mathscr{I}_i \subseteq \mathscr{J}_i$ a dense subset of $\mathscr{J}_i$ of cardinality κ_i, $\langle \kappa_i : i < \kappa \rangle$ increasing with limit κ, and $\lambda_j > \sum_{i<j} 2^{\kappa_i}$ $(\geq \sum_{i<j} \lambda_i)$, by [Sh 430, 26,3.4].

Let $\mathbf{B}_i$ be Intalg$(\mathscr{J}_i)$, the Boolean algebra of closed-open intervals of $\mathscr{J}_i$. Let $\mathbf{B}$ be the free product of $\{\mathbf{B}_i : i < \kappa\}$, so $\mathbf{B}$ extends each $\mathbf{B}_i$ and each element of $\mathbf{B}$ is a Boolean combination of finitely many elements of $\bigcup\limits_{i<\kappa} \mathbf{B}_i$. It is straight to check $\mathbf{B}$ is as required:

- $(*)_1$ $|\mathbf{B}| = \sum_{i<\kappa} |\mathbf{B}_i| + \aleph_0 = \sum_{i<\kappa} \lambda_i = \sum_{i<\kappa} 2^{\kappa_i} = 2^{<\kappa}$,
- $(*)_2$ $\mathbf{B}$ satisfies the κ–c.c.

[Why? Let $a_i \in \mathbf{B} \backslash \{0\}$ for $i < \kappa$, so let $a_i = \tau_i(b_{i,0}, \ldots, b_{i,n_i-1})$ for $i < \kappa$, $b_{i,\ell} \in \mathbf{B}_{\alpha_{i,\ell}}$. As we can replace a_i by any a'_i, $0 < a'_i \leq a_i$ without loss of generality $a_i = \bigcap\limits_{\ell<n_i} b_{i,\ell}$, $b_{i,\ell} \in \mathbf{B}_{\alpha_{i,\ell}} \backslash \{0\}$. So without loss of generality $\alpha_{i,0} < \alpha_{i,1} < \cdots <$

α_{i,n_i}. As $\kappa > \aleph_0$ is regular and as we can replace $\langle a_i : i < \kappa\rangle$ by $\langle a_i : i \in X\rangle$ whenever $X \in [\kappa]^\kappa$, without loss of generality for some m, $\bigwedge_{\ell<m} \alpha_{i,\ell} = \alpha_\ell$ and $i < j$ & $\{\ell,k\} \subseteq [m,n] \Rightarrow \alpha_{i,\ell} < \alpha_{j,k}$. Let $a'_i = \bigcap_{\ell<m} b_{i,\ell}$, so clearly

$$a'_i \cap a'_j \neq 0 \Leftrightarrow a_i \cap a_j \neq 0 \Leftrightarrow \bigwedge_{\ell<m} b_{i,\ell} \cap b_{j,\ell} \neq 0.$$

But $\mathbf{B}_i$ satisfies the κ–Knaster condition (as $\kappa = \mathrm{cf}(\kappa) > \mathrm{density}(\mathscr{J}_i)$), so can we finish.]

$(*)_3$ $\mathbf{B}$ has no independent subset of cardinality κ^+.

[Why? Let $a_i \in \mathbf{B}$ for $i < \kappa^+$, let $a_i = \tau_i(b_{i,0},\dots,b_{i,n_i-1})$ and let $b_{i,\ell} \in \mathbf{B}_{\alpha_{i,\ell}}\backslash\{0,1\}$. We can replace $\langle a_i : i < \kappa^+\rangle$ by $\langle a_i : i \in X\rangle$ for $X \in [\kappa^+]^{\kappa^+}$, so without loss of generality $\tau_i = \tau$, $n_i = n$ and $\alpha_{i,\ell} = \alpha_\ell$. Let $b_{i,\ell} = \bigcup_{k\in u_{i,\ell}} [x_{i,\ell,k}, x_{i,\ell,k+1})$ where $\bar{x}^{i,\ell} = \langle x_{i,\ell,k} : k \le k_{i,\ell}\rangle$ is an increasing sequence of elements of $\{-\infty\} \cup \mathscr{J}_i \cup \{\infty\}$, $x_{i,\ell,0} = -\infty$, $x_{i,\ell,k_{i,\ell}} = \infty$, $u_{i,\ell} \subseteq k_{i,\ell}$. We can find $y_{i,\ell,k} \in \mathscr{I}_i$ such that $x_{i,\ell,k} < y_{i,\ell,k} < x_{i,\ell,k+1}$. Without loss of generality $k_{i,\ell} = k_\ell$, $y_{i,\ell,k} = y_{\ell,k}$, $u_{i,\ell} = u_\ell$.

Without loss of generality $y_{i,\ell,k} = y_{\ell,k}$. For a finite $A \subseteq \mathbf{B}$ let at $(A) = \mathrm{at}(A,\mathbf{B})$ be the number of atoms in the Boolean subalgebra of $\mathbf{B}$ which A generates (all this was mainly for clarity). Now for any finite $u \subseteq \kappa^+$

$$\begin{aligned}&\mathrm{at}(\{a_i : i \in u\},\mathbf{B})\\ &\quad\le \mathrm{at}(\{b_{i,\ell} : i \in u, \ell < n\},\mathbf{B}) \le \prod_{\ell<n} \mathrm{at}(\{b_{i,\ell} : i \in u\},\mathbf{B}_{\alpha_{i,\ell}})\\ &\quad\le \prod_{\ell<n} \mathrm{at}(\{x_{i,\ell,k} : i \in u, k < k_\ell\},\mathbf{B}_{\alpha_{i,\ell}}\}) \le \prod_{\ell<n}\Big(\sum_{k<k_\ell}(|u|+1)\Big) \le k^* \times |u|^n\end{aligned}$$

for $k^* = \max\{k_\ell + 1 : \ell < n\}$. So if u is large enough this is $< 2^{|u|}$, showing non independence. $\square_{6.10}$

6.11 Claim. *Let* $\mathbf{B}$ *be the completion of* $\mathrm{FBA}(\chi)$

(1) λ *is not a free caliber of* $\mathbf{B}$ *if*

 $(*)$ $\lambda = \mu^+ = 2^\mu$, $\mu \le \chi$, μ *strong limit singular of cofinality* $\aleph_0$,

(2) λ *is a free caliber of* $\mathbf{B}$ *if*

 $(*)$ $\mu = \mu^{\aleph_0} < \lambda = \mathrm{cf}(\lambda) \le 2^\mu$, $\chi \ge \lambda$, *or at least*

 $(*)'$ $\chi \ge \mu, \mu < \lambda = \mathrm{cf}(\lambda) \le 2^\lambda$, μ *strong limit singular of cofinality* $\aleph_0$ *and the* $(< \aleph_1)$*–box product topology on* ${}^\chi\omega$ *has density* $< \lambda$.

Proof. 1) By 6.4, 6.5's proofs.
2) If $(*)$ use 6.7, if $(*)'$ the proof is similar. $\square_{6.11}$

6.12 Remark. We can deal with singular cardinals similarly as in the earlier proofs.

6.13 Claim. *In the earlier claims if*

$(*)_1$ $\lambda = \mu^{++}$, *or at least if*

$(*)_2$ $\mu < \lambda$, *and* $[\alpha < \lambda \Rightarrow \mathrm{cf}([\alpha]^\theta, \subseteq) < \lambda]$, $\chi = \sup_{i<\theta} \chi_i$

then "in the $(\le \theta^+)$*-box product topology,* ${}^\chi\theta$ *has density* $< \lambda$*" can be replaced by "in the* $(< \theta^+)$*-box product topology,* ${}^\mu\theta$ *has density* $< \lambda$*".*

6.14 Conclusion. 1) Let $\ell \in \{1,2\}$ for simplicity. The following questions cannot be answered in ZFC (assuming the consistency of large cardinals).

Assume $\beth_\omega^{+\ell} \le \beth_{\omega+1}$

$(a)_\ell$ Does $\prod\limits_{n<\omega} \mathrm{FBA}(\beth_m)$ have free caliber $\beth_\omega^{+\ell}$?

$(b)_\ell$ Does the completion of $\mathrm{FBA}(\beth_\omega)$ have free caliber $\beth_\omega^{+\ell}$?

$(c)_\ell$ Does the completion of $\mathrm{FBA}(\beth_\omega^{+\ell})$ have free caliber $\beth_\omega^{+\ell}$?

2) Moreover we can add

$$\text{for } x \in \{a,b,c\} \text{ even } (*)_1 + (*)_2, \text{ and } \neg(*)_1 + \neg(*)_2.$$

Proof. 1) Let $\ell = 2$. By Gitik and Shelah [GiSh 597] it is consistent with ZFC that with the $(< \aleph_1)$-box product topology, ${}^{(\beth_\omega)}\omega$ has density $\le \beth_\omega^+$, so we can use 6.4, 6.6(i) (using 6.14 of course). For the other direction by Gitik and Shelah [GiSh 597] the necessary assumptions for 6.3, 6.11(i) are consistent.

For $\ell = 1$, if $\beth_\omega^+ = 2^{\beth_\omega}$ then the answer is NO by 6.3, 6.11.

To get consistency for $\lambda = \beth_\omega^+$ we need dual: in ${}^\mu\omega$, for every μ^+ open sets there is a point belonging to μ^+ of them (this is phrased in 6.15 below). This too is proved consistent in [GiSh 597].
2) Similarly. $\square_{6.14}$

6.15 Definition. $\mathrm{Pr}_{\theta,\sigma}(\lambda,\mu)$ means:

> if f_α is a partial function from μ to θ such that $|\mathrm{Dom}(f_\alpha)| < \sigma$ for $\alpha < \lambda$, <u>then</u> some $f \in {}^\mu\theta$ extends λ of the functions f_α.

If $\sigma = \theta$ we may omit it.

6.16 Claim. *In Claim 6.14 the assumption on the density of box products can be replaced by cases of Definition 6.15:*

- (a) *2.1 Assume* $\mathbf{B} = \mathscr{B}(\chi)$ *is a Maharam measure algebra of dimension* $\chi, \mathrm{cf}(\lambda) > 2^{\aleph_0}$ *and* $\mathrm{cf}(\lambda) > \beth_2 \vee \lambda = \mathrm{cf}(\lambda) \vee \boxtimes^-_{\lambda,\aleph_0}$. *If* $\mathrm{Pr}_{\aleph_0}(\lambda,\chi)$ <u>*then*</u> $\mathbf{B}$ *has* λ *as a free caliber*
- (b) *6.6 Assume* $2^\theta < \lambda = \mathrm{cf}(\lambda)$, $\chi = \sup_{i<\theta}\chi_i$. *If* $\mathrm{Pr}_\theta(\lambda,\chi)$ <u>*then*</u> $\prod\limits_{i<\theta} \mathrm{FBA}(\chi_i)$ *has free caliber* λ.

Proof. Straight. $\square_{6.16}$

In fact cases of Pr are essentially necessary and sufficient conditions.

6.17 Claim. *1) Assume* $\lambda = \mathrm{cf}(\lambda) > 2^{\aleph_0}$, *and* χ_n *are cardinal. The following conditions are equivalent*

- (a) $\prod\limits_{n<\omega} \mathrm{FBA}(\chi_n)$ *has free caliber* λ;
- (b) *if for* $\alpha < \lambda$, $i < \omega$, (u^α_i, v^α_i) *is a pair of disjoint finite subsets of* χ_i <u>*then*</u> *for some* $X \in [\lambda]^\lambda$ *we have*

$$i < \omega \quad \Rightarrow \quad \bigcup_{\alpha\in X} u^\alpha_i \cap \bigcup_{\alpha\in X} v^\alpha_i = \emptyset,$$

i.e., if f^α_i *is a finite function from* χ_i *to* $\{0,1\}$ *for* $i < \omega$, $\alpha < \lambda$, *then for some* $\langle f_i : i < \omega\rangle$

$$(\exists^\lambda \alpha < \lambda)(\forall i < \omega) f^\alpha_i \subseteq f_i.$$

Proof. Straight. $\square_{6.17}$

6.18 Discussion. For measure, the parallel seems cumbersome. We still may like to be more concrete on the dependencies appearing. Note

$\otimes_1$ in 3.7, we can have $\bar{x} = \langle x_\alpha : \alpha < \lambda \rangle$ satisfies

$(*)_{\mathbf{B},\bar{x}}$ for every $X \in [\lambda]^\lambda$, $m < \omega$, and $\beta(\alpha, k) < \lambda$ for $\alpha < \lambda$, $k < 2m$ pairwise distinct, for every n large enough there are pairwise distinct $\alpha_0, \dots, \alpha_{2n-1} \in X$ such that

$$0 = \bigcap_{\ell < n} \Big(\bigcup_{k < m} \Big(x_{\beta(\alpha_{2\ell}, k)} \Delta x_{\beta(\alpha_{2\ell+1}, k)} \Big) \Big),$$

$\otimes_2$ if $(*)_{\mathbf{B},\bar{x}}$ holds then the Boolean algebra $\mathbf{B}' = \langle x_\alpha : \alpha < \lambda \rangle_{\mathbf{B}}$ has no independent subset of cardinality λ. Moreover, if $x'_\alpha \in \mathbf{B}'$ for $\alpha < \lambda$ are distinct, then $(*)_{\mathbf{B}',\langle x'_\alpha : \alpha < \lambda \rangle}$.

§7 A nice subfamily of functions exists

We expand and continue on [Sh 430, 6.6D], [Sh 513, 6.1].

7.1 Claim. *Assume*

(A) $\lambda = \mathrm{cf}(\lambda) \geq \mu > 2^\kappa$,

(B) $\mathscr{D}$ *is a* μ*–complete*[2] *filter on* λ

(C) $f_\alpha : \kappa \to \mathrm{Ord}$ *for* $\alpha < \lambda$,

(D) $\mathscr{D}$ *contains the co-bounded subsets of* λ.

Then
0) We can find $w \subseteq \kappa$ *and* $\bar\beta^* = \langle \beta^*_i : i < \kappa \rangle$ *such that:* $i \in \kappa \setminus w \Rightarrow \mathrm{cf}(\beta^*_i) > 2^\kappa$ *and for every* $\bar\beta \in \prod\limits_{i \in \kappa \setminus w} \beta^*_i$ *for* λ *ordinals* $\alpha < \lambda$ *(even a set in* $\mathscr{D}^+$*) we have* $\bar\beta < f_\alpha \restriction (\kappa \setminus w) < \bar\beta^* \restriction (\kappa \setminus w)$, $f_\alpha \restriction w = \bar\beta^* \restriction w$, *and* $\sup\{\beta^*_j : \beta^*_j < \beta^*_i\} < f_\alpha(i) < \beta^*_i$.
1) We can find a partition $\langle w^*_\ell : \ell < 2 \rangle$ *of* κ, $X \in \mathscr{D}^+$ *and* $\langle A_i : i < \kappa \rangle$, $\langle \bar\lambda_i : i < \kappa \rangle$, $\langle h_i : i < \kappa \rangle$, $\langle n_i : i < \kappa \rangle$ *such that:*

(a) $A_i \subseteq \mathrm{Ord}$,

(b) $\bar\lambda_i = \langle \lambda_{i,\ell} : \ell < n_i \rangle$ *and* $2^\kappa < \lambda_{i,\ell} \leq \lambda_{i,\ell+1} \leq \lambda$ *and* $2^\kappa <$ *cf*$(\lambda_{i,\ell})$,

(c) h_i *is an order preserving function from* $\prod\limits_{\ell < n_i} \lambda_{i,\ell}$ *onto* A_i *so* $n_i = 0 \Leftrightarrow |A_i| = 1$. *(The order on* $\prod\limits_{\ell < n_i} \lambda_{\ell,i}$ *being lexicographic,* $<_{\ell x}$*),*

(d) $i < \kappa \ \& \ \alpha \in X \quad \Rightarrow \quad f_\alpha(i) \in A_i$, *and we let* $f^*_\alpha(i,\ell) = [h^{-1}_i(f_\alpha(i))](\ell)$, *so* $f^*_\alpha \in \prod\limits_{\substack{i<\kappa \\ \ell<n_i}} \lambda_{i,\ell}$,

(e) $i \in w^*_0 \Leftrightarrow n_i = 0$ *(so* $|A_i| = 1$*),*

(f) *if* $i \in w^*_1$ *then* $|A_i| \leq \lambda$, *hence* $|\bigcup\limits_{i \in w^*_1} A_i| \leq \lambda$,

(g) *if* $g \in \prod\limits_{\substack{i<\kappa \\ \ell<n_i}} \lambda_{i,\ell}$ *then* $\{\alpha \in X : g < f^*_\alpha\} \in \mathscr{D}^+$ *and letting* $\beta^*_j = \sup \mathrm{Rang}(h_i)$, *part (0) holds where* w^*_0 *plays the roll of* w *and* w^*_1 *of* $\kappa \setminus w$

(h) *if* $\mathscr{D}$ *is* $(|\alpha|^\kappa)^+$*-complete for any* $\alpha < \mu_1$ *then* $\mu_1 \leq \sup\{\lambda_{i,\ell} : i \in w^*_1;$ *and* $\ell < n_i\} \leq \lambda$ *when* $w^*_1 \neq \emptyset$ *(so, e.g., if* $\mu = \lambda$ *and assuming GCH*

$$\sup\{\mathrm{cf}(\lambda_{i,\ell}) : i \in w^*_1 \text{ and } \ell < n_i\} = \lambda).$$

---
[2] in parts (0), (1), $\mu = (2^\kappa)^+$ is O.K.

2) In part (1) we can add $(*)_1$ to the conclusion if (E) below holds,

$(*)_1$ *if $\lambda_{i,\ell} \in [\mu, \lambda)$ then $\lambda_{i,\ell}$ is regular.*

(E) *For any set $\mathfrak{a}$ of $\leq \kappa$ singular cardinals from the interval (μ, λ), we have $\max \mathrm{pcf}\{\mathrm{cf}(\chi) : \chi \in \mathfrak{a}\} < \lambda$.*

3) Assume in part (1) that (F) below holds. Then we can demand $(*)_2$.

$(*)_2$ *$\lambda^i_\ell \geq \mu_1$ for $i \in w_1$, $\ell < n_i$.*

(F) *$\mathrm{cf}(\mu_1) > \kappa$ and $\alpha < \mu_1 \Rightarrow \mathscr{D}$ is $[|\alpha|^{\leq\kappa}]^+$–complete.*

4) If in part (1) in addition (G) below holds, then we can add

$(*)_3$ *$\lambda \in \mathrm{pcf}_{\sigma\text{-complete}}\{\lambda^i_\ell : i \in w^*_1$; and $\ell < n_i\}$ if $w^*_1 \neq \emptyset$, moreover*

$(*)_4$ *if $\ell_i < n_i$ for $i \in w^*_1$ then $\lambda \in \mathrm{pcf}_{\sigma\text{-complete}}\{\mathrm{cf}(\lambda^i_{\ell_i}) : i \in w^*_1\}$.*

$(G)(i)$ *$(\forall\alpha < \lambda)(|\alpha|^{<\sigma} < \lambda)$ and $\sigma = \mathrm{cf}(\sigma) > \aleph_0$,*

(ii) *$\mathscr{D}$ is λ-complete*

(iii) *$f_\alpha \neq f_\beta$ for $\alpha \neq \beta$ (or just $\alpha \neq \beta \in X$ for some $X \in D^+$).*

5) If in part (1) in addition (H) below holds then we can add

$(*)_5$ *if $m < m^*$, $A \in J_m$ and $\ell_i < n_i$ for $i \in \kappa \setminus A$ (so $w^*_0 \subseteq A$) then $\lambda \in \mathrm{pcf}\{\lambda^i_{\ell_i} : i \in \kappa \setminus A\}$.*

$(H)(i)$ *$m^* < \omega$ and J_m an $\aleph_1$-complete ideal on κ for $m < m^*$,*

(ii) *$\mathscr{D}$ is λ-complete.*

6) If in part (1) in addition (I) holds then we can[3] add:

$(*)_6$ *if $\alpha < \beta$ are from X then $f^*_\alpha <_J f^*_\beta$*

(I) (i) *$\lambda\mu^+$, $cf(\mu) \leq 2^\kappa$*

(ii) *$J = \{S : S \subseteq S_* = \{(i, f) : i < \kappa, \ell < n_i\}$ and $\max\ pcf\{\lambda_{i,\ell} : (i,\ell) \in S\} < \mu$*

(iii) *$\mathscr{D}$ is the filter of co-bounded subsets of λ.*

[3] Clearly the demand (I)(iii) can be weakened. Also instead of (I)(i) we can use (E) from part (2).

Remark. 1) If $\lambda_{i,\ell}$ is singular we can replace it with a sequence $\langle \gamma_{i,\ell_1} : \zeta < \operatorname{cf}(\lambda_{i,\ell})\rangle$, and the index set $\langle\langle\alpha\rangle : \alpha < \lambda_{i,\ell}\rangle$ by $\langle(\zeta,\gamma) : \zeta < \operatorname{cf}(\lambda_{i,\ell})$ and $\gamma < \gamma_{i,\ell_2}\rangle$, and γ_{i,ℓ_1}, are replaced by sequences of regular cardinals. Not clear if all this helps.
2) The reader may concentrate on the case (F) + (G)(ii) holds.

Proof. 0) By part (1).
1) Let χ be regular large enough. Choose N such that

- (i) $N \prec (\mathscr{H}(\chi), \in)$,
- (ii) $2^\kappa + 1 \subseteq N$ and $\|N\| = 2^\kappa$,
- (iii) $\kappa, \mu, \lambda, \mathscr{D}$ and $\langle f_\alpha : \alpha < \lambda\rangle$ belong to N,
- (iv) $N^\kappa \subseteq N$.

Next choose $\delta(*) < \lambda$ which belongs to $B^* = \bigcap\{B \in \mathscr{D} : B \in N\}$, which is the intersection of $\leq 2^\kappa < \mu$ members of $\mathscr{D}$. Necessarily $B^* \in \mathscr{D}$ so such $\delta(*)$ exists. For each $i < \kappa$ let

$$Y_i =: \{A \in N : A \text{ a set of ordinals and } f_{\delta(*)}(i) \in A\},$$

clearly $Y_i \neq \emptyset$ as $\bigcup\limits_{\gamma<\lambda} (f_\gamma(i) + 1) \in N$, hence there is a set $A_i \in Y_i$ of minimal order type. As $N^\kappa \subseteq N$ clearly $\bar{A} =: \langle A_i : i \in \kappa\rangle$ belongs to N.

Let us define:

$$w_0^* =: \{i < \kappa : |A_i| = 1\}$$

$$w_1^* =: \{i < \kappa : |A_i| \neq 1\}.$$

Hence the order type of A_i for $i \in w_1^*$ is necessarily a limit ordinal.

Now note

$(*)_1$ $A_i \neq \emptyset$ and $\langle w_0^*, w_1^*\rangle$ is a partition of κ.

[Why? Recall $A_i \in Y_i$ hence $f_{\delta(*)}(i) \in A_i$ so $A_i \neq \emptyset$ indeed. Also $\langle w_0^*, w_1^*\rangle$ is a partition of κ by their choice.]

$(*)_2$ $|A_i| = 1$ iff $A_i = \{f_{\delta(*)}(i)\}$ iff $f_{\delta(*)}(i) \in N$ (iff $i \in w_0^*$).

[Why? Think.]

$(*)_3$ Without loss of generality $A_i \subseteq \{f_\alpha(i) : \alpha < \lambda\}$.

[Why? As $\{f_\alpha(i) : \alpha < \lambda\} \in Y_i$ and $A_i \cap \{f_\alpha(i) : \alpha < \lambda\} \in Y_i$ has order type $\leq$ otp(A_i).]
Hence

$(*)_4$ If $i \in \kappa \backslash w^*_0$ then $|A_i| \leq \lambda$.

Let for $i \in w^*_1$

$$\begin{aligned} K_i = \big\{(\bar\lambda, \bar\beta) \in N : & \text{for some } n,\ \bar\lambda = \langle \lambda_\ell : \ell < n\rangle \in N, \text{ and} \\ & \bar\beta = \langle \beta_\eta : \eta \in \prod_{\ell<n} \lambda_\ell\rangle \in N \text{ and } \beta_\eta \in \{f_\alpha(i) : \alpha < \lambda\} \text{ and} \\ & f_{\delta(*)}(i) \in \{\beta_\eta : \eta \in \prod_{\ell<n} \lambda_\ell\} \subseteq A_i \text{ and} \\ & \text{for any } \eta <_{\ell x} \nu \text{ from } \prod_{\ell<n} \lambda_\ell \text{ then } \beta_\eta \leq \beta_\nu\big\}. \end{aligned}$$

Note that we really mean just $\beta_\eta \leq \beta_\nu$, so necessarily $(\forall \eta \in \Pi\bar\lambda)(\exists \nu \in \Pi\bar\lambda)(\eta <_{\ell x} \nu \wedge \beta_\eta < \beta_\nu)$.

Clearly $(\bar\lambda, \bar\beta) \in N \Rightarrow$ otp$\{\beta_\eta : \eta \in \prod_\ell \lambda_\ell\} \leq \lambda_{\ell g(\eta)-1} \times \lambda_{\ell g(\eta)-2} \times \ldots \times \lambda_0$ Cartesian product.

We define a partial order $<^*$ on $\bigcup_{i<\kappa} K_i$.

$(\bar\lambda^1, \bar\beta^1) <^* (\bar\lambda^2, \bar\beta^2)$ iff: (a) + (b) holds where

- (a) $\{\beta^1_\eta : \eta \in \prod_\ell \lambda^1_\ell\} \subseteq \{\beta^2_\eta : \eta \in \prod_\ell \lambda^2_\ell\}$
- (b) one of the following clauses holds
  - (α) otp$(\prod_\ell \lambda^1_\ell, \leq_{\ell x}) <$ otp$(\prod_\ell \lambda^2_\ell, \leq_{\ell x})$
  - (β) otp$(\prod_\ell \lambda^1_\ell, \leq_{\ell x}) =$ otp$(\prod_\ell \lambda^2_\ell, \leq_{\ell x})$ and $\ell g(\bar\lambda^1) < \ell g(\bar\lambda^2)$
  - (γ) otp$(\prod_\ell \lambda^1_\ell, \leq_{\ell x}) =$ otp$(\prod_\ell \lambda^2_\ell, \leq_{\ell x}), \ell g(\bar\lambda^1) = \ell g(\bar\lambda^2)$ and $\bigvee_{k<\ell g(\bar\lambda^1)} [\lambda^1_{\ell g(\bar\lambda^1)-1-k} < \lambda^2_{\ell g(\bar\lambda^2)-1-k}$ and $\bigwedge_{\ell<k} \lambda^1_{\ell g(\bar\lambda^1)-1-\ell} = \lambda^2_{\ell g(\bar\lambda^2)-1-\ell}]$.

$(*)_5$ $(K_i, \leq^*) \subseteq N$ is a partial order which is a well quasi order (i.e., no strictly decreasing ω–chains).

[Why? Reflect.]

$(*)_6$ otp$(A_i) \leq |A_i|^n$ for some $n < \omega$.

[Why? By Dushnik–Milner [DM], we can find $A_{i,n} \subseteq A_i$ for $n < \omega$ such that $A_i = \bigcup_{n<\omega} A_{i,n}$ and $\text{otp}(A_{i,n}) \le |A_i|^n$. So as $A_i \in N$ there is such sequence $\langle A_{i,n} : n < \omega \rangle$ in N so $A_{i,n} \in N$ hence for some n we have $f_{\delta(*)}(i) \in A_{i,n} \in N$, so by the choice of A_i clearly $\text{otp}(A_i) \le |A_i|^n$.]

$(*)_7$ For each $i < \kappa$, there is $(\bar\lambda, \bar\beta) \in K_i$ such that $\bigwedge_{\ell<\lg(\bar\lambda)} \lambda_\ell \le |A_i|$.

Why? As said above $\alpha_i := \text{otp}(A_i)$ is a limit ordinal, now let $\lambda_0 = \text{cf}(\alpha_i)$, let n be as in $(*)_6$ and let $\lambda_1 = \ldots = \lambda_n$ be $|A_i|$, so $\bar\lambda$ is a sequence of cardinals $\le |A_\ell|$ of length $n+1$. Choose $\bar\gamma = \langle \gamma_\varepsilon : \varepsilon < \lambda \rangle$ be an increasing continuous sequence of ordinals with limit α_i such that $\gamma_0 = 0$.

For each $\varepsilon < \lambda_0$ and $\gamma \in [\gamma_\varepsilon, \gamma_{\varepsilon+1}]$ there is a unique $\eta \in \prod_{\ell \le n} \lambda_\ell$ such that:

$(a)_{\gamma,\eta}$ $\eta(0) = \varepsilon$

$(b)_{\gamma,\eta}$ the order type of the following set is $\gamma - \gamma_\varepsilon$
$(\{\nu \in {}^n\lambda : \nu <_{\ell x} (\eta(1), \ldots, \eta(n))\}, <_{\ell x})$.

In this case we let $\eta = \eta_\gamma$ and $\beta_\eta = \gamma$; clearly

- if $\varepsilon < \lambda_0$ and $\rho_\varepsilon := \langle \varepsilon \rangle \hat{}\, \langle (0)_n \rangle$ then $\beta_{\rho_\varepsilon} = \gamma_\varepsilon$.

For $\eta \in (\prod_{\ell \le n} \lambda_\ell) \backslash \{\eta_\beta : \beta < \alpha_i\}$ let $\beta_1 = \beta_1(\eta)$ be the minimal ordinal $\beta_1 \in A_i$ such that $\eta <_{\ell x} \eta_{\beta_1}$, (it is well defined because $\{\rho_\varepsilon : \varepsilon < \lambda_0\}$ is cofinal in $(\{\eta_\gamma : \gamma < \alpha_i\}, <_{\ell x}))$ and let $\beta_\eta = \beta_1(\eta)$.

Now check.

So we can find a $<^*$-minimal $(\bar\lambda^i, \bar\beta^i) \in K_i$ in $\{(\bar\lambda, \bar\beta) \in K_i : \ell < \ell g(\bar\lambda) \Rightarrow \lambda_\ell \le |A_i|\}$ and let $n_i = \lg(\bar\lambda^i)$. Note:

$(*)_8$ we can above in the choice of A_i demand $A_i = \{\beta^i_\eta : \eta \in \prod_{\ell < n_i} \lambda^i_\ell\}$,

$(*)_9$ $\lambda^i_\ell \le \lambda^i_{\ell+1} \le \lambda$ for $\ell < n_i$.

[Why? The second inequality by $(*)_4$ and the choice of $(\bar\lambda^i, \bar\beta^i)$, the first inequality as otherwise by renaming we can omit $\lambda^i_{\ell+1}$ and contradict the $<^*$–minimality of $(\bar\lambda^i, \bar\beta^i)$.]

Let $\langle \eta^*_i : i < \kappa \rangle$ be such that $\beta^i_{\eta^*_i} = f_{\delta(*)}(i)$ and $\eta^*_i \in \prod_{\ell < n_i} \lambda^i_\ell$.

$(*)_{10}$ $\lambda^i_\ell > 2^\kappa$; moreover $\text{cf}(\lambda^i_\ell) > 2^\kappa$.

[Why? Trivial or see $(*)_{12}$].

Let $Y = \{\alpha < \lambda : \text{for every } i < \kappa \text{ we have } f_\alpha(i) \in A_i\}$, as $\bar f \in N$ and $\langle A_i : i < \kappa\rangle \in N$ necessarily $Y \in N$. Also $Y \in \mathscr{D}^+$ because $\delta(*) \in Y$ and the choice of $\delta(*)$. So for $\alpha \in Y$ we let $\langle \eta^\alpha_i : i < \kappa\rangle$ be such that $\eta^\alpha_i \in \prod\limits_{\ell < n_i} \lambda^i_\ell$ and $f_\alpha(i) = \beta^i_{\eta^\alpha_i}$ and η^α_i is $<_{\ell x}$-minimal under those restrictions (so it is uniquely determined).

We now define $f^*_\alpha \in \prod\limits_{\substack{i<\kappa \\ \ell < n_i}} \lambda^i_\ell$ for $\alpha < \lambda$ by $f^*_\alpha(i,\ell) = \eta^\alpha_i(\ell)$.

Note:

$(*)_{11}$ $\langle \bar\lambda^i : i < \kappa\rangle$, $\langle \bar\beta^i : i < \kappa\rangle$ and $\bar f$, hence $\langle\langle \eta^\alpha_i : i < \kappa\rangle : \alpha < \lambda\rangle$ and $\bar f^* = \langle f^*_\alpha : \alpha \in Y\rangle$ belong to N.

$(*)_{12}$ $\eta^{\delta(*)}_i(\ell) = f^*_{\delta(*)}(i,\ell) \in [\sup(N \cap \lambda^i_\ell), \lambda^i_\ell)$ and $\alpha \in Y \ \Rightarrow f^*_\alpha(i,\ell) < \lambda^i_\ell$.

[Why? $f^*_\alpha(i,\ell) < \lambda^i_\ell$ as $\eta^\alpha_i \in \prod\limits_{n<n_i} \lambda^i_n$ and if for some $f^*_{\delta(*)}(i,\ell) < \sup(N \cap \lambda^i_\ell)$ as if for some $\beta^* \in N \cap \lambda^i_n, \beta^* > f^*_{\delta(*)}(i,\ell)$ then $f^*_{\delta(*)}(i) \in A'_i =: \{\gamma \in A_i : \gamma < \beta^*\} \in N$, easily we get contradiction to the choice of $(\bar\lambda^i, \bar\beta^i)$ as $\mathrm{otp}(A'_i) < \mathrm{otp}(A_i)$.]

$(*)_{13}$ for every $g \in \prod\limits_{\substack{i<\kappa \\ \ell < n_i}} \lambda^i_\ell$ and $X \in [Y]^\lambda \cap N$ such that $\delta(*) \in X$ <u>there is</u> $\alpha \in X$ such that

$$g < f^*_\alpha \text{ i.e. } i < \kappa \ \& \ \ell < n_i \ \Rightarrow \ g(i,\ell) < f^*_\alpha(i,\ell).$$

[Why? If not, there is such g, so as $\langle (\bar\lambda^i, \bar\beta^i) : i < \kappa\rangle$, $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ and X, Y belong to N also $\bar f^* = \langle f^*_\alpha : \alpha \in X\rangle$ belongs to N, so all the requirements on g are first order with parameters from N, so without loss of generality $g \in N$. Now $\delta(*) \in X$ cannot satisfy the requirement hence there are $i < \kappa$, $\ell < n_i$ such that $g(i,\ell) > f^*_{\delta(*)}(i,\ell)$ contradicting $(*)_{12}$.]

Let

$$Z_i = \{\eta \in \prod_{i<n_i} \lambda^i_\ell : \text{if } \nu \in \prod_{\ell<n_i} \lambda^i_\ell \text{ and } \nu <_{\ell x} \eta; \text{ then } \beta^i_\nu < \beta^i_\eta\},$$

$$Z^+_i = \{\eta \restriction k : \eta \in Z_i; \text{ and } k \le n_i\}.$$

As $(\bar\lambda^i, \bar\beta^i) \in N$ clearly also Z_i, $Z^+_i \in N$.

$(*)_{14}$ If $i < \kappa$, $k < n_i$ then $\lambda^i_k = \mathrm{otp}\{\eta(k) : (\eta^*_i \restriction k) \triangleleft \eta \in Z_i\}$.

[Why? Let $Z'_i = \{\eta \in Z_i : \lambda^i_k > \text{ otp}\{\nu \in Z_i : \eta \restriction k \triangleleft \nu \in Z_i\}\}$. So $\eta^*_i \in Z'_i \in N$, by renaming

$$\eta \in Z'_i \Rightarrow \lambda^i_k > \sup\{\nu(k)_i : \eta \restriction k \triangleleft \nu \in Z_i\},$$

and we get a contradiction to $(*)_8$ as in the proof of $(*)_9$ if $\lambda^i_{k-1} = \lambda^i_k$ and as in $(*)_{12}$ if $\lambda^i_{k-1} < \lambda^i_k$.]
Hence

$(*)_{15}$ without loss of generality $\langle \beta^i_\eta : \eta \in \prod\limits_{\ell < n_i} \lambda^i_\ell \rangle$ is increasing (with $<_{\ell x}$ not just $\leq_{\ell x}$).

[Why? Use $(*)_{14}$ for every $\nu \in Z_{n_i - 1}$ and rename.]

$(*)_{16}$ $\mu \leq \max \text{ pcf}\{\lambda^i_\ell : i \in w^*_1 \text{ and } \ell < n_i\}$.

[Why? Otherwise let $\mu > \mu_0 = \max \text{pcf}\{\lambda^i_\ell : i \in w^*_1 \text{ and } \ell < n_i\}$, and so $B^* =: \{\beta^i_\eta : i < \kappa, \eta \in \prod\limits_{\ell < n_i} \lambda^i_\ell\}$ has cardinality μ_0 so there is $\mathscr{P} \in N$, $|\mathscr{P}| < \lambda$, $\mathscr{P} \subseteq [\mu_0]^{\leq \kappa}$ and $\mathscr{P}$ is cofinal in $([\mu_0]^{\leq \kappa}, \subseteq)$. (Why? By assumption (D)). Note that if for some $X \in (\mathscr{D} + Y)^+$, $\bar{f} \restriction X$ is constant we are done. Otherwise

$$a \in \mathscr{P} \Rightarrow \{\alpha < \lambda : \text{ Rang}(f_\alpha) \subseteq a\} = \emptyset \text{ mod } \mathscr{D}$$

but $\mathscr{D}$ is μ-complete hence

$$X^* =: \{\alpha \in Y : (\exists a \in \mathscr{P})[\text{ Rang}(f_\alpha) \subseteq a]\} = \emptyset \text{ mod } \mathscr{D}$$

and $X^* \in N$ and $\delta(*) \in X^*$, contradicting the choice of X^*.]

$(*)_{17}$ $\max \text{ pcf}\big\{\lambda^i_\ell : i \in w^*_i \text{ and } \ell < n_i\big\} \leq \lambda$.

[Why? By $(*)_{13}$.]

$(*)_{18}$ λ^i_ℓ has cofinality $> 2^\kappa$.

[Why? Otherwise by $(*)_{12}$ we get a contradiction.]
The conclusion can now be checked easily.
2) Let $\mathfrak{a} = \{\text{cf}(\lambda^i_\ell) : \lambda^i_\ell \text{ is singular and } \mu \leq \lambda^i_\ell < \lambda\}$ and use (E).
3) Easy.
[Clearly $\mathscr{D}$ is μ_2^+-complete where $\mu_2 = \mu_2^\kappa = \Sigma\{|\alpha|^\kappa : \alpha < \mu_1\}$, so choose N as above of cardinality μ_2.]
4) Without loss of generality in clause (iii) of (G) we have $\alpha < \beta < \lambda \Rightarrow f_\alpha \neq f_\beta$

(otherwise replace $\mathscr{D}$ by $\mathscr{D}+X$ and change f_α for $\alpha\in\lambda\backslash X$ in quite an arbitrary way).
Assume that the desired conclusion fails. For this we choose not just one model N but an $(\omega+1)$-tree of models. More precisely, we choose by induction on $i\le\omega$ a sequence $\langle N_\eta : \eta\in T_i\rangle$ such that

(a) $T_i\subseteq {}^i\lambda$,
(b) $j<i \ \& \ \eta\in T_i\Rightarrow \eta\restriction j\in T_j$,
(c) $|T_i|<\lambda$,
(d) $N_\eta\prec(\mathscr{H}(\chi),\in)$ satisfies (i)–(iv) from the proof of part (1),
(e) for $\eta\in T_i$ we have $\eta\in N_\eta$ and $\langle N_\nu : \nu\in\bigcup\limits_{j<i}T_j\rangle\in N_\eta$ and

$$\nu\triangleleft\eta\Rightarrow N_\nu\prec N_\eta \ \& \ N_\nu\in N_\eta,$$

(f) if $i=0$, then $T_i=\{\langle\rangle\}$,
(g) if i is ω, then $T_i=\{\eta\in{}^i\lambda : (\forall j<i)(\eta\restriction j\in T_i)$,
(h) if $i=j+1,\eta\in T_j$ and $\langle a_{\eta,\varepsilon} : \varepsilon<\varepsilon_\eta<\lambda\rangle$ list $[\sup(N_\eta\cap\lambda)]^{<\sigma}$, then

$$\{\nu\in T_i : \eta\triangleleft\nu\}=\{\eta\hat{\ }\langle\alpha\rangle : \alpha<\varepsilon_\eta\},$$

and $a_{\eta,\varepsilon}\in N_{\eta\hat{\ }\langle\varepsilon\rangle}$,
(i) $T=\bigcup\limits_{i\le\omega}T_i$.

There is no problem to carry out the definition (note that $\varepsilon_\eta<\lambda$ by assumption (G)(i) and $|T_{m+1}|<\lambda$ as in addition λ is regular, and $|T_\omega|<\lambda$ by assumption (G)(i) as $\sigma>\aleph_0$). Now

$$B^*=\bigcap\{B\in\mathscr{D} : \text{for some } \eta\in T \text{ we have } B\in N_\eta\}$$

being the intersection of $\le|T|+2^\kappa<\lambda$ sets in $\mathscr{D}$, belongs to $\mathscr{D}$ (using assumption (G)(ii)), so choose $\delta(*)\in B^*$. Now we choose by induction on $k<\omega$, $\eta_k\in T_k$ and w_0^k, w_1^k, $\langle(\bar\lambda^{i,k},\bar\beta^{i,k}) : i<\kappa\rangle\in N_{\eta_k}$ as in the proof of (1) for N_{η_k}, such that $w_0^k\subseteq w_0^{k+1}$, $\eta_k\triangleleft\eta_{k+1}$ and $(\forall i\in w_1^k)[(\bar\lambda^{i,k+1},\bar\beta^{i,k+1})<^*(\bar\lambda^{i,k},\bar\beta^{i,k})]$. The last assertion can be satisfied in the choice of the $k+1$ step by the assumption toward contradiction and basic pcf.

If $\bigcup\limits_{k<\omega}w_0^k=\kappa$, then $f_{\delta(*)}\in N_{\bigcup\limits_k\eta_k}$, hence $\delta(*)\in N_{\bigcup\limits_k\eta_k}$, contradiction. If $i\in\kappa\setminus\bigcup\limits_{k<\omega}w_0^k$, then $\langle(\bar\lambda^{i,k},\bar\beta^{i,k}) : k<\omega\rangle$ is strictly decreasing in K_i by $<^*$ (more exactly in $\bigcup_{k<\omega} K_i[N_{\eta_k}])$, contradicting a parallel of $(*)_{11}$.

(5) We choose by induction on $t \in \omega$ the objects N_t, δ_t, $\bar{A}^t = \langle A^t_i : i < \kappa\rangle$, $\langle(\bar{\lambda}^t_i, \bar{\beta}^t_i) : i < \kappa\rangle$, $\langle h^t_i : i < \kappa\rangle$, K^t_i such that

- (a) for each t, they are as required in the proof of part (1),
- (b) $N_t \in N_{t+1}$, $K^t_i \subseteq K^{t+1}_i$ and $(\bar{\lambda}^{t+1}_i, \bar{\beta}^{t+1}_i) \le^* (\bar{\lambda}^t_i, \bar{\beta}^t_i)$ in K^{t+1}_i,
- (c) for each t for some $m_t < m^*$ we have

$$\{i < \kappa : (\bar{\lambda}^{t+1}_i, \bar{\beta}^{t+1}_i) <^* (\bar{\lambda}^t_i, \bar{\beta}^t_i)\} = \kappa \bmod J_{m_t}.$$

No problem to carry it out by assumption toward contradiction. So for some m, $\{t : m_t = m\}$ is infinite, contradicting "J_m is $\aleph_1$–complete, and for each $i < \kappa$, $\bigcup_t K^t_i$ well ordered by $<^*$".

6) Clearly $(\Pi\{\lambda_{i,\ell} : (i,\ell) \in S_*\}, <_J)$ has true cofinality λ so let $\langle g_\alpha : \alpha < \lambda\rangle$ be $<_J$-increasing cofinal in it. We choose $\alpha_\varepsilon, \beta_\varepsilon < \alpha$ by induction on $\varepsilon < \lambda$ such that:

- $(*)$ (a) α_ε is minimal such that $\zeta < \varepsilon, f^*_{\beta_\zeta} <_J g_{\alpha_\varepsilon}$
  - (b) $\beta_\varepsilon \in X$ is minimal such that $g_{\alpha_\varepsilon} < f^*_{\beta_*}$.

Now $X' = \{\beta_\varepsilon : \varepsilon < \lambda\}$ is as required. $\square_{7.1}$

See section 9 for actually some consequences.

7.2 Notation.. If f is a function from, say, θ to the ordinals, and $\bar{g}$ is a sequence of length θ of functions from the ordinals to the ordinals, then $f^* = f^{\bar{g}}$ is a function from the ordinals to the ordinals defined by $f^*(i) = g_i(f(i))$.

We spell out a special case of 7.1

7.3 Fact. Assume

$$(*) \qquad 2^\theta < \mu,\ \mathrm{cf}(\mu) = \theta \quad \text{and } (\forall\alpha < \mu)(|\alpha|^\theta < \mu),$$

and $\lambda = \mu^+$.

Then:

(1) For every sequence $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$ of functions from θ to the ordinals, we can find $u^* \in [\theta]^\theta$ and $\bar{\beta}^* = \langle \beta^*_i : i \in u^*\rangle$ such that one of the following cases occurs:

$(*)_1$ for some $X \in [\lambda]^\lambda$, $f_\alpha \restriction u^* = \bar{\beta}^*$ for $\alpha \in X$,

$(*)_2(\alpha)$ if $\theta > \aleph_0$ <u>then</u> β^*_i is a limit ordinal (for every $i \in u^*$), and $\langle \mathrm{cf}(\beta^*_i) : i \in u^*\rangle$ is strictly increasing with limit μ and $\lambda = \mathrm{tcf}\Big(\prod\limits_{i \in u^*} \mathrm{cf}(\beta^*_i)/J^{bd}_{u^*}\Big)$ and for every $\bar\gamma \in \prod\limits_{i \in u^*} \beta^*_i$ for λ ordinals $\alpha < \lambda$ we have

$$(\forall i \in u^*)(\gamma_i < f_\alpha(i) < \beta^*_i),$$

$(*)_2(\beta)$ if $\theta = \aleph_0$ <u>then</u> for some strictly increasing sequence $\bar\lambda = \langle \lambda_i : i \in u^*\rangle$ of regular cardinals with limit μ, $\lambda = \mathrm{tcf}(\prod\limits_{i \in u^*} \lambda_i/J^{bd}_{u^*})$ and for some $\bar g = \langle g_i : i < \theta\rangle$, $g_i : \mathrm{Ord} \to \lambda_i$, we have: for every $\bar\gamma \in \prod\limits_{i \in u^*} \lambda_i$ for λ ordinals $\alpha < \lambda$ we have

$$i \in u^* \quad \Rightarrow \quad \gamma_i < f^{\bar g}_\alpha(i) < \lambda_i,$$

$(*)_3$ β^*_i is a limit ordinal of cofinality λ for $i \in u^*$ and for some $X \in [\lambda]^\lambda$ we have: $i \in u^* \;\Rightarrow\; \langle f_\alpha(i) : \alpha \in X\rangle$ is strictly increasing with limit β^*_i and for $\alpha \in X$, the interval $[f_\alpha(i), \beta^*_i)$ is disjoint to

$$\{f_\beta(j) : \beta \in X; \text{ and } j \in u^*\backslash\{i\} \;\&\; \beta_j \neq \beta_i \text{ or } \beta < \alpha \text{ and } j \in u^*\}.$$

2) Assume $\theta > \aleph_0$. For every sequence $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ of pairwise distinct functions from θ to ${}^{\omega>}\mathrm{Ord}$ such that $|\{f_\alpha(i) : \alpha < \lambda\}| < \lambda$ for $i < \theta$, we can find $u^* \in [\theta]^\theta$ and $n(*) \in [1, \omega)$ and $v \subseteq n(*)$ non-empty and $\bar\beta^* = \langle \beta^*_{\ell,i} : \ell < n(*), i \in u^*\rangle$ such that for each i

(a) for $\ell \in v$ we have that $\beta^*_{\ell,i}$ is a limit ordinal, $\langle \mathrm{cf}(\beta^*_{\ell,i}) : i \in u^*\rangle$ is strictly increasing with limit μ and $\lambda = \mathrm{tcf}(\prod\limits_{i \in u^*} \mathrm{cf}(\beta^*_{\ell,i})/J^{bd}_{u^*})$, and also for $i < j$ in u^*, and $\ell, k \in v$ we have $\mathrm{cf}(\beta^*_{\ell,i}) < \mathrm{cf}(\beta^*_{k,j})$,

(b) for every $\bar\gamma \in \prod_{\ell,i} \beta^*_{\ell,i}$ for λ ordinals $\alpha < \lambda$ we have

$$(\forall i < u^*)(\forall \ell \in v)[\gamma_{\ell,i} < (f_\alpha(i))(\ell) < \beta^*_{\ell,i}] \quad \text{and}$$

$$(\forall i \in u^*)(\forall \ell \in n(*)\backslash v)[f_\alpha(i))(\ell) = \beta^*_{\ell,i}].$$

3) In part (2) if $\theta = \mathrm{cf}(\theta) > \aleph_0$, we can replace $u^* \in [\theta]^\theta$ by $u \in J^+$ for any normal ideal J on θ. Moreover if $\{\delta < \theta : (\forall \alpha < \mathrm{cf}(\delta))(|\alpha|^{<\sigma} < \mathrm{cf}(|\delta|)\}$ is stationary then $\mathrm{Rang}(f_\alpha) \subseteq {}^{\sigma>}\mathrm{Ord}$ is fine. If we omit the assumption $|\{f_\alpha(i) : \alpha < \lambda\}| < \lambda$, instead of v we have a partition (v_1, v_2, v_3) of $\{\ell : \ell < n^*\}$ such that clause (a) holds for $\ell \in 2$, clause (b) holds with $\ell \in v_2 \cup v_3$, $\ell \in v_1$ instead of $\ell \in v$, $\ell \in n^* \setminus v$, and the parallel of $(*)_3$ holds for $\ell \in v_3$.

Proof. 1) By 7.1(0),(1),(2) we know that

$\otimes$ there is $\langle \beta^*_i : i < \theta\rangle$ and $w^* \subseteq \theta$ such that letting $u^* = \theta \backslash w^*$ we have:

(a) for every $\bar\gamma \in \prod\limits_{i \in u^*} \beta^*_i$ for λ ordinals $\alpha < \lambda$ we have

$$i \in w^* \Rightarrow f_\alpha(i) = \beta^*_i,$$

$$i \in u^* \Rightarrow \gamma_i < f_\alpha(i) < \beta^*_i,$$

and moreover (w^*_0, w^*_1), $\langle \lambda^\ell_i : i \in w^*_1, \ell < n_i\rangle, X, \langle f^*_\alpha : \alpha < \lambda\rangle, \bar h = \langle h_i : i < \theta\rangle$ are as there (so $w^*_1 = u^*$ and $w^*_0 = w^*$); clearly $\lambda_{i,\ell} \le \lambda$.

Case 1. $|w^*| = \theta$.
So for some $X \in [\lambda]^\lambda$ we have $\langle f_\alpha \restriction w^* : \alpha \in X\rangle$ is constant. Easily $(*)_1$ holds.

Case 2. For some unbounded subset u' of θ and $\langle m_i : i \in u'\rangle$, $m_i < n_i$ we have $i \in u' \Rightarrow \lambda_{i,m_i} = \lambda$.
Clearly $(*)_3$ holds and we get X by "thinning": choose by induction on $\gamma < \lambda$ the γ-th member $\alpha_\gamma < \lambda$ of X, fixing $\langle h^{-1}_i(f_\alpha(i)) \restriction m_i : i \in u'\rangle$.

Case 3. For some unbounded $u' \subseteq \theta$ we have $\mu_* =: \sup\{\lambda^\ell_i : i \in u' \text{ and } \ell < n_i\}$ is $< \mu$.
So $\{f^*_\alpha \restriction \{(i,\ell) : i \in u', \ell < n_i\} : \alpha \in X\}$ has cardinality $\le \mu^\theta_* < \mu < \lambda$ so for some unbounded $X' \subseteq X$ we have $\langle f_\alpha \restriction u' : \alpha \in X'\rangle$ is constant so $(*)_1$ holds.

Case 4. Neither case 1 nor case 2 nor case 3.
Let $\mu = \sum_{i<\theta} \mu_i$, $\mu_i < \mu$ increasing with i. Choose $j_i \in u^*$ such that j_i is the minimal $j > \bigcup\limits_{\zeta<i} j_\zeta$ satisfying $\lambda > \lambda^{n_i-1}_j > \mu_i + \sum_{\zeta<i} \lambda_{j_\zeta}$, and let $m_{j_i} < n_{j_i}$ be the minimal m such that $\lambda^m_{j_i} > \mu_i + \sum_{\zeta<i} \lambda_{j_\zeta}$.
Assume $\theta > \aleph_0$, replacing $\langle u_i : i < \theta\rangle$, without loss of generality $\mu_{j_i} = \mu_i$ and by Fodor lemma, replacing $\langle j_i : i < \theta\rangle$ by a subsequence, without loss of generality $\mu^* =:$

$\sup\{\lambda^m_{j_i} : i < \theta, m < m_{j_i}\} < \mu$, and without loss of generality $\langle h^{-1}_{j_i}(f_\alpha(j_i)) \restriction m_{j_i} : i < \theta\rangle = x$ is the same for all $\alpha \in X$.

Choose $u^* = \{j_i : i < \theta\}$, $\lambda_i = \lambda^{m_i}_{j_i}$, which is regular by 7.1(2) as then assumption (E) is trivial. Now, $\langle \lambda_j : j \in u^*\rangle$ is a strictly increasing sequence of regular cardinals with limit μ, and hence $\prod\limits_{j\in u^*} \lambda_j/J^{bd}_{u^*}$ is μ–directed and hence λ–directed. But, by 7.1, $\{\langle h^{-1}_{j_i}(f_\alpha(i))(m_i) : i \in u^*\rangle : \alpha \in X\}$ is unbounded in it (or use "max pcf$\{\lambda_{i,\ell} : i < \theta, \ell < n_i\} \le \lambda$"). So $\lambda = \text{tcf}(\prod\limits_{j\in u^*} \lambda_i/J^{bd}_{u^*})$. Let g_j be defined by $g_j(\gamma) = (h^{-1}_j(\gamma))(m_j)$, and we are done. We leave the case $\theta = \aleph_0$ to the reader.
2) First without loss of generality $\ell g(f_\alpha(i)) = n^*$, i.e., does not depend on α, secondly, e.g., by successive applications of part (1).
3) Similar. $\square_{7.3}$

7.4 Conclusion. For
1) In 7.3(1), $(*)_2$ and $(*)_3$ implies

$(*)'_2$ there are u^*, $\beta^* = \langle \beta^*_i : i \in u^*\rangle$ and X such that

- (a) $u^* \in [\theta]^\theta$,
- (b) $X \in [\lambda]^\lambda$,
- (c) $\langle f_\alpha \restriction u^* : \alpha \in X\rangle$ is $<_{J^{bd}_{u^*}}$-increasing if $\theta > \aleph_0$, and $\langle f^{\bar g}_\alpha \restriction u^* : \alpha \in X\rangle$ is $<_{J^{bd}_{u^*}}$–increasing if $\theta = \aleph_0$ (for appropriate $\bar g$),
- $(d)(\alpha)$ if $\theta > \aleph_0$ then for every $\bar\gamma \in \prod\limits_{i\in u^*} \beta^*_i$ there are λ ordinals $\alpha \in X$ such that
$$i \in u^* \quad \Rightarrow \quad \gamma_i < f_\alpha(i) < \beta^*_i,$$
- $(d)(\beta)$ if $\theta = \aleph_0, \lambda_i =$ Rang(g_i) then for every $\bar\gamma \in \prod\limits_{i\in u^*} \lambda_i$ there are λ ordinals $\alpha \in X$ such that
$$i \in u^* \quad \Rightarrow \quad \gamma_i < f^{\bar g}_\alpha(i) < \lambda_i,$$
- (e) if $(*)_3$ <u>then</u>:
  - (i) $\alpha < \beta$ from $X \quad \Rightarrow \quad f_\alpha \restriction u^* < f_\beta \restriction u^*$,
  - (ii) if $i \neq j$ are in u^* and $\beta^*_i < \beta^*_j$ then $\alpha \in X \ \Rightarrow\ f_\alpha(j) > \beta^*_i$,
  - (iii) if $i, j \in u^*$, $\beta^*_i = \beta^*_j$ and $\alpha < \beta$ are from X then $f_\alpha(i) < f_\beta(j)$.

2) Similarly for 7.3(2), getting $\circledcirc$ from the proof of 6.7. [Saharon copied!]

$\circledcirc$ there are u^*, m^*, v, $\bar\beta^*$, X such that

(a) $u^* \in [\theta]^\theta$ and $X \in [\lambda]^\lambda$,

(b) $i \in u^* \;\Rightarrow\; m(i) = m^*$,

(c) $v \subseteq m^*$ but $v \neq m^*$,

(d) $\bar\beta^* = \langle \beta^*_{\ell,i} : \ell < m^*, i \in u^* \rangle$,

(e) $\ell \in v \Rightarrow \langle f^{[\ell]}_\alpha \restriction u^* : \alpha \in X \rangle$ is $<_{J^{bd}_{u^*}}$-increasing and cofinal in $\prod\limits_{i \in u^*} \beta^*_{\ell,i}$,

(f) $\ell \in m^* \backslash v \;\Rightarrow\; f^{[\ell]}_\alpha \restriction u^* = \langle \beta^*_{\ell,i} : i \in u^* \rangle$,

(g) for every $\bar\gamma \in \prod\limits_{\substack{\ell \in v \\ i \in u^*}} \beta^*_{\ell,i}$ for λ ordinals $\alpha \in X$ we have,

$$i \in u^* \;\;\&\;\; \ell \in v \quad \Rightarrow \quad \gamma_{\ell,i} < f^{[\ell]}_\alpha(i) < \beta^*_{\ell,i},$$

(h) if $\ell \in v$, $\alpha \in X$, $i \in u^*$ then $f^{[\ell]}_\alpha(i) > \sup\big\{\beta^*_{\ell_1,i_1} : \beta^*_{\ell_1,i_1} < \beta^*_{\ell,i}$ where $\ell_1 < m^*$ and $i_1 < \theta\big\}$ and $\alpha < \beta \in X$ implies: for every $i \in u^*$ large enough we have $f^{[\ell]}_\beta(i) > \max\big\{f^{[\ell_1]}_\alpha(i_1) : \beta^*_{\ell_1,i_1} = \beta^*_{\ell,i}$ and $\ell_1 < m^*$ and $i_1 < \theta\big\}$ (the interesting case is $i_1 = i$).

Proof. Straight. Choose the γ–th member of X for $\gamma < \lambda$, by induction on γ. $\square_{7.4}$

Similarly we can prove

7.5 Claim. *Assume*

(A) $\lambda = \mathrm{cf}(\lambda) > 2^\theta$,

(B) $\mu = \min\{\mu : \mu^\theta \geq \lambda\}, \mathrm{cf}(\mu) = \theta > \aleph_0$,

(C) *if* $\mathfrak{a} \subseteq \mathrm{Reg} \cap \mu \setminus 2^\theta$, $|\mathfrak{a}| \leq \theta$, $\lambda \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{a})$, *then for some* $\mathfrak{b} \subseteq \mathfrak{a}$, $\lambda = \mathrm{tcf}(\prod \mathfrak{b}/[\mathfrak{b}]^{<\theta})$.

Then the conclusions of 7.3, 7.4 hold.

7.6 Remark. Concerning clause (C) of 7.5.

(Note: this holds if

$$\mathfrak{d} \subseteq \mathrm{Reg} \setminus 2^\theta \;\;\&\;\; |\mathfrak{d}| \leq \theta \Rightarrow |\mathrm{pcf}(\mathfrak{d})| \leq \theta.$$

Why? Now $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ is well defined and $\lambda \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{a})$ so letting $\mathrm{pcf}(\mathfrak{a}) \cap \lambda$ be $\langle \theta_\zeta : \zeta < \theta\rangle$, choose $\mu_\zeta \in \mathfrak{b}_\lambda[\mathfrak{a}] \setminus \bigcup\limits_{\xi<\zeta} \mathfrak{b}_{\theta_\xi}[\mathfrak{a}]$, and let $\mathfrak{b} = \{\mu_\zeta : \zeta < \theta\}$).

Proof. Similar [Fill?]. $\square_{7.5}$

7.7 Fact. 1) Assume

- (A) $\lambda = \mathrm{cf}(\lambda) > 2^\theta$ and $n < \omega$,
- (B) $f^\ell_\alpha \in {}^\theta\mathrm{Ord}$ for $\ell < n$, $\alpha < \lambda$,
- (C) $\alpha \neq \beta \Rightarrow \langle f^\ell_\alpha : \ell < n\rangle \neq \langle f^\ell_\beta : \ell < n\rangle$,
- (D) $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$.

Then we can find an ultrafilter $\mathscr{D}$ on θ (possibly a principal one) and $X \in [\lambda]^\lambda$, $v \subseteq n$ and $f_\ell \in {}^\theta\mathrm{Ord}$ for $\ell < n$ such that

- (a) for $\ell \in n\backslash v$ and $\alpha \in X$ we have $f^\ell_\alpha/\mathscr{D} = f_\ell/\mathscr{D}$,
- (b) for $\alpha < \beta$ from X and ℓ, $m \in v$ such that $f_\ell/\mathscr{D} = f_m/\mathscr{D}$ (e.g., $\ell = m$) we have $f^\ell_\alpha/\mathscr{D} < f^m_\beta/\mathscr{D}$,
- (c) if $\ell, m < n$ and $f_\ell/\mathscr{D} < f_m/\mathscr{D}$ and α, β are from X then $f^\ell_\alpha/\mathscr{D} < f^\ell_\beta/\mathscr{D}$.

2) Assume

- (a) $\lambda = \mathrm{cf}(\lambda) > 2^\theta$ and $(\forall \alpha < \lambda)(|\alpha|^{<\sigma} < \lambda)$ and $\aleph_1 + |\varepsilon(*)|^+ \leq \sigma = \mathrm{cf}(\sigma)$ and
- (b) $f^\varepsilon_\alpha \in {}^\theta\mathrm{Ord}$ for $\varepsilon < \varepsilon(*)$ and $\alpha < \lambda$,
- (c) I is a σ-complete ideal on θ,
- (d) $\mathscr{D}$ is a λ-complete filter on λ to which all cobounded subsets of λ belong.

Then we can find X, v, f_ε (for $\varepsilon < \varepsilon(*)$) and $\bar{w}$, J such that

- (α) $X \in [\lambda]^\lambda$,
- (β) $f_\varepsilon \in {}^\theta\mathrm{Ord}$ for $\varepsilon < \varepsilon(*)$,
- (γ) J is a σ-complete ideal on θ extending I,
- (δ) $\bar{w} = \langle w_\varepsilon : \varepsilon < \varepsilon(*)\rangle$, $w_\varepsilon \subseteq \theta$,
- (ε) if $\alpha \in X$ and $\varepsilon < \varepsilon(*)$ then $f^\varepsilon_\alpha \restriction w_\varepsilon = f_\varepsilon \restriction w_\varepsilon$,
- (ζ) if $\alpha < \beta$ are from X then $\varepsilon < \varepsilon(*) \ \Rightarrow\ f^\varepsilon_\alpha < f^\varepsilon_\beta \bmod (J + w_\varepsilon)$, moreover
$$\{i < \theta : \text{for some } \zeta, \xi < \varepsilon(*) \text{ we have } i \notin w_\zeta, i \notin w_\xi \text{ and}$$
$$f_\zeta(i) \leq f_\xi(i) \text{ but } f^\zeta_\alpha(i) \geq f^\xi_\beta(i)\} \in J,$$
- (η) if $\alpha \in X$ and $i < \theta$, ζ, $\xi < \varepsilon(*)$, $f_\zeta(i) < f_\xi(i)$ then $f_\zeta(i) < f^\xi_\alpha(i)$,
- (θ) if $2^{|\varepsilon(*)|} < \sigma$ then $\varepsilon < \varepsilon(*) \ \Rightarrow\ w_\varepsilon \in J \ \vee\ \theta \setminus w_\varepsilon \in J$.

3) We can combine 7.1(1) with part (2) (having $\langle \lambda^\varepsilon_{i,\ell} : \ell < \ell^\varepsilon_i \rangle$).

Remark. We can prove also the parallel of 7.1(5).

Proof. 1) Like the proof of 7.1(4) or by part (2) for $\sigma = \aleph_1$.
2) We repeat the proof of 7.1(4) except that $T \subseteq \bigcup\limits_{i<\sigma} {}^i\lambda$. After defining $B^* \in \mathscr{D}$ and choosing $\delta(*)$, for $\eta \in T$, $\varepsilon < \varepsilon(*)$ and $i < \theta$ we let $\beta_{\varepsilon,i,\eta} = \min[N_\eta \cap \text{ Ord } \setminus f^\varepsilon_{\delta(*)}(i)]$ and $w_{\varepsilon,\eta} = \{i < \theta : f^\varepsilon_{\delta(*)}(i) \in N_\eta\}$.

So clearly

$(*)_1$ $\eta \triangleleft \nu \in T \ \Rightarrow \ (\forall \varepsilon < \varepsilon(*))(\forall i < \theta)(\beta_{\varepsilon,i,\eta} \geq \beta_{\varepsilon,i,\nu} \ \& \ w_{\varepsilon,\eta} \subseteq w_{\varepsilon,\nu})$

and

$(*)_2$ $i \notin w_{\varepsilon,\eta} \Rightarrow \ \text{cf}(\beta_{\varepsilon,i,\eta}) > 2^\theta$.

Let J_η is the σ–ideal on θ generated by

$$I \cup \{w \subseteq \theta : \text{for some } \varepsilon < \varepsilon(*) \text{ we have } w \subseteq \theta \setminus w_{\varepsilon,\eta} \text{ and}$$
$$\lambda > \ \max \text{pcf}\{\text{cf}(\beta_{\varepsilon,i,\eta}) : i < w\}\}.$$

If for some η, $\theta \notin J_\eta$ then we are done.

Letting $\omega_\varepsilon := \omega_{\varepsilon,\eta}, J := J_\eta$ and choosing the α-th member of X' by induction on α, to satisfy $\forall \varepsilon < \varepsilon' \forall \alpha < \beta : f^\varepsilon_\alpha < f^\varepsilon_\beta \mod (J + \omega_\varepsilon)$.

For all $\varepsilon < \varepsilon(*)$, use 7.3(1) when ω_ε here stands for ω there, θ here for κ there and f^ε_α here stands for f_α there $\forall \alpha < \lambda$.

Letting $f_\varepsilon : \theta \to \text{Ord}$ be $f_\varepsilon(i) = \beta^*_i$ and choose $X \subseteq X'$ such that $\forall \alpha \in X \forall i < \theta \forall \zeta, \xi < \varepsilon(*)$: if $f_\zeta(i) < f_\xi(i)$ then $f_\zeta(i) < f^\xi_\alpha(i)$ and $\{i < \theta : f_\zeta(i) = f_\xi(i)$ for some $\zeta, \xi\} \in J$.

In addition, if $f_\zeta(i) < f_\xi(i)$ then $f_\zeta(i) < f^\xi_\beta(i) \leq f^\xi_\alpha(i) \leq f_\zeta(i)$, a contradiction so the rest of 7.10(2))(ζ) holds.

So assume that $\eta \in T \Rightarrow \theta \in J_\eta$. We now choose by induction on $\zeta < \sigma$, a sequence $\eta_\zeta \in T_i$ such that $\xi < \zeta \ \Rightarrow \ \eta_\xi = \eta_\xi \restriction j$ and

$$\zeta = \xi + 1 \Rightarrow \{i < \theta : (\exists \varepsilon < \varepsilon(*))(\beta_{\varepsilon,i,\eta_\zeta} > \beta_{\varepsilon,i,\eta_\xi})\} = \theta \mod I.$$

For some $\varepsilon < \varepsilon(*)$ and infinite $Y \subseteq \theta$ we have:

$$\xi \in Y \Rightarrow Z_\xi = \{i < \theta : \beta_{\varepsilon,i,\xi} > \beta_{\varepsilon,i,\xi+1}\} = \theta \mod I.$$

But for $\xi < \zeta$ we have $\beta_{\varepsilon,i,\xi+1} \geq \beta_{\epsilon,i,\zeta}$ by $(*)_1$. Without loss of generality $\mathrm{otp}(Y) = \omega$. As I is σ–complete and $\sigma \geq \aleph_0$, there is an $i \in \bigcap\{Z_\xi : \xi \in Y\}$, and $\langle \beta_{\varepsilon,i,\xi} : \xi \in Z\rangle$ is strictly decreasing, a contradiction.

Now for $\zeta = 0$, ζ limit there are no "serious" demands and for ζ successor ordinal we use $\theta \in J_\eta$.

3) Left to the reader (and not used). $\square_{7.7}$

7.7.4A Fact. Assume

(A) $\lambda = \mu^+$, $\mu > 2^\theta$, $\theta = \mathrm{cf}(\mu) > \aleph_0$,

(B) $|\epsilon(*)|^+ + \aleph_0 < \theta$,

(C) $f^\varepsilon_\alpha \in {}^\theta\mathrm{Ord}$ for $\varepsilon < \varepsilon(*)$, $\alpha < \lambda$,

(D) $(\forall \alpha < \mu)(|\alpha|^\theta < \mu)$.

Then we can find a stationary $S \subseteq \{\delta < \theta : \mathrm{cf}(\delta) \geq |\varepsilon(*)|^+ + \aleph_0\}$ and unbounded subset X' of λ and $S_\varepsilon \subseteq S$ and $f_\varepsilon \in {}^S\mathrm{Ord}$ for $\varepsilon < \varepsilon(*)$

(a) for $\varepsilon < \varepsilon(*)$ we have $\alpha \in X \Rightarrow f^\varepsilon_\alpha \restriction S_\varepsilon = f_\varepsilon \restriction S_\varepsilon$,

(b) for $\varepsilon_1 < \varepsilon(*)$ and $\alpha < \beta$ from X if $S_{\varepsilon,\zeta} = \{i \in S : f_\varepsilon(i) \leq f_\zeta(i)\} \setminus S_\varepsilon \setminus S_\zeta$ is unbounded in θ then $f^\varepsilon_\alpha \restriction S_{\varepsilon,\zeta} < f^\zeta_\beta \bmod J^{bd}_{S_{\varepsilon,\zeta}}$,

(c) if $\zeta, \varepsilon < \varepsilon(*)$, $f_\zeta(i) < f_\varepsilon(i)$, and $\alpha \in X$ then $f_\zeta(i) < f^\varepsilon_\alpha(i)$,

(d) if $2^{|\varepsilon(*)|} < \theta$ then $\varepsilon < \varepsilon(*) \Rightarrow S_\varepsilon \in \{\emptyset, S\}$.

Proof. Let $\bar f^\varepsilon = \langle f^\varepsilon_\alpha : \alpha < \lambda\rangle$, let χ be large enough and $\langle \lambda_\varepsilon : \varepsilon < \theta\rangle$ be increasing continuous with limit μ, and choose by induction on $\zeta < \theta$, an elementary submodel N_ζ of $(\mathscr{H}(\chi), \in, <^*_\zeta)$ of cardinality $(\lambda_\zeta)^\theta$ such that $(\lambda_\zeta)^\theta \subseteq N_\zeta$, ${}^\theta(N_\zeta) \subseteq N_\zeta$, $\{\bar f^\varepsilon : \varepsilon < \varepsilon(*)\} \in N_\zeta$, and $\langle N_\xi : \xi < \zeta\rangle \in N_\zeta$.

Choose $\delta(*) \in \lambda \setminus \bigcup\limits_{\zeta<\theta} N_\zeta$, possible as $|\bigcup\limits_{\zeta<\theta} N_\zeta| = |\sum_{\varepsilon<\theta}(\lambda_\varepsilon)^\theta| = \mu < \lambda$. For each $\zeta < \theta$, $\varepsilon < \varepsilon(*)$ and $i < \theta$ let $\beta^*_{\varepsilon,i,\zeta} = \min(N_\zeta \cap \mathrm{Ord} \setminus f^\varepsilon_{\delta(*)}(i))$.

For each limit $i < \theta$ of cofinality $> |\varepsilon(*)|$ look at $\langle \beta^*_{\varepsilon,i,\zeta} : \zeta < i\rangle$, it is a non–increasing sequence of ordinals, hence it is constant on some end segment, i.e., for some $j_{\varepsilon,i} < i$ we have

$$j_{\varepsilon,i} \leq \zeta < i \Rightarrow \beta^*_{\varepsilon,i,\zeta} = \beta^*_{\varepsilon,i,j_{\varepsilon,i}}.$$

As $\mathrm{cf}(i) > |\varepsilon(*)|$, necessarily $j_i = \sup\{j_{\varepsilon,i} : \varepsilon < \varepsilon(*)\}$ is $< i$, hence for some $j(*) < \theta$ the set

$$S = \{i < \theta : \mathrm{cf}(i) > |\varepsilon(*)|, i \text{ a limit ordinal}\}$$

is stationary. The rest should be clear. $\square_?$

$\longrightarrow$ scite{7.7.4A} undefined

Remark. We can demand $S \subseteq S^*$ in 7.7 if $S^* \subseteq \{\delta < \theta : \mathrm{cf}(\delta) \geq |\varepsilon(*)|^+ + \aleph_0\}$ is stationary.

* * * * * * *

7.8 Discussion. We may wonder what occurs for ultraproducts of free Boolean algebras $\prod\limits_{i<\theta} \mathrm{FBA}(\chi_i)/\mathscr{D}$ (or even reduced products, recall $\mathrm{FBA}(\chi_i)$ is the free Boolean algebra generated, say, by $\{x_\alpha : \alpha < \chi_i\}$ freely). Now

$(*)_1$ if $\mathscr{D}$ is $\aleph_1$–complete, the situation is as in the $\theta > \aleph_0$ case for products;

$(*)_2$ if

$$(\exists A_0, A_1, \dots)\Big(\bigwedge_{n<\omega} A_n \in \mathscr{D} \ \ \& \ \ \bigcap_{n<\omega} A_n = \emptyset\Big),$$

the situation is as in the $\theta = \aleph_0$ case.

7.9 Claim. *Assume*

(A) $\lambda = \mu^{++}$, $\mu > 2^\theta$,

(B) $f_\alpha : \theta \to \mathrm{Ord}$ *for* $\alpha < \lambda$.

<u>Then</u> we can find $\bar{u}^* = \langle u^*_0, u^*_1, u^*_2\rangle$, $\bar{\beta}^*$, X *such that*

(a) $\langle u^*_0, u^*_1, u^*_2\rangle$ *is a partition of* θ,

(b) $\bar{\beta} = \langle \beta^*_i : i < \theta\rangle$,

(c) $X \in [\lambda]^\lambda$ *(we can use an appropriate ideal* J *on* λ *and demand* $X \in J^+$*),*

(d) $\alpha \in X \ \Rightarrow \ f_\alpha \restriction u^*_0 = \langle \beta^*_i : i \in u^*_0\rangle$,

(e) *if* $i \in u^*_1$ *<u>then</u>* $\langle f_\alpha(i) : \alpha \in X\rangle$ *is strictly increasing with limit* β^*_i *(so* $\mathrm{cf}(\beta^*_i) = \lambda$*),*

(f) $i \in u^*_2 \Rightarrow 2^\theta < \mathrm{cf}(\beta^*_i) < \mu$,

(g) *for every* $\bar{\gamma} \in \prod\limits_{i \in u^*_2} \beta^*_i$ *for* λ *ordinals* $\alpha \in X$ *we have*

$$i \in u^*_2 \quad \Rightarrow \quad \gamma_i < f_\alpha(i) < \beta^*_i,$$

(h) *there are* $\langle \lambda_{i,\ell} : i < \theta,\ \ell < n_i\rangle$, $\langle A_i : i < \theta\rangle$, $\langle h_i : i < \theta\rangle$ *as in 7.1 satisfying*

- *(α)* $i \in u^*_0 \Leftrightarrow \ n_i = 0$,
- *(β)* $i \in u^*_1 \Leftrightarrow (n_i > 0 \ \& \ \lambda_{i,0} = \lambda) \Leftrightarrow (n_i > 0 \ \& \ (\forall \ell)(\lambda_{i,\ell} = \lambda))$,
- *(γ)* *if* $u^*_0 \neq \theta$, *then* $\lambda = \max \mathrm{pcf}\{\lambda_{i,\ell} : i < \theta, \ell < n_i\}$ *and* $\mu^+ \notin \max \mathrm{pcf}\{\lambda_{i,\ell} : i < \theta,\ \ell < n_i\}$.

Proof. Let $\bar{C} = \langle C_\alpha : \alpha < \mu^+ \rangle$ be such that $\mathrm{otp}(C_\alpha) \le \theta^+$, $[\beta \in C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta]$, C_α a set of successor ordinals and the set

$$S^* = \{\delta < \lambda^+ : \ \mathrm{cf}(\delta) = \theta^+ \text{ and } \alpha = \sup(C_\alpha)\}$$

is stationary (exists by [Sh 420, §1]).

Let $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ be given. Let χ be strong limit such that $\bar{f} \in \mathscr{H}(\chi)$. We choose M_α by induction on $\alpha < \mu^+$ such that

$(*)_1$ (α) $M_\alpha \prec (\mathscr{H}(\chi), \in, <^*_\chi)$,

(β) $\|M_\alpha\| = 2^\theta$ and $2^\theta + 1 \subseteq M_\alpha$ and ${}^{\theta \ge}(M_\alpha) \subseteq M_\alpha$,

(γ) $\lambda, \bar{f}, \bar{C}$ and α belong to M_α,

(δ) $\langle M_\beta : \beta < \alpha \rangle$ belongs to M_α and $\beta \in C_\alpha \Rightarrow M_\beta \prec M_\alpha$.

Now for every $\beta \in \lambda \backslash \bigcup_{\alpha < \mu^+} M_\alpha$ we define a function $g^\beta_\alpha \in {}^\theta(M_\alpha \cap \text{ Ord})$ for $\alpha < \mu^+$ and a function F_β from μ^+ to μ^+, as follows

$$(*)_2 \qquad g^\beta_\alpha(i) = \min(M_\alpha \cap \chi \backslash f_\beta(i)).$$

[Why is it well defined? As $\bar{f} \in M_\alpha$ also $\bigcup\{f_\gamma(i) + 1 : \gamma < \lambda\} \in M_\alpha \cap \lambda^+ < \chi$ and $f_\beta(i)$ is smaller than that ordinal.]

For $\beta < \lambda, \alpha < \mu^+$ we let

$(*)_3$ (a) $u^\beta_{\alpha,0} = \{i < \theta : f_\beta(i) \in M_\alpha\}$,

(b) $u^\beta_{\alpha,1} = \{i < \theta : f_\beta(i) \notin M_\alpha \text{ and } \mathrm{cf}(g^\beta_\alpha(i)) = \lambda\}$,

(c) $u^\beta_{\alpha,2} = \{i < \theta : \mathrm{cf}(g^\beta_\alpha(i)) \le \mu^+ \text{ and } f_\beta(i) \notin M_\alpha\}$.

Note. $f_\beta(i) \notin M_\alpha \Rightarrow \lambda \ge \mathrm{cf}(g_\beta(i)) > 2^\theta$.
[Why? If $i \in \theta \backslash u^\beta_{\alpha,0}$ and $\lambda < \mathrm{cf}(g^\beta_\alpha(i))$, then

$$\bigcup\{f_\gamma(i) : \gamma < \lambda \text{ and } f_\gamma(i) < g^\beta_\alpha(i)\}$$

belongs to M_α and contradicts the choice of $g^\beta_\alpha(i)$. If $i \in \theta \backslash u^\beta_{\alpha,0}$ and $\mathrm{cf}(g^\beta_\alpha(i)) \le 2^\theta$ then $g^\beta_\alpha(i) = \sup(M_\alpha \cap g^\beta_\alpha(i))$.]

Similarly choose $\langle A^\beta_{\alpha,i} : i < \theta\rangle$, $\langle h^\beta_{\alpha,i} : i < \theta\rangle$, $\langle \lambda^\beta_{\alpha,i,\ell} : i < \theta,\ \ell < n^\beta_{\alpha,i}\rangle$, $\langle f^{*,\beta}_{\alpha,\gamma} : \gamma \in X\rangle$ as in 7.1(1). Let $U^\beta_\alpha = \{(i,\ell) : i < \theta, \ell < n_i\}$; this is $\mathrm{Dom}(f^{*,\beta}_{\alpha,\gamma})$ for $\gamma \in X^\beta_\alpha$. Let

$$J = J_{\beta,\alpha} = \{u \subseteq U^\beta_\alpha : \mu^+ > \ \max \mathrm{pcf}\{\mathrm{cf}(\lambda^\beta_{\alpha,i,\ell}) : (i,\ell) \in u\}\}.$$

By the pcf theorem ([Sh:g, VIII,2.6]) there is $W^\beta_\alpha \subseteq U^\beta_\alpha$ such that:

$$\mu^+ \notin \mathrm{pcf}\{\mathrm{cf}(\lambda^\beta_{\alpha,i,\ell}) : (i,\ell) \in U^\beta_\alpha \setminus W^\beta_\alpha\},$$

$$\mu^+ \geq \ \max \mathrm{pcf}\{\mathrm{cf}(\lambda^\beta_{\alpha,i,\ell}) : (i,\ell) \in W^\beta_\alpha\}.$$

If $W^\beta_\alpha \notin J$ let $\bar h_{\beta,\alpha} = \langle h_{\alpha,\beta,\gamma} : \gamma < \mu^+\rangle \in M_\alpha$ be $<_{J \restriction W^\beta_\alpha}$–increasing and cofinal in $\prod\limits_{(i,\ell)\in W^\beta_\alpha} \lambda^\beta_{\alpha,i,\ell}$. Then for some $\gamma = \gamma(\alpha,\beta) < \mu^+$,

$$f^*_\beta < h_{\alpha,\beta,\gamma(\alpha,\beta)} \bmod J.$$

In fact any $\gamma' \in [\gamma(\alpha,\beta),\mu^+)$ will do, and now we let $F_\beta(\alpha) = \gamma(\alpha,\beta)$. If $W^\beta_\alpha \in J_{\beta,\alpha}$ we let $F_\beta(\alpha) = \alpha + 1$.

So the set $E_\beta = \{\delta < \mu^+ : \delta$ a limit ordinal such that $(\forall \alpha < \delta) F_\beta(\alpha) < \delta\}$ is a club of μ^+. Hence there is $\delta = \delta_\beta \in S^* \cap \ \mathrm{acc}(E_\beta)$ (i.e., $\delta = \sup(E_\beta \cap \delta)$ and $\delta \in S^*$). Now for each $(i,\ell) \in U^\beta_\alpha$ the sequence $\langle g^\beta_\alpha(i,\ell) : \alpha \in C_{\delta_\beta}\rangle$ is non-increasing as $\langle M_\alpha : \alpha \in C_{\delta_\beta}\rangle$ is increasing. Hence it is eventually constant, and similarly $(\bar\lambda^\beta_{\alpha,i}, h^\beta_{\alpha,i})$, $A^\beta_{\alpha,i}$ as in 7.1(2) (any freedom left – choose the $<^*_\chi$–first), so easily $\langle(\bar\lambda^\beta_{\alpha,i}, h^\beta_{\alpha,i}) : \alpha \in C_{\delta_\beta}\rangle$ is eventually constant; say for $\alpha \in C_{\delta_\beta} \backslash \alpha^*(\beta,i)$. But $\mathrm{otp}(C_{\delta_\beta}) \leq \theta^+$ so $\alpha^*(\beta) = \sup\{\alpha^*(\beta,i) : i < \theta\}$ is $< \delta_\beta$, and reflection shows that

$$\alpha \in C_{\delta_\beta} \setminus (\alpha^*(\beta)+1) \Rightarrow W^\beta_\alpha \in J_{\beta,\alpha}.$$

Choose such $\alpha^\otimes_\beta$. So for some $\alpha^\otimes$, $\delta^\otimes$, $\langle(\bar\lambda_i, h_i) : i < \theta\rangle$ we have

$$X = \Big\{\beta < \lambda : \beta \notin \bigcup_{\alpha<\mu^+} M_\alpha \ \text{ and } \ \alpha^\otimes_\beta = \alpha^\otimes,\ \delta_\beta = \delta^\otimes, \\ \bar\lambda^\beta_{\alpha^\otimes,i} = \bar\lambda_i, h^\beta_{\alpha^\otimes,i} = h_i \text{ for } i < \theta\Big\}$$

belongs to $[\lambda]^\lambda$. Now we continue as in 7.1. $\square_{7.7}$

7.10 Claim. *1) In 7.9 we can replace $\lambda = \mu^{++}$, by $\lambda = \tau^+$, $\tau = \mathrm{cf}([\tau]^{\leq\mu}, \subseteq)$ using [Sh 420, §2].*
2) Also if λ is weakly inaccessible $> \beth_\omega$, $(\forall \alpha < \lambda)$ $[\lambda > \mathrm{cf}([\alpha]^{\leq\mu}, \subseteq)]$ we can get 7.9.

§8 Consistency of "$\mathscr{P}(\omega_1)$ has a free caliber" and discussion of pcf

This solves [M2, Problem 37].

8.1 Claim. *Assume for simplicity GCH and $\mathbb{P}$ is adding $\aleph_{\omega_1}$ Cohen reals. In $\mathbf{V}^{\mathbb{P}}$ we have $2^{\aleph_0} = \aleph_{\omega_1}$, $2^{\aleph_1} = \aleph_{\omega_1+1}$ and*

- $(*)$ *there is no complete Boolean algebra $\mathbf{B}$ of cardinality $2^{\aleph_1}$ such that FreeCal($\mathbf{B}$) $= \emptyset$. In fact for any complete Boolean algebra $\mathbf{B}$ of cardinality $2^{\aleph_1}$ we have $\aleph_{\omega_1+1} \in$ FreeCal($\mathbf{B}$).*

Proof. Clearly (as if the Boolean algebra $\mathbf{B}$ has cardinality $2^{\aleph_1} = \aleph_{\omega_1+1}$ and satisfies the ccc then $(*)$ holds, i.e., $\aleph_{\omega_1+1} \in$ FreeCal($\mathbf{B}$), because $\mathbf{V}^{\mathbb{P}} \models (\aleph_{\omega_1})^{\aleph_0} = \aleph_{\omega_1}$, and otherwise we can reduce to the case $\mathbf{B} = \mathscr{P}(\omega_1)$) it is enough to show

$(*)_1$ $\mathbf{V}^{\mathbb{P}} \models \aleph_{\omega_1+1} \in$ FreeCal($\mathscr{P}(\omega_1)$).

So let $p^* \in \mathbb{P}$

$$p^* \Vdash_{\mathbb{P}} \text{“}\langle \underset{\sim}{a}_\alpha : \alpha < \aleph_{\omega_1+1}\rangle \text{ is a sequence of distinct elements of } \mathscr{P}(\omega_1)\text{”}.$$

Note: $P = \{f : f \text{ is finite function from } \aleph_{\omega_1} \text{ to } \{0,1\}\}$. So $\mathbb{P}_A = \{f \in \mathbb{P} : \mathrm{Dom}(f) \subseteq A\} \lessdot \mathbb{P}$ for any $A \subseteq \aleph_{\omega_1}$.

For each $\alpha < \aleph_{\omega_1+1}$ and $i < \omega_1$ there is a maximal antichain $\langle f_{\alpha,i,n} : n < \omega\rangle$ of $\mathbb{P}$ and sequence of truth values $\langle \mathbf{t}_{\alpha,i,n} : n < \omega\rangle$ such that

$$f_{\alpha,i,n} \Vdash_{\mathbb{P}} \text{“}i \in \underset{\sim}{a}_\alpha \text{ \underline{iff} } \mathbf{t}_{\alpha,i,n}\text{”}.$$

Let $A_\alpha = \bigcup\limits_{i<\omega_1, n<\omega} \mathrm{Dom}(f_{\alpha,i,n}) \cup \mathrm{Dom}(p^*)$, so $A_\alpha \in [\aleph_{\omega_1}]^{\le \aleph_1}$. Let $A_\alpha = \{\gamma_{\alpha,j} : j < j_\alpha\}$, $\gamma_{\alpha,j}$ strictly increasing with j.

As $\mathbf{V} \models 2^{\aleph_1} < \aleph_{\omega_1+1}$, without loss of generality

$(*)_2$ (a) $j_\alpha = j^*$

(b) the truth value of "$\gamma_{\alpha,j} \in \mathrm{Dom}(f_{\alpha,i,n})$" and the value of $f_{\alpha,i,n}(\gamma_{\alpha,j})$ do not depend on α.

Let $\underset{\sim}{a}$ be the Mostowski collapse of the name, i.e., $\underset{\sim}{a} = \mathrm{OP}_{j^*,A_\alpha}(\underset{\sim}{a}_\alpha)$ for each α (without loss of generality it does not depend on α). [Remember $\mathrm{OP}_{A,B}(\beta) = \alpha$ iff $\alpha \in A$, $\beta \in B$, $\mathrm{otp}(\beta \cap B) = \mathrm{otp}(\alpha \cap A)$.] We apply 7.1(1),(6) to $f_\alpha : j^* \to \aleph_{\omega_1}$ where f_α is defined by $f_\alpha(j) = \gamma_{\alpha,j}$ with $(\aleph_{\omega_1+1}, (2^{\aleph_1})^+, \aleph_1)$ here standing for (λ,μ,κ)

there; and get $\langle w^*_\ell : \ell < 2\rangle$, $X \in [\aleph_{\omega_1+1}]^{\aleph_{\omega_1+1}}$ and $\langle \lambda_{j,\ell} : j < j^*,\ \ell < n_j\rangle$, and $\bar h = \langle h_j : j < j^*\rangle$ and J as there. In particular, now each $\lambda_{j,\ell}$ is regular $\in (2^{\aleph_1}, \aleph_{\omega_1})$ (which is not really nec). For $i < \omega_1$ let $w_i = \{j < j^* : j \in w^*_0$ or $\lambda_{j,n_j-1} \leq \aleph_i\}$. Clearly $J = \{w \subseteq \{(i,\ell) : \ell < n_i, i < \aleph_1\} : w \subseteq \{(j,\ell) : i \in w_i\}$ for some i.

We call $\langle (g^0_i, g^1_i, \xi_i) : i < \omega_1\rangle$ a witness above f^* if:

$\boxtimes_1(i)$ $f^*, g^0_i, g^1_i \in \mathbb{P}$ and $p^* \leq f^*$,

(ii) $f^* \leq g^0_i$,

(iii) $f^* \leq g^1_i$,

(iv) $\mathrm{Dom}(g^\ell_i) \subseteq j^*$,

(v) $\langle \mathrm{Dom}(g^0_i) \cup \mathrm{Dom}(g^1_i) \backslash \mathrm{Dom}(f^*) : i < \omega_1\rangle$ are pairwise disjoint,

(vi) $g^0_i \Vdash$ "$\xi_i \in \underset{\sim}{a}$",

(vii) $g^1_i \Vdash$ "$\xi_i \notin \underset{\sim}{a}$",

$(viii)$ $\xi_i < \omega_1$ and $\xi_i \neq \xi_j$ for $i \neq j$, recalling 7.3(6) or repeating its proof.

Shrinking X (still unbounded in $\aleph_{\omega_1+1}$) we get:

$\boxtimes_2$ if $\alpha < \beta$ are from X then there is $i < \omega_1$ such that

$$j \in j^* \setminus w_i \wedge \lambda_{j,n_j-1} > \aleph_i \Rightarrow (h^{-1}_j(\gamma_{\alpha,j}))(m) < (h^{-1}_j(\gamma_{\beta,j}))(m),$$

and

$$j \in j^* \setminus w_i \wedge j_1 < j^* \Rightarrow \gamma_{\alpha,j_1} \neq \gamma_{\beta,j}.$$

Let

$$u_{\alpha,i} = \ \mathrm{OP}_{A_\alpha,j^*}\big(\mathrm{Dom}(g^0_i) \cup \mathrm{Dom}(g^1_i)\backslash \mathrm{Dom}(f^*)\big)$$

and

$$g^\ell_{\alpha,i} =: g^\ell_i \circ \ \mathrm{OP}_{j,A_\alpha}.$$

8.2 Fact. There are f^* and a witness $\langle (g^0_i, g^1_i, \xi_i) : i < \omega_1\rangle$ above f^* and $X \subseteq \aleph_{\omega_1+1}$ unbounded and an ideal $J \supseteq J^{\mathrm{bd}}_{\omega_1}$ on ω_1 such that:

$\oplus$ if $\alpha \neq \beta$ are in X then

$$\{i : \text{the functions } g^0_{\alpha,i}, g^1_{\beta,i} \text{ are compatible}\} \in J.$$

We show how to finish the proof assuming the fact, and then we shall prove the fact. For some unbounded $X \subseteq \aleph_{\omega_1+1}$ we have $\alpha \in X \Rightarrow f^{**} = \mathrm{OP}_{A_\alpha, j^*}(f^*)$ i.e., does not depend on $\alpha \in X$. [Why? As there are $\leq |\mathbb{P}| = \aleph_{\omega_1} < \aleph_{\omega_1+1}$ possibilities.]

We shall prove

$$f^{**} \Vdash_{\mathbb{P}} \text{“}\langle \underset{\sim}{a}_\alpha : \alpha \in X\rangle \text{ is independent (as a family of subsets of } \omega_1), \text{ even modulo } J^{\mathrm{bd}}_{\omega_1}\text{”}.$$

This is more than enough.

If not then for some $n < \omega$ and pairwise distinct $\alpha_1, \dots, \alpha_{2n} \in X$, we have:

$$\neg(f^{**} \Vdash_{\mathbb{P}} \text{“}\bigcap_{\ell=1}^{n} \underset{\sim}{a}_{\alpha_\ell} \cap \bigcap_{\ell=n+1}^{2n} (\omega_1 \backslash \underset{\sim}{a}_{\alpha_\ell}) \text{ is unbounded in } \omega_1\text{”})$$

So for some f^1, $f^{**} \leq f^1 \in \mathbb{P}$, and $\zeta < \omega_1$ we have

$$\boxtimes_3 \qquad f^1 \Vdash_{\mathbb{P}} \text{“}\bigcap_{\ell=1}^{n} \underset{\sim}{a}_{\alpha_i} \cap \bigcap_{\ell=n+1}^{n} (\omega_1 \backslash \underset{\sim}{a}_{\alpha_i}) \subseteq \zeta\text{”}.$$

Now letting
$g^0_{\alpha,i} = g^0_i \circ \mathrm{OP}_{j^*, A_\alpha}$ and
$g^1_{\alpha,i} = g^1_i \circ \mathrm{OP}_{j^*, A_\alpha}$
we have

$$\mathrm{Dom}(g^0_{\alpha,i}) \cup \mathrm{Dom}(g^1_{\alpha,i}) \subseteq \{\gamma_{\alpha,j} : j < j^*\}.$$

Let

$$B =: \{i < \omega_1 : \xi_i < \zeta\},$$

$$B_{\ell,m} =: \{i : u_{\alpha_\ell,i} \cap u_{\alpha_m,i} \neq \emptyset\} \text{ for } \ell \neq m,$$

$$B_\ell =: \{i : \mathrm{Dom}(f^1) \cap (\mathrm{Dom}(g^0_{\alpha_\ell,i}) \cup \mathrm{Dom}(g^1_{\alpha_\ell,i})) \neq \mathrm{Dom}(f^{**})\}.$$

Now

$\circledast_1$ $B \in J$.
[Why? By clause $(viii)$ of $\boxtimes_1$ above and the choice of J.]

$\circledast_2$ $B_{\ell,m} \in J$ for $\ell \neq m \in \{1, \ldots, 2n\}$.
[Why? By the Fact 8.2 which we are assuming.]

$\circledast_3$ $B_\ell \in J$ for $\ell \in \{1, \ldots, 2n\}$
[by clause (v) of $\boxtimes_1$ above (in fact is finite).]

So we can find $i \in \omega_1 \backslash \bigcup\limits_{\ell \neq m} B_{\ell,m} \backslash \bigcup\limits_{\ell} B_\ell \backslash B$ (because the set of inappropriate i's is in J).

So $f^2 = f^1 \cup \bigcup\limits_{\ell=1}^{n} g^0_{\alpha_\ell,i} \cup \bigcup\limits_{\ell=n+1}^{2n} g^1_{\alpha_\ell,i} \in \mathbb{P}$ forces that the intersection from $\boxtimes$ is not $\subseteq \zeta$, contradicting the choice of f^1. $\square$

Proof of the Fact 8.2. We divide the proof into two cases, depending on the answer to:

<u>Question</u>: Is there $\zeta < \omega_1$ such that: for no g^0, $g^1 \in \mathbb{P}_{j^*}$ above p^* and $\xi \in [\zeta, \omega_1)$ do we have

$$g^0 \restriction w_\zeta = g^1 \restriction w_\zeta, \quad g^0 \Vdash \text{“}\xi \in \underset{\sim}{a}\text{”}, g^1 \Vdash \text{“}\xi \notin \underset{\sim}{a}\text{”?}$$

Case A: The answer is YES.

Let $\zeta < \omega_1$ exemplify the yes. As GCH holds in $\mathbf{V}$, clearly for some unbounded $X \subseteq \aleph_{\omega_1+1}$ and $\langle \gamma^{**}_j : j \in w_\zeta \rangle$ we have

$$j \in w_\zeta \ \& \ \alpha \in X \quad \Rightarrow \quad \gamma_{\alpha,j} = \gamma^{**}_j.$$

So $\underset{\sim}{a}$ is actually a $\mathbb{P}_{\{\gamma^{**}_j : j \in w_\zeta\}}$-name. So for $\alpha \in X$, $\underset{\sim}{a}_\alpha$ depends only on $\{f \in \underset{\sim}{G}_{\mathbb{P}} : \operatorname{Dom}(f) \subseteq \{\gamma_{\alpha,i} : i \in w_\zeta\}\}$. So $p^* \Vdash_{\mathbb{P}}$ “$\underset{\sim}{a}_\alpha = \underset{\sim}{a}_\beta$ for $\alpha, \beta \in X$, a contradiction.

Case B: The answer is NO.

So for every $\zeta < \omega_1$, we have $\langle \xi^*_\zeta, g^0_\zeta, g^1_\zeta \rangle$ giving the counterexample for ζ, without loss of generality $\operatorname{Dom}(g^0_\zeta) = \operatorname{Dom}(g^1_\zeta)$. As $\langle w_\zeta : \zeta < \omega_1 \rangle$ is increasing continuous, by Fodor's lemma we can find $S \subseteq \omega_1$ stationary and $\zeta^* < \omega_1$ and n^* such that

$$\zeta \in S \Rightarrow (\operatorname{Dom}(g^0_\zeta) \cap w_\zeta) \cup (\operatorname{Dom}(g^1_\zeta) \cap w_\zeta) \subseteq w_{\zeta^*},$$

and $\langle \mathrm{Dom}(g^0_\zeta) \cup \mathrm{Dom}(g^1_\zeta) : \zeta \in S\rangle$ forms a Δ–system with heart v, and $g^0_\zeta \restriction v = g^1_\zeta \restriction v$ (being included in w_ζ) does not depend on ζ, and we call it f^*. Also $\mathrm{Dom}(g^\ell_\zeta \setminus w_{\zeta^*})$ has n^* elements and $\zeta_1 < \zeta \in S \Rightarrow \xi^*_{\zeta_1} < \zeta$.

Let $\langle \varepsilon(i) : i < \omega_1\rangle$ be a (strictly) increasing sequence listing S, and $\xi_i = \xi^*_{\varepsilon(i)}$. For $\ell < n^*$, $\alpha \in X'$ and $i < \omega_1$ we let $f^\ell_\alpha(i)$ be the ℓ-th member of $\{\gamma_{\alpha,j} : j \in \mathrm{Dom}(g^0_{\varepsilon(i)}) \setminus w_{\zeta^*}\}$. Shrinking X without loss of generality $\langle \gamma_{\alpha,j} : j \in w_{\zeta^*}\rangle$ does not depend on $\alpha \in X$ (by $\boxtimes_2$); $J = J^{\mathrm{bd}}_{\omega_1}$ and X are as required. $\square_{8.2,8.1}$

8.3 Discussion. 1) Clearly we can replace $\aleph_1$, $\aleph_{\omega_1+1}$ by any θ, λ as in 7.7.
2) Normally if μ is strong limit singular of cofinality θ, (at least large enough), we can find long intervals $\mathfrak{a}_i$ of the Reg $\cap\mu$ for $i < \theta$, $i < j \Rightarrow \sup(\mathfrak{a}_i) < \min(\mathfrak{a}_j)$ such that $(\forall \bar\lambda \in \prod_i \mathfrak{a}_i)$ [max pcf $(\mathrm{Rang}(\bar\lambda)) = \lambda^*$] for some $\lambda^* \in [\mu, 2^\mu]$, usually $\mathrm{cf}(2^\mu)$. This is a strong indication that $\langle I^n_{\sup(\mathfrak{a}_i),\min(\mathfrak{a}_i)} : i < \theta\rangle$ will have a λ-sequence, so for example there is a $(2^\theta)^+$-c.c. Boolean algebra of cardinality λ having no independent subset of cardinality λ, for which even λ-Knaster property fails.

To make this happen for no μ, we need a very special pcf structure in the universe. But we do not know even if the following simple case is consistent.

<u>8.4 Question</u>: Is it consistent that

- $(*)$ for every set $\mathfrak{a}$ of odd (or even) regular cardinals with $|\mathfrak{a}| < \mathrm{Min}(\mathfrak{a})$ we have max pcf$(\mathfrak{a})$ is odd (or even respectively) (we may moreover ask $(\forall\alpha)$ $2^{\aleph_\alpha} = \aleph_{\alpha+2}$)?

Essentially by [Sh 430, §5]:

8.5 Lemma. *Assume $\mu > \theta = \mathrm{cf}(\mu)$, μ strong limit, $\mu = \sum_{i<\theta}\mu_i$, $\mu_i < \mu$ strong limit, $\mathrm{cf}(\mu_i) = \sigma_i$ and $2^{\mu_i} = \mu_i^+$, $\mu_i = \sum_{\zeta<\sigma_i}\mu_{i,\zeta}$, $n_{i,\zeta} < \omega$, $\lambda = \mathrm{tcf}(\prod\limits_{i<\theta}\mu_i^+/J^*)$, $J^{bd}_\theta \subseteq J^*$.*

Let $I_{i,\zeta} = \mathrm{ERI}^{n_{i,\zeta}}_{\beth_{n_{i,\zeta}}(\mu_{i,\zeta})^+,\mu^+_{i,\zeta}}$ and

$$J = \sum_{J^*} J^{\mathrm{bd}}_{\sigma_i}.$$

<u>*Then*</u>

- (a) *there is a (J,λ)-sequence $\bar\eta$ for*

$$\langle I_{i,\zeta} : i < \theta, \zeta < \sigma_i\rangle.$$

- (b) *if $i < \theta \Rightarrow \sigma_i = \theta$ then we can find $\zeta(i) < \theta$ for $i < \theta$ such that there is a λ-sequence $\bar\eta$ for $\langle I_{i,\zeta(i)} : i < \theta\rangle$.*

8.6 Remark. So if $S = \{\mu : \mu \text{ strong limit }, \text{cf}(\mu) = \aleph_0, 2^\mu = \mu^+\}$ is unbounded, then for a class of cardinals μ which is closed unbounded

$(*)(a)$ μ strong limit and $\mu = \sup(S \cap \mu)$

(b) if $\text{cf}(\mu) = \aleph_0$ then we can find $\lambda \in (\mu, 2^\mu] \cap \text{Reg}$ and $\mu_n < \mu = \sum_n \mu_n$, $\mu_n < \mu_{n+1}$ and there is a λ-sequence $\bar\eta$ for $\langle I^n_{\beth_n(\mu_n)^+,\mu_n^+} : n < \omega\rangle$.

§9 Having a λ-sequence for a sequence of non-stationary ideals

9.1 Lemma. *Assume*

(a) μ *is a strong limit singular of cofinality* θ,
(b) $\lambda = 2^\mu = \mathrm{cf}(\lambda)$,
(c) λ_i *regular increasing for* $i < \delta$ *with limit* μ, $\delta < \mu$ *(usually* $\delta = \theta$*)*,
(d) J *is an ideal on* δ *extending* J^{bd}_δ,
(e) $\lambda = \mathrm{tcf}(\prod\limits_{i<\delta} \lambda_i / J)$,
(f) $\langle A_\zeta : \zeta < \zeta(*) \rangle$ *is a partition of* δ *(so* A_δ *pairwise disjoint) each* A_ζ *in* J^+ *(otherwise not interesting)*,
(g) $|\delta| < \sigma = \mathrm{cf}(\sigma) < \lambda_0$.

<u>Then</u> *there is a sequence* $\bar\eta = \langle \eta_\alpha : \alpha < \lambda \rangle$, $\eta_\alpha \in \prod\limits_{i<\delta} \lambda_i, \mathrm{cf}(\eta_\alpha(i)) = \sigma$, *satisfying*

(*) *For any sequence* $\langle F_{\zeta,i} : \zeta < \zeta(*), i < \delta \rangle$ *of functions, for every large enough* $\alpha < \lambda$ *we have*
(**) *if* $\zeta < \zeta(*)$, $F_{\zeta,i}(\eta_\alpha \restriction \bigcup_{\xi<\zeta} A_\xi)$ *is a club of* λ_i *for* $i < \delta$ *(really* $i \in A_\zeta$*)*,

<u>then</u>

$$\{i \in A_\zeta : \eta_\alpha(i) \notin F_{\zeta,i}(\eta_\alpha \restriction \bigcup_{\xi<\zeta} A_\zeta)\} \in J.$$

Moreover

$(**)^+$ *if* $\zeta < \zeta(*)$, $n < \omega$ *and* $\beta_0, \dots, \beta_{n-1} < \alpha$, *and for each* $i \in A_\zeta$ *we have:* $F_{\zeta,i}(\eta_{\beta_0}, \beta_0 \dots, \eta_{\beta_{n-1}}, \beta_{n_1}, \eta_\alpha \restriction \bigcup_{\xi<\zeta} A_\xi)$ *is a club of* λ_i, *then*

$$\{i \in A_\zeta : \eta_\alpha(i) \notin F_{\zeta,i}(\eta_{\beta_0}, \beta_0 \dots, \eta_{\beta_{n-1}}, \beta_{n-1}, \eta_\alpha \restriction \bigcup_{\xi<\zeta} A_\xi)\} \in J.$$

9.2 Discussion. For a given μ as in (a), clause (b) may fail, but then we will have another lemma. What about (e)?

If $\theta > \aleph_0$ there are such $\langle \lambda_i : i < \theta \rangle$ even for $J = J^{\mathrm{bd}}_\theta$ (see [Sh:g, VIII,§1]. If $\theta = \aleph_0$ we do not know, but we know that the failures are "rare". E.g.,

$$\{\delta < \omega_1 : \beth_\delta \text{ fails } (e) \text{ i.e. } \neg[\beth_{\delta+1} =^+ \ \mathrm{pp}(\beth_\delta)]\}$$

is not stationary. About $\mathrm{pp}_{J^{bd}_\omega}$ e.g. if $|\mathfrak{a}| \le \aleph_0 \;\Rightarrow\; |\mathrm{pcf}(\mathfrak{a})| \le \aleph_0$ we then can get it, see [Sh:g, XI,§5].

9.3 Remark. 1) This can be rephrased as having a (λ, J)-sequence for $\langle \prod J^{\mathrm{nst},\sigma}_{\lambda_{i,n}} : i < \delta\rangle$ with $\lambda_{i,n}$ decreasing.

So compared to earlier theorems, the λ, λ_i for which the Lemma applies are fewer, but the result is stronger: nonstationary ideal and we get also the "super" version see $(**)$.
2) Of course another variant is to start with $I_i = J^{\mathrm{nst},\sigma}_{\lambda_i}$ and get $J = J^{\mathrm{nst},\sigma}_\lambda$.
3) Considering functions with finitely many η_β's, $\beta < \alpha$ as parameters (i.e., $(**)^+$); thinning $\langle f_\alpha : \alpha < \lambda\rangle$ the conclusion follows.
4) In $(**)^+$ instead $n < \omega$ we can ask $n < \sigma$ if $(\forall \alpha < \lambda)(|\alpha|^{<\sigma} < \lambda)$

Proof of 9.1. For simplicity we concentrate on $(**)$ (in 10.1 we concentrate on the parallel of $(**)^+$). List the possible $\langle F_{\zeta,i} : i < \delta, \zeta < \zeta(*)\rangle$, i.e., sequence with each $F_{\zeta,i}$ having the "right" domain and range, which are clear from the statement, as $\big\langle \langle F^\beta_{\zeta,i} : i < \delta, \zeta < \zeta(*)\rangle : \beta < \lambda\big\rangle$. Let us define $\eta_\alpha \in \prod\limits_i \lambda_i$ by induction on α.

For a given α we choose $\eta_\alpha \restriction A_\zeta$ by induction on $\zeta < \zeta(*)$.

Define for $i \in A_\zeta$, $\beta < \alpha$

$$C^\beta_i = \begin{cases} F^\beta_{\zeta,i}(\eta_\alpha \restriction \bigcup\limits_{\xi<\zeta} A_\zeta) & \text{if this set is a club of } \lambda_i, \\ \lambda_i & \text{otherwise.} \end{cases}$$

So we need

Fact. There is $\eta \in \prod\limits_{i \in A_\zeta} \lambda_i$ such that

$$\bigwedge_{\beta<\alpha} \{i \in A_\zeta : \eta(i) \notin C^\beta_i\} \in J, i \in A_\zeta \Rightarrow\; \mathrm{cf}(\eta(i)) = \sigma.$$

Proof of the Fact. We shall choose by induction on $\varepsilon < \sigma$ a function $g_\varepsilon \in \prod\limits_{i \in A_\zeta} \lambda_i$ such that $\varepsilon_1 < \varepsilon \Rightarrow g_{\varepsilon_1} < g_\varepsilon$ (in all coordinates) and

$$(\forall \beta < \alpha)(\forall^J i \in A_\zeta)[(g_\varepsilon(i), g_{\varepsilon+1}(i)) \cap C^\beta_i \neq \emptyset].$$

Why is this enough?

Let $\nu = \eta \restriction A_\zeta$ be defined by

$$\nu(i) = \bigcup_{\varepsilon<\sigma} g_\varepsilon(i).$$

Now $\nu(i) < \lambda_i$ as $g_\varepsilon(i) < \lambda_i$ and $\sigma < \lambda_i = \mathrm{cf}(\lambda_i)$. (We can also say something for $\sigma \geq \mu$, but not now.) Also $\langle g_\varepsilon(i) : \varepsilon < \sigma\rangle$ is strictly increasing, so $\mathrm{cf}(\nu(i)) = \sigma$.

Now let $\beta < \alpha$ and define

$$B^*_\beta = \{i \in A_\zeta : \nu(i) \notin C^\beta_i\}.$$

We would like to have $B^*_\beta \in J$. For each $i \in B^*_\beta$, the sequence $\langle g_\varepsilon(i) : i < \sigma\rangle$ is a strictly increasing sequence of ordinals with limit not in C^β_i.

So for some $\varepsilon_{\beta,i} < \sigma$

$$C^\beta_i \cap (g_{\varepsilon_{\beta,i}}(i), \nu(i)) = \emptyset.$$

So

$$\bigwedge_{\varepsilon\geq\varepsilon_{\beta,i}} (g_{\varepsilon(i)}, g_{\varepsilon+1}(i)) \cap C_{\beta,i} = \emptyset.$$

Let $\varepsilon_\beta = \sup_{i<\delta} \varepsilon_{\beta,i}$.

Now $\varepsilon_{\beta,i} < \sigma$ & $\sigma = \mathrm{cf}(\sigma) > |\delta| \geq |A_\zeta|$, so $\varepsilon_\beta < \sigma$. So

$$\bigwedge_{i\in B^*_\beta} (g_{\varepsilon_\beta}(i), g_{\varepsilon_{\beta+1}}(i)) \cap C^\beta_i = \emptyset,$$

and hence $B^*_\beta \in J$ as required, i.e., ν is the required η.

Why is the choice of the g_ε possible?

Construction.

$\varepsilon = 0$. Trivial.

ε limit. $g_\varepsilon(i) = \bigcup_{\varepsilon_1<\varepsilon} g_{\varepsilon_1}(i) < \lambda_i$ (as $\varepsilon < \sigma < \lambda_i = \mathrm{cf}(\lambda_i)$).

$\varepsilon + 1$. For $\beta < \alpha$ define $h_{\beta,\varepsilon} \in \prod_{i\in A_\zeta} \lambda_i$ by

$$h_{\beta,\varepsilon}(i) =: \min\{\gamma < \lambda_i : (g_\varepsilon(i), \gamma) \cap C^\beta_i \neq \emptyset\}.$$

So $\{h_{\beta,\varepsilon} : \beta < \alpha\}$ is a subset of $\prod\limits_{i \in A_\zeta} \lambda_i$ of cardinality $< \lambda$, but $\prod\limits_{i<\delta} \lambda_i/J$ hence $\prod\limits_{i \in A_\zeta} \lambda_i/(J \restriction A_\zeta)$ has true cofinality λ (as if $A_\zeta \in J$ there is nothing to prove). So there is $g'_\varepsilon \in \prod\limits_{i \in A_\zeta} \lambda_i$ which is a $<_{J \restriction A_\zeta}$-upper bound of $\{h_{\beta,\varepsilon} : \beta < \alpha\}$.

Let $g_{\varepsilon+1}(i) = \max\{g'_\varepsilon(i), g_\varepsilon(i) + 1\}$, clearly it is as required. $\square_{9.1}$

9.4 Claim. *1) Assume*

(a) *$\bar\eta = \langle \eta_\alpha : \alpha < \lambda \rangle$, where $\eta_\alpha \in \prod\limits_{i \in \mathrm{Dom}(J)} \mathrm{Dom}(I_i)$ and J is an ideal on δ extending J^{bd}_δ, each I_i and ideal and I an ideal on λ extending J^{bd}_λ,*

(b) *$\langle A_\zeta : \zeta < \zeta(*) \rangle$ is a partition of $\mathrm{Dom}(J), A_\zeta \notin J$,*

(a) *for every $\bar F = \langle F_i : i \in \mathrm{Dom}(J) \rangle$, for the I-majority of $\alpha < \lambda$, for every $\zeta < \zeta(*)$ <u>if</u> $F_i(\eta_\alpha \restriction \bigcup\limits_{\xi<\zeta} A_\xi) \in I_i$ for $i \in A_\zeta$, <u>then</u>*

$$(\forall^J i \in A_\zeta)[(\eta_\alpha(i) \notin F_i\big(\eta_\alpha \restriction \bigcup_{\xi<\zeta} A_\xi\big)].$$

(d) *$I^*_j = \prod\limits_{\ell<n_j} I_{i(j,\ell)}$ for $j < \delta^*$, where $i(j,\ell) < \delta$*

(e) *$J^* = \big\{A \subseteq \delta^* :$ for some $B \subseteq \delta, \bigwedge\limits_\zeta (B \cap A_\zeta) \in J$ and $\bigwedge\limits_{i \in A} \bigvee\limits_{\ell<n_j} i(j,\ell) \in B\}$ is an ideal on δ^*,*

(f) *η^*_α is defined by*

$$\eta^*_\alpha(j) = \langle \eta_\alpha(i(j,\ell)) : \ell < n_j \rangle.$$

<u>Then</u>[4] $\langle \eta^*_\alpha : \alpha < \lambda \rangle$ is a (λ, J^*, I)-sequence for $\langle I^*_j : j < \delta \rangle$.
2) If we strengthen clause (c) to the parallel of $(**)^+$ in 9.1, <u>then</u> $\langle \eta^*_\alpha : \alpha < \lambda \rangle$ is a super (λ, J^*, I)-sequence for $\langle I^*_i : i < \delta \rangle$.

Proof. Straightforward. $\square_{9.4}$

9.5 Conclusion. Assume (a)–(g) of 9.1 (see 9.2) and (a), (e) of 9.4. Then there is a super (λ, J^*)-sequence for $\langle I^*_j : j < \delta \rangle$.

9.6 Conclusion. Assume $\mu >$ $\mathrm{cf}(\mu) = \aleph_0$ is a strong limit, and

[4] so we have dealt here with the case of $J^{\mathrm{bd}}_{\bar\lambda}$, $\bar\lambda$ decreasing

$$\lambda = 2^\mu = \operatorname{cf}(2^\mu) = \operatorname{tcf}\Big(\prod_{n<\omega} \lambda_n / J^{bd}_\omega\Big),$$

λ_n regular $< \mu$. Let $\langle k_n : n < \omega\rangle$ be such that

$$(\forall k)(\exists^\infty n)(k_n = k),$$

and, e.g., $\theta = (2^{\aleph_0})^+$.

For $n < \omega$ and $k < k_n$ let $\ell(n,k) = \sum\{k_m : m < n\} + k$ and let

$$I_n = \prod_{k<k_n} J^{\mathrm{nst},\theta}_{\lambda_{\ell(n,k)}}$$

$$J = \{A \subseteq \omega : \sup_{n\in A} k_n < \omega\}.$$

<u>Then</u> there is a (λ, J)-sequence for $\langle I_n : n < \omega\rangle$ (even a super one).

Proof. By Lemma 9.1 and Claim 9.4, we choose in 9.4 the parameters $\delta = \omega$, $\zeta(*) = \omega$ and let

$$A_\zeta = \Big\{\sum_{m\leq n} k_m - \zeta : k_n > \zeta\Big\}.$$

$\square_{9.6}$

We may wonder on the "tcf" assumption; at the expense of using "some J" this can be overcome:

9.7 Claim. *Assume $\mu >$ cf$(\mu) = \aleph_0$ strong limit singular,*

$$\lambda = 2^\mu = \operatorname{cf}(2^\mu) \in \operatorname{pcf}\{\lambda_n : n < \omega\},$$

$$\lambda_n = \operatorname{cf}(\lambda_n) < \mu,$$

and $\langle k_n : n < \omega\rangle$ is as in 9.6. <u>Then</u> we can find $i(n,\ell) < \omega$, $\ell < k_n$ with no repetitions,

$$i(n,0) > i(n-1, k_{n-1} - 1) > \cdots > i(n-1, 0),$$

and letting

$$I_n = \prod_{\ell<k_n} J^{\mathrm{nst},\theta}_{\lambda_{i(n,\ell)}},$$

we have: for some ideal $J \supseteq J^{\mathrm{bd}}_\omega$ on ω, there is a (λ, J)-sequence for $\langle I_n : n < \omega\rangle$.

Proof. Let

$$\begin{aligned}\mathrm{pcf}_{J^{bd}_\omega}(\{\lambda_n : n<\omega\}) = \{\chi : &\mathrm{cf}(\chi)=\chi= \ \mathrm{tcf}(\prod_{n\in A}\lambda_n/J^{bd}_A) \\ &\text{for some infinite } A\subseteq\omega\}.\end{aligned}$$

By a pcf claim:

9.8 Fact. We can find increasing $\langle \chi_\varepsilon : \varepsilon < \varepsilon(*)\rangle$, $\varepsilon(*) < \omega_1$, a limit ordinal, J^* an ideal $\supseteq J^{\mathrm{bd}}_{\varepsilon(*)}$, such that

$$\chi_\varepsilon \in \ \mathrm{pcf}_{J^{\mathrm{bd}}_\omega}(\{\lambda_n : n<\omega\}),$$

say

$$\chi_\varepsilon = \ \mathrm{tcf}\Big(\prod_{n\in B_\varepsilon}\lambda_n/J^{bd}_{B_\varepsilon}\Big),$$

$\langle B_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ is a partition of ω, and

$$\lambda = \ \mathrm{tcf}\Big(\prod_{\varepsilon<\varepsilon(*)}\lambda_\varepsilon/J^*\Big).$$

Continuation of the proof of 9.7. Let $\langle k_n : n<\omega\rangle$ be as before. Choose

$$\langle i(n,\ell) : \ell < k_n\rangle \text{for each } n$$

such that

(*a*) $i(n,\ell) > i(n,\ell+1), i(n,\ell_1) < i(n+1,\ell_2)$, and
(*b*) for every k and $\varepsilon_0,\dots,\varepsilon_{k-1}$, for infinitely many n we have

$$k_n = k, \qquad i(n,\ell)\in B_{\varepsilon_\ell}.$$

Let

$$A_\ell = \{i(n,\ell) : n < \omega, k_n > \ell\}.$$

So

> $\langle A_\ell : \ell < \omega\rangle$ is a sequence of pairwise disjoint subsets of ω such that $|A_\ell \cap B_{\varepsilon_\ell}| = \aleph_0$.

We apply 9.1 for

$$\langle A_n : n < \omega\rangle, \ \ \langle \lambda_n : n < \omega\rangle, \lambda, \mu.$$

$\square_{9.7}$

9.9 Remark. If $\mu >$ cf$(\mu) > \aleph_0$, 2^μ regular, the parallel to 9.6 always occurs. $\square_{9.7}$

If we use $\bar{A} = \langle A_0\rangle$, $A_0 = \delta$ in 9.1:

9.10 Conclusion. In 9.1 we get:

> there is a (λ, J)-sequence for $\langle I_i : i < \delta\rangle$, even a super one.

9.11 Remark. By the proofs in [Sh 420, §1] we can replace $\langle S^{\lambda_i}_\theta : i < \delta\rangle$, $S^\lambda_\theta = \{\delta < \lambda_i : \ \mathrm{cf}(\delta) = \theta\}$ by some large enough $\bar{S} = \langle S_i : i < \delta\rangle$, where $S_i \in I[\lambda_i]$, see below.

Also if $\langle f_\alpha : \alpha < \lambda\rangle$ is $<_J$-increasing cofinal in $\prod\limits_{i<\delta} \lambda_i/J$, continuous when it can be, then for some club E of λ we have $\langle f_\delta : \delta \in E$, $\mathrm{cf}(\delta) = \theta$, $\bar{f} \restriction \delta$ has an exact least upper bound lub$\rangle$ is OK. Probably more interesting is to strengthen $I^{nst,\theta}_{\lambda_i}$ by club guessing, as follows.

9.12 Definition. For $\bar{C} = \langle C_\delta : \delta \in S\rangle, S \subseteq \lambda$, stationary

$$\begin{aligned}\mathrm{id}^a(\bar{C}) = \{A \subseteq \lambda :& \text{for some club } E \text{ of } \lambda \text{ the set}\\ & \{\delta \in S : C_\delta \subseteq E\} \text{ is not stationary}\\ & (\text{so as we can shrink } E, \text{ equivalently, empty})\}.\end{aligned}$$

9.13 Lemma. *Assume*

(a) μ *is a strong limit singular* θ,

(b) $\lambda = 2^\mu = \mathrm{cf}(\lambda)$,

(c) λ_i *regular increasing for* $i < \delta$ *with limit* $\mu, \delta < \mu$ *(usually* $\delta = \mathrm{cf}(\mu)$*)*,

(d) J *is an ideal on* δ *extending* J^{bd}_δ,

(e) $\lambda = \mathrm{tcf}(\prod\limits_{i<\delta} \lambda_i/J)$,

(f) $\langle A_\zeta : \zeta < \zeta(*)\rangle$ *is a partition of* δ *(so pairwise disjoint)*,

(g) $\sigma = \mathrm{cf}(\sigma) < \mu$, *moreover* $\sigma < \lambda_0$ *and satisfies*

$\otimes^{\sigma,\delta}_J$ *we have* $\sigma > \delta$ *(or at least if* $A_\varepsilon \in J$ *for* $\varepsilon < \sigma$ *then*

$$\{i < \delta : i \in A_\varepsilon \text{ for every large enough } \varepsilon < \sigma\} \in J).$$

Then

(1) *For* $\theta \in \mathrm{Reg} \cap (\sigma, \lambda_0)$ *we can find* $\langle S_i : i < \delta\rangle$, $\langle \bar C^i : i < \delta\rangle$, $\bar I = \langle I_i : i < \delta\rangle$, $\bar\eta = \langle \eta_\alpha : \alpha < \lambda\rangle$ *such that*

(α) $S_i \in I[\lambda_i]$ *is stationary, and* $\delta \in S_i \Rightarrow \mathrm{cf}(\delta) = \sigma$,

(β) $\bar C^i = \langle C^i_\delta : \delta \in S_i\rangle$, C^i_δ *a club of* δ,

(γ) $I_i = \mathrm{id}^a(\bar C^i) = \big\{A \subseteq \lambda_i :$ *for some club*E *of* λ_i *we have:*$\delta \in S \cap A_i$ *implies* $\sup(C^i_\delta \backslash E) < \delta\big\}$,

(δ)(∗) *For any sequence* $\langle F_{\zeta,i} : \zeta < \zeta(*), i < \delta\rangle$ *of functions, for every large enough* $\alpha < \lambda$ *we have*

(∗∗) *if* $\zeta < \zeta(*)$, $F_{\zeta,i}(\eta_\alpha \restriction \bigcup\limits_{\xi<\zeta} A_\xi)$ *a member of* $\mathrm{id}^a(\bar C^i)$ *for* $i < \delta$ *(really* $i \in A_\zeta$*)*, *then*

$$\{i \in A_\zeta : \eta_\alpha(i) \in F_{\zeta,i}(\eta_\alpha \restriction \bigcup_{\xi<\zeta} A_\zeta)\} \in J.$$

Moreover

$(**)^+$ *if* $\zeta < \zeta(*)$, $n < \omega$ *and* $\beta_0, \dots, \beta_{n-1} < \alpha$ *and for each* $i \in A_\zeta$ *we have:* $F_{\zeta,i}(\eta_{\beta_0}, \beta_0 \dots, \eta_{\beta_{n-1}}, \beta_{n-1}, \eta_\alpha \restriction \bigcup\limits_{\xi<\zeta} A_\xi)$ *in a member of* $\mathrm{id}^a(\bar C^i)$ *then*

$$\big\{i \in A_\zeta : \eta_\alpha(i) \in F_{\zeta,i}(\eta_{\beta_0}, \beta_0, \dots, \eta_{\beta_{n-1}}, \beta_{n-1}, \eta_\alpha \restriction \bigcup_{\xi<\zeta} A_\xi)\big\} \in J.$$

Remark. 1) Included in the proof are imitations of proofs from [Sh 420, §1] and of 9.1.
2) We have a bit of flexibility in the proof.
3) In $(**)^+$, we can replace $n < \omega$ by $n < \tau$ when $(\forall \alpha < \lambda)$ $(|\alpha|^{<\tau} < \lambda)$.

Proof. Let $\theta = 2^\sigma$. By [Sh 420, §1] we can find $\bar{e}^i$ such that:

- (i) for $i < \delta, \bar{e}^i = \langle e^i_\alpha : \alpha \in S_i \rangle$, $S_i \in I[\lambda]$,
- (ii) e^i_α a club of α of order type σ such that $\alpha \in S_i \Rightarrow \mathrm{cf}(\alpha) = \sigma$,
- (iii) for χ large enough, $x \in \mathscr{H}(\chi)$, we can find $\langle N_i : i \leq \sigma \rangle$ such that $x \in N_\varepsilon \prec (\mathscr{H}(\chi), \in, <^*_\chi)$, $\langle N_\zeta : \zeta \leq \varepsilon \rangle \in N_{\varepsilon+1}$, N_ε increasing continuous, $\|N_\varepsilon\| = \theta$, $\theta + 1 \subseteq N_\varepsilon$ and

$$i < \delta \quad \Rightarrow \quad \sup e^i_{\sup(N_\sigma \cap \lambda_i)} \in S_i.$$

For $\bar{d} \in \bigcup \big\{ \prod\limits_{i<\delta} e_i : e_i \text{ a club of } \sigma \big\}$ let $\bar{e}^{i,\bar{d}} = \langle e^{i,\bar{d}}_\alpha : \alpha \in S_i \rangle$, $e^{i,\bar{d}}_\alpha = \langle \beta \in e^i_\alpha : \mathrm{otp}(e^i_\alpha \cap \beta) \in d_i \rangle$. For each such $\bar{d}$ we repeat the proof of 9.1, so we choose $\eta_\alpha = \eta^{\bar{d}}_\alpha$ by induction on $\alpha < \lambda$, and for each α, choose $\eta_\alpha \restriction \big(\bigcup\limits_{\varepsilon<\zeta} A_\varepsilon \big)$ by induction on $\zeta \leq \zeta(*)$. If we succeed fine, so assume we fail. So for some $\alpha = \alpha[\bar{d}]$, $\zeta = \zeta[\bar{d}]$ the situation is: $\langle \eta^{\bar{d}}_\beta : \beta < \alpha \rangle$ and $\eta^{\bar{d}}_\alpha \restriction \big(\bigcup\limits_{\varepsilon<\zeta} A_\varepsilon \big)$ are defined, but we cannot define $\eta^{\bar{d}}_\alpha \restriction A_\zeta$ and as there we can compute a family $\mathbf{E} = \mathbf{E}^i_{\bar{d}}$ of cardinality $< \lambda$ whose members has the form $\bar{B} = \langle B_i : i < \delta \rangle$, $B_i \in \mathrm{id}^a(\bar{e}^{i,d})$ and let $E^i_{B_i}$ be a club of λ_i exemplifying $B_i \in \mathrm{id}^a(\bar{e}^{i,d})$; let $\mathbf{E}^i_{\bar{d}} = \{ \langle E^i_{B_i} : i < \delta \rangle : \bar{B} = \langle B_i : i < \delta \rangle \in \mathbf{E} \}$. Let $\langle N_i : i \leq \sigma \rangle$ be as in $\otimes(iii)$ for $x = \big\{ \langle \langle \mathbf{E}^1_{\bar{d}}, \bar{d} \rangle : d_i \subseteq \sigma \text{ a club for } i < \delta \rangle, \bar{\lambda}, \langle \bar{e}^i : i < \delta \rangle \big\}$.

As in the proof of 9.1 quite easily:

$$\varepsilon \leq \sigma \;\; \& \;\; \bar{B} = \langle B_i : i < \delta \rangle \in \bigcup_{\bar{d}} \mathbf{E}_{\bar{d}} \quad \Rightarrow \{ i < \delta : \sup(N_\varepsilon \cap \lambda_i) \notin E^i_{B_i} \} \in J.$$

Let $d_i = \big\{ \mathrm{otp}(e^i_{\sup(N_\sigma \cap \lambda_i)} \cap \sup(N_\varepsilon \cap \lambda_\ell)) : \varepsilon < \sigma \text{ and } \sup(N_\varepsilon \cap \lambda_i) \in e^i_{\sup(N_\sigma \cap \lambda_i)} \big\}$. Clearly d_i is a club of σ and let $\bar{d} = \langle d_i : i < \delta \rangle$. Now $\langle \sup(N_\sigma \cap \lambda_i) : i \in A_{\zeta[\bar{d}]}$ is as required. $\square_{9.13}$

* * *

9.14 Conclusion. 1) In 9.13 we get:

for some function $c : [\lambda]^2 \longrightarrow \sigma$, for every $X, Y \in [\lambda]^\lambda$ and $\zeta < \sigma$, for some $\alpha \in X$, $\beta \in Y$ we have $\alpha > \beta$ and $c(\{\alpha, \beta\}) = \zeta$.

2) In 9.13 we can add:

if e.g. $\chi = (2^\lambda)^+$, for every $X \subseteq 2^\mu$ for every $\alpha < \lambda$ large enough, for $\zeta < \zeta(*)$, there is a sequence $\langle N_\varepsilon : \varepsilon < \sigma\rangle$ as in the proof of 9.13 such that

$$(\boxtimes) \qquad \{i \in A_\zeta : \eta_\alpha(i) \neq \sup(N_\delta \cap \lambda_1)\} \in J.$$

9.15 Remark. In 9.14(1) we get even $Pr_1(\lambda, \lambda, \sigma, \sigma)$.

Proof. 1) We relay on part 2).
2) For $\alpha < \beta$ let $c(\{\alpha, \beta\}) = \zeta$ if

$$\{i \in A_0 : f_\beta(i) \geq f_\alpha(i) \ \text{ or } \ f_\beta(i) < f_\alpha(i) \ \& \ \zeta \neq \ \text{otp}(e^i_{f_\alpha(i)} \cap \beta)\} \in J,$$

and zero if there is no such ζ.
Let $X, Y \in [\lambda]^\lambda$. take $\alpha \in X$ large enough, so that we can find $\langle N_\varepsilon : \varepsilon \leq \sigma\rangle$ as there, with $(\boxtimes)$ for part (2). We can find $\beta \in N_{\zeta+1} \cap Y$ such that $\langle \sup(N_\zeta \cap \lambda_i) : i < \delta\rangle <_J \eta_\beta$ (as $Y \cap N_{\zeta+1}$ is unbounded in $\lambda \cap N_{\zeta+1}$). Now $\alpha > \beta$ are as required. $\square_{9.14}$

9.16 Claim. *In 9.1*

(1) *Instead of "$\mu > \theta = \text{cf}(\theta) > |\delta|$" we can assume only*
$\otimes_1$ $\mu > \theta = \text{cf}(\theta)$ *and if* $\langle u_\zeta : \zeta < \theta\rangle$ *is a sequence of members of* J *then*
$$\big\{i < \delta : \theta = \sup\{\zeta : i \notin u_\zeta\}\big\} = \delta \ \textit{mod} \ J.$$

(2) *Weakening the conclusion of 9.1 to "weak (J, λ)-sequence", we can replace "$\theta = \text{cf}(\theta) > |\delta|$" by*
$\otimes_2$ $\theta = \text{cf}(\theta)$ *and if* $\langle u_\zeta : \zeta < \theta\rangle$ *is a sequence of members of* J *then*
$$\big\{i < \delta : \theta = \sup\{\zeta : i \notin u_\zeta\}\big\} \in J^+.$$

(3) *In part (1) and (2), if $\theta > \aleph_0$, then we can find $\bar{C}^i = \langle C^i_\delta : \delta \in S^{\lambda_i}_\theta\rangle$ with C^i_δ a club of δ such that: we can replace $I^{\text{nst},\theta}_{\lambda_i}$ by $\text{id}^a_{\lambda_i}(\bar{C}^i)$, see 9.13. above.*

§10 The power of a strong limit singular is itself singular: existence

10.1 Lemma. *Assume*

(*a*) μ *strong limit singular,*

(*b*) 2^μ *is singular,* $\lambda = \mathrm{cf}(2^\mu)$ *(so* $2^\mu > \lambda > \mu$*),*

(*c*) $\mu > \sigma = \mathrm{cf}(\sigma) > \mathrm{cf}(\mu)$,

(*d*) $2^\mu = \mathrm{pp}(\mu)$ *(see discussion in §9).*

<u>Then</u>

(*α*) *we can find* J, J^*, $\bar\theta^i = \langle \theta^i_\zeta : \zeta < \mathrm{cf}(\mu)\rangle$ *for* $i < \lambda$ *and* $\bar\lambda$ *such that*

  (*i*) $\bar\theta^i$ *is an increasing sequence of regular cardinals* $< \mu$ *with limit* μ *for* $i < \lambda$,

  (*ii*) $\bar\lambda = \langle \lambda_\alpha : \alpha < \lambda\rangle$ *is an increasing sequence of regulars* $\in (\mu + \lambda, 2^\mu)$ *with limit* 2^μ,

  (*iii*) $J \subseteq J^*$ *are ideals on* $\mathrm{cf}(\mu), \mathrm{cf}(\mu)$*-complete,*

  (*iv*) $\lambda_\alpha = \mathrm{tcf}\big(\prod\limits_\zeta \theta^\alpha_\zeta / J\big)$,

  (*v*) $\langle \bar\theta^\alpha : \alpha < \lambda\rangle$ *is* $<_{J^*}$*-increasing, i.e.* $\alpha < \beta \to \{\zeta < \mathrm{cf}(\mu) : \theta^\alpha_\zeta \geq \theta^\beta_\zeta\} \in J^*$*, with* $<_{J^*}$*-exact upper bound* $\langle \theta^*_\zeta : \zeta < \mathrm{cf}(\mu)\rangle$ *and (*θ^*_ζ *is a cardinal* $< \mu$*, normally singular)* $\mu = \lim\langle \theta^*_\zeta : \zeta < \mathrm{cf}(\mu)\rangle$ *and*

$$\bigwedge_{\substack{\alpha<\lambda \\ \zeta < \mathrm{cf}(\mu)}} \theta^\alpha_\zeta < \theta^*_\zeta,$$

  (*vi*) *if* $J \neq J^{\mathrm{bd}}_{\mathrm{cf}(\mu)}$*, then* $\mathrm{cf}(\mu) = \aleph_0$ *and* $\mathrm{pp}_{J^{\mathrm{bd}}_{\mathrm{cf}\mu}}(\mu) < 2^\mu$ *and* J *as in 9.7 so for most such* μ *we have the conclusion of (1), see [Sh:E12] and §4.*

(*β*) *If* J, $\bar\theta^\alpha (\alpha < \lambda)$, $\bar\lambda$ *are as in clause (α)* <u>*then*</u> *we can find* $\bar\eta = \langle \eta_\alpha : \alpha < \lambda\rangle$ *such that*

  (*i*) $\bar\eta = \langle \eta_\alpha : \alpha < \lambda\rangle$, $\eta_\alpha \in \prod\limits_{\zeta < \mathrm{cf}(\mu)} \theta^*_\zeta \subseteq {}^{\mathrm{cf}}(\mu)\mu$*. Moreover,* $\eta_\alpha \in \prod\limits_{\zeta < \mathrm{cf}(\mu)} \theta^\alpha_\zeta$ *and* $\sigma = \mathrm{cf}(\eta_\alpha(i))$ *for* $\alpha < \lambda$, $i < \mathrm{cf}(\mu)$

  (*ii*) *If* $\bar C = \langle C_\zeta : \zeta < \mathrm{cf}(\mu)\rangle$, $\theta^\alpha_\zeta \cap C_\zeta$ *a club of* θ^α_ζ *for* $\alpha < \lambda$, $\zeta < \mathrm{cf}(\mu)$, <u>*then*</u> *for some* $\alpha^* = \alpha^*_{\bar C}$ *we have*

$$\alpha \in [\alpha^*, \lambda) \Rightarrow (\forall^J \zeta < \mathrm{cf}(\mu))[\eta_\alpha(\zeta) \in C_\zeta].$$

(γ) *Assume* $\langle A_\varepsilon : \varepsilon < \varepsilon^* \rangle$ *is a partition of* $\mathrm{cf}(\mu)$ *to sets not in* J. *Then we can add*

(*ii*)$^+$ *For any sequence of functions*

$$F = \langle F_\zeta : \zeta < \mathrm{cf}(\mu) \rangle,$$

for some $\alpha^* = \alpha^*_{\bar F}$, *for every* $\alpha \in [\alpha^*, \lambda)$ *we have*

$(*)$ *if* $\varepsilon < \varepsilon^*$, $n < \omega$, $\beta_\ell < \alpha$ *for* $\ell < n$ *then*

$$\zeta < \mathrm{cf}(\mu) : F_\zeta(\ldots, \beta_\ell, \eta_{\beta_\ell}, \ldots, \eta_\alpha \restriction \bigcup_{\xi < \varepsilon} A_\zeta) \cap \theta^\alpha_\zeta$$

is a club of θ^α_ζ *but*

$$\eta_\alpha(\zeta) \notin F_\zeta(\beta_\ell, \eta_{\beta_\ell}, \ldots, \eta_\alpha \restriction \bigcup_{\xi < \varepsilon} A_\xi) \cap \theta^\alpha_\zeta$$

belongs to J. *(If we use constant* F *this reduces to (ii)).*

Proof of clause (α).

First choose $\langle \lambda^0_\alpha : \alpha < \lambda \rangle$ as demanded in clause (ii) (but we will manipulate it later, possible by clause (e)). Now as in 9.6, for each α there are

$$J_\alpha, \bar\theta^\alpha = \langle \theta^\alpha_\zeta : \zeta < \mathrm{cf}(\mu) \rangle$$

as there, so satisfying (i), (iii), (iv), (vi).

As $\lambda = \mathrm{cf}(\lambda) > \mu > 2^{\mathrm{cf}(\mu)}$, we can replace $\bar\lambda$ by a subsequence, so without loss of generality $J \subseteq J^*$, so J^* is $\mathrm{cf}(\mu)$-complete and $\bar\theta^\alpha$ is $<_J$-increasing, see 7.1. So $\langle \bar\theta^\alpha : \alpha < \lambda \rangle$ has $<_{J^*}$-exact upper bound $\bar\theta^*$, without loss of generality

$$\bigwedge_{\alpha,\zeta} \theta^\alpha_\zeta < \theta^*_\zeta.$$

So clause (v) holds.

<u>Note</u>: If $\mathrm{cf}(\mu) > \aleph_0$ we have $J = J^{bd}_\mu$.

<u>Proof of (β) + (γ)</u>: (Here $\mathrm{cf}(\mu)$ can be replaced by any $\delta \le \mu$ such that $\mathrm{cf}(\delta) = \mathrm{cf}(\mu)$.)

List all relevant $\bar F = \langle F_\zeta : \zeta < \delta \rangle$ with values subsets of μ. So there are $\le 2^\mu$ of them, list them as $\langle \bar F^i : i < 2^\mu \rangle$ with

$$\bar{F}^i = \langle F^i_\zeta : \zeta < \delta \rangle.$$

We choose $\eta_\alpha \in \prod_{\zeta < \operatorname{cf}(\mu)} \theta^*_\zeta$ by induction on α.

For a given $\alpha < \lambda$ we choose $\eta_\alpha \restriction A_\varepsilon$ by induction on $\varepsilon < \varepsilon^*$. We will choose $\eta_\alpha \restriction A_\varepsilon$ such that

(∗) if $n < \omega$, $\beta_0, \beta_1, \ldots, \beta_{n-1} < \alpha$ and $i < \sup\{\lambda_\beta : \beta < \alpha\}$ (necessarily $< \lambda_\alpha$),

$$\zeta \in A_\varepsilon : F^i_\zeta(\ldots \beta_\ell, \eta_{\beta_\ell}, \ldots, \eta_\alpha \restriction \bigcup_{\xi<\varepsilon} A_\xi) \cap \theta^\alpha_\zeta$$

$$\text{is a club of } \theta^\alpha_\zeta \text{ but } \eta_\alpha(\zeta) \text{ does not belong to it } \in J.$$

But in 9.1's proof we have shown that this is possible. $\square_{10.1}$

* * * * * * * * *

We have conclusions variants similar to the case 2^μ is regular.

§11 Preliminaries to the construction of ccc Boolean algebras with no large independent sets

Monk [M2] asks:

Problem 33. Assume $\mathrm{cf}(\mu) \le \kappa < \mu < \lambda \le \mu^{\mathrm{cf}(\mu)}$. Is it possible in ZFC that there is a Boolean algebra of cardinality λ, satisfying the κ-cc with no independent subset of cardinality λ?

This is closely related to the problem of "is λ a free caliber of such Boolean algebra" (see also in Monk [M2]).

Why in ZFC? Because of earlier results under "μ strong limit, $2^\mu = \mu^+$", I think.

The real problem seems to me is for λ regular, and we shall prove that "almost always" there is such a Boolean algebra, so we prove the consistency of failure.

We shall use $\langle J^{bd}_{\langle \lambda_{i,0},\lambda_{i,1}\rangle} : i < \delta\rangle$ with regular $\lambda_{i,0} > \lambda_{i,1}$, but we use Boolean algebras whose existence is only consistent.

So we shall use $\bar\eta$ a (λ, J)-sequence for $\langle J^{bd}_{\langle \lambda_{i,0},\lambda_{i,1}\rangle} : i < \delta\rangle$, if $\delta = \omega$ the Boolean algebra $\mathbf{B}$ will have a dense subalgebra $\mathbf{B}^*$ which will be the free product of $\{\mathbf{B}_n : n < \omega\}$, $x^-_t, x^+_t \in \mathbf{B}_n$ for $t \in \mathrm{Dom}(I_n)$ and $\mathbf{B} = \langle \mathbf{B}^*, y_\alpha : \alpha < \lambda\rangle$ where $y_\alpha \in$ completion of $\mathbf{B}^*$ is defined from $\langle x^-_{\eta_\alpha(n)}, x^+_{\eta_\alpha(n)} : n < \omega\rangle$. We need special properties of $\mathbf{B}_n$, x^-_t, x^+_t ($t \in \mathrm{Dom}(I_n)$). The construction continues [RoSh 534, §3]. Concerning the parallel to 6.16 see later.

For the case μ strong limit we can use instead subalgebras of the measure algebra. See §2. Now we have consistency (and independence) for λ, $\mu < \lambda \le 2^\mu$, μ strong limit singular, hence we concentrate on the other case where the behavior is different i.e. when for some χ we have $\mathrm{cf}(\mu) \le \kappa < \chi = \chi^{<\kappa} < \mu < \lambda < \mu^{\mathrm{cf}(\mu)} \le 2^\chi$. The proof here uses ideals which are "easier" and can be generalized to get "non-n-independent subset of $\mathbf{B}$ of cardinality λ for some specific n". For this we need to start with "there is a λ_n-complete uniform filter $\mathscr{D}_n$ on λ_n^{+n}".

11.1 Definition. We say $(\mathbf{B}_1, \bar x^+, \bar x^-)$ witness $(I, \mathscr{T})$ if

- (a) $\mathscr{T}$ is a set of Boolean terms written as $\tau = \tau(x_1, \dots, x_{n_\tau})$
- (b) I is an ideal
- (c) $\mathbf{B}$ is a Boolean algebra
- (d) $\bar x^+ = \langle x^+_t : t \in \mathrm{Dom}(I)\rangle, x^+_t \in \mathbf{B}$
- (e) $\bar x^- = \langle x^-_t : t \in \mathrm{Dom}(I)\rangle, x^-_t \in \mathbf{B}$
- (f) $x^-_t < x^+_t$
- (g) If $X \in I^+$ and $\mathbf{B} \subseteq \mathbf{B}'$ and

$$\mathbf{B}' \models x^-_t \le y_t \le x^+_t \text{ for } t \in X$$

then for some $\tau(x_1, \dots, x_n) \in \mathscr{T}$ and pairwise distinct $t_1, \dots, t_n \in X$ we have

$$\mathbf{B}' \vDash \tau(y_{t_1}, y_{t_2}, \dots, y_{t_n}) = 0.$$

11.2 Explanation. We think of having $\bar\eta$ a (λ, J)-sequence for $\langle I_i : i < \delta\rangle$, and having $(\mathbf{B}_i, \bar x_i^+, \bar x_i^-)$ witnessing $(I_i, \mathscr{T})$ for $i < \delta$ and using the sequence of intervals $\langle (x^-_{i,\eta_\alpha(i)}, x^+_{i,\eta_\alpha(i)}) : i < \delta\rangle$ as a sequence of approximations for an element x_α of the desired Boolean algebra $\mathbf{B}$ of cardinality λ.

But we may think not only of "$\{x_\alpha : \alpha < \lambda\}$ has no independent subset of cardinality λ" but of other subsets of $\mathbf{B}$. So sometimes we use

11.3 Definition. 1) We say that $(\mathbf{B}, \bar x^-, \bar x^+)$ strongly witnesses $(I, \mathscr{T})$ if: (a)-(f) as before, and

$(g)^+$ If $\mathbf{B} \subseteq \mathbf{B}'$,

$$\mathbf{B}' \vDash x_t^- \le y_t \le x_t^+ \text{for } t \in \text{ Dom}(I),$$

$\langle b_\ell : \ell \le m\rangle$ is a sequence of pairwise disjoint non-zero members of $\mathbf{B}'$, $m < \omega$ and

$$X \in \Big(\prod_{\ell=1}^m I\Big)^+,$$

and $u \subseteq [1, m]$, then we can find n, $\tau(x_1, \dots, x_n) \in \mathscr{T}$ and distinct $\bar t^1, \dots, \bar t^n \in X$, so $\bar t^r = \langle t^r_\ell : \ell = 1, \dots, m\rangle$, such that $\tau(c_{\bar t^1}, \dots, c_{\bar t^n}) = 0$ where

$$c_{\bar t} = b_0 \cup \bigcup_{\substack{\ell \in [1,m] \\ \ell \in u}} (b_\ell \cap y_{t_\ell}) \cup \bigcup_{\substack{\ell \in [1,m] \\ \ell \notin u}} (b_\ell - y_{t_\ell}.)$$

2) We say that $(\mathbf{B}, \bar x^+, \bar x^-)$ witness $(I, \mathscr{T})$ m-strongly if we restrict ourselves to this m. Similarly $[m_1, m_2]$-strongly.

Next we need our specific $(\mathbf{B}, \bar x^-, \bar x^+, I)$. The following is essentially from [Sh 126, p.244-246].

11.4 Claim. *1) If $\mu = 2^\lambda = \lambda^+$, (or just $\mu \nrightarrow [\mu]^2_\mu$) and $2^\mu = \mu^+$, then we can find $\bar F = \langle F_\alpha : \alpha < \mu^+\rangle$ such that:*

$(*)^\mu_{\bar F}(a)$ *$F_\alpha : [\mu]^2 \to \alpha \times \mu$ is one to one*

(b) *If $A \in (J^{bd}_{\langle \mu^+, \mu\rangle})^+$, then for some $(\alpha, i_0), (\alpha, i_1), (\beta, i_2) \in A$ we have $F_\alpha(\{i_0, i_1\}) = (\beta, i_2)$.*

We write this also as

$$F(\{\alpha, i_0\}, \{\alpha, i_1\}) = (\beta, i_2).$$

We can add that for every β *we have* $\mathrm{Rang}(F_\alpha) \cap (\{\beta\} \times \mu)| \leq 1$ *for* $\alpha > \mu$. *We do not strictly distinguish* $\bar{F}$ *from* F.
2) The property $(*)^\mu_{\bar{F}}$ *is preserved by forcing notions which have the* $(3, J^{\mathrm{bd}}_{\langle \mu^+, \mu \rangle})^+$*-c.c. (see 11.6 below).*
3) Let $\mathbf{B} = \mathbf{B}_{\bar{F}}$ *be the Boolean algebra freely generated by*

$$x^+_{\alpha,i} = x^+_{(\alpha,i)}; x^-_{\alpha,i} = x^-_{(\alpha,i)} (\text{ for } (\alpha, i) \in \mu^+ \times \mu)$$

except that $x^-_{\alpha,i} \leq x^+_{\alpha,i}$ *and*

$$x^+_{(\alpha,i)} \cap x^+_{(\alpha,j)} \cap x^+_{F_\alpha(i,j)} = 0.$$

<u>*Then*</u>

(*i*) $(\mathbf{B}, \bar{x}^+, \bar{x}^-)$ *witness* $(J^{\mathrm{bd}}_{\langle \mu^+, \mu \rangle}, \{x_0 \cap x_1 \cap x_2 = 0\})$
(*ii*) $\mathbf{B}$ *satisfies the c.c.c.*

11.5 Remark. On more general Boolean algebras generated by such equations see Hajnal, Juhasz, Szemintklossy [HaJuSz].

11.6 Definition. For an ideal J and forcing notion $\mathbb{P}$, we say that $\mathbb{P}$ satisfies the (n, J)-c.c. if for $\langle p_t : t \in A \rangle$, $A \in J^+$, there is $B \subseteq A$, $B \in J^+$ such that any n conditions in $\{p_t : t \in B\}$ have a common upper bound.

11.7 Fact. If $\mathbb{P}$ is the forcing notion $\mathbb{P}_{\chi,\theta}$ of adding χ Cohens for θ and $\lambda^{<\theta} = \lambda$ then P satisfies (n, J)-c.c. for $n < \omega$, $J = J_{\langle \lambda^{++}, \lambda^+ \rangle}$.

Proof of 11.4. 1) Let $\{A_\alpha : \alpha < \mu^+\}$ list all subsets A of $\mu^+ \times \mu$ of cardinality μ such that for every $\beta < \mu^+$ we have $|A \cap (\{\beta\} \times \mu)| \leq 1$. For every α such that $\mu < \alpha < \mu^+$ choose $H_\alpha : [\mu]^2 \to \alpha$ such that $(\forall X \in [\mu]^\mu)$ $[H''_\alpha([X]^2) = \alpha]$. For each α, choose $F_\alpha(i,j) \in \{\beta^\alpha_{\{i,j\}}\} \times \mu$ by induction on $<^\otimes$, where $\{i,j\} <^\otimes \{i',j'\}$ iff $\max\{i,j\} < \max\{i',j'\} \vee (\max\{i,j\} = \max\{i',j'\}$ & $\min\{i,j\} < \min\{i',j'\})$, with $\beta^\alpha_{i,j}$ with no repetition so that

$$F_\alpha(i,j) \in \alpha \times \mu^+ \backslash \cup \{\{\beta^\alpha_{i',j'}\} \times \mu : \{i',j'\} <^\otimes \{i,j\}\},$$

and if possible

$$F_\alpha(i,j) \in A_{H_\alpha(\{i,j\})},$$

which occurs if $A_{H_\alpha(\{i,j\})} \subseteq \alpha \times \mu$.
2) Trivial. Let $\mathbb{P}$ be the forcing notion. Let $p^* \Vdash$ "$\underset{\sim}{A} \in (J^{bd}_{\langle \mu^+,\mu\rangle})^+$ and it exemplifies a contradiction to $(*)^\mu_{\bar{F}}$". Let $A =: \{(\alpha,i) : p^* \nVdash (\alpha,i) \notin \underset{\sim}{A}\}$. So $A \subseteq \mu^+ \times \mu$ and,

$$p^* \Vdash \text{“}A \supseteq \underset{\sim}{A}, \underset{\sim}{A} \in (J^{\rm bd}_{\langle \mu^+,\mu\rangle})^{+}\text{”},$$

hence

$$A \in (J^{\rm bd}_{\langle \mu^+,\mu\rangle})^+.$$

For $(\alpha,i) \in A$ there is $p_{(\alpha,i)} \geq p^*$ such that

$$p_{(\alpha,i)} \Vdash \text{“}(\alpha,i) \in \underset{\sim}{A}\text{”}.$$

Apply the $(3, J^{\rm bd}_{\langle \mu^+,\mu\rangle})$-cc to $\langle p_{(\alpha,i)} : (\alpha,i) \in A\rangle$, and obtain B as in Definition 11.6. As $B \in (J^{bd}_{\langle \mu^+,\mu\rangle})^+$, by $(*)^\mu_{\bar{F}}$ we can find (α,i_0), (α,i_1), $(\beta,i_2) \in B$ such that

$$F_\alpha(\{i_0,i_1\}) = (\beta,i_2).$$

But by the choice of B there is $q \in \mathbb{P}$ such that

$$q \geq p_{(\alpha,i_0)}, p_{(\alpha,i_1)}, p_{(\beta,i_2)}$$

(hence $q \geq p^*$). So

$$q \Vdash \text{“}(\alpha,i_0),(\alpha,i_1),(\beta,i_2) \in \underset{\sim}{A} \text{ and } F_\alpha(\{i_0,i_1\}) = (\beta,i_2)\text{”}.$$

But this contradicts the assumption on $p^*, \underset{\sim}{A}$.
3) For clause (i), read the definition. For clause (ii): Call $\mathscr{X} \subseteq \mu^+ \times \mu$ <u>closed</u> if $F(t_1,t_2) = t_3$ & $|\{t_1,t_2,t_2\} \cap \mathscr{X}| > 1 \Rightarrow \{t_1,t_2,t_3\} \subseteq \mathscr{X}$. Now

- $(*)$ if $F(t_i,s_i) = r_i$ for $i = 0,1$ <u>then</u> $\{t_0,s_0,r_0\} \cap \{t_1,s_1,r_1\}$ has ≤ 1 or 3 elements.

[Why? As each F_α is one to one and

$$F = \bigcup_{\alpha<\mu^+} F_\alpha \restriction (\{\alpha\} \times [\mu]^2)$$

and

$$\langle \{\alpha\} \times [\mu]^2 : \alpha < \mu^+ \rangle \text{ are pairwise disjoint]}$$

(∗∗) if $\mathscr{X} \subseteq \mu^+ \times \mu$, and $\mathbf{B}_{\mathscr{X}}$ is defined naturally: it is freely generated by $\{x_t^+, x_t^- : t \in \mathscr{X}\}$ except the equations explicitly demanded on those variables, <u>then</u> $\mathbf{B}_{\mathscr{X}} \subseteq \mathbf{B}$ (even if $\mathscr{X}$ is not closed).

[Why? If $f : \{x_t^-, x_t^+ : t \in \mathscr{X}\} \to \{0,1\}$ preserves the equations, and we define

$$f^* : \{x_t^-, x_t^+ : t \in \mu^+ \times \mu\} \to \{0,1\}$$

by

$$f^*(y) =: \begin{cases} f(y) & \text{if} \quad y = x_t^{\pm}, t \in \mathscr{X} \\ 0 & \text{if} \quad y = x_t^{\pm}, t \notin \mathscr{X}, \end{cases}$$

then f^* preserves the equations.]

(∗∗∗) $\mathbf{B} \models$ ccc.

[Why? Let $\langle a_\zeta : \zeta < \omega_1 \rangle$ be a sequence of non-zero elements. We can find finite $\mathscr{X}_\zeta$ such that $a_\zeta \in \mathbf{B}_{\mathscr{X}_\zeta}$. Let $f_\zeta : \mathbf{B}_{\mathscr{X}_\zeta} \to \{0,1\}$ be a homomorphism such that $f_\zeta(a_\zeta) = 1$. Let

$$\mathscr{X}_\zeta^+ =: \mathscr{X}_\zeta \cup \cup \{\{t_1,t_2,t_3\} : F(t_1,t_2) = t_3, \text{ and } \{t_1,t_2,t_3\} \cap \mathscr{X}_\zeta > 1\}.$$

Without loss of generality $\langle \mathscr{X}_\zeta^+ : \zeta < \omega_1 \rangle$ is a Δ-system with heart $\mathscr{X}^+$.

Without loss of generality $f_\zeta \restriction \{x_t^+ : t \in \mathscr{X}^+\}$ is constant.

Without loss of generality $\mathscr{X}_\zeta \cap \mathscr{X}^+$ is constant.

So

$(*)_4$ If $\zeta \neq \xi < \omega_1$

$$F(t_1,t_2) = t_3 \text{ and } \{t_1,t_2,t_3\} \subseteq \mathscr{X}_\xi \cup \mathscr{X}_\zeta,$$

<u>then</u>

$$\{t_1,t_2,t_3\} \subseteq \mathscr{X}_\zeta \text{ or } \{t_1,t_2,t_3\} \subseteq \mathscr{X}_\xi.$$

[Why? Without loss of generality

$$|\{t_1, t_2, t_3\} \cap \mathscr{Z}_\zeta| \geq 2.$$

So

$$\{t_1, t_2, t_3\} \subseteq \mathscr{Z}^+_\zeta.$$

Now if $t_i \in \mathscr{Z}^+_\zeta \backslash \mathscr{Z}_\zeta$, then $t_i \notin \mathscr{Z}_\xi$ (otherwise $t_i \in \mathscr{Z}^+_\zeta \cap \mathscr{Z}^+_\xi$, hence $t_i \in \mathscr{Z}^+$, but $\mathscr{Z}_\zeta \cap \mathscr{Z}^+$ is constant). So $\{t_1, t_2, t_3\} \subseteq \mathscr{Z}_\zeta$.]

Now $f_\zeta \cup f_\xi$ preserves the equations on $\mathscr{Z}_\zeta \cup \mathscr{Z}_\xi$ and by the homomorphism it induces, $a_\zeta \cap a_\xi$ is mapped to 1, so $\mathbf{B}_{Z_\zeta \cup Z_\xi} \models$ "$a_\zeta \cap a_\xi \neq 0$" hence by $(**)$ we have $\mathbf{B} \models$ "$a_\zeta \cap a_\xi \neq 0$".] $\square_{11.4}$

11.8 Fact. Assume

(a) $(\mathbf{B}, \bar{x}^-, \bar{x}^+)$ is a witness for $(I, \mathscr{T})$
(b) $y^-_t = -x^+_t$, $y^+_t = -x^-_t$ for $t \in \operatorname{Dom}(I_i), \bar{y}^- = \langle y^-_t : t \in \operatorname{Dom}(I)\rangle$, $\bar{y}^+ = \langle y^+_t : t \in \operatorname{Dom}(I)\rangle$
(c) $\mathscr{T}' = \{-\tau(-x_0, \dots, -x_{n-1}) : \tau(x_0, \dots, x_{n-1}) \in \mathscr{T}\}$.

Then $(\mathbf{B}, \bar{y}^-, \bar{y}^+)$ is a witness for $(I, \mathscr{T}')$ (and is called the dual of $(\mathbf{B}, \bar{x}^-, \bar{x}^+)$).

We may consider

11.9 Definition. 1) Let $(*)^\mu_{\bar{F},\bar{H}}$ mean

(a) $\bar{F} = \langle F_\alpha : \alpha < \mu^+\rangle, F_\alpha$ is a partial function from $[\mu]^2$ into $\alpha \times \mu$
(b) $\bar{H} = \langle H_\alpha : \alpha < \mu^+\rangle, H_\alpha$ is a partial function from $[\mu]^2$ into $\{0, 1\}$
(c) if $A \in (J^{\mathrm{bd}}_{\langle\mu^+,\mu\rangle})^+$ and $\ell < 2$ then for some (α, i_0), $(\alpha, i_1) \in A$ we have $F_\alpha(i_0, i_1) \in A$ and $H_\alpha(i_0, i_1) = \ell$
(d) the Boolean algebra $\mathbf{B}_{\bar{F},\bar{H}}$ defined below satisfies the c.c.c. We may write $F =: \bigcup_{\alpha<\mu^+} F_\alpha$, $H =: \bigcup_{\alpha<\mu^+} H_\alpha$ instead of $\bar{F}$, $\bar{H}$ respectively.

2) $\mathbf{B}_{\bar{F},\bar{H}}$ is the Boolean algebra generated freely by $\{x^-_t, x^+_t : t \in \mu^+ \times \mu\}$ except that $x^-_t \leq x^+_t$ and $x^+_{t_0} \cap x^+_{t_1} \cap x^+_{t_2} = 0$ when $F(t_0, t_1) = t_2$, $H(t_0, t_1) = 0$ and $(-x^-_{t_0}) \cap (-x^-_{t_1}) \cap (-x^-_{t_2}) = 0$ when $F(t_0, t_1) = t_2$, $H(t_0, t_1) = 1$.

11.10 Remark. Of course, $\mathbf{B}_{\bar{F},\bar{H}}$ is defined from two sets of triples, which are disjoint and no distinct two have > 1 element in common.

11.11 Claim. . *Assume $(*)^\mu_{F_0}$ of 11.4(1) and e.g., $\mu = \lambda^+$, $\lambda^{<\theta} = \lambda$.*
1) For some $(\theta^{<\theta})^+$-c.c., θ-complete, forcing notion P of cardinality $\le \mu^+$ we have

$$\Vdash_P \text{“}(*)^\mu_{F,H} \textit{ for some } F, H\text{”}.$$

2) If $(*)^\mu_{F,H}$ and $\mathbb{Q}$ is a forcing notion satisfying the $(3, J^{bd}_{\langle \mu^+,\mu\rangle})$-c.c. then in $\mathbf{V}^{\mathbb{Q}}$ we have $(*)^\mu_{F,H}$. If $V = V_0^{\mathbb{P}}, \mathbb{P}$ as above it is enough that $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ satisfies the $(3, J^{\rm bd}_{\langle \mu^+,\mu\rangle})$-c.c.

Proof. 1) Let

$$\begin{aligned} P = \{(f,h) :& \text{for some } u = u_{(f,h)} \subseteq \mu^+ \times \mu \text{ of cardinality } < \theta \text{ we have :} \\ & f, h \text{ are partial functions, } \mathrm{Dom}(f) = \mathrm{Dom}\,(h) \subseteq (\mathrm{Dom}\, F) \cap [u]^2, \\ & f \subseteq F_0 \text{ and } \mathrm{Rang}(h) \subseteq \{0,1\} \text{ and } \mathbf{B}_{f,h} \text{ satisfies the c.c.c.}\} \end{aligned}$$

where $\mathbf{B}_{f,h}$ is defined as in 11.9(2) (and see 11.10).
The order $(f_1, h_1) \le (f_2, h_2)$ iff

(*i*) $u_{(f_1,h_1)} \subseteq u_{(f_2,h_2)}$,
(*ii*) $f_1 = f_2 \restriction [u_{(f_1,h_1)}]^2$
(*iii*) $h_1 = h_2 \restriction [u_{(f_1,h_1)}]^2$
(*iv*) $\mathbf{B}_{(f_1,h_1)} \subseteq \mathbf{B}_{(f_2,h_2)}$ moreover $\mathbf{B}_{(f_1,h_1)} \lessdot \mathbf{B}_{(f_2,h_2)}$.

The reader can check $\square_{11.11}$

11.12 Claim. *Assume $2^{\lambda^{+\ell}} = \lambda^{+\ell+1}$ for $\ell < n$ and let $\lambda_\ell = \lambda^{n-\ell+1}$*

(1) *We can find W such that*
   (*a*) $W \subseteq \Big[\prod\limits_{\ell<n} \lambda_\ell\Big]^n$
   (*b*) *if $u_1 \ne u_2$ belongs to W then $|u_1 \cap u_2| \le 1$*
   (*c*) *if $A \in (J^{\rm bd}_{\langle \lambda_\ell:\ell<n\rangle})^+$ then $[A]^n \cap W \ne \emptyset$*
   (*d*) *$\langle \lambda_\ell : \ell < n\rangle$ is a decreasing sequence of regulars*

(2) *there is a forcing notion* $\mathbb{Q}$ *of cardinality* λ^{+n}, λ^+*-complete satisfying the* λ^+*-c.c. and even the* $(n, J^{\mathrm{bd}}_{\langle\lambda_\ell:\ell<m\rangle})$*-c.c. and adding* W *satisfying (a), (b), (c) of part 1 and*

(e) W *is locally finite: if* $A \subseteq \prod\limits_{\ell<n} \lambda_\ell$ *is finite, then for some finite* B, $A \subseteq B \subseteq \prod\limits_{\ell<n} \lambda_\ell$ *and* $w \in W$ *&* $|w \cap B| \geq 2 \Rightarrow\ w \subseteq B$

(3) *if* P *is adding* χ *many* θ*-Cohen reals,* $\lambda = \lambda^\theta$ *and in* V, $\overline{W}$ *satisfies (a), (b), (c), (d) and (e), then in* V^P *still clause (c) holds (and trivially the other demands on* W*). (See [Sh 126].)*

Proof. 1) We prove by induction on n that for any such λ satisfying $\ell < n \Rightarrow 2^{\lambda^{+\ell}} = \lambda^{+(\ell+1)}$ we can find (W, F) such that (a), (b), (c) of 11.11(1) hold for W, $\langle\lambda^{+(\ell+1)} : \ell < n\rangle$ and

(f) $F : W \to \lambda^+$ satisfies: if $A \in (J^{\mathrm{bd}}_{\langle\lambda^{+(\ell+1)}:\ell<n\rangle})^+$, then $\mathrm{Rang}(F \restriction [A]^n) = \lambda^+$.

The induction step is as in the previous proof.
2) Similar to the proof of 11.11
3) Because $\mathbb{P}$ satisfies the $(n, J^{\mathrm{bd}}_{\langle\lambda_\ell:\ell<n\rangle})$-c.c. $\square_{11.12}$

11.13 Claim. *Assume*

(A) $W, \langle\lambda_\ell : \ell < n\rangle$ *satisfy (a), (b), (c), (d) and (e) of Claim 11.12(1)*

(B) $3 \leq m < n/2$, $n > 6$

(C) $\mathbf{B}$ *is the Boolean algebra generated by* $\{x^-_t, x^+_t : t \in \prod\limits_{\ell<n} \lambda_\ell\}$ *freely except:*

$(*)_1$ $x^-_t \leq x^+_t$

$(*)_2$ *if* $w = \{t_0, \dots, t_{n-1}\} \in W$*, where* t_ℓ *is increasing in the lexicographic order, and* $u \subseteq n$, $|u| \geq m$ *and* $n - |u| > m$*, then*

$$\bigcap_{\ell\in u} x^+_{t_\ell} \cap \bigcap_{\ell<n,\ell\notin u} (-x^-_{t_\ell}) = 0.$$

(D) $\mathscr{T} = \mathscr{T}_{n,m} = \{\bigcap_{\ell\in u} x_\ell \cap \bigcap_{\ell<n,\ell\notin u}(-x_\ell) : u \subseteq n, m \leq |u| \leq n - m\}$.
<u>Then</u>

(i) $\mathbf{B} \models$ "$x^-_t < x^+_t$ & $x^-_s \nleq x^+_t$" *for* $t \neq s$ *in* $\prod\limits_{\ell<n} \lambda_\ell$

(ii) $(\mathbf{B}, \bar{x}^-, \bar{x}^+)$ *is a witness for* $(J^{\mathrm{bd}}_{\prod_{\ell<n} \lambda_\ell}, \mathscr{T})$

(iii) $\mathbf{B}$ *satisfies the ccc.*

Proof. Clearly $\mathbf{B} \vDash x_t^- \leq x_t^+$ by the equation in $(*)_1$ and $\mathbf{B} \vDash$ "$x_t^- \neq x_t^+$" because the function f_0 given by,

$$f_0(x_s^-) = 0, f_0(x_s^+) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases}$$

preserves all the required equations (as $2 \leq m$). Taken together, $\mathbf{B} \vDash x_t^- < x_t^+$. Also $\mathbf{B} \vDash x_t^- \not\leq x_s^+$ when $t \neq s$ using f_1 defined by

$$f_1(x_r^+) = f_1(x_r^-) = \begin{cases} 1 & \text{if } r = t \\ 0 & \text{if } r \neq t \end{cases}$$

So clause (i) of the conclusion holds. Clause (ii) holds easily by the equation in $(*)_2$ and assumption (A) i.e. (c) of 11.12(1).

We are left with verifying clause (iii), i.e., the c.c.c. So let $a_\zeta \in \mathbf{B} \backslash \{0\}$ for $\zeta < \omega_1$. For every ζ we can find a finite set $Z_\zeta \subseteq \prod\limits_{\ell<n} \lambda_\ell$ such that $a_\zeta \in \langle x_t^-, x_t^+ : t \in Z_\zeta \rangle$. By 11.12, i.e., by clause (A), without loss of generality

(∗) if $w \in W$ & $|w \cap Z_\zeta| \geq 2 \Rightarrow w \subseteq Z_\zeta$.

Let $f_\zeta^* : \{x_t^-, x_t^+ : t \in Z_\zeta\} \to \{0,1\}$ be such that it preserves all the equations (from $(*)_1 + (*)_2$) on these variables and so the homomorphism it induces from $\mathbf{B}_{Z_\zeta}$ to $\{0,1\}$, $\hat{f}_\zeta^*$ maps a_ζ to 1. Without loss of generality $\langle Z_\zeta : \zeta < \omega_1 \rangle$ is a Δ-system with heart Z and $f_\zeta^* \restriction \{x_t^-, x_t^+ : t \in Z\}$ is constant.

Let $\zeta(1) < \zeta(2) < \omega_1$ and define f_2

$$f_2(x_t^-) = \begin{cases} f_{\zeta(1)}^*(x_t^-) & \underline{\text{if}}\ t \in Z_{\zeta(1)} \\ f_{\zeta(2)}^*(x_t^-) & \underline{\text{if}}\ t \in Z_{\zeta(2)} \\ 0 & \text{otherwise.} \end{cases}$$

$$f_2(x_t^+) = \begin{cases} f_{\zeta(1)}^*(x_t^+) & \underline{\text{if}}\ t \in Z_{\zeta(1)} \\ f_{\zeta(2)}^*(x_t^+) & \underline{\text{if}}\ t \in Z_{\zeta(2)} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly it is well defined and with the right domain. Does f_2 preserve all the equations?

Case 1. $x^-_t \leq x^-_t$ if $t \notin Z_{\zeta(1)} \cup Z_{\zeta(2)}$ trivial (both get value zero), and if $t \in Z_{\zeta(\ell)}$ then trivial (as $f^*_{\zeta(\ell)}$ preserves this equation).

Case 2. $\bigcap_{\ell \in u} x^+_{t_\ell} \cap \bigcap_{\substack{\ell<n \\ \ell \notin u}} (-x^-_{t_\ell}) = 0$.

If $\ell \in \{1,2\}$ and $\{t_0,\dots,t_{n-1}\} \subseteq Z_{\zeta(1)}$ this holds as $f^*_{\zeta(\ell)}$ preserves this equation. So assume this fails for $\ell = 1,2$ so $|\{t_0,\dots,t_{n-1}\} \cap Z_{\zeta(\ell)}| \leq 1$ hence $2 \geq |\{t_0,\dots,t_{n-1}\} \cap (Z_{\zeta(1)} \cup Z_{\zeta(2)})|$ so $\{\ell : t_\ell \notin Z_{\zeta(1)} \cup Z_{\zeta(2)}\}$ necessarily includes members of u, hence the equation holds. $\square_{11.13}$

11.14 Comment. 1) If in addition we have κ-complete maximal ideals $I_{n,\ell}$ on $\lambda_{n,\ell}$ extending $J^{\rm bd}_{\lambda_{n,\ell}}$ and $\langle \lambda_{n,\ell} : \ell < n\rangle$ as above for $\bar\eta$ a (λ, J)-sequence, e.g., for $\langle I^*_n : n < \omega\rangle$ where $I^*_n = \prod J_{\langle \lambda_{n,\ell}:\ell<n\rangle}$, we are in a powerful situation as it can be applied to n-tuples rather than each one separately. But above we prepare the proof for not using it by having strong equations.
2) We can waive the "locally finite" demand proving as in the proof of $(***)$ in the proof of 11.4.

§12 Constructing c.c.c. Boolean Algebras with no large independent sets

On such constructions see Rosłanowski Shelah [RoSh 534, §3].

12.1 Construction's Hypothesis. We assume

(a) $\bar\eta$ is a normal (λ, J)-sequence for $\langle I_i : i < \delta\rangle$

(b) $(\mathbf{B}_i, \bar x_i^-, \bar x_i^+)$ is a witness for $(I_i, \mathscr{T}_i)$, $\|\mathbf{B}_i\| = |\operatorname{Dom}(I_i)|$

(c) $\lambda = \operatorname{cf}(\lambda)$, $\sum\limits_{i<\delta} |\operatorname{Dom}(I_i)| < \lambda$

12.2 Remark. Actually $\mathscr{T}_i$ do not influence the construction, only the properties of the Boolean algebra constructed. Similarly, the normality and the fact that $\|\mathbf{B}_i\| = |\operatorname{Dom}(I_i)|$, as well as clause (c).

We define a Boolean algebra $\mathbf{B}$ and $y_\alpha \in \mathbf{B}$ $(\alpha < \lambda)$ as follows:

12.3 The construction.

Case 1. $\delta = \omega$.

Let $\mathbf{B}_*$ be the free product of $\{\mathbf{B}_i : i < \delta\}$ (so $\mathbf{B}_n = *_{i<n}\mathbf{B}_i$, $\mathbf{B}^n \subseteq \mathbf{B}^{n+1} \subseteq \mathbf{B}_*$, so $\mathbf{B}_* = \langle \bigcup\limits_{n<\omega} \mathbf{B}_n\rangle_{\mathbf{B}_*}$).

Let $\mathbf{B}_*^c$ be the completion of $\mathbf{B}_*$.

For each $i < \delta$ and $\eta \in \{\eta_\alpha \restriction i : \alpha < \lambda\} \subseteq \prod\limits_{j<i} \operatorname{Dom}(I_j)$ we define $y_\eta^- < y_\eta^+$ in $\mathbf{B}^i$. This is done by induction on i.

$\underline{i = 0}$. $y_\eta^- = 0$, $y_\eta^+ = 1$.

$\underline{i = j+1}$. $y_\eta^- = y^-_{\eta\restriction j} \cup (y^+_{\eta\restriction j} \cap x^-_{i,\eta(j)})$, $y_\eta^+ = y^-_{\eta\restriction j} \cup (y^+_{\eta\restriction j} \cap x^+_{i,\eta(j)})$.

So easily

$$j < i \quad\Rightarrow\quad y^-_{\eta_\alpha\restriction j} \le y^-_{\eta_\alpha\restriction i} < y^+_{\eta_\alpha\restriction i} \le y^+_{\eta_\alpha\restriction j}.$$

Now let y_α be $\operatorname{lub}\{y^-_{\eta_\alpha\restriction i} : i < \delta\}$. (Note: If $\mathbf{B}_i \models$ "$0 < x^-_{i,t} < x^+_{i,t} < 1$" for $t \in \operatorname{Dom}(I_i)$, then also $y_\alpha =$ maximal lower bound of $\{y^+_{\eta_\alpha\restriction i} : i < \delta\}$. This will not be used.)

[Otherwise, the difference contains some member of $\mathbf{B}_*$, hence of some $\mathbf{B}^i$ $(i < \delta)$, but there is none.]

Lastly $\mathbf{B} = \mathbf{B}_{\bar\eta, \bar I, \langle(\mathbf{B}_i, \bar x_i^-, \bar x_i^+): i<\delta\rangle}$ is the subalgebra of $\mathbf{B}_*^c$ generated by $\mathbf{B}_* \cup \{y_\alpha : \alpha < \lambda\}$ (by the finitary operations, so it is not complete).

Case 2. $\delta > \omega$.

We find by induction on $i < \delta$, $\mathbf{B}^i$, $\{(y^-_\eta, y^+_\eta) : \eta \in \{\eta_\alpha \restriction i : \alpha < \lambda\}\}$ such that

(*i*) $\mathbf{B}^i$ increasing (by $\subseteq$, even $\lessdot$)

(*ii*) $\mathbf{B}^i \models y^-_\eta < y^+_\eta$ (when $\bigvee_\alpha \eta = \eta_\alpha \restriction i$)

$$j < i \Rightarrow \mathbf{B}^i \models y^-_{\eta\restriction j} \le y^-_\eta \le y^+_\eta \le y^+_{\eta\restriction j}.$$

(*iii*) $\mathbf{B}^0$ is the trivial Boolean algebra

(*iv*) if $i = j+1$ then $\mathbf{B}^i = \mathbf{B}^j * \mathbf{B}_j$ (free product) and for $y \in \{\eta_\alpha \restriction i : \alpha < \lambda\}$

$$y^-_\eta = y^-_{\eta\restriction j} \cup (y^+_{\eta\restriction j} \cap x^-_{j,\eta(j)})$$

$$y^+_\eta = y^-_{\eta\restriction j} \cup (y^+_{\eta\restriction j} \cap x^+_{j,\eta(j)})$$

(*v*) For i limit, $\mathbf{B}^i$ is generated freely by

$$\bigcup_{j<i} \mathbf{B}^j \cup \{y^-_\eta, y^+_\eta : \eta \in \{\eta_\alpha \restriction i : \alpha < \lambda\}$$

except: the equations in $\mathbf{B}$ and

$$y^-_{\eta\restriction j} \le y^-_\eta \le y^+_\eta \le y^+_{\eta\restriction j} \text{ for } j < i, \eta \text{ as above.}$$

Lastly, $\mathbf{B} \subseteq$ completion $\left(\bigcup_{i<\delta} \mathbf{B}^i\right)$ is defined as in case 1 using $y_\alpha =: y^-_{\eta_\alpha}$.

12.4 The construction. A variant

$$\bar{x}^\pm_i = \langle x^\pm_{i,\eta} : \eta \in \{\eta_\alpha \restriction (i+1) : \alpha < \lambda\}\rangle$$

so we use $x_{i,\eta_\alpha\restriction(i+1)}$ instead of $x_{i,\eta_\alpha(i)}$.

12.5 The construction. A variant. It is like 12.4 but we are given $(\mathbf{B}^\pm_i, \langle x_{i,\alpha} : \alpha < \lambda_i\rangle)$ and we define by induction on i, $\mathbf{B}^i$ (increasing with i), and follows:

Case 1: $i = 0$: $\mathbf{B}^i$ is the trivial Boolean algebra, $y^-_{\eta_\alpha\restriction i} = 0$, $y^+_{\eta_\alpha\restriction i} = 1$.

Case 2: $i = j+1$: $\mathbf{B}^i$ is generated by $\mathbf{B}^j \cup \{x^-_{\eta_\alpha\restriction i}, x^+_{\eta_\alpha\restriction i} : \alpha < \lambda\}$ freely except the equations in $\mathbf{B}^j$ and

$$\tau(\dots x^-_{\eta_{\alpha_\ell}\restriction i}, x^+_{\eta_{\alpha_\ell}\restriction i}, \dots)_{\ell<n} = 0$$

whenever $\mathbf{B}_i \vDash \tau(\dots, x^-_{\eta_{\alpha_\ell}(j)}, x^+_{\eta_{\alpha_\ell}(j)}, \dots)_{\ell<n} = 0$; lastly defines

$$y^-_{\eta_\alpha\restriction i} = y^-_{\eta_\alpha\restriction j} \cup \left(y^+_{\eta_\alpha\restriction j} \cap x^-_{j,\eta_\alpha(j)}\right)$$

$$y^+_{\eta_\alpha\restriction i} = y^-_{\eta_\alpha\restriction j} \cup \left(y^+_{\eta_\alpha\restriction j} \cap x^-_{j,\eta_\alpha(j)}\right)$$

Case 3: i limit

$\mathbf{B}^i$ is generated by $\bigcup_{j<i} \mathbf{B}^j \cup \{y^-_{\eta_\alpha\restriction i}, y^+_{\eta_\alpha\restriction i} : \alpha < \lambda\}$ freely except the equations in $\mathbf{B}^j$ for $j < i$ and $y^-_{\eta_\alpha\restriction j} \le y^-_{\eta_\alpha\restriction i} \le y^+_{\eta_\alpha\restriction i} \le y^+_{\eta_\alpha\restriction j}$ for $\alpha < \lambda$.

12.6 Comment. Clearly 12.4 includes 12.5 as a special case, but mostly there is no real difference in the uses. The reader may concentration on 12.5.

12.7 Discussion. Usually the conclusions are of the form: among any λ elements of $\mathbf{B}$, something occurs. The first need is $\|\mathbf{B}\| = \lambda$, a trivial thing.

12.8 Fact. $(*)_3 \Rightarrow (*)_2 \Rightarrow (*)_1$, where

$(*)_1$ $\|\mathbf{B}\| = \lambda$

$(*)_2$ for every $\alpha < \beta < \lambda$

$$\{i : \mathbf{B}_i \vDash \neg(\exists y)(x^-_{\eta_\alpha(i)} \le y \le x^+_{\eta_\alpha(i)} \wedge x^-_{\eta_\beta(i)} \le y \le x^+_{\eta_\beta(i)})\} \neq \emptyset$$

i.e.

$$\{i : \mathbf{B}_i \vDash x^-_{\eta_\alpha(i)} \le x^+_{\eta_\beta(i)} \vee x^-_{\eta_\beta(i)} \not\le x^+_{\eta_\alpha(i)}\} \neq \emptyset$$

$(*)_3$ if $t \neq s$ are in $\mathrm{Dom}(I_i)$ for some $i < \delta$, <u>then</u>

$$\mathbf{B}_i \vDash x^-_t \not\le x^+_s \vee x^-_s \not\le x^+_t.$$

Proof. Easy. $\square_{12.7}$

12.9 Remark. If not said otherwise, all examples satisfy $(*)_3$.

We will also be interested in stronger properties. In section 15 we will be interested in the case $(\mathbf{B}, \bar{x}^-, \bar{x}^+)$ the pairs (x^-_η, x^+_η), (x^-_ν, x^+_ν) were independent.

12.10 Claim. *Assume*

(∗) $a_\alpha \in \mathbf{B}$ *for* $\alpha < \lambda$.

Then we can find in $\mathbf{B}$ *a sequence* $\langle b_\ell : \ell \le m\rangle$ *a* $\mathbf{B}$*-partition of* 1 *(i.e., a sequence of disjoint non-zero elements with union 1),* $m \ge 0$, *and* $X \in [\lambda]^\lambda$ *and* $c \le b_0$ *in* $\mathbf{B}$ *and* n, *and Boolean terms* τ_ℓ *for* $\ell = 1, \dots, m$ *with* n *variables and ordinals* $\gamma_{\alpha,k} \in X$ *for* $\alpha \in X$, $k < n$ *and* γ_k *for* $k \in [n, n^*)$, *where* $n^* \ge n$ *and* $i^* < \delta$, ν_k *for* $k < n^*$ *such that*

(i) $n = 0$ *iff* $m = 0$ *iff* $\langle a_\alpha : \alpha \in X\rangle$ *constant*

(ii) $\gamma_{\alpha,0} < \gamma_{\alpha,1} < \dots < \gamma_{\alpha,n-1}$ *and* $\gamma_n < \gamma_{n+1} < \dots < \gamma_{n^*-1} < \gamma_{\alpha,0}$

(iii) *if* $\alpha < \beta$ *are in* X *then* $\gamma_{\alpha,n-1} < \gamma_{\beta,0}$

(iv) *if* $\alpha \in X$ *then* $a_\alpha \le \bigcup_{\ell \le m} b_\ell, a_\alpha \cap b_0 = c$ *and* $[\ell \in [1,m] \Rightarrow a_\alpha \cap b_\ell = \tau_\ell(y_{\gamma_{\alpha,0}}, y_{\gamma_{\alpha,1}}, \dots, y_{\gamma_{\alpha,n-1}})]$, *and* $[\ell \in [1,m] \Rightarrow 0 < a_\alpha \cap b_\ell < b_\ell]$ *(so* τ_ℓ *non-trivial)*

(v) $\eta_{\gamma_{\alpha,k}} \restriction i^* = \nu_k$ *for* $k < n$

(vi) $\{b_\ell : \ell \le m\} \subseteq \langle \mathbf{B}_i \cup \{y_{\gamma_k} : k \in [n,n^*)\}\rangle$ *and* $\eta_{\gamma_k} \restriction i^* = \nu_k$ *for* $k \in [n,n^*)$

(vii) $\langle \nu_k : k < n^*\rangle$ *is with no repetition.*

Proof. By the Δ-system lemma and Boolean algebra manipulation. $\square_{12.10}$

12.11 Claim. . *A sufficient condition to*

$\otimes_0$ $\mathbf{B}$ *has no independent subset of cardinality* λ *is*

$\otimes_1$ *if* a_α, X, n, m, τ_ℓ, $\gamma_{\alpha,k}$, $b_\ell(\alpha \in X,\ k < n,\ \ell < n)$ *are as above in 12.10, and* $c_0 = 0$, $m = 1$, *then* $\{a_\alpha : \alpha \in X\}$ *is not independent, which follows from:*

$\otimes_2$ *if* a_ℓ, X, n, m, τ_ℓ, $\gamma_{\alpha,k}$ $(\alpha \in X, k < n)$ *are as above in 12.10,* $c_0 = 0$, $m = 1$, *then*

> *for every* $A, \mathbf{B}'$ *and* y_t, *if* $A \in ((I_i)^n)^+, \mathbf{B}_i \subseteq \mathbf{B}', \mathbf{B}' \models x_t^- \le y_t \le x_t^+$ *for* $t \in$ *Dom*(I_i), *then* $\langle \tau_1(y_{t_0}, \dots, y_{t_{n-1}}) : \langle t_0, \dots, t_{n-1}\rangle \in A\rangle$ *is not strongly independent*

12.12 Remark. If we ask more on $\bar\eta$, we can weaken $\otimes_2$, like:
if $n < \omega$, $\langle \gamma_{\alpha,k} : k < n\rangle$ increasing $\alpha < \beta \Rightarrow \gamma_{\alpha,n-1} < \gamma_{\alpha,0}$, then letting $\eta'_\alpha = \langle\langle \eta_{\gamma_{\alpha,k}}(i) : k < n\rangle : i < \delta\rangle$, gives that $\bar\eta' = \langle \eta'_\alpha : \alpha < \lambda\rangle$ is a (λ, J)-sequence for $\langle (I_i)^n : i < \delta\rangle$ as well as some weaker versions.

Proof of 12.11. $\underline{\otimes_1 \Rightarrow \otimes_0}$.

We choose by induction on $\ell \le m$ a sequence $\langle (\tau_\ell, \gamma^\ell_{\alpha,0}, \dots, \gamma^\ell_{\alpha,m(\ell)-1}) : \alpha < \lambda \rangle$ such that

(*i*) $\tau_\ell = \tau_\ell(x_i, \dots, x_{m(\ell)-1})$ is a Boolean term

(*ii*) $\gamma^\ell_{\alpha,0} < \gamma^\ell_{\alpha,1} < \dots < \gamma^\ell_{\alpha,m(\ell)-1} < \lambda$

(*iii*) $\alpha < \beta < \lambda \Rightarrow \gamma^\ell_{\alpha,m(\ell)-1} < \gamma^\ell_{\beta,0}$ when they are well defined

(*iv*) $\tau_\ell(a_{\gamma^\ell_{\alpha,0}}, \dots, \gamma^\ell_{\alpha,m(\ell)-1}) \cap \bigcup_{\ell_1 \le \ell} b_{\ell_1} = 0$.

<u>For $\ell = 0$</u>: Let $\tau_\ell(x_0, x_1) = x_0 - x_1$, so $m(\ell) = 2$

$$\gamma^0_{\alpha,0} = 2\alpha, \qquad \gamma^\ell_{\alpha,1} = 2\alpha + 1.$$

<u>For $\ell + 1$</u>. For each $\alpha(*) < \lambda$, apply $\otimes_1$ with $1 - b_{\ell+1}, b_{\ell+1}, \langle a^\ell_{\alpha(*)+\alpha} : \alpha < \lambda \rangle$, where $a^\ell_\alpha =: \tau^\ell_\alpha(a_{\gamma^\ell_{\alpha,0}}, \dots, a_{\gamma^\ell_{\alpha,m(\ell)-1}})$ here standing for $b_0, b_1, \langle a_\alpha : \alpha < \lambda \rangle$ there, and get a Boolean term $\tau^{\ell+1}_{\alpha(*)}(x_0, \dots, x_{m(\ell+1,\alpha(*))-1})$, and ordinals $\beta^\ell_{\alpha^*(*),0} < \dots < \beta^\ell_{\alpha(*),m(\ell+1,\alpha(*))-1}$, all in the interval $[\alpha(*), \lambda)$, such that

$$\tau^{\ell+1}_{\alpha(*)}(a^\ell_{\beta^\ell_{\alpha(*),0}}, a^\ell_{\beta^\ell_{\alpha(*),1}}, \dots, a^\ell_{\beta^\ell_{\alpha(*),m(\ell+1,\alpha(*)-1)}}) = 0.$$

Let $X \in [\lambda]^\lambda$ be such that

(*a*) $\alpha \in X \Rightarrow \tau^{\ell+1}_\alpha = \tau^*_\ell$, $m(\ell,\alpha) = m(\ell,*)$

(*b*) X is thin enough, i.e. if $\alpha < \beta$ are in X then $\beta^\ell_{\alpha,0}, \dots, \beta^\ell_{\alpha,m(\ell,*)} < \beta$.

Now if ε is the ζ-th element of X we let

$$u^{\ell+1}_\zeta = \{\gamma^\ell_{\beta,m} : m < m(\ell) \text{ and } \beta \in \{\beta^\ell_{\varepsilon,0}, \dots, \beta^0_{\varepsilon,m(\ell,*)-1}\}\}.$$

So $|u^{\ell+1}_\zeta| = m(\ell) \times m(\ell,*)$ let $m(\ell+1) = m(\ell) \times m(\ell,*)$ let $\gamma^{\ell+1}_{\zeta,0} < \gamma^{\ell+1}_{\zeta,1} < \dots < \zeta^{\ell+1}_{\zeta,m(\ell+1)-1}$ list $u^{\ell+1}_\zeta$, and it should be clear what is $\tau_{\ell+1}$. For $\ell = m$ we have finished.

$\underline{\otimes_2 \Rightarrow \otimes_1}$. Straight. $\square_{12.11}$

12.13 Fact. 1) In 12.10 we can add (so in $\otimes_2$ of 12.11 we can assume) that

(*viii*) $\tau_1(x_0, \dots, x_{n-1}) \in \{x_k, -x_k : k < n\}$ <u>if</u>

- $(*)$ for a set of $i < \delta$ from J^+ we have $\langle x^+_{i,t} - x^-_{i,t} : t \in \operatorname{Dom}(I_i) \rangle$ is a sequence of pairwise disjoint (nonzero) elements of $\mathbf{B}_i$.

2) Assume

$(*)^+$ for every $i < \delta$ we have $\langle x^+_{i,t} - x^-_{i,t} : t \in \mathrm{Dom}(I_i)\rangle$ is a sequence of pairwise disjoint (non zero) elements of $\mathbf{B}_i$.

Then

(a) in 12.10 above we can add:

$$b_0, \dots, b_m = \bigcup_{i<\delta} B^i.$$

(b) Under 12.5 we can add: for $k \in [1, m)$, if i is large enough, if $\alpha_0, \dots, \alpha_{n-1} \in X$ letting b^ℓ_k be the projection of b_α in $\mathbf{B}^{i+1}$ (i.e. any element b satisfying

$$(\forall x \in \mathbf{B}^{i+1})(x \le b_k \to b \le b_k, x \ge b_k \to b \ge b_k)$$

(there is a minimal and maximal such b^i_k and they are in $\langle \mathbf{B}^i \cup \{\rho : \rho = f_\alpha \restriction (x+1)$ for some $i, \neg(\nu \triangleleft \rho)\}\rangle$), $f_{\alpha_\ell} \restriction i = f_{\alpha_0} \restriction i$, $\langle f_{\alpha_\ell}(i) : \ell \le s\rangle$ is with no repetitions and $\tau(x_0, \dots, x_{s-1})$ is a Boolean term then

$$\mathbf{B} \restriction b_k \models \tau(b_k \cap y_{\alpha_0}, \dots, b_k \cap y_{\alpha_{s-1}}) = 0 \Rightarrow$$
$$\mathbf{B}^{i+1} \models \tau(b^i_k \cap y_{\alpha_0}, \dots, b^i_k \cap y_{\alpha_{s-1}}) = 0$$

(we can even be more explicit).

Proof. Straightforward. $\square_{12.13}$

We can now phrase sufficient conditions for having free caliber λ (for $\mathscr{T}$) and for having no $\mathscr{T}$-free subset of $\mathbf{B}$ of cardinality λ.

12.14 Claim. *1) Sufficient conditions for "$\mathbf{B}$ satisfies the κ-c.c." are (κ is regular uncountable and):*

$(*)_1$ *$\delta = \omega$ and each $\mathbf{B}_i$ satisfies the κ-Knaster condition*

$(*)_2$ *each $\mathbf{B}_i$ satisfies the κ-Knaster condition and $(\forall \alpha < \kappa)$ $(|\alpha|^{|\delta|} < \kappa)$*

$(*)_3$ *each $\mathbf{B}_i$ satisfies the κ-Knaster condition, $\kappa > \delta$ and for every $A \in [\lambda]^\kappa$, and limit ordinal $\delta' \le \delta$ for some $B \in [A]^\kappa$ and $i < \delta$ we have*

$$\alpha \in B, \beta \in B,\ \eta_\alpha \restriction \delta' \ne \eta_\beta \restriction \delta' \Rightarrow \ell g(\eta_\alpha \cap \eta_\beta) = i$$

(follows from "$\bar\eta$ is κ^+-free", see 1.20 and Definition 1.21).

12.15 Claim. *Assume*

(*A*) $\bar\eta$ *is a normal* (λ, J)*-sequence for* $\langle I_i : i < \delta\rangle$

(*B*) $(\mathbf{B}_i, \bar x_i^-, \bar x_i^+)$ *is a witness for* $(I_i, \{x_0 \cap x_1 \cap x_2 = 0\})$

(*C*) $\mathbf{B}$ *is as constructed in 12.1, 12.3.*

Then

(*α*) λ *is not a free caliber of* $\mathbf{B}$

(*β*) $\mathbf{B}$ *has cardinality* λ *and satisfies the* κ*-c.c. if* κ *is as in 12.14.*

Proof. Straightforward. $\square_{12.15}$

12.16 Conclusion. Assume for simplicity that $V \models \mathrm{GCH}$, $\theta = \theta^{<\theta} < \chi = \chi^{<\chi}$ and P is the forcing notion of adding χ θ-Cohen subsets of θ, i.e.

$$P = \{f : f \text{ is a partial function from } \chi \text{ to } \{0,1\} \text{ with domain of cardinality } < \theta\}.$$

Then (cardinal arithmetic on V^P is well known) and

(∗) if $\mathrm{cf}(\mu) < \theta < \mu < \chi$ then there is a $(2^{\mathrm{cf}(\mu)})^+$-c.c. Boolean algebra $\mathbf{B}$ of cardinality $\lambda = \mu^+$ such that λ is not a free caliber of $\mathbf{B}$ (and even satisfying the κ-c.c. if κ is as in 12.14).

Proof. Use 12.15 and §11. $\square_{12.16}$

The problem of "$\mathbf{B}$ with no independent subset of cardinality λ" is somewhat harder.

12.17 Claim. *Assume*

(*A*) $\bar\eta$ *is a normal* (λ, J)*-sequence for* $\langle I_i : i < \delta\rangle$

(*B*) $(\mathbf{B}_i, \bar x_i^-, \bar x_i^+)$ *is a witness for* $(I_i, \mathscr{T}_{n_i,m_i})$ *(on* $\mathscr{T}_{n_i,m_i}$ *see 11.13 clause (D))*

(*C*) $3 \le m_i < n_i/2$

(*D*) *for every* $k < \omega$, $\{i : km_i < n_i\} \in J^+$

(*E*) $\mathbf{B}$ *is as in construction 12.1, 12.3.*

Then

(i) $\mathbf{B}$ *does not have a free subset of cardinality* λ

(ii) $\mathbf{B}$ *has cardinality* λ *and satisfies the* κ*-c.c.* κ *is as in 12.14.*

Proof. Straightforward (using the criterion in 12.11). $\square_{12.17}$

12.18 Conclusion. Assume for simplicity $\mathbf{V} \models$ GCH, and $\theta = \theta^{<\theta} < \chi = \chi^{<\chi}$ is the forcing notion of adding χ θ-Cohen reals. Then cardinal arithmetic in $\mathbf{V}^{\mathbb{P}}$ is well known and

$(*)$ if $\mathrm{cf}(\mu) < \theta < \mu < \chi$ then there is a $(2^{\mathrm{cf}(\mu)})^+$-c.c. Boolean algebra $\mathbf{B}$ of cardinality $\lambda = \mu^+$ without an independent subset of cardinality λ

$(**)$ we can demand that $\mathbf{B}$ satisfies the $\mathrm{cf}(\mu)^+$-c.c. if

$$V \models \text{“}\{\delta < \mu^+ : \mathrm{cf}(\delta) = \mathrm{cf}(\mu)\} \in I[\lambda]\text{”}$$

Proof. By 12.17, where $(\mathbf{B}_i, \bar{x}_i^-, \bar{x}_i^+)$ is provided by 11.13 (and W for it by 11.12). For $(**)$ see 1.20(2). $\square_{12.18}$

* * *

We would also like sufficient condition for inequalities, for simplicity $n = 2$.

12.19 Claim. *1) Assume 12.1, 12.3 and* $(*)$ *of 12.13 and* $n < \omega$ *and* $\tau^0 = \tau^0(x_0, \dots, x_{n-1})$ *a Boolean term and* $\tau^1 = \tau^1(-x_0, \dots, -x_{n-1})$*. Then* $(*)_1 \Rightarrow (*)_2$*, where*

$(*)_1$ *if* $\ell < 2$ *for a set of* $i < \delta$ *from* J^+ *we have: if* $X \in I_i^+$ *then for some* $t_0, \dots, t_{n-1} \in X$*, pairwise distinct, we have*

$$\mathbf{B}_i \models \tau^\ell(x_{i,t_0}, \dots, x_{i,t_{n-1}}) = 0,$$

$(*)_2$ *if* $a_\alpha \in \mathbf{B}$ *for* $\alpha < \lambda$ *then for some* $k < \omega$ *and* $\alpha_{\ell,m} < \lambda$ *for* $\ell < k$, $m < n$ *we have* $\alpha_{\ell,0} < \alpha_{\ell,1} < \dots < \alpha_{\ell,m-1} < \alpha_{\ell+1,0}$ *(for* $\ell < k$*) and for some* $i(\ell) \in \{0,1\}$ *for* $\ell < k$ *we have*

$$\mathbf{B}_i \models \bigcap_{\ell<k} \tau^{i(\ell)}(a_{\ell,0}, \dots, a_{\ell,m-1}) = 0.$$

2) Assume 12.1, 12.3 (using 12.5) and $(*)$ of 12.10 and for simplicity $I_i = J^{\rm bd}_{\lambda_i}$ <u>and</u> assume further $n < \omega$, $\mathbf{t}$ a function from $\{0, \dots, n-1\}$ to $\{+1, -1\}$ and $\tau^0 = \tau^0(x_0, \dots, x_{n-1})$ a Boolean term, increasing in x_ℓ if $\mathbf{t}(\ell) = +1$, decreasing with x_ℓ if $\mathbf{t}(\ell) = -1$. Let $\tau^1(x_0, \dots, x_{n-1}) = \tau^0(-x_0, \dots, -x_{n-1})$. Assume also $\tau_0(-x_0, \dots, -x_{n-1}) = 0$ if $x_\ell \in \{0, 1\}$ and $\bigwedge_\ell (x_\ell = 1 \equiv \mathbf{t}(\ell) = 1)$ or $\bigwedge_\ell x_\ell = 1 \equiv \mathbf{t}(\ell) = -1$. Then $(*)_3 \Rightarrow (*)_4$, where

$(*)_3$ *for a set of $i < \delta$ which belongs to J^+ the following holds: if $\gamma_{\alpha,\ell} < \lambda_i$ and*

$$[\alpha < \beta < \lambda_i \Rightarrow \max_{\ell<n} \gamma_{\alpha,\ell} < \min_{\ell<n} \gamma_{\alpha,\ell}$$

then for some $\alpha(0) < \cdots < \alpha(n-1)$ we have, for every $\ell < n$:

$$\tau^0(x^{\mathbf{t}(\ell)}_{\gamma_{\alpha(0),\ell}}, x^{\mathbf{t}(\ell)}_{\gamma_{\alpha(1),\ell}}, \dots, x^{\mathbf{t}(\ell)}_{\gamma_{\alpha(n-1),\ell}}) = 0$$

$$\tau^1(x^{-\mathbf{t}(\ell)}_{\gamma_{\alpha(0),\ell}}, x^{-\mathbf{t}(\ell)}_{\gamma_{\alpha(1),\ell}}, \dots, x^{-\mathbf{t}(\ell)}_{\gamma_{\alpha(n-1),\ell}}) = 0$$

$(*)_4$ *if $a_\alpha \in \mathbf{B}$ for $\alpha < \lambda$ then for some $\alpha_0 < \cdots < \alpha_{n-1}$ we have $\tau^0(a_{\alpha_0}, \dots, a_{\alpha_{n-1}}) = 0$.*

Proof. Easy.

* * * * * * * * * * *

12.20 Comments. 1) This concludes the proof of the consistency of the existence, answering a part of Monk's problem 33.
2) We can get "$\mathbf{B} \models (\mathrm{cf}(\mu))^+$-c.c." when 12.14 provides one.
3) We may still like to get "no k-independent set" for some specific k as done in 12.19. Probably also 11.14 will help but we have not really looked into it.

Clearly it is supposed to have, for a J^+-set of i's:

$(*)_i$ for some function F, if $m < \omega$, and $X \subseteq (\mathrm{Dom}\ I_i)^m$ is F-large (i.e., if $k < \omega$, $\bar t^0, \dots, \bar t^{k-1} \in X$ and $F(\bar t^0, \dots, \bar t^{k-1}) \in I$ then for some $\bar t \in X$, Rang $\bar t \cap F(\bar t^0, \dots, \bar t^{k-1}) = \emptyset$).

<u>Then</u> for some distinct $\bar t^0, \dots, \bar t^{n-1} \in X$, we have

$$\ell < m \Rightarrow \tau(t^0_\ell, t^1_\ell, \dots, t^{n-1}_\ell) = 0.$$

See more in 13.12, 13.13.

The singular case

We continue to deal with problem 33 of Monk [M2]. This time we concentrate on the case λ is singular. Though a priori this looked to be the side issue, we can get quite a coherent picture.
Note: If $\kappa >$ cf(λ) there is such a Boolean algebras (the disjoint sum of cf(λ) Boolean algebras each of cardinality $< \lambda$). Moreover

12.21 Claim. *Assume*

$(*)$ $\lambda > \mathrm{cf}(\lambda) = \theta$ *and* $(\forall \alpha < \lambda)$ $(|\alpha|^{<\kappa} < \lambda)$ *and* $\lambda > \kappa = \mathrm{cf}(\kappa) > \aleph_0$.

1) The following conditions are equivalent:

(A) *there are* $\mathbf{B}$ *and* a_ζ *such that*

- (a) $\mathbf{B}$ *is a* κ*-c.c. Boolean algebra*
- (b) $a_\zeta \in \mathbf{B} \backslash \{0\}$ *for* $\zeta < \theta$
- (c) *if* $\langle w_\zeta : \zeta < \theta \rangle$ *is a sequence of pairwise disjoint finite subsets of* θ *<u>then</u> for some finite* $u \subseteq \theta$ *we have*

$$\bigcap_{\zeta \in u} \bigcup_{\xi \in w_\zeta} a_\xi = 0.$$

(B) *there is a Boolean algebra* $\mathbf{B}$ *of cardinality* λ *with no independent subset of cardinality* λ.

2) The following conditions are equivalent

$(A)'$ *there are* $\mathbf{B}$, a_ζ *such that*

- (a) $\mathbf{B}$ *is a* κ*-c.c. Boolean algebra*
- (b) $a_\zeta \in \mathbf{B} \backslash \{0\}$ *for* $\zeta < \theta$
- (c) *for any* $X \in [\theta]^\theta$ *for some finite* $w \subseteq X$ *we have* $\bigcap\limits_{\zeta \in w} a_\zeta = 0$

$(B)'$ *there is a Boolean algebra* $\mathbf{B}$ *of cardinality* $\geq \lambda$ *which does not have* λ *as a free caliber.*

Proof. 1) $(A) \Rightarrow (B)$. The case $\theta = \aleph_0$ is easier, so we leave it to the reader.

Without loss of generality $\mathbf{B}$ has cardinality θ. Let $\lambda = \kappa + \theta + \sum_{\zeta < \theta} \lambda_\zeta$ where $\lambda > \lambda_\zeta > \kappa + \theta + \sum_{\xi < \zeta} \lambda_\xi$. Let $\mathbf{B}^*$ be the Boolean algebra freely generated by $\mathbf{B} \cup \{x_{\zeta,\alpha} : \zeta < \theta, \alpha < \lambda_\zeta^+\}$ except for the equations in $\mathbf{B}$ and

$$x_{\zeta,\alpha} \leq a_\zeta (\text{for } \zeta < \theta, \alpha < \lambda^+_\zeta).$$

Clearly $\mathbf{B} \subseteq \mathbf{B}^*$ and assume that $\{b_\gamma : \gamma < \lambda\} \subseteq \mathbf{B}^*$ is independent. Then for each γ there are $n(\gamma) < \omega$ and Boolean terms τ_γ and $\zeta_{\gamma,\ell} < \theta$, $\alpha_{\gamma,\ell} < \lambda_{\zeta_{\gamma,\ell}}$ for $\ell < n_\zeta$ and $c_{\gamma,\ell} \in \mathbf{B}$ for $\ell < m(\gamma)$ such that $b_\gamma = \tau_\gamma(x_{\zeta_{\gamma,0},\alpha_{\gamma,0}}, \dots, x_{\zeta_{\gamma,n(\gamma)-1},\alpha_{\gamma,n(\gamma)-1}}, c_{\gamma,0}, \dots, c_{\gamma,m(\gamma)-1})$. As $\mathrm{cf}(\lambda) = \theta > \aleph_0$, without loss of generality $\tau_\gamma = \tau$, $n(\gamma) = n(*)$ and $m(\gamma) = m(*)$. Also for each $\varepsilon < \theta$ there is $X_\varepsilon \in [\lambda^+_\varepsilon]^{\lambda^+_\varepsilon}$ such that

(∗) $\gamma \in X_\varepsilon$ implies $\zeta_{\gamma,\ell} = \zeta_{\varepsilon,\ell}(*) < \theta, c_{\gamma,\ell} = c^*_{\varepsilon,\ell} \in \mathbf{B}$.

Without loss of generality, $\langle \zeta_{\varepsilon,\ell} : \ell < n(*)\rangle$ is nondecreasing. We can find $Y \in [\theta]^\theta$ such that $\langle\langle \zeta_{\varepsilon,\ell}(*) : \ell < n\rangle : \varepsilon \in Y\rangle$ is a Δ-system. In fact for some $n'(*) \leq n(*)$ we have

$(*)_1$ $\varepsilon \in Y \ \& \ \ell < n'(*) \Rightarrow \zeta_{\varepsilon,\ell}(*) = \zeta_\ell(*)$

$(*)_2$ $\varepsilon_1 \in Y \ \& \ \varepsilon_2 \in Y \ \& \ \varepsilon_1 < \varepsilon_2 \Rightarrow \zeta_{\varepsilon_1,n(*)-1}(*) < \zeta_{\varepsilon_2,n'(*)}(*)$.

By renaming, without loss of generality $X_\varepsilon = [\lambda_\varepsilon, \lambda^+_\varepsilon]$ for $\varepsilon \in Y$. Let $w_\varepsilon = \{\zeta_{\varepsilon,\ell}(*) : n'(*) \leq \ell < n(*)\}$, so let u be as required in clause (A)(c), so $u \subseteq \theta$ is finite.

Let for $\varepsilon \in u$, $\gamma_{\varepsilon,1} < \gamma_{\varepsilon,2}$ be members of X_ε.

Clearly

$$b_{\gamma_{\varepsilon,1}} \triangle b_{\gamma_{\varepsilon,2}} \leq \bigcup_{\ell \in [n'(*),n(*)} a_{\zeta_{\varepsilon,\ell}}$$

hence

$$\bigcap_{\varepsilon \in u} (b_{\gamma_{\varepsilon,2}} \triangle b_{\gamma_{\varepsilon,2}}) \leq \bigcap_{\varepsilon \in u} \Big(\bigcup_{\ell \in [n'(*),n(*))} a_{\zeta_{\varepsilon,\ell}} \Big) = \bigcup_{\varepsilon \in u} \bigcap_{\xi \in w_\varepsilon} a_\xi = \emptyset,$$

so $\langle b_\gamma : \gamma < \lambda\rangle$ is not independent.

$\underline{\neg(A) \Rightarrow \neg(B)}$.

Like [Sh:92] (in short: Let $\lambda = \sum_{\zeta<\theta} \lambda_\zeta$, $(\forall \alpha < \lambda)$ $(|\alpha|^{<\kappa} < \lambda_\zeta)$, $\lambda_\zeta = \mathrm{cf}(\lambda_\zeta) > \kappa + \theta + \sum_{\xi<\zeta} \lambda_\xi$. Let $S_\zeta = \{\delta < \lambda_\zeta : \mathrm{cf}(\delta) \geq \kappa\}$. Remember that by [Sh:92]:

$\boxtimes_{\lambda_\zeta}[\mathbf{B}]$ Let $\mathbf{B}$ be a κ-c.c. Boolean algebra. Then:

(∗) for any $\bar{x} = \langle x_\alpha : \alpha < \lambda_\zeta\rangle$ pairwise distinct $x_\alpha \in \mathbf{B}$, there are $a^- < a^+$ in $\mathbf{B} \backslash \{0\}$, such that: if $\langle \mathbf{B}_\alpha : \alpha < \lambda_\zeta\rangle$ is an increasing continuous sequence of subalgebras of $\mathbf{B}$ of cardinality $< \lambda_\zeta$ satisfying $x_\alpha \in \mathbf{B}_{\alpha+1}, \{a^-, a^+\} \subseteq \mathbf{B}_0$, we have $a^- \leq x_\delta \leq a^+$ and$(\forall y)[0 < y \leq a^+ - a^- \ \& \ y \in B_\delta \to (x_\delta - a^-) \cap y \neq 0 \ \& \ (a^+ - x_\delta) \cap y \neq 0]$

is stationary.

So fix $\bar{x} = \langle x_\gamma : \gamma < \lambda \rangle$, sequence of distinct elements of $\mathbf{B}$, for each $\zeta < \theta$ let $a^-_\zeta - a^+_\zeta$ be as in

(∗) (for $\bar{x} \restriction \lambda_\zeta$), and let $a_\zeta = a^+_\zeta - a^-_\zeta \in \mathbf{B}^+$.

Let $\mathbf{B}^\zeta_\alpha$ be the subalgebra generated by $\{x_\gamma : \gamma < \max\{\alpha, \bigcup\limits_{\xi<\zeta} \lambda_\xi\}\} \cup \{a_\xi : \xi < \theta\}$ for $\alpha < \lambda_\zeta$ and for each $\zeta < \theta$ let S_ζ be as above.

As $\neg(A)$, necessarily there is a sequence of pairwise disjoint finite subsets of θ, say $\bar{u} = \langle u_\varepsilon : \varepsilon < \theta \rangle$ with any finite intersection of members $\langle \bigcup\limits_{\zeta \in u_\varepsilon} a_\zeta : \varepsilon < \theta \rangle$ is not zero.

Now we can manipulate, choosing by induction on $\varepsilon < \theta$, $\bar{t}^{\varepsilon,\alpha} \in \prod\limits_{\zeta \in u_\varepsilon} S_\zeta$ and defining

$$a^*_{\varepsilon,\alpha} = \bigcup_{\zeta \in u_\varepsilon} \Big((a_\zeta - \bigcup_{\xi \in u_\varepsilon \setminus (\zeta+1)} a_\xi) \cap x_{t^{\varepsilon,\alpha}_\zeta} \Big).$$

2) Similarly. $\square_{12.21}$

12.22 Discussion. 1) Note: if $\theta < \kappa$, clearly $(A)_\theta$ & $(A)'_\theta$.
2) Note if $(\forall \alpha < \theta)(|\alpha|^{<\kappa} < \theta)$, then $\neg(A)_\theta$ & $\neg(A)'_\theta$.
3) Note that if $\chi = \chi^{<\chi} < \chi(*) = \chi(*)^{<\chi(*)}$ then for some χ^+-c.c. $(<\chi)$-complete forcing notion of cardinality $\chi(*)$ in V^P we have $\neg(A)_\theta$ & $\neg(A)'_\theta$ when $\theta = \mathrm{cf}(\theta) \in (\chi, \chi(*))$.
4) It is natural to get $\mathrm{CON}(\kappa < \chi = \chi^{<\chi} < \theta = \mathrm{cf}(\theta) < 2^\chi + (A)_\theta$ & $\neg(A)'_\theta)$. This is well connected to our problems but we have not looked at it.

12.23 Claim. *In 11.3 the condition*

(∗) $(\forall \alpha < \lambda)(|\alpha|^{<\kappa} < \lambda)$

can be replaced by the weaker one

$(*)^-$ *for arbitrarily large regular $\lambda' < \lambda$ we have $\boxtimes_{\lambda'}[\mathscr{B}]$ for any κ-c.c. Boolean Algebra (see 12.21's proof).*

Getting free caliber for regular cardinals

Remember that λ is a free caliber of a Boolean algebra $\mathbf{B}$ if for any $X \in [\mathbf{B}]^\lambda$ there is an independent $Y \in [X]^\lambda$; of course, we can replace a Boolean algebra by a locally compact topological space (which is a slightly more general case, but the proof is not really affected).

Monk asks whether there is a κ-cc Boolean algebra $\mathbf{B}$ of cardinality $\geq \lambda$ with no independent subset of cardinality λ, and μ such that

$$\mu < \lambda < \mu^\kappa, \qquad (\forall \alpha < \mu)(|\alpha|^\kappa < \lambda).$$

Here we deal with the case of λ regular and give a sufficient set-theoretic condition on κ such that any κ-cc Boolean algebra of cardinality $\geq \lambda$ has λ as a free caliber, so the consistency of a negative answer follows, but we do not directly force. So this section is complementary to sections 12 and 11.

12.24 Hypothesis.

(a) $\lambda = \operatorname{cf}(\lambda) > 2^\kappa$, but for simplicity we assume

$$\lambda = \mu^+, \quad \mu = \sum_{i < \operatorname{cf}(\mu)} \lambda_i, \quad \lambda_i = \lambda_i^{<\kappa}, \quad \operatorname{cf}(\mu) < \kappa.$$

We shall use it to shorten proofs when helpful, and, later, will show what can be done without it

(b) $\mathbf{B}^*$ is a κ-cc. Boolean algebra, $a_\alpha \in \mathbf{B}$ for $\alpha < \lambda$ are pairwise distinct.

Let $\bar{a} =: \langle a_\alpha : \alpha < \lambda \rangle$. We would like to find $X \in [\lambda]^\lambda$ such that $\{a_\alpha : \alpha \in X\}$ is independent.

12.25 Definition. For $\mathbf{B} \subseteq \mathbf{B}^*, x \in \mathbf{B}^*$ let

$$\operatorname{Proj}^0(x, \mathbf{B}) =: \{y \in \mathbf{B} : y \leq x\}$$

$$\operatorname{Proj}^1(x, \mathbf{B}) =: \{y \in \mathbf{B} : y \cap x = 0\}$$

$$\operatorname{Proj}^2(x, \mathbf{B}) =: \{y \in B : y = 0 \text{ or } (\forall z)(0 < z \leq y \ \& \ z \in B \Rightarrow 0 < z \cap x < z)\}.$$

12.26 Fact. Let $\mathbf{B} \subseteq \mathbf{B}^*$, $x \in B^*$

(1) If $y_\ell \in \operatorname{Proj}^\ell(x,\mathbf{B})$ for $\ell < 3$, <u>then</u> $\langle y_\ell : \ell < 3\rangle$ are pairwise disjoint
(2) $\bigcup\limits_{\ell<3} \operatorname{Proj}^\ell(x,\mathbf{B})$ is dense in $\mathbf{B}$
(3) $\operatorname{Proj}^\ell(x,\mathbf{B})$ is an ideal on $\mathbf{B}$
(4) $\operatorname{Proj}^\ell(x,\mathbf{B})$ is complete inside $\mathbf{B}^*$ i.e., if in $\mathbf{B}^*$ we have x is $\leq$ lub of $\{x_\alpha : \alpha < \alpha^*\}$ and $\{x_\alpha : \alpha < \alpha^*\} \subseteq \operatorname{Proj}^\ell(x,\mathbf{B})$ and $x \in B$ <u>then</u> $x \in \operatorname{Proj}^\ell(x,\mathbf{B})$.

12.27 Definition.

$$\chi = \chi_{\bar a} = \operatorname{Min}\{\|\mathbf{B}\| : \mathbf{B} \subseteq \mathbf{B}^*, |W_{\mathbf{B}}| = \lambda\},$$

where

$$\begin{aligned} W_{\mathbf{B}} = W_{\mathbf{B},\bar a} = \{\alpha :& \operatorname{Proj}^2(a_\alpha, \langle \mathbf{B} \cup \{a_\beta : \beta < \alpha\}\rangle_{\mathbf{B}^*}) = \{0\} \\ & \text{and } \operatorname{Proj}^\ell(a_\alpha,\mathbf{B}) \text{ is dense in} \\ & \operatorname{Proj}^\ell(a_\alpha, \langle \mathbf{B} \cup \{a_\beta : \beta < \alpha\}\rangle_{\mathbf{B}^*}) \text{ for } \ell = 0,1\} \end{aligned}$$

12.28 Remark.

(1) $\operatorname{Proj}^2(a_\alpha, B) = \{0\}$ is close to saying, $a_\alpha =$ the lub in $\mathbf{B}^*$ of $\operatorname{Proj}^0(a_\alpha,\mathbf{B})$, but not the same (holds if $\mathbf{B} \lessdot \mathbf{B}^*$).
Could have worked with a variant as indicated.
(2) Trivially $\chi \leq \lambda$, use $\mathbf{B} = \langle a_\alpha : \alpha < \lambda\rangle_{\mathbf{B}^*}$.

12.29 Fact. If $\chi = \lambda$, <u>then</u> for some $X \in [\lambda]^\lambda$, $\langle a_\alpha : \alpha \in X\rangle$ is independent.

Proof. Let $\mathbf{B}_\alpha =: \langle a_\beta : \beta < \alpha\rangle_{\mathbf{B}^*}$, so $\mathbf{B}_\alpha$ are increasing continuous in α, $\|\mathbf{B}_\alpha\| \leq \aleph_0 + |\alpha| < \lambda$. Let

$$S =: \{\alpha < \lambda : \operatorname{Proj}^2(a_\alpha, \mathbf{B}_\alpha) = \{0\}\}$$

$$S' =: \{\alpha \in S : \operatorname{cf}(\alpha) \geq \kappa\}.$$

Now

$(*)$ S' is not stationary.

[Why? For $\delta \in S'$, $\ell < 2$ let $\mathscr{I}_{\delta,\ell} \subseteq \operatorname{Proj}^\ell(a_\delta, \mathbf{B}_\delta)\backslash\{0\}$ be an antichain, maximal under the conditions defining Proj^ℓ. So $|\mathscr{I}_{\delta,\ell}| < \kappa$, as $\mathbf{B}^* \models \kappa$-cc. Hence for some $f(\delta) < \delta$ we have

$$\mathscr{I}_{\delta,0} \cup \mathscr{I}_{\delta,1} \subseteq \mathbf{B}_{f(\delta)}.$$

So if S' is stationary, by Fodor lemma, for some $\alpha^* < \lambda$, $S^* = \{\delta \in S' : f(\delta) = \alpha^*\}$ is stationary.

We would like to show:

$$(**)\ \ \delta \in S^* \Rightarrow \operatorname{Proj}^2(a_\delta, \mathbf{B}_{\alpha^*}) = \{0\}.$$

If so, we have gotten that $\mathbf{B}_{\alpha^*}$, S^* exemplify $\chi \leq \|\mathbf{B}_{\alpha^*}\|$, contradiction. For proving $(**)$, let $\delta \in S^*$, assume $b \in \operatorname{Proj}^2(a_\delta, \mathbf{B}_{\alpha^*})\backslash\{0\}$.

So, by 12.26, (for $\mathbf{B}_{\alpha^*}, a_\delta$) we have $(\forall x \in \mathscr{I}_{\delta,0} \cup \mathscr{I}_{\delta,1})\ x \cap b = 0$.

Now, $b \notin \operatorname{Proj}^2(a_\delta, \mathbf{B}_\delta)$, as the latter is $\{0\}$. So, there is c such that $\mathbf{B}_\delta \models$ "$0 < c \leq b$ and $c \cap a_\delta = 0 \vee c \leq a_\delta$", that is $c \in \operatorname{Proj}^0(a_\delta, \mathbf{B}_\delta) \cup \operatorname{Proj}^1(a_\delta, \mathbf{B}_\delta)$, but as $c \leq b$ we have

$$(\forall x \in \mathscr{I}_{\delta,0} \cup \mathscr{I}_{\delta,1})(x \cap c = 0).$$

So c contradicts the maximality of $\mathscr{I}_{\delta,0}$ (if $c \in \operatorname{Proj}^0(a_\delta, \mathbf{B}_\delta)$) or of $\mathscr{I}_{\delta,1}$ (if $c \in \operatorname{Proj}^1(a_\delta, \mathbf{B}_\delta)$).

The contradiction proves $(**)$ and $(*)$.]

So $\lambda\backslash S$ is stationary. For $\delta \in \lambda\backslash S$ choose $b_\delta \in \operatorname{Proj}^2(a_\delta, \mathbf{B}_\delta)\backslash\{0\}$. So by Fodor's lemma, for some $b^* \in \bigcup_{\alpha<\lambda} \mathbf{B}_\alpha$ we have

$$S^{**} =: \{\delta : \delta \in \lambda\backslash S, b_\delta = b^*\} \text{ is stationary.}$$

Now we know that $\langle a_\delta : \delta \in S^*\rangle$ is independent. $\square_{12.29}$

12.30 Remark. In the characteristic case, $\mathbf{B}^*$ is the completion of a Boolean algebra of smaller cardinality $\mathbf{B}'$, so $\chi \leq \|\mathbf{B}'\|$.

12.31 Claim. *Now, without loss of generality*

$\boxtimes$ $\mathbf{B}^* = \langle \mathbf{B} \cup \{a_\alpha : \alpha \in W_{\mathbf{B}}\}\rangle$ *for some* $\mathbf{B} \subseteq \mathbf{B}^*$, $\|\mathbf{B}\| = \chi$, $W_{\mathbf{B}} = \lambda$.

Proof. $\mathbf{B} \subseteq \mathbf{B}^*$ exemplifies the value of χ, let $\mathbf{B}^c$ be the completion of $\mathbf{B}$, and we can let for $\alpha \in W_{\mathbf{B}}$

$$a'_\alpha = \text{ lub in } \mathbf{B}^c \text{ of } \operatorname{Proj}^0(a_0, \mathbf{B}).$$

Now if $Y \in [W_{\mathbf{B}}]^\lambda$, $\langle a'_\alpha : \alpha \in Y\rangle$ is independent in $\mathbf{B}^c$ then $\{a^*_\alpha : \alpha \in y\}$ is independent in $\mathbf{B}^*$. Alternatively use $\langle \mathbf{B} \cup \{a_i : \alpha \in W_{\mathbf{B}}\}\rangle_{\mathbf{B}^*}$.

(Remember: $\mathbf{B}$ is not necessarily a complete subalgebra of $\mathbf{B}^*$.)

12.32 Definition. Let

$$\begin{aligned} K =: \{\bar{\mathbf{B}} : \bar{\mathbf{B}} =& \langle \mathbf{B}_i : i \le \chi\rangle \\ & \text{is an increasing continuous sequence of subalgebras of a} \\ & \mathbf{B}^*, \|\mathbf{B}_i\| < \aleph_0 + |i|^+ \text{ and } W_{\mathbf{B}_\chi} \in [\lambda]^\lambda, \mathbf{B}_\chi \supseteq \mathbf{B} \\ & (\text{of } \boxtimes \text{ of } 12.31)\}. \end{aligned}$$

(so $W_{\mathbf{B}_\chi}$ is cobounded in λ, in fact if $\mathbf{B}_\chi \subseteq \langle \mathbf{B} \cup \{a_\beta : \beta < \alpha\}\rangle_{\mathbf{B}^*}$ then $|W_{\mathbf{B}_\chi} \supseteq (\alpha, \lambda)|$.)

12.33 Fact. 1) $\operatorname{cf}(\chi) < \kappa$.
2) $\operatorname{cov}(\chi, \chi, \kappa, 2) \ge \lambda$, meaning:

$$\lambda \le \min\{|\mathscr{P}| : \mathscr{P} \subseteq [\chi]^{<\chi} \ \& \ (\forall A \in [\chi]^{<\kappa})(\exists B \in \mathscr{P})(A \subseteq B).$$

Proof. 1) By (2).
2) Assume not. Remember $\mathbf{B} \subseteq \mathbf{B}^*$, $|W_{\mathbf{B}}| = \lambda$, $\|\mathbf{B}\| = \chi$.
For each $\alpha \in W_{\mathbf{B}}$ choose $\mathscr{I}_{\alpha,\ell} \subseteq \operatorname{Proj}^\ell(a_\alpha, \mathbf{B})$ for $\ell < 2$ as in the proof of 12.29. Let $\mathscr{P} \subseteq [\mathbf{B}]^{<\chi}$, $|\mathscr{P}| < \lambda$ and

$$(\forall A \in [\chi]^{<\kappa})(\exists B \in \mathscr{P})(A \subseteq B).$$

So for each $\alpha \in W_{\mathbf{B}}$, there is $A_\alpha \in \mathscr{P}$ such that $\mathscr{I}_{\alpha,0} \cup \mathscr{I}_{\alpha,1} \subseteq A_\alpha$. So for some $A^* \in \mathscr{P}$

$$W = \{\alpha \in W_{\mathbf{B}} : \mathscr{I}_{\alpha,0} \cup \mathscr{I}_{\alpha,1} \subseteq A^*\} \in [\lambda]^\lambda.$$

(exists as we divide $W_{\mathbf{B}}$ into $|\mathscr{P}|$ sets, so at least one has size λ, as $|\mathscr{P}| < \lambda = \operatorname{cf}(\lambda)$). Now $\chi \le |\langle A^*\rangle_{\mathbf{B}}|$, contradiction, as in the proof of 12.29 (to the definition of χ.

$\square_{12.33}$

12.34 Definition. For $\bar{\mathbf{B}} \in K$ and $\alpha \in W_{\mathbf{B}_\chi}$ let

$$u(\alpha, \bar{B}) =: \{i < \chi : \text{for some } \ell < 2, \text{Proj}^\ell(a_\alpha, \mathbf{B}_i) \text{ is not a predense} \\ \text{subset of } \text{Proj}^\ell(a_\alpha, \mathbf{B}_{i+1})\}.$$

12.35 Discussion. We may consider $\bar{\mathbf{B}}' = \langle \mathbf{B}'_i : i \leq \chi \rangle \in \kappa$ when

$$\mathbf{B}'_i = \langle \mathbf{B}_i \cup X \rangle, X \text{ fixed countable } \subseteq \mathbf{B}^*.$$

Possibly

$$u(\alpha, \bar{\mathbf{B}}) \neq u(\alpha, \bar{\mathbf{B}}')$$

or just for some i, $\text{Proj}^\ell(a_\alpha, \mathbf{B}_i)$ is not dense in $\text{Proj}^\ell(a_\alpha, \mathbf{B}'_i)$. We think of the set of such α as bad, and put them all in one λ-complete ideal. But maybe λ belongs to it. So we will try to find some $\bar{\mathbf{B}}$ for which this does not occur.

This will help in that we eventually try to choose $\alpha_\zeta \in W_{\bar{\mathbf{B}}}$ for $\zeta < \lambda$ by induction on λ such that $\langle a_{\alpha_\zeta} : \zeta < \lambda \rangle$ is independent.

So in stage ζ we consider all

$$X \in [\{\alpha_\xi : \xi < \zeta\}]^{<\aleph_0}.$$

The existence of $\bar{\mathbf{B}}$ requires some properties of λ which certainly hold in the main case (with $\lambda = \mu^+ \ldots$).

So to ease the proof instead of every $i < \chi$, we use "every $i < \chi$ large enough".

12.36 Definition. 1) We define a partial order on $K : \bar{\mathbf{B}}^1 \leq \bar{\mathbf{B}}^2$ <u>if</u> for every i large enough

$$i \leq \chi \Rightarrow \mathbf{B}^1_i \subseteq \mathbf{B}^2_i.$$

2) We say $\bar{\mathbf{B}}^2$ is finitely generated over $\bar{\mathbf{B}}^1$ <u>if</u> for some finite X

$$\mathbf{B}^2_i = \langle \mathbf{B}^1_i \cup X \rangle_{\mathbf{B}^*} \text{ for } i < \chi \text{ large enough.}$$

In this case we let $\bar{\mathbf{B}}^1[X] = \langle \mathbf{B}^1_i[X] : i \leq \chi \rangle$ be $\bar{\mathbf{B}}^2$.
3) For $\bar{\mathbf{B}}^1 \leq \bar{\mathbf{B}}^2$ let

$$\begin{aligned}\mathrm{Bad}(\mathbf{B}^1,\mathbf{B}^2) = \{\alpha :& \text{if } \alpha \in W_{\mathbf{B}^1_\chi} \cap W_{\mathbf{B}^2_\chi}, \\ & \text{then for arbitrarily large } i < \chi, \text{ for some } \ell < 2, \\ & \mathrm{Proj}^\ell(a_\alpha, \mathbf{B}^1_i) \text{ is not dense in } \mathrm{Proj}^\ell(a_\alpha, \mathbf{B}^2_i)\}.\end{aligned}$$

4) $J_{\bar{\mathbf{B}}^1}$ is the λ-complete ideal on λ generated by all $\mathrm{Bad}(\bar{\mathbf{B}}^1, \bar{\mathbf{B}}^2)$, where $\bar{\mathbf{B}}^1 \leq \bar{\mathbf{B}}^2$ and $\bar{\mathbf{B}}^2$ is finitely generated over $\bar{\mathbf{B}}^1$.

What do we need to carry a proof?

12.37 Lemma. *There is $\bar{\mathbf{B}}^\otimes \in K$ such that $\lambda \notin J_{\bar{B}^\otimes}$.*

12.38 Remark. We may like to have $J \supseteq J_{\bar{B}^\otimes}$ normal extending $I^{\mathrm{nst},\theta}_\lambda$ (and $\lambda \notin J$), then we need more work.

Proof in the case $\lambda = \chi^+$. (Enough, see 12.24(a)). Assume there is no such $\bar{\mathbf{B}} = \bar{\mathbf{B}}^\otimes$. We choose by induction on $\zeta < \chi$, $\bar{\mathbf{B}}^\zeta \in K$, such that $\bar{\mathbf{B}}^\zeta$ is increasing with ζ and: for each ζ, as $\lambda \in J_{\bar{\mathbf{B}}^\zeta}$ we can find $\langle X_{\zeta,\epsilon} : \epsilon < \epsilon_\zeta\rangle$ witnessing it, i.e. $X_{\zeta,\epsilon} \in [\mathbf{B}^*]^{<\aleph_0}$, $\epsilon_\zeta < \lambda$ (so without loss of generality $\epsilon_\zeta \leq \chi$)

$$\lambda = \bigcup_{\epsilon<\epsilon_\zeta} \mathrm{Bad}(\bar{\mathbf{B}}^\zeta, \bar{\mathbf{B}}^\zeta[X_{\zeta,\epsilon}])$$

where

$$\mathbf{B}^\zeta_i[X_{\zeta,\epsilon}] = \langle \mathbf{B}^\zeta_i \cup X_{\zeta,\epsilon}\rangle_{\mathbf{B}^*}.$$

Now easily $(K, \leq)$ is χ^+-directed, so we demand

$$\bigwedge_{\epsilon<\epsilon_\zeta} \bar{\mathbf{B}}^\zeta \leq \bar{\mathbf{B}}^\zeta[X_{\zeta,\epsilon}] \leq \bar{\mathbf{B}}^{\zeta+1}.$$

Also $i \in [\zeta, \lambda)$ & $\zeta < \xi \leq \chi \Rightarrow \mathbf{B}^\zeta_i \subseteq \mathbf{B}^\xi_i$. Let $\delta^* < \lambda$ be such that

$$\bigwedge_{\zeta<\chi} \mathbf{B}^\zeta_\chi \subseteq \langle \mathbf{B} \cup \{a_\alpha : \alpha < \delta^*\}\rangle_{\mathbf{B}^*}.$$

So for each $\zeta < \chi$ we have

$$\delta^* \in \bigcup_{\epsilon<\epsilon_\zeta} \mathrm{Bad}(\bar{\mathbf{B}}^\zeta, \bar{\mathbf{B}}^\zeta[X_{\zeta,\epsilon}]),$$

hence there is $\xi(\zeta) < \epsilon_\zeta$ such that

$$\delta^* \in \ \mathrm{Bad}(\bar{\mathbf{B}}^\zeta, \bar{\mathbf{B}}^\zeta[X_{\zeta,\xi(\zeta)}]).$$

For each $\zeta < \chi$, there is $i(\zeta) < \chi$ such that $X_{\zeta,\xi(\zeta)} \subseteq \mathbf{B}^{\zeta+1}_{i(\zeta)}$, hence

$$(\forall i)[i(\zeta) \leq i \leq \chi \Rightarrow \mathbf{B}^\zeta_i[X_{\zeta,\zeta(\xi)}] \subseteq \mathbf{B}^{\zeta+1}_i$$

because

$$X_{\zeta,\xi(\zeta)} \subseteq \mathbf{B}^{\zeta+1}_{i(\zeta)} \subseteq \mathbf{B}^{\zeta+1}_i.$$

We restrict ourselves to $\xi < \kappa$. So without loss of generality

$$\bigwedge_{\zeta_1<\zeta_2\leq\kappa} \ \bigwedge_{\alpha\in[\kappa^+,\chi]} \mathbf{B}^{\zeta_1}_\alpha \subseteq \mathbf{B}^{\zeta_2}_\alpha,$$

and if ζ is a limit and $\alpha \in [\kappa^+, \chi]$, then $\mathbf{B}^\zeta_\alpha = \bigcup_{\xi<\zeta} \mathbf{B}^\xi_\alpha$. As $\mathrm{cf}(\chi) < \kappa$, there is $i(*) < \chi$ such that $Z = \{\zeta < \kappa : i(\zeta) \leq i(*)\}$ is unbounded (we can demand more).

Now the set $u(\delta^*, \bar{\mathbf{B}}^\kappa)$ has cardinality $< \kappa$ because $\mathbf{B}^*$ satisfies the c.c.c.

Remember,

$$u(\delta^*, \bar{\mathbf{B}}^\kappa) = \{i < \chi : \bigcup_{\ell=0,1} \mathrm{Proj}^\ell(a_{\delta^*}, \mathbf{B}^\kappa_i) \text{ is not predense in } \bigcup_{\ell=0,1} \mathrm{Proj}^\ell(a_{\delta^*}, \mathbf{B}^\kappa_{i+1})\}.$$

Choose for $i \in u(\delta^*, \bar{\mathbf{B}}^\kappa) \cup \{\kappa^+\}$ and $\ell = 0, 1$ a predense subset $\mathscr{I}^{\delta^*,\ell}_{\kappa,i}$ of $\mathrm{Proj}^\ell(a_{\delta^*}, \mathbf{B}^\kappa_{i+1})$ of cardinality $< \kappa$.

Now, for $i \in u(\delta^*, \bar{\mathbf{B}}^\kappa) \cup \{\kappa^+\} \backslash \kappa^+$ the sequence $\langle \mathbf{B}^\zeta_{i+1} : \zeta \leq \kappa \rangle$ is increasing continuous. So for some $\zeta_i < \kappa$

$$\mathscr{I}^{\delta^*,0}_{\kappa,i} \cup \mathscr{I}^{\delta^*,1}_{\kappa,i} \subseteq \mathbf{B}^{\zeta_i}_{i+1}.$$

Let

$$\zeta(\delta^*) =: \sup_i \zeta_i < \kappa.$$

So clearly

(∗) if $i \in [\kappa^+, \chi]$, $\ell < 2$, then

$$\text{Proj}^\ell(a_{\delta^*}, \mathbf{B}_i^{\zeta[\delta^*]}) = \text{ Proj}^\ell(a_{\delta^*}, \mathbf{B}_i^\kappa) \cap \mathbf{B}_i^{\zeta[\delta^*]}$$

is a predense subset of $\text{Proj}^\ell(a_{\delta^*}, \mathbf{B}_i^\kappa)$.

[Why? By induction on i. If $i = \kappa^+$ directly. If i is a limit - trivial. If $i = j+1 \geq \kappa^+$, $j \notin u(\delta^*, \bar{\mathbf{B}}^\kappa)$, then by transitivity of being predense in. If $i = j+1$, $j \in u(\delta^*, \bar{\mathbf{B}}^\kappa)$, using $\mathscr{I}_j^{\delta^*,\ell}$.]

Now, clearly

$$[\zeta(\delta^*), \kappa) \Rightarrow \bigwedge_{\ell<2} \bigwedge_{i \in [\kappa^+, \chi)} (\text{Proj}^\ell(a_{\delta^*}, \mathbf{B}_i^\zeta) \text{ is predense in } \text{Proj}^\ell(a_{\delta^*}, \mathbf{B}_i^{\zeta+1})).$$

This follows from (∗). Choose $\zeta \in Z \backslash \zeta(\delta^*)$ so we contradict the choice of $\bar{\mathbf{B}}^{\zeta+1}$. $\square_{12.37}$

12.39 Convention.. We fix $\bar{\mathbf{B}}^\otimes \in K$ such that $\lambda \notin J_{\bar{\mathbf{B}}^\otimes}$.

12.40 Fact. $\{\alpha < \lambda : u(\alpha, \bar{\mathbf{B}}^\otimes) \text{ bounded in } \chi\}$ is bounded in λ.

Proof. By the choice of χ as minimal. $\square_{12.40}$

12.41 Convention.. Let f_α be an increasing function from $\text{otp}(u(\alpha, \bar{\mathbf{B}}^\otimes))$ onto $u(\alpha, \bar{\mathbf{B}}^\otimes)$.

12.42 Fact. For some $j^* < \kappa$

$$Y_0 = \{\alpha < \lambda : \text{Dom}(f_\alpha) = j^*\} \in (J_{\bar{\mathbf{B}}^\otimes})^+.$$

So without loss of generality $(\forall \alpha)[\text{Dom}(f_\alpha) = j^*]$.

12.43 Claim. *We can find $\langle \gamma^*_j : j < j^* \rangle$, $w^* \subseteq j^*$ such that:*

$(*)_1$ *<u>if</u> $\bar\gamma = \langle \gamma_j : j < j^* \rangle$, $\gamma_j \le \gamma^*_j$,*

$$\gamma_j = \gamma^*_j \Leftrightarrow j \in w^*,$$

<u>then</u> the set of $\alpha \in Y_0$ satisfying the following, is in $(J_{\mathbf{B}^\otimes})^+$:

$$j \in w^* \Rightarrow f_\alpha(j) = \gamma^*_j$$

$$j \in j^* \backslash w^* \Rightarrow \gamma_j < f_\alpha(j) < \gamma^*_j.$$

Also

$(*)_2$ $j \in j^* \backslash w^* \Rightarrow \mathrm{cf}(\gamma^*_j) > 2^\kappa$ *and*

$$\lambda = \max \mathrm{pcf}\{\mathrm{cf}(\gamma^*_j) : j \in j^* \backslash w^*\}.$$

$(*)_3$ *Moreover if we fix $\mu = \mu^{<\kappa} < \lambda$ we can demand*

$$j \in j^* \backslash w^* \Rightarrow \mathrm{cf}(\gamma^*_j) > \mu.$$

$(*)_4$ *if $j^* = \sup(J^* \backslash w^*)$, and E is the equivalence relation on $j^* \backslash w$ defined by $j_1 E j_1 \Leftrightarrow \gamma^*_{j_1} = \gamma^*_{j_2}$ (so the equivalence classes are convex) then J is an ideal on j^* such that $J^{\mathrm{bd}}_{j^*} \subseteq J$, $w^* \in J$,*

$$A \in J \Rightarrow \bigcup\{j/E : j \in A\} \in J,$$

and

(α) *$\prod\limits_{j<j^*} \gamma^*_j / J$ has true cofinality λ, so possibly shrinking Y_0, for $\alpha < \beta$ in Y_0, $f_\alpha <_J f_\beta$.*

Proof. By 7.1(0) (or [Sh 430, 6.6D] or [Sh 513, 6.1]), as $j^* < \kappa$, so $2^{|j^*|} < \lambda$. $\square_{12.43}$

12.44 Observation. $\langle \gamma^*_j : j < j^* \rangle$ is non-decreasing, with limit χ, and $\gamma^*_j < \chi$ and, of course, $\mathrm{cf}(j^*) = \mathrm{cf}(\chi)$.

Proof. As $\mathrm{Rang}(f_\alpha) \subseteq \chi$, and the fact, $\gamma^*_j < \chi$ if $j \in w^*$, $\gamma^*_j \le \chi$ if $j \notin w^*$, but then

$$\mathrm{cf}(\gamma^*_j) \ge 2^\kappa > \kappa > \mathrm{cf}(\chi).$$

$\square_{12.44}$

12.45 Comment on the Claim. *1) For it, possibly $\wedge_\alpha f_\alpha = f^*$, so then we get $w^* = j^*$. Also possibly $f_\alpha(j) < \alpha$, so $w^* = \emptyset$ and $J = \{\phi\}$.*
2) If the ideal $J_{\bar{\mathbf{B}}^\otimes}$ is normal enough, for some $X \in (J_{\bar{\mathbf{B}}^\otimes})^+, \langle f_\alpha : \alpha \in X\rangle$ is $<_J$-increasing.
3) If $(\forall \alpha < \lambda)(|\alpha|^{|j^*|} < \lambda)$, then necessarily

$$j \in j^* \backslash w^*, \mathrm{cf}(\gamma^*_j) = \lambda$$

(like the $\triangle$-system lemma). BUT for the interesting case, and in particular by our assumptions, this is not the case: as $\gamma^*_j \le \chi < \lambda$, hence $J \supseteq [j^*]^{<\aleph_0}$.

12.46 Hypothesis. Each $\mathbf{B}^\otimes_i$ is the union of μ filters $\langle \mathscr{D}_{i,\beta} : \beta < \mu\rangle$, $\mu = \mu^{<\kappa}$ (we can use somewhat less), this of course is only a consistent assumption.

12.47 Claim. *For some*

$$\bar{\iota} = \langle \iota_j : j < j^*\rangle \in {}^{j^*}\mu$$

we can restrict ourselves to

$$Y_1, \alpha < \lambda = \begin{cases} j \in w^* \Rightarrow f_\alpha(j) = \gamma^*_j, \\ j \in j^* \backslash w^* \Rightarrow \gamma^{**}_j < f_\alpha(j) < \gamma^*_j \text{ and} \\ \bigwedge\limits_{j<j^*} (\mathrm{Proj}^2(a_\alpha, \bar{\mathbf{B}}^\otimes_{\gamma^{**}_j}) \cap \mathscr{D}_{\gamma^{**}_j, \iota_j} \neq \{0\}) \end{cases}$$

where

$$\gamma^{**}_j = \begin{cases} \gamma^*_j & \text{if } j \in w^* \\ \cup\{\gamma^*_i : \gamma^*_i < \gamma^*_j\} & \text{otherwise.} \end{cases}$$

in particular $Y_1 \notin J_{\bar{B}^\otimes}$.

Proof. As $\mu^{<\kappa} < \lambda$, $J_{\bar{\mathbf{B}}^\otimes}$ λ-complete and $j \in j^* \backslash w^* \Rightarrow \mathrm{cf}(\gamma^*_j) > \mu$. $\square_{12.47}$

12.48 Claim. *For some $X \in [X]^\lambda$, the sequence $\langle a_\alpha : \alpha \in X\rangle$ is independent.*

Proof.
Case 1. w^* is unbounded in j^*: We choose by induction on $\beta < \lambda$,

$$N_\beta \prec (\mathscr{H}((2^\lambda)^+), \in, <^*_{(2^\lambda)^+})$$

increasing continuous, $\|N_\beta\| < \lambda$, $N_\beta \cap \lambda \in \lambda$, $\langle N_{\beta_1 : \beta_1 \le \beta}\rangle \in N_{\beta+1}$ and $\bar{\mathbf{B}}^\otimes$,$\mathbf{B}^*$, $\bar{a} \in N_0$. Let $\alpha_\beta = \alpha(\beta)$ be the first $\alpha < \lambda$ such that $\alpha \in Y_1$, $\alpha \notin \cup(J_{\bar{\mathbf{B}}^\otimes} \cap N_\beta)$

$$(\text{so } \bigwedge_{j \in w^*} f_\alpha(j) = \gamma^*_j).$$

Clearly $\alpha_\beta \in \lambda \cap N_{\beta+1} \backslash N_\beta$, $\langle \alpha_{\beta_1} : \beta_1 < \beta\rangle \in N_{\beta+1}$. Let $n < \omega$, $\beta_1 < \cdots < \beta_n$ and we will prove that $\langle \alpha_{\alpha(\beta_\ell)} : \ell = 1, \ldots, n\rangle$ is independent.

Now

$$j \in w^* \Rightarrow \text{ there is } b_j \in \bigcap_{\ell=1}^{n} \mathrm{Proj}^2(a_{\alpha_{\beta_\ell}}, \bar{\mathbf{B}}^\otimes_{\gamma^*_j}) \backslash \{0\}.$$

[Why? As $\alpha_{\beta_1}, \ldots, \alpha_{\beta_n} \in Y_1$, so $\mathscr{D}_{\gamma^*_j, \iota_j} \cap \mathrm{Proj}^2(a_{\alpha_{\beta_\ell}}, \mathbf{B}^\otimes_{\gamma^*_j}) \neq \emptyset$. Choose $b_{j,\ell}$ there, so $b_j = \bigcap_{\ell=1}^n b_{j,\ell}$ is OK.]

Consider

$$\mathrm{Bad}(\mathbf{B}^\otimes, \bar{\mathbf{B}}^\otimes[\{a_{\alpha(\beta_1)}, \ldots, a_{\alpha(\beta_\ell)}\}]) \in J_{\bar{\mathbf{B}}^\otimes},$$

it belongs to $N_{\beta_{\ell+1}}$. So

$$\alpha_{\beta_{\ell+1}} \notin \mathrm{Bad}(\mathbf{B}^\otimes, \mathbf{B}^\otimes[\{a_{\alpha(\beta_1)}, \ldots, a_{\alpha(\beta_\ell)}\}]).$$

So for each ℓ for some $i_\ell < \chi, k < 2$ & $i \in [i_\ell, \chi) \Rightarrow \mathrm{Proj}^k(a_{\alpha_{\beta_{\ell+1}}}, \mathbf{B}^\otimes_i)$ is predense in $\mathrm{Proj}^k(a_{\alpha_{\beta_{\ell+1}}}, \langle \mathbf{B}^\otimes_i \cup \{a_{\alpha_{(\beta_1)}}, \ldots, a_{\alpha_{(\beta_\ell)}}\}\rangle)$.

So if $j \in w^*$, $\gamma^*_j > \sup_{\ell=1,\ldots,n} i_\ell$ (exists) and $\eta \in {}^{[1,n]}2$, we prove by induction on ℓ that

$$b^\ell_j = b_j \cap \bigcap_{k=1}^{\ell} (a_{\alpha_{\beta_k}})^{[\eta(k)]}.$$

For $\ell = 0$ trivial.

For $\ell > 0$, $b^{\ell-1}_j \in \langle \mathbf{B}^\otimes_{\gamma^*_j} \cup \{a_{\alpha_{\beta_1}}, \ldots, a_{\alpha_{\beta_{\ell-1}}}\}\rangle$ is > 0, is in

$$\mathrm{Proj}^2(a_{\alpha_\beta}, \langle \mathbf{B}^\otimes_{\gamma^*_j} \cup \{a_{\alpha(\beta_1)}, \ldots a_{\alpha(\beta_{\ell-1}}\}\rangle)$$

as it is below b_j and $b_j \in \mathrm{Proj}^2(a_{\alpha(\beta_\ell)}, \mathbf{B}^\otimes_{\gamma^*_j})$ by its choice and j is $> i_\ell$, so $b_j \in \mathrm{Proj}^2(a_{\alpha_{\beta_\ell}}, \langle \bar{\mathbf{B}}^\otimes \cup \{a_{\alpha_{\beta_1}}, \ldots, a_{\alpha_{\beta_{\ell-1}}}\}\rangle)$.

We use implicitly

12.49 Fact. For $\alpha < \lambda$ large enough,

$$i < \chi \Rightarrow \text{ Proj}^2(a_\alpha, \bar{\mathbf{B}}_i^\otimes) \neq \{0\}.$$

Proof. By χ's minimality. $\square_{12.49}$

Case 2. Not 1, i.e., w^* bounded in j^* <u>or</u> just $j^* = \sup(J^*\backslash w)$. Similarly using $(*)_2$ of 14.17 find $j \in j^* \setminus w^*$ such that if $j_\ell \in j/E$ for $\ell = 1, \dots, n$ then $f_{\alpha_{\beta_1}}(f_1) < f_{\alpha_{\beta_2}}(j_2) < \cdots < f_{\alpha_{\beta_n}}(j_n)$. $\square_{12.48}$

* * *

12.50 Conclusion. If $\mu = \mu^{<\mu} < \theta = \theta^{<\theta}$ then for some μ-complete μ^+-c.c. forcing notion of cardinality θ, in V^P:

<u>If</u> $\mathbf{B}$ is a κ-c.c. Boolean algebra of cardinality $\geq \lambda$, $\mu = \mu^{<\kappa}$, $\lambda = \text{ cf}(\lambda) \in (\mu, \theta]$ <u>then</u> λ is a free caliber of $\mathbf{B}$.

Proof. By 12.24 - 12.49 above and [Sh 80]. $\square_{12.50}$

12.51 Claim. *The following implications hold:* $(*)_1 \Rightarrow (*)_2 \Rightarrow (*)_3 \Rightarrow (*)_4$ *where*

$(*)_1(a)$ $\mu^{2^{<\kappa}} = \mu < \lambda = \text{ cf}(\lambda)$

(b) *if a Boolean algebra* $\mathbf{B}$ *satisfies the* $(2^{<\kappa})^+$*-c.c. and* $|\mathbf{B}| < \lambda$, <u>*then*</u> $\mathbf{B}$ *is the union of* μ *filters*

$(*)_2(a)$ $\kappa < \lambda = \text{ cf}(\lambda)$

(b) *if a Boolean algebra* $\mathbf{B}$ *satisfies the* κ*-c.c., for* $i < \lambda$, $F_i \subseteq \mathbf{B}\backslash\{0\}$ *is a set of* $< \kappa$ *members closed under intersection* <u>*then*</u> *we can find* $< \lambda$ *filters* $\mathscr{D}_\alpha$ $(\alpha < \alpha^* < \lambda)$ *of* $\mathbf{B}$ *such that* $(\forall i < \lambda)$ $(F_i \subseteq \mathscr{D}_\alpha)$

$(*)_3(a)$ $\kappa < \lambda = \text{ cf}(\lambda)$

(b) *if a Boolean algebra* $\mathbf{B}$ *satisfies the* κ*-c.c.,* $\mathscr{D}$ *a* λ*-complete uniform filter on* λ, $\theta = \text{ cf}(\theta) < \kappa$ *and for* $i < \lambda$, F_i *is a decreasing sequence of elements of* $\mathbf{B}\backslash\{0\}$ *of length* θ <u>*then*</u> *for some* $X \in \mathscr{D}^+$, $\bigcup_{i \in X} F_i$ *belongs to some ultrafilter on* $\mathbf{B}$

$(*)_4(a)$ $\kappa < \lambda = \text{ cf}(\lambda)$

(b) *if* $\mathbf{B}$ *is a* κ*-c.c. Boolean algebra of cardinality* $\geq \lambda$ <u>*then*</u> λ *is a free caliber of* $\mathbf{B}$.

Proof. Should be clear from the proof in §14. $\square_{12.51}$

§15 On irr: The invariant of the ultraproduct bigger than the ultraproduct of invariants

We solve here some of the questions of Monk [M2] on the possibility that

$$\mathrm{inv}(\prod_{\zeta<\kappa} \mathbf{B}_\zeta/\mathscr{D}) > \prod_{\zeta<\kappa} \mathrm{inv}(\mathbf{B}_\zeta)/\mathscr{D}.$$

In 13.1 - 13.11 we deal with the irredundance number irr (getting consistency of the above and solving [MS, Problem 26]). We then prove the existence of such examples in ZFC (improving Rosłanowski Shelah [RoSh 534]) for inv $= s$, hd, hL, Length solving [M2, Problems 46, 51, 55, 22], respectively. See more in [Sh 641].

13.1 Hypothesis. $\lambda = \lambda^{<\lambda}$, $n(*) < \omega$.

13.2 Definition. $\mathbb{P} = \mathbb{P}^{n(*)}_\lambda$ is the set of $p = (u, \mathbf{B}, \bar{\mathscr{F}}) = (u^p, \mathbf{B}^p, \bar{\mathscr{F}}^p)$ such that

- (a) $u \in [\lambda^+]^{<\lambda}$
- (b) $\mathbf{B}$ is a Boolean algebra generated by $\{x_\alpha : \alpha \in u\}$
- (c) $\alpha \in u \Rightarrow x_\alpha \notin \langle\{x_\beta : \beta \in u \cap \alpha\}\rangle_{\mathbf{B}}$
- (d) in $\mathbf{B}, \{x_\alpha : \alpha \in u\}$ is $n(*)$-independent, i.e., any nontrivial Boolean combination of $\le n(*)$ members of $\{x_\alpha : \alpha \in u\}$ is not zero (in $\mathbf{B}$)
- (e) $\bar{\mathscr{F}} = \langle \mathscr{F}_\ell : \ell \le n(*)\rangle$ and $\mathscr{F}_{\ell+1} \subseteq F_\ell$
- (f) $\mathscr{F}_\ell$ is a non-empty family of functions from $\{x_\alpha : \alpha \in u\}$ to $\{0,1\}$ respecting the equations holding in $\mathbf{B}$. Call the homomorphism (from $\mathbf{B}$ to $\{0,1\}$) such that f induces, $\hat{f}$
- (g) if $f \in \mathscr{F}_{\ell+1}, \ell < n(*)$ and $\alpha \in u$ <u>then</u> for some $f' \in \mathscr{F}_\ell$ we have
$$f' \restriction (\alpha \cap u) = f \restriction (\alpha \cap u), f'(\alpha) \neq f(\alpha).$$
- (h) if $f : u \to \{0,1\}$ and $(\forall v \in [u]^{<\aleph_0})(f \restriction u \in \mathscr{F}_\ell)$ then $f \in \mathscr{F}_\ell$
- (i) if $a \in \mathbf{B}\backslash\{0\}$ then for some $f \in F_0$, we have $\hat{f}(a) = 1$.

The order is: $p \le q$ <u>iff</u>

- (α) $u^p \subseteq u^q$
- (β) $\mathbf{B}^p$ is a subalgebra of $\mathbf{B}^q$
- (γ) $\mathscr{F}^p_\ell = \{f \restriction u^p : f \in \mathscr{F}^q_\ell\}$.

Let $\underset{\sim}{\mathbf{B}}$ = the direct limit of $\{\mathbf{B}^p : p \in \underset{\sim}{G}_P\}$.

Note. We can ignore $\mathbf{B}^p$ at it is reconstructible from $\mathscr{F}_0^p$. Also clause (d) follows from the rest.

13.3 Notation. We let $p \restriction \alpha = (u^p \cap \alpha, \langle x_\beta : \beta \in u^p \cap \alpha\rangle_{\mathbf{B}}, \langle \mathscr{F}_\ell \restriction \alpha : \ell \leq n(*)\rangle)$ where for $u \subseteq \lambda^+$ we let $\mathscr{F}_\ell \restriction u = \{f \restriction u : f \in \mathscr{F}_\ell\}$ and $f \restriction u = f \restriction (u \cap \text{ Dom}(f))$.

13.4 Fact. $(p \restriction \alpha) \leq p$ for $p \in \mathbb{P}$.

13.5 Fact. In $\mathbb{P}$, every increasing sequence of length $< \lambda$ has a lub: essentially the union.

Proof. Trivial (use compactness and clause (h) of Definition 13.2). $\square_{13.5}$

13.6 Fact. For $\alpha < \lambda$, $\{p \in P : \alpha \in u^p\}$ is dense open.

Proof. If $p \in \mathbb{P}$ let us define $q = (u^q, \mathbf{B}^q, \mathscr{F}^q)$, $u^q = u^p \cup \{\alpha\}$, $\mathbf{B}^q$ is $\mathbf{B}^p$ if $\alpha \in u^p$, and is the free extension of $\mathbf{B}$ by x_α otherwise, $\mathscr{F}_\ell^q = \{f \in {}^{u^q}2 : f \restriction u^p \in \mathscr{F}_\ell^p\}$. $\square_{13.6}$

13.7 Fact. 1) If $p \in \mathbb{P}$, $p \restriction \alpha \leq q$ and $u^q \subseteq \alpha$ <u>then</u> p, q are compatible.
2) $\mathbb{P}$ satisfies the λ^+-c.c. and even in λ^+-Knaster.
3) Moreover, if $p_\alpha \in \mathbb{P}$ for $\alpha < \lambda^+$ then for some club E of λ^+ and regressive function h on E we have $\alpha \in E \wedge \beta \in E \wedge h(\alpha) = h(\beta) \wedge \text{ cf}(\alpha) = \lambda = (\beta) \Rightarrow p_\alpha, p_\beta$ are compatible.

Proof. 1) Let us define $r = (u^r, \mathbf{B}^r, \bar{\mathscr{F}}^r)$ by:

$$u^r = u^p \cup u^q, \mathscr{F}_\ell^r = \{Ff : f \in {}^{u^r}2 \text{ and } f \restriction u^p \in \mathscr{F}_\ell^p, f \restriction u^q \in \mathscr{F}_\ell^q\}.$$

Now

$(*)_1$ $\mathscr{F}_\ell^p = \mathscr{F}_\ell^r \restriction u^p$.

[Why? If $f \in \mathscr{F}_\ell^p$, then $f \restriction \alpha = f \restriction (\alpha \cap u) \in \mathscr{F}_\ell^{p \restriction \alpha}$ but $p \restriction \alpha \leq q$. Hence by the definition on the order of $\mathbb{P}$ there is $g \in \mathscr{F}_\ell^q$ such that $f \restriction \alpha \subseteq g$, so $f \cup g \in \mathscr{F}_\ell^r$, $(f \cup q) \restriction u^p = f$, so $\mathscr{F}_\ell^p \subseteq \mathscr{F}_\ell^r \restriction u^p$. The other direction holds by the choice of $\mathscr{F}_\ell^r$.]

$(*)_2$ $\mathscr{F}_\ell^q = \mathscr{F}_\ell^r \restriction u^p$.

[Why? Similarly using 13.4.]

$(*)_3$ $\mathscr{F}^r_{\ell+1} \subseteq \mathscr{F}^r_\ell$.

[Why? As $\mathscr{F}^p_{\ell+1} \subseteq \mathscr{F}^p_\ell, \mathscr{F}^q_{\ell+1} \subseteq \mathscr{F}^q$.]

$(*)_4$ if $f \in \mathscr{F}^r_{\ell+1}$ and $\beta \in u^r$ then for some $g \in \mathscr{F}^r_\ell$ we have $f \restriction \beta \subseteq g$ & $f(\beta) \neq g(\beta)$.

[Why? The proof splits into two cases:

Case 1. $\beta \in u^q$.

So $f \restriction \alpha \in \mathscr{F}^q_{\ell+1} \restriction \alpha$ but $q \in P$ so there is $g_0 \in \mathscr{F}^q_\ell$ such that $(f \restriction \alpha) \restriction \beta \subseteq g_0$, $(f \restriction \alpha)(\beta) \neq g_0(\beta)$ hence $g_0 \in \mathscr{F}^q_\ell = \mathscr{F}^p_\ell \restriction \alpha$ therefore $g_0 \restriction (u^p \cap \alpha) \in \mathscr{F}^{p \restriction \alpha}_\ell$ hence there is g_1 such that

$$g_0 \restriction (u^p \cap \alpha) \subseteq g_1 \in F^p_\ell.$$

So $g_0 \cup g_1 \in \mathscr{F}^r_\ell$ is as required.

Case 2. $\beta \notin u^q$.

So $\beta \in u^p \backslash \alpha$. Now $f \restriction u^p \in \mathscr{F}^p_{\ell+1}$ hence there is $f' \in \mathscr{F}^p_\ell$ such that

$$f' \restriction (u^p \cap \beta) = f \restriction (u^p \cap \beta), \quad f'(\beta) \neq f(\beta).$$

Now $f \restriction \alpha \in \mathscr{F}^q_{\ell+1}$ hence by clause (e), we have $f \restriction \alpha \in \mathscr{F}^q_\ell$ hence

$$(f \restriction \alpha) \cup f' \in \mathscr{F}^r_\ell \text{is as required.}$$

By $\mathscr{F}^r_\ell$ we can define $\mathbf{B}^r$ and is as required.]

2), 3) Follows from (1). $\square_{13.7}$

13.8 Claim. *Assume that $k > 2n(*) + 1$, $\langle \delta_\ell : \ell < k \rangle$ is increasing, $\delta_\ell < \lambda$; we stipulate $\delta_k = \lambda^+$, for $\ell < k$, $p_\ell \in \mathbb{P}$, $p_\ell \restriction \delta_\ell = p^*$, $u^{p_\ell} \subseteq \delta_{\ell+1}$ and for $\ell, m < k$, $\mathrm{OP}_{u^{p_m}, u^{p_\ell}} : u^{p_\ell} \to u^{p_m}$ maps p_ℓ to p_m (the natural meaning $\mathrm{otp}(u^{p_\ell}) = \mathrm{otp}(u^{p_m})$ and*

$$\mathscr{F}^{p_\ell}_n = \{f \circ \mathrm{OP}_{u^{p_\ell}, u^{p_m}} : f \in \mathscr{F}^{p_m}_n\}$$

so $\mathrm{OP}_{u^{p_\ell}, u^{p_m}}$ induces an isomorphism $\mathrm{OP}_{p_\ell, p_m}$ from $\mathbf{B}^{p_\ell}$ onto $\mathbf{B}^{p_m}$). Then there is $q \in \mathbb{P}$ such that

(a) $\bigwedge_{m<k} p_m \leq q$

(b) *if* $b \in \mathbf{B}^{p_0}$ *then* $\mathbf{B}^q \models$ "$b = \bigcup\limits_{\substack{u \subseteq (0,k) \\ |u| > n(*)}} \bigcap\limits_{m \in u} \mathrm{OP}_{p_m,p_0}(b))$".

Proof. Let us define q: put $u^q = \bigcup\limits_{m<k} u^{p_m}$ and

$$\begin{aligned}\mathscr{F}^q_\ell = \{f \in {}^{(u^q)}2 : & n(*) - \ell \geq |\{m \in (0,k) : (\exists \alpha \in u^{p_0} \backslash u^{p^*}) \\ & [f(\mathrm{OP}_{u^{p_m},u^{p_0}}(\alpha)) \neq f(\alpha)]\}| \\ & \text{and } f \restriction u^{p_m} \in \mathscr{F}^{p_m}_\ell \text{ for } m < k\}.\end{aligned}$$

Now note

$(*)_1$ $\mathscr{F}^{p_m}_\ell = \mathscr{F}^q_\ell \restriction u^{p_m}$.

[Why? If $f \in \mathscr{F}^q_\ell$ then $f \restriction u^{p_m} \in \mathscr{F}^{p_m}_\ell$ by the definition of $\mathscr{F}^q_\ell$. If $f \in \mathscr{F}^{p_m}_\ell$ then for $m_1 < k$ we let $f_{m_1} = f \circ \mathrm{OP}_{u^{p_{m_1}},u^{p_m}}$, so $\bigcup\limits_{m_1<k} f_{m_1} \in \mathscr{F}^q_\ell$ and we are done.]

$(*)_2$ if $f \in \mathscr{F}^q_{\ell+1}, \alpha \in u^q$ <u>then</u> for some $g \in \mathscr{F}^q_\ell$

$$g \restriction \alpha = f \restriction \alpha, \;\; g(\alpha) \neq f(\alpha).$$

[Why? If $\alpha \in u^{p_0}$ we have $f \restriction u^{p_0} \in \mathscr{F}^{p_0}_{\ell+1}$ and there is $g_0 \in \mathscr{F}^{p_0}_\ell$, such that $g_0 \restriction \alpha = f \restriction \alpha$, $g_0(\alpha) \neq f(\alpha)$. Let $g_m = \mathrm{OP}_{u^{p_0},u^{p_m}} \circ g_0$. Then $g = \bigcup\limits_{m<k} g_m$ is as required.

If not, $\alpha \in u^{p_m} \backslash u^{p^*}$ for some $m > 0$, so $\alpha \geq \delta_m$ and $f \restriction u^{p_m} \in \mathscr{F}^{p_m}_{\ell+1}$ so there is $g \in \mathscr{F}^{p_m}_{\ell+1}, g \restriction \alpha = f \restriction \alpha$, $g(\alpha) \neq f(\alpha)$. Now $g^* = g \cup (f \restriction (\bigcup\limits_{\substack{m_1<k \\ m_1 \neq m}} u^{p_{m_1}})$ is as required.] So

$(*)_3$ $q \in \mathbb{P}$ and $p_m \leq q$.

So (a) of the conclusion of the claim holds. By clause (i) of Definition 13.2 and the choice of q also clause (b) holds. $\square_{13.8}$

13.9 Conclusion. $\Vdash_{\mathbb{P}^{n(*)}_x}$ "$\underset{\sim}{\mathbf{B}}$ is a Boolean algebra generated by $\{x_\alpha : \alpha < \lambda^+\}$, which is $n(*)$-free hence $\mathrm{irr}_{n(*)}(\underset{\sim}{\mathbf{B}}) \geq \lambda^+$ but $\mathrm{irr}_{2n(*)+1}(\underset{\sim}{\mathbf{B}}) = \lambda$".

Proof. Putting together the claims. $\square_{13.9}$

13.10 Conclusion. If $\lambda = \lambda^{<\lambda} > \aleph_0$ and the forcing notion $\mathbb{P}$ is $\mathbb{P} = \prod\limits_n \mathbb{P}^n_\lambda$ (where $\mathbb{P}^n_\lambda$ is from Definition 13.2) <u>then</u>

- $(*)$ $\mathbb{P}$ is a λ-complete λ^+-c.c. forcing notion, and in $\mathbf{V}^{\mathbb{P}}$ for some Boolean algebras $\mathbf{B}_n (n < \omega)$ we have
  - (a) $\mathrm{irr}_n(\mathbf{B}_n) = \lambda^+$, $\mathrm{irr}_{2n+1}(\mathbf{B}_n) = \lambda$
  - (b) for $\mathscr{D}$ a nonprincipal ultrafilter on ω, $\lambda^+ \le \mathrm{irr}(\prod\limits_{n<\omega} \mathbf{B}_n/\mathscr{D}, \prod\limits_{n<\omega} (\mathrm{irr}(\mathbf{B}_n))/\mathscr{D} = \lambda^\omega/\mathscr{D} = \lambda$
  - (c) So $\mathrm{irr}(\prod\limits_{n<\omega} \mathbf{B}_n/\mathscr{D}) > \prod\limits_{n<\omega} \mathrm{irr}(\mathbf{B}_n)/\mathscr{D}$

Proof. The λ^+-c.c. follows from 13.7(3) as $\lambda = \lambda^{\aleph_0}$. The $\mathbf{B}_n$ are from 13.2. The proof that $\Vdash_{\mathbb{P}}$ "$\mathrm{irr}_n(\underset{\sim}{\mathbf{B}}_n) = \lambda^+$ but $\mathrm{irr}_{2n+2}(\underset{\sim}{\mathbf{B}}) = \lambda'$ is like the proof of 13.9.

Concerning $\mathrm{irr}((\prod\limits_{n<\omega} \underset{\sim}{\mathbf{B}}_n/\mathscr{D}) = \lambda^+$ use $x^*_\alpha = \langle x^n_\alpha : n < \omega\rangle/\mathscr{D} \in \prod\limits_{n<\omega} \underset{\sim}{\mathbf{B}}_n/\mathscr{D}$. $\square_{13.10}$

13.11 Comment. 1) Surely in 13.9 we can fix exactly the n such that $\mathrm{irr}_n(\mathbf{B}) = \lambda^+$, $\mathrm{irr}_{n+1}(\mathbf{B}) = \lambda$ (the assertion outline in [Sh 620] is wrong. We first note an approximation

- $\circledast$ if $p \in \mathbb{P}$ and $\alpha_0 < \alpha_1 < \ldots < \alpha_{2n(*)-1}$ are in u^p, then there are $f', f'' \in \mathscr{F}^p_0$ satisfying $f' \restriction (u^p \cap \alpha_0) = f'' \restriction (u^p \cap \alpha_0)$ and $f'(\alpha_0) \ne f''(\alpha_0)$ but $f' \restriction \{\alpha_1, \ldots, \alpha_{2n(*)-1}\} = f'' \restriction \{\alpha_1, \ldots, \alpha_{2n(*)-2}\}$.

[Why? By clause (g) of Definition 13.2 (as $\mathscr{F}_{n(*)} \ne \emptyset$ by clause (f) of Definition 13.2) there are $f'_{n(*)-1} \in \mathscr{F}_{n(*)-1}, f''_{n(*)} \in \mathscr{F}_{n(*)}$ such that $f'_{n(*)-1} \restriction (u^p \cap \alpha_0) = f''_{n(*)-1}(u^p \cap \alpha_0)$ and $f'_{n(*)-1}(\alpha_0) \ne f''_{n(*)-1}(\alpha_0)$. Now we choose by downward induction on $\ell \le n(*) - 1$, a member f'_ℓ of $\mathscr{F}_\ell$ such that $f'_\ell \restriction \{\alpha_1, \ldots, \alpha_{n(*)-1-\ell}\} = f''_{n(*)-1} \restriction \{\alpha_1, \ldots, \alpha_{n(*)-1-\ell}\}$ and $f' \restriction (u^p \cap \alpha_1) = f' \restriction (u^p \cap \alpha_1)$, clearly possible: for $\ell = n(*) - 1$, the function $f'_{n(*)-1}$ has already been chosen, for $\ell - 1$ use clause (g) of Definition 13.2. Next we choose by downward induction on $\ell \le n(*)$ a member f''_ℓ of $\mathscr{F}^p_\ell$ such that $f''_\ell \restriction (\alpha_{n(*)} \cap u^p) = f'' \restriction (\alpha_{n(*)} \cap u^p)$ and $f''_\ell \restriction \{\alpha_{n(*)}, \alpha_{n(*)+1}, \ldots, \alpha_{n(*)+n(*)-\ell-1}\} \subseteq f'_0$.
Now f'_0, f''_0 are as required.]

So $\Vdash_P$ "in $x_\alpha : \alpha < \lambda^+\} \subseteq \underset{\sim}{\mathbf{B}}, x_\alpha \notin \langle\{x_\beta : \beta \in u \cup \alpha\}\rangle$ if $u \in [\lambda^+]^{\leq 2n(*)-1}$ and $\alpha \in \lambda^+ \backslash u$".

We now note that

$$\boxtimes \Vdash_{\mathbb{P}} \text{"irr}_n(\underset{\sim}{\mathbf{B}} = \lambda^+ \text{ iff } n \leq 2n(*)\text{"}.$$

First note that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{Y} \subseteq \lambda^+$ has cardinaltiy λ^+" where $\underset{\sim}{Y} = \{\alpha < \lambda^+$: for some (equivalently every) $p \in \underset{\sim}{G}_{\mathbb{P}}$ satisfying $\alpha \in u^p$ we have: $\{0,1\} = \{f(x_\alpha) : f \in \mathscr{F}^p_{n(*)}\}$. Now like $\circledast$

$$\circledast_1 \Vdash_{\mathbb{P}} \text{"}\{x_\alpha : \alpha \in \underset{\sim}{Y}\} \text{ exemplifies irr}_{2n(*)}(\underset{\sim}{\mathbf{B}} = \lambda^{+}\text{"}.$$

We can change slightly the definition of $\mathbb{P} = \mathbb{P}^{n(*)}_\lambda$, demanding that $f \in \mathscr{F}_{n(*)} \Rightarrow f$ is constantly zero, then we get $\Vdash$ $\text{irr}_n(\underset{\sim}{\mathbf{B}}) = \lambda^+$ iff $n \leq 2n(*) - 1$.

13.12 Claim. *Assume*

- (A) $\lambda = \text{tcf}(\prod\limits_{i<\delta} \lambda_i/J)$
- (B) $\bar\lambda = \langle\lambda_i : i < \delta\rangle$ *is a sequence of regular cardinals* $> |\delta|$
- (C) $\lambda_i >$ max pcf$\{\lambda_j : j < i\}$, *so necessarily* $J^{\text{bd}}_\delta \subseteq J$
- (D) $\langle A_\zeta : \zeta < \kappa\rangle$ *is a sequence of pairwise disjoint members of* J^+
- (E) $\mathscr{D}$ *is a uniform ultrafilter on* κ.

Then, we can find a Boolean algebra $\mathbf{B}_\zeta$ *for* $\zeta < \kappa$ *such that for* inv $\in \{s, hd, hL\}$ *(see Monk [M2])*

- (a) $inv^+(\mathbf{B}_\zeta) \leq \lambda$ *so* $\lambda = \chi^+ \Rightarrow inv(\mathbf{B}_\zeta) \leq \chi$ *(moreover* $inv^+_2(\mathbf{B}_3) \leq \lambda$*; see [RoSh 534])*
- (b) $inv^+(\prod\limits_{\zeta<\kappa} \mathbf{B}_\zeta/\mathscr{D}) = \lambda$ *(so if* $\lambda = \chi^+$ *then* $inv(\prod\limits_{\zeta<\kappa} \mathbf{B}_\zeta/\mathscr{D}) \geq \lambda)$

Proof. Let $\bar\eta = \langle\eta_\alpha : \alpha < \lambda\rangle$ be a $<_J$-increasing cofinal sequence of members of $\prod\limits_{i<\delta} \lambda_i$ such that

$$\zeta < \kappa \Rightarrow \lambda_\zeta > |\{\eta_\alpha \restriction \zeta : \alpha < \lambda\}|$$

(such $\bar\eta$ exists by [Sh:g, II,3.5]). We define $(\mathbf{B}^*_{\zeta,i}\bar x^-_{\zeta,i}, \bar x^+_{\zeta,i})$ for $\zeta < \kappa$, $i < \delta$ as follows. Let $I_i = J^{\text{bd}}_{\lambda_i}$, and $\bar x^-_{\zeta,i} = \langle x^-_{\zeta,i,\alpha} : \alpha < \lambda_i\rangle$, $\bar x^+_{\zeta,i} = \langle x^+_{\zeta,i,\alpha} : \alpha < \lambda_i\rangle$.

Case 1. $i \notin \cup\{A_\varepsilon : \varepsilon \in [\zeta, \kappa)\}$.

Let $\mathbf{B}_{\zeta,i}$ be the Boolean algebra generated by $\{x^-_{\zeta,i,\alpha}, x^+_{\zeta,i,\alpha} : \alpha < \lambda_i\}$ freely except that $x^-_{\zeta,i,\alpha} \le x^+_{\zeta,i,\alpha}$, and $(x^+_{\zeta,i,\alpha} - x^-_{\zeta,i,\alpha}) \cap (x^+_{\zeta,i,\beta} - x^-_{\zeta,i,\beta}) = 0$ when $\alpha < \beta < \lambda_i$.

Case 2. $i \in \cup\{A_\varepsilon : \varepsilon \in [\zeta, \kappa)\}$.

Let $\mathbf{B}_{\zeta,i}$ be the Boolean algebra generated by $\{x^-_{\zeta,i,\alpha}, x^+_{\zeta,i,\alpha} : \alpha < \lambda_i\}$ freely except that

$$\alpha < \beta \Rightarrow x^-_{\zeta,i,\alpha} \le x^+_{\zeta,i,\alpha} \le x^-_{\zeta,i,\beta} \le x^+_{\zeta,i,\beta}$$

(e.g. $\mathbf{B}_{\zeta,i} \subseteq \mathscr{P}(\lambda_i)$, $x^-_{\zeta,i,\alpha} = [0, 4\alpha+1)$, $x^+_{\zeta,i,\alpha} = [0, 4\alpha+2)$).

Let $\mathbf{B}_\zeta$ be constructed as in 12.1, 12.3 from $\bar\lambda$, $\langle I_i : i < \delta\rangle, (\mathbf{B}_{\zeta,i}, \bar x^-_{\zeta,i}, \bar x^+_{\zeta,i})$ for $i < \delta$, and let $y^\zeta_\alpha, y^\zeta_\eta$ be as there.

Now $\mathrm{inv}^+\Big(\prod\limits_{\zeta<\kappa} \mathbf{B}_\zeta/\mathscr{D}\Big) > \lambda$ is exemplified by $\langle y^*_\alpha : \alpha < \lambda\rangle$ where $y^*_\alpha = \langle y^\zeta_\alpha : \zeta < \kappa\rangle/\mathscr{D}$, because for $\alpha < \lambda$, $u \subseteq \lambda \setminus \{\alpha\}$ finite, for some $\zeta^* < \kappa$, we have $\beta \in u \;\Rightarrow\; \ell g(\eta_\alpha \cap \eta_\beta) \in \delta \setminus \bigcup\limits_{\varepsilon\in[\zeta,\kappa)} A_\varepsilon$, hence

$$\zeta \in [\zeta_{\alpha,\beta}, \kappa) \Rightarrow \mathbf{B}_\zeta \models y^\zeta_\alpha - \bigcap_{\beta\in u} y^\zeta_\beta > 0.$$

Hence $\{\zeta < \kappa : \mathbf{B}_\zeta \models y^\zeta_\alpha \cap y^\zeta_\beta = 0\} \supseteq [\zeta_{\alpha,\beta}, \kappa) \in \mathscr{D}$ and therefore $\prod\limits_{\zeta<\kappa} \mathbf{B}_\zeta/\mathscr{D} \models$ "$y^*_\alpha - \bigcap\limits_{\beta\in u} y^*_\beta > 0$".

Lastly, $\mathrm{inv}^+_{(2)}(\mathbf{B}_\zeta) \le \lambda$ follows by 12.19(2) for $\tau(x_0, x_1, x_2) = (x_1 - x_0 \cup x_2)$ with the variables permuted according to the particular inv. $\square_{13.12}$

13.13 Claim. *Claim 13.12 holds for Length too.*

Proof. We repeat the proof of 13.12, but in the definition of $\mathbf{B}_{\zeta,i}$ just interchange the two cases.

Case 1. $i \notin \cup\{A_\varepsilon : \varepsilon \in [\zeta, \kappa]\}$.

Let $\mathbf{B}_{\zeta,i}$ be as $\mathbf{B}_{\zeta,i}$ in case 2 in the proof of 13.12.

Case 2. $i \in \cup\{A_\varepsilon : \varepsilon \in [\zeta, \kappa)\}$.

As in Case 1 in the proof of 13.12 or just let $\mathbf{B}_{i,\zeta}$ be generated by $\{x^-_\alpha, x^+_\alpha : \alpha < \lambda_i\}, \{x^-_{\zeta,i,\alpha}, x^+_{\zeta,i,\alpha} : \alpha < \lambda_i\}$ freely except $x^-_{\zeta,i,\alpha} \le x^+_{\zeta,i,\alpha}$.

Now for $\alpha < \beta < \lambda$, letting $i(\alpha,\beta) = \operatorname{Min}\{i : \eta_\alpha(i) \neq \eta_\beta(i)\}$ and $\zeta_{\alpha,\beta} = \operatorname{Min}\{\zeta : i(\alpha,\beta) \notin \cup\{A_\varepsilon : \varepsilon \in [\zeta,\kappa)\}$ we have

$$\zeta \in [\zeta_{\alpha,\beta},\kappa) \Rightarrow \mathbf{B}_{\zeta,i} \models \text{“}y^\zeta_\alpha \le y^\zeta_\beta \text{ or } y^\zeta_\beta < y^\zeta_\alpha\text{”}.$$

hence

$$\prod_{\zeta<\kappa} \mathbf{B}_\zeta \models \text{“}y^*_\alpha < y^*_\beta \text{ or } y^*_\beta < y^*_\alpha\text{”},$$

where $y^*_\alpha = \langle y^\zeta_\alpha : \zeta < \kappa\rangle/\mathscr{D}$.

As for $\mathrm{Length}^+(\mathbf{B}_\zeta) \le \lambda$, it is by 12.9(1). $\square_{13.13}$

13.14 Conclusion. 1) If $\mathscr{D}$ is a uniform ultrafilter on κ, then for a class of cardinals $\chi = \chi^\kappa$ and Boolean algebras $\mathbf{B}_i$ for $i < \kappa$ such that, for $\mathrm{inv} \in \{s, hL, hd\}$ we have:

(a) $\mathrm{inv}(\mathbf{B}_i) \le \chi$ hence $\prod\limits_{i<\kappa} \mathrm{inv}(\mathbf{B}_i) \le \chi$

or

(b) $\mathrm{inv}(\prod\limits_{i<\kappa} \mathbf{B}_i/\mathscr{D}) = \chi^+$.

2) Similarly with inv = Length.

Proof. Let χ be any strong limit singular cardinal of cofinality $> \kappa$. So by [Sh:g, II,§1] we can find $\langle \lambda_i : i < \mathrm{cf}(\chi)\rangle$, strictly increasing sequence of regular cardinals $< \chi$ with $\mathrm{tcf}(\prod\limits_{i<\mathrm{cf}(\chi)} \lambda_i/J^{\mathrm{bd}}_{\mathrm{cf}(\chi)}) = \chi^+$. Without loss of generality $\prod\limits_{i<j} \lambda_i < \lambda_j$ and let for $i < \kappa$, $A_i = \{\alpha\kappa + i : \alpha < \mathrm{cf}(\chi)\}$. So we can apply 13.12 (for part (1)) or 13.13 (for part (2)). $\square_{13.14}$

Remark. For cellularity similar results hold (in ZFC), i.e., $c(\mathbf{B}_n) \le \lambda$, $c(\prod\limits_{n<\omega} \mathbf{B}_n) > \lambda$, see on it in Monk [M2, p.61-62]; by [Sh:g, III,4.11,p.181,4.12] so this applies to $\lambda = \mu^+$ for $\lambda > \aleph_1$ by [Sh:g, II,4.1], [Sh 572], to λ inaccessible not Mahlo by [Sh:g, III,4.8](2),p.177 and for many Mahlo cardinals (see [Sh:g, III,4.10A,p.178]. For incomparability number (Inc) similar results are proved “almost in ZFC”, see [Sh 462].

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