Ultrafilters on ω —
— their ideals and their cardinal characteristics

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Abstract

For a free ultrafilter $\mathcal{U}$ on ω we study several cardinal characteristics which describe part of the combinatorial structure of $\mathcal{U}$. We provide various consistency results; e.g. we show how to force simultaneously many characters and many π–characters. We also investigate two ideals on the Baire space ω^ω naturally related to $\mathcal{U}$ and calculate cardinal coefficients of these ideals in terms of cardinal characteristics of the underlying ultrafilter.

1991 Mathematics subject classification. 03E05 03E35

Key words and phrases. ultrafilter, P–point, Ramsey ultrafilter, character, π–character, ideal, Ramsey null sets (nowhere Ramsey sets), cardinal coefficient, Mathias forcing, Laver forcing, Easton forcing.

⋆ Research partially supported by DFG–grant Nr. Br 1420/1–1.

⋆⋆ Supported by the German–Israeli Foundation for Scientific Research & Development Grant No. G-294.081.06/93.

† Publication 642

Introduction

Let $\mathcal{U}$ be a non–principal ultrafilter on the natural numbers ω. Recall that $\mathcal{U}$ is a *P–point* iff for all countable $\mathcal{A} \subseteq \mathcal{U}$ there is $U \in \mathcal{U}$ with $U \setminus A$ being finite for all $A \in \mathcal{A}$. $\mathcal{U}$ is said to be *rapid* iff for all $f \in \omega^\omega$ there is $U \in \mathcal{U}$ with $|U \cap f(n)| \leq n$ for all $n \in \omega$. $\mathcal{U}$ is called *Ramsey* iff given any partition $\langle A_n;\ n \in \omega \rangle$ of ω, there is either $n \in \omega$ with $A_n \in \mathcal{U}$ or $U \in \mathcal{U}$ with $|A_n \cap U| \leq 1$ for all $n \in \omega$. It is well–known (and easily seen) that Ramsey ultrafilters are both rapid and P–point.

With $\mathcal{U}$ we can associate ideals on the real numbers (more exactly, on the Baire space ω^ω) in various ways. One way of doing this results in the well–known ideal $r^0_{\mathcal{U}}$ of *Ramsey null sets* with respect to $\mathcal{U}$ (see § 2 for the definition). Another, less known, ideal related to $\mathcal{U}$ was introduced by Louveau in [Lo] and shown to coincide with both the meager and the nowhere dense ideals on ω^ω with respect to a topology somewhat finer than the standard topology (see § 3 for details). This ideal which we call $\ell^0_{\mathcal{U}}$ is related to *Laver forcing with* $\mathcal{U}$, $\mathbb{L}_{\mathcal{U}}$ [Bl 1], in a way similar to the connection between $r^0_{\mathcal{U}}$ and *Mathias forcing with* $\mathcal{U}$, $\mathbb{M}_{\mathcal{U}}$. Furthermore, $\ell^0_{\mathcal{U}}$ and $r^0_{\mathcal{U}}$ coincide in case $\mathcal{U}$ is a Ramsey ultrafilter [Lo], as do $\mathbb{L}_{\mathcal{U}}$ and $\mathbb{M}_{\mathcal{U}}$ [Bl 1].

A natural problem which has, in fact, been studied for many ideals $\mathcal{I}$ on the reals [BJ 1] is to figure out the relationship between certain *cardinal coefficients* of $\mathcal{I}$ as well as to determine their possible values. An example of such a cardinal coefficient is the *additivity* of $\mathcal{I}$, $\mathsf{add}(\mathcal{I})$, that is, the size of the smallest subfamily of $\mathcal{I}$ whose union is not in $\mathcal{I}$; another one, the *uniformity* of $\mathcal{I}$, $\mathsf{non}(\mathcal{I})$, is the cardinality of the least set of reals which does not belong to $\mathcal{I}$ (see § 2 for more such coefficients). One of the goals of this work is to carry out such an investigation for $\mathcal{I}$ being either $\ell^0_{\mathcal{U}}$ or $r^0_{\mathcal{U}}$. (In fact, this was the original motivation for this paper.)

In sections 2 and 3 of the present paper we reduce this problem to a corresponding problem about cardinal characteristics of the underlying ultrafilter $\mathcal{U}$, by actually calculating the ideal coefficients in terms of the latter as well as of two other cardinal invariants of the continuum, the *unbounding number* $\mathfrak{b}$ and the *dominating number* $\mathfrak{d}$ (see § 1 for the definitions). Here, by a *cardinal characteristic* of $\mathcal{U}$, we mean a cardinal number describing part of the combinatorial structure of $\mathcal{U}$, such as the *character* of $\mathcal{U}$, $\chi(\mathcal{U})$, that is, the size of the smallest subfamily $\mathcal{F}$ of $\mathcal{U}$ such that each member of $\mathcal{U}$ contains some member of $\mathcal{F}$ — or the π*–character* of $\mathcal{U}$, $\pi\chi(\mathcal{U})$, the cardinality of the least $\mathcal{F} \subseteq [\omega]^\omega$ such that each element of $\mathcal{U}$ contains an element of $\mathcal{F}$ (see § 1 for details). We show for example that $\mathsf{non}(r^0_{\mathcal{U}}) = \pi\chi(\mathcal{U})$ (Theorem 1(c) in § 2) or that the uniformity of $\ell^0_{\mathcal{U}}$ can be expressed as the maximum of $\mathfrak{d}$ and some cardinal closely related to $\pi\chi(\mathcal{U})$ (Theorem 2(c) in § 3). The interest of such characterizations lies in the fact that, unlike the ideal coefficients, the ultrafilter characteristics have been studied previously, in particular in connection with ongoing research on $\beta\omega$ (see e.g. [vM]) but also in investigations of the cofinality of ultraproducts of the form $\omega^\omega/\mathcal{U}$, and so already established results on the latter can be used to show something on the former. Furthermore, the ultrafilter characteristics as well as the classical cardinal invariants of the continuum are combinatorially simpler objects than the ideal coefficients and thus easier to calculate in any given model of set theory. Accordingly, we investigate the ultrafilter characteristics in the remainder of our work (§§ 1, 4 – 7).

It turns out that only rather elementary facts about these characteristics and their relationship to other cardinal invariants can be proved in ZFC. Most of these results which we expound in section 1 are well–known. To make our paper self–contained, we include proofs. (For the consequences of these ZFC–results on the ideal coefficients, see the corollaries in sections 2 and 3.)

This leaves the field wide open for independence results of various sorts to which the main body of the present paper (sections 4 to 7) is devoted. — First, we deal with distinguishing between different coefficients for a fixed Ramsey ultrafilter $\mathcal{U}$. Most questions one would ask in this direction have been solved long ago (see §§ 1 and 4). The remaining case, to force a Ramsey ultrafilter $\mathcal{U}$ with $\pi\chi(\mathcal{U}) < \chi(\mathcal{U})$, is taken care of in a rather straightforward construction in Theorem 3 in section 4. — Next, we are concerned with producing simultaneously many different ultrafilters for which a fixed cardinal characteristic assumes many different values. For one of our cardinals, this has been done by Louveau ([Lo], see also § 1) under MA long ago. For the others, it is a much more difficult problem which we tackle in sections 5 and 6. For example we show that given a set of uncountable cardinals R in a model of GCH, we can force that for each $\lambda \in R$ there is an ultrafilter $\mathcal{U}$ with $\pi\chi(\mathcal{U}) = \lambda$ (Theorem 4(a) and Corollary 5.5). Similarly, given a set of cardinals of uncountable cofinality R, such a model can be extended to one which has an ultrafilter (even a P–point) $\mathcal{U}$ with $\chi(\mathcal{U}) = \lambda$ for all $\lambda \in R$ (Theorem 5 and Corollary 6.1). For quite many years, R. Frankiewicz, S. Shelah and P. Zbierski have planned to write a paper proving this for regulars (i.e. for any set of regulars R, there is a forcing extension with a P–point with character λ for each $\lambda \in R$). The proof of Theorem 5 can be extended in various ways, e.g. to make all the ultrafilters Ramsey (Corollary 6.2) or to prove a dual result (Theorem 7). It is an elegant combination of a *ccc*–iteration and an Easton product. Results on characters and π–characters like those described in sections 5 and 6 are interesting not just because they shed light on the ideal coefficients studied in section 2 and 3, but also because χ and $\pi\chi$ play a role in the topological investigation of $\beta\omega$ (see [vM]). — Finally, we explore in section 7 the connection between the ultrafilter characteristics and the *reaping* and *splitting numbers* $\mathfrak{r}$ and $\mathfrak{s}$ (see § 1 for the definitions). Using iterated forcing we show (Theorem 8) that a result of Balcar and Simon ([BS], see also Proposition 7.1) which says that $\mathfrak{r}$ is the minimum of the π–characters cannot be dualized to a corresponding statement about $\mathfrak{s}$. The main technical device of the proof is a careful analysis of $\mathbb{L}_{\mathcal{U}}$–names for reals where $\mathcal{U}$ is a Ramsey ultrafilter.

We close with a list of open problems in section 8.

All sections of this work from section 2 onwards depend on section 1, but can be read independently of each other; however, § 3 uses the basic definitions of § 2; and sections 5 and 6 are closely intertwined.

Notational remarks and some prerequisites. We refer to standard texts like [Je] or [Ku] for any undefined notion. $\mathfrak{c}$ stands for the cardinality of the continuum. $cf(\kappa)$ is the cofinality of the cardinal κ. Given a function f, $dom(f)$ is its domain, $rng(f)$ its range, and if $A \subseteq dom(f)$, then $f{\restriction}A$ is the restriction of f to A and $f[A] := rng(f{\restriction}A)$ is the image of A under f. $\forall^\infty n$ means *for all but finitely many* n, and $\exists^\infty n$ is used for *there are infinitely many* n.

$[\omega]^\omega$ ($[\omega]^{<\omega}$, respectively) denotes the infinite (finite, resp.) subsets of ω; $\omega^{\uparrow\omega}$ ($\omega^{\uparrow<\omega}$, resp.) stands for the strictly increasing functions from ω to ω (for the strictly increasing finite sequences of natural numbers, resp.). Identifying subsets of ω with their increasing enumerations naturally identifies $[\omega]^\omega$ and $\omega^{\uparrow\omega}$. We reserve letters like σ, τ for elements of $\omega^{<\omega}$ and $\omega^{\uparrow<\omega}$, and letters like s, t for elements of $[\omega]^{<\omega}$. $\hat{}$ is used for concatenation of sequences (e.g., $\sigma\hat{\ }\langle n\rangle$). Given a tree $T \subseteq \omega^{<\omega}$, we denote by *stem*$(T)$ its stem, and by $[T] := \{f \in \omega^\omega;\ \forall n\ (f\restriction n \in T)\}$ the set of its branches. Given $\sigma \in T$, we let $T_\sigma := \{\tau \in T;\ \tau \subseteq \sigma\ \vee\ \sigma \subseteq \tau\}$, the restriction of T to σ, and $succ_T(\sigma) := \{n \in \omega;\ \sigma\hat{\ }\langle n\rangle \in T\}$. For $A, B \subseteq \omega$, we say $A \subseteq^* B$ (A *is almost included in* B) iff $A \setminus B$ is finite. If $\mathcal{A} \subseteq [\omega]^\omega$ and $B \in [\omega]^\omega$ satisfies $B \subseteq^* A$ for all $A \in \mathcal{A}$, we call B a *pseudointersection* of $\mathcal{A}$. A sequence $\mathcal{T} = \langle T_\alpha;\ \alpha < \kappa\rangle$ is called a κ*–tower* (or: *tower of height* κ) iff $T_\beta \subseteq^* T_\alpha$ for $\beta \geq \alpha$ and $\mathcal{T}$ has no pseudointersection.

Concerning forcing, let $\mathbb{P}$ be a p.o. in the ground model V. $\mathbb{P}$–names are denoted by symbols like $\dot{f}$, $\dot{X}$, ..., and for their interpretations in the generic extension $V[G]$, we use $f = \dot{f}[G]$, $X = \dot{X}[G]$... We often confuse Boolean–valued models $V^{\mathbb{P}}$ and the corresponding forcing extensions $V[G]$ where G is $\mathbb{P}$–generic over V. $\mathbb{P}$ is called σ*–centered* iff there are $P_n \subseteq \mathbb{P}$ with $\bigcup_n P_n = \mathbb{P}$ and, for all n and $F \subseteq P_n$ finite, there is $q \in \mathbb{P}$ with $q \leq p$ for all $p \in F$. $\star$ is used for two–step iteration (e.g. $\mathbb{P} \star \dot{\mathbb{Q}}$). If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha;\ \alpha < \kappa\rangle$ (where κ is a limit ordinal) is an iterated forcing construction with limit $\mathbb{P}_\kappa$ (see [B] or [Je 1] for details) and G_κ is $\mathbb{P}_\kappa$–generic, we let $G_\alpha = G_\kappa \cap \mathbb{P}_\alpha$ be the restriction of the generic, and $V_\alpha = V[G_\alpha] = V^{\mathbb{P}_\alpha}$ stands for the intermediate extension. In V_α, $\mathbb{P}_{[\alpha,\kappa)}$ denotes the rest of the iteration. $\mathbb{C}_\kappa$ (where κ is any ordinal) stands for the p.o. adding κ Cohen reals. For sections 5 and 6, we assume familiarity with Easton forcing (see [Je] or [Ku]) and the ways it can be factored. In particular, we use that if $\mathbb{P}$ is *ccc* and $\mathbb{Q}$ is ω_1–closed (in V), then $\mathbb{P}$ is still *ccc* in $V^{\mathbb{Q}}$ and $\mathbb{Q}$ is ω_1–distributive in $V^{\mathbb{P}}$. Recall that a p.o. $\mathbb{Q}$ is called λ*–distributive* iff the intersection of fewer than λ open dense subsets of $\mathbb{Q}$ is open dense. In section 7, we shall need basic facts about club sets in ω_1: that the intersection of a countable family of clubs is club, that given clubs $\{C_\alpha;\ \alpha < \omega_1\}$, their *diagonal intersection* $\{\beta \in \omega_1;\ \forall \alpha < \beta\ (\beta \in C_\alpha)\}$ is club, and that if $\mathbb{P}$ is *ccc* and $\Vdash_{\mathbb{P}}$"$\dot{C}$ is club", then there is a club D in the ground model such that $\Vdash_{\mathbb{P}}$"$D \subseteq \dot{C}$" (see [Ku, chapter II, § 6 and chapter VII, (H1)].

More notation will be introduced when needed.

On the genesis of this paper and acknowledgements. The first author is very much indebted to the members of the logic group at Charles University, Praha: to Bohuslav Balcar, Petr Simon and Egbert Thümmel for introducing him to the world of characters and π–characters; to the latter for explaining him how the cardinal characteristics of $r^0_{\mathcal{U}}$ could be read off from those of $\mathcal{U}$ in case $\mathcal{U}$ is a Ramsey ultrafilter. He gratefully acknowledges support from the Center for Theoretical Study for his stay in January/February 1995, and thanks Bohuslav Balcar for having him invited. A preliminary version of this paper, by the first author only, was circulated late in 1995. It consisted of sections 2 to 4 and 7 of the present work and one more section the results of which have been superseded. Unfortunately, it contained several inaccuracies, and a few basic results were not mentioned.

The main bulk of the important results in sections 5 and 6 were proved by the second author in September 1996 while the first author was visiting him at Rutgers University. Section 1 is joint work. We thank Alan Dow, Martin Goldstern and Claude Laflamme for comments. We also thank the referee for many valuable suggestions and for detecting a gap in the original proof of Theorem 3.

1. Setting the stage — some cardinal characteristics of ultrafilters

Let $\mathcal{U}$ be a non–principal ultrafilter on the natural numbers ω. We define the following four cardinal invariants associated with $\mathcal{U}$.

$$
\begin{aligned}
\mathfrak{p}(\mathcal{U}) &= \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq \mathcal{U}\ \wedge\ \neg\exists B \in \mathcal{U}\ \forall A \in \mathcal{A}\ (B \subseteq^* A)\}\\
\pi\mathfrak{p}(\mathcal{U}) &= \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq \mathcal{U}\ \wedge\ \neg\exists B \in [\omega]^\omega\ \forall A \in \mathcal{A}\ (B \subseteq^* A)\}\\
\pi\chi(\mathcal{U}) &= \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq [\omega]^\omega\ \wedge\ \forall B \in \mathcal{U}\ \exists A \in \mathcal{A}\ (A \subseteq^* B)\}\\
\chi(\mathcal{U}) &= \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq \mathcal{U}\ \wedge\ \forall B \in \mathcal{U}\ \exists A \in \mathcal{A}\ (A \subseteq^* B)\}
\end{aligned}
$$

The definition of $\mathfrak{p}$ is dual to the one of χ; similarly $\pi\mathfrak{p}$ and $\pi\chi$ are dual. Therefore we can expect a strong symmetry when studying these cardinals. Note that $\mathfrak{p}(\mathcal{U}) \geq \omega_1$ is equivalent to saying $\mathcal{U}$ *is a P–point*. Ultrafilters with $\pi\mathfrak{p}(\mathcal{U}) \geq \kappa$ are called *pseudo–P_κ–points* in [Ny]. $\pi\chi(\mathcal{U})$ is referred to as *π–character*, and $\chi(\mathcal{U})$ is known as the *character* of the ultrafilter $\mathcal{U}$. Furthermore, a family $\mathcal{A}$ which has the property in the definition of $\pi\chi(\mathcal{U})$ ($\chi(\mathcal{U})$, respectively) is called a *π–base* (*base*, resp.) of $\mathcal{U}$. Both these cardinals have been studied intensively, see e.g. [BK], [BS], [BlS], [Ny] and [vM].

It is easy to see that for any ultrafilter $\mathcal{U}$, the following hold: $\omega \leq \mathfrak{p}(\mathcal{U}) \leq \pi\mathfrak{p}(\mathcal{U})$, $\pi\chi(\mathcal{U}) \leq \chi(\mathcal{U}) \leq \mathfrak{c}$, and $\omega_1 \leq \pi\mathfrak{p}(\mathcal{U})$. Furthermore, $\mathfrak{p}(\mathcal{U})$ is a regular cardinal, and we have $cf(\pi\chi(\mathcal{U})) \geq \mathfrak{p}(\mathcal{U})$. (The same holds with $\pi\chi$ replaced by χ, see Proposition 1.4 below for a stronger result.) To obtain more restrictions on the possible values, and on the possible cofinalities, of these cardinals, we need to introduce some classical cardinal coefficients of the continuum. For $f, g \in \omega^\omega$, we say g *eventually dominates* f ($f \leq^* g$, in symbols) iff $f(n) \leq g(n)$ holds for almost all $n \in \omega$. If $\mathcal{U}$ is an ultrafilter, we say g *$\mathcal{U}$–dominates* f ($f \leq_{\mathcal{U}} g$, in symbols) iff $\{n;\ f(n) \leq g(n)\} \in \mathcal{U}$.

$$
\begin{aligned}
\mathfrak{b} &= \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \omega^\omega\ \wedge\ \forall g \in \omega^\omega\ \exists f \in \mathcal{F}\ (f \not\leq^* g)\}\\
\mathfrak{d} &= \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \omega^\omega\ \wedge\ \forall g \in \omega^\omega\ \exists f \in \mathcal{F}\ (g \leq^* f)\}\\
\mathfrak{s} &= \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq [\omega]^\omega\ \wedge\ \forall B \in [\omega]^\omega\ \exists A \in \mathcal{A}\ (|A \cap B| = |(\omega \setminus A) \cap B| = \omega)\}\\
\mathfrak{r} &= \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq [\omega]^\omega\ \wedge\ \forall B \in [\omega]^\omega\ \exists A \in \mathcal{A}\ (A \subseteq^* B\ \vee\ A \subseteq^* \omega \setminus B)\}\\
\mathfrak{p} &= \min\nolimits_{\mathcal{U}} \pi\mathfrak{p}(\mathcal{U})\\
\mathsf{cof}(\omega^\omega/\mathcal{U}) &= \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \omega^\omega\ \wedge\ \forall g \in \omega^\omega\ \exists f \in \mathcal{F}\ (g \leq_{\mathcal{U}} f)\}
\end{aligned}
$$

$\mathfrak{b}$ and $\mathfrak{d}$ are dual, and so are $\mathfrak{s}$ and $\mathfrak{r}$. $\mathfrak{b}$ is called *(un)bounding number*, $\mathfrak{d}$ is referred to as *dominating number*, $\mathfrak{s}$ is known as *splitting number*, $\mathfrak{r}$ is called either *reaping number* or *refinement number*, and $\mathfrak{p}$ is the *pseudointersection number*. $\mathsf{cof}(\omega^\omega/\mathcal{U})$ which is self–dual is called the *cofinality* of the ultraproduct $\omega^\omega/\mathcal{U}$. Families like $\mathcal{F}$ and $\mathcal{A}$ in the defining clauses of the first four of these numbers are referred to as *unbounding*, *dominating*, *splitting* and *reaping families*, respectively. It is known that $\mathfrak{p}$ and $\mathfrak{b}$ are regular, that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{b} \leq cf(\mathfrak{d})$, that $\mathfrak{p} \leq \mathfrak{s} \leq \mathfrak{d} \leq \mathfrak{c}$, and that $\mathfrak{b} \leq \mathfrak{r} \leq \mathfrak{c}$ (see [vD] and [Va]). Also recall that $\mathfrak{p} = \mathfrak{c}$ is equivalent to MA(σ–centered) [Be], Martin's axiom for σ–centered p.o.'s; thus all these cardinals equal $\mathfrak{c}$ under MA.

Concerning the relationship to the ultrafilter invariants, we see easily that $\pi\mathfrak{p}(\mathcal{U}) \leq \mathfrak{s}$ and $\mathfrak{r} \leq \pi\chi(\mathcal{U})$ for all ultrafilters $\mathcal{U}$. Also, MA implies $\pi\mathfrak{p}(\mathcal{U}) = \mathfrak{c}$ for all $\mathcal{U}$, while there are (under MA) Ramsey ultrafilters

$\mathcal{U}$ with $\mathfrak{p}(\mathcal{U}) = \kappa$ for all regular $\omega_1 \leq \kappa \leq \mathfrak{c}$ [Lo, Théorèmes 3.9 et 3.12]. Furthermore, $\mathsf{cof}(\omega^\omega/\mathcal{U})$ is regular and $\mathfrak{b} \leq \mathsf{cof}(\omega^\omega/\mathcal{U}) \leq \mathfrak{d}$; for more results on $\mathsf{cof}(\omega^\omega/\mathcal{U})$ see [Bl], [Ca], [Ny], [SS] and the recent [BlM]. The following proposition which relates the cofinality of $\omega^\omega/\mathcal{U}$ to other invariants is well–known. We include a proof for completeness' sake.

Proposition 1.1. (Nyikos [Ny, Theorem 1 (i) and 3 (i)], see also [Bl, Theorem 16]).
(a) If $\pi\chi(\mathcal{U}) < \mathfrak{d}$, *then* $\mathsf{cof}(\omega^\omega/\mathcal{U}) = \mathfrak{d}$. *Equivalently,* $\max\{\pi\chi(\mathcal{U}), \mathsf{cof}(\omega^\omega/\mathcal{U})\} \geq \mathfrak{d}$.
(b) If $\pi\mathfrak{p}(\mathcal{U}) > \mathfrak{b}$, *then* $\mathsf{cof}(\omega^\omega/\mathcal{U}) = \mathfrak{b}$. *Equivalently,* $\min\{\pi\mathfrak{p}(\mathcal{U}), \mathsf{cof}(\omega^\omega/\mathcal{U})\} \leq \mathfrak{b}$.

Proof. Given $f \in \omega^\omega$ and $A \in [\omega]^\omega$ define $f_A \in \omega^\omega$ by

$$f_A(n) := \min\{f(k);\ k \geq n \text{ and } k \in A\},$$

and note that if $g \in \omega^\omega$ is strictly increasing with $g \leq_{\mathcal{U}} f$ then $g \leq^* f_A$ for any $A \subseteq^* \{n;\ g(n) \leq f(n)\} \in \mathcal{U}$. $(\star)$

(a) If $\{f^\alpha;\ \alpha < \mathsf{cof}(\omega^\omega/\mathcal{U})\}$ is cofinal modulo $\mathcal{U}$ and $\{A_\beta;\ \beta < \pi\chi(\mathcal{U})\}$ is a π–base, then $\{f^\alpha_{A_\beta};\ \alpha < \mathsf{cof}(\omega^\omega/\mathcal{U})$ and $\beta < \pi\chi(\mathcal{U})\}$ is dominating by $(\star)$.

(b) If $\kappa < \min\{\pi\mathfrak{p}(\mathcal{U}), \mathsf{cof}(\omega^\omega/\mathcal{U})\}$ and $\{g^\alpha;\ \alpha < \kappa\} \subseteq \omega^\omega$ are strictly increasing, then find $f \in \omega^\omega$ with $g^\alpha \leq_{\mathcal{U}} f$ for all α. Put $A_\alpha = \{n;\ g^\alpha(n) \leq f(n)\} \in \mathcal{U}$, and find $A \subseteq^* A_\alpha$ for all α. By $(\star)$, we get $g^\alpha \leq^* f_A$ for all α, and the g^α are not unbounded. $\square$

Since we always have $\pi\mathfrak{p}(\mathcal{U}) \leq \mathfrak{d}$ and $\pi\chi(\mathcal{U}) \geq \mathfrak{b}$, we infer immediately

Corollary 1.2. (Nyikos [Ny, Theorem 3 (viii)]) *For any ultrafilter* $\mathcal{U}$, *we have either* $\pi\mathfrak{p}(\mathcal{U}) \leq \mathfrak{b}$ *or* $\pi\chi(\mathcal{U}) \geq \mathfrak{d}$. $\square$

Corollary 1.3. $\pi\mathfrak{p}(\mathcal{U}) \leq \pi\chi(\mathcal{U})$ *holds for any ultrafilter* $\mathcal{U}$. $\square$

We thus see that the four ultrafilter characteristics defined at the beginning are, in fact, linearly ordered.

⋆ ⋆ ⋆

Unfortunately, we shall need some more ultrafilter coefficients whose definition is not as nice as the one of the four above. The reason for introducing these cardinals will become clear in §§ 2 and 3.

$$\begin{aligned}
\mathfrak{p}'(\mathcal{U}) &= \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq \mathcal{U}\ \wedge\ \forall \bar{B} \in [\mathcal{U}]^\omega\ \exists A \in \mathcal{A}\ \forall B \in \bar{B}\ (B \not\subseteq^* A)\}\\
\pi\chi_\sigma(\mathcal{U}) &= \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq [\omega]^\omega\ \wedge\ \forall \bar{B} \in [\mathcal{U}]^\omega\ \exists A \in \mathcal{A}\ \forall B \in \bar{B}\ (A \subseteq^* B)\}\\
\chi_\sigma(\mathcal{U}) &= \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq [\mathcal{U}]^\omega\ \wedge\ \forall \bar{B} \in [\mathcal{U}]^\omega\ \exists \bar{A} \in \mathcal{A}\ \forall B \in \bar{B}\ \exists A \in \bar{A}\ (A \subseteq^* B)\}
\end{aligned}$$

There is again some symmetry. For example, the cardinal which is dual to $\mathfrak{p}'(\mathcal{U})$ can be defined as

$$\chi'(\mathcal{U}) = \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq [\mathcal{U}]^\omega\ \wedge\ \forall B \in \mathcal{U}\ \exists \bar{A} \in \mathcal{A}\ \exists A \in \bar{A}\ (A \subseteq^* B)\}$$

Of course, we have $\chi'(\mathcal{U}) = \chi(\mathcal{U})$, and thus get nothing new. Similarly, the primed version of $\pi\chi(\mathcal{U})$, as well as the σ–versions of $\mathfrak{p}(\mathcal{U})$ and $\pi\mathfrak{p}(\mathcal{U})$, give us nothing new. One could define a primed version of $\pi\mathfrak{p}(\mathcal{U})$, but we won't need it. Concerning the possible values of the primed cardinal, we note that $\omega_1 \leq \mathfrak{p}'(\mathcal{U}) \leq \pi\mathfrak{p}(\mathcal{U})$ as well as $\mathfrak{p}(\mathcal{U}) \leq \mathfrak{p}'(\mathcal{U})$. Furthermore, $\mathfrak{p}'(\mathcal{U})$ is regular, and we have the following result which might be folklore:

PROPOSITION 1.4. *$cf(\chi(\mathcal{U})) \geq \mathfrak{p}'(\mathcal{U})$. In particular $\chi(\mathcal{U})$ has uncountable cofinality.*

Proof. First note that if $\langle \mathcal{F}_n;\ n \in \omega \rangle$ is a strictly increasing sequence of proper filters on ω, then $\mathcal{F} = \bigcup_n \mathcal{F}_n$ is not an ultrafilter. To see this, choose a strictly decreasing sequence $\langle A_n;\ n \in \omega \rangle$ of subsets of ω such that $A_0 = \omega$ and $A_{n+1} \in \mathcal{F}_{n+1} \setminus \mathcal{F}_n$ for all n. Let $B = \bigcup_n (A_{2n+1} \setminus A_{2n+2})$ and $C = \bigcup_n (A_{2n} \setminus A_{2n+1})$. Thus $B \cup C = \omega$. Assume that $B \in \mathcal{F}$. Then $B \in \mathcal{F}_n$ for some n. Hence also $A_n \cap B \in \mathcal{F}_n$ and $A_{n+1} \cap B \in \mathcal{F}_{n+1}$. If n is even we see $A_n \cap B \subseteq A_{n+1} \notin \mathcal{F}_n$; if n is odd, we have $A_{n+1} \cap B \subseteq A_{n+2} \notin \mathcal{F}_{n+1}$, a contradiction in both cases. Therefore $B \notin \mathcal{F}$. Similarly we show $C \notin \mathcal{F}$, and $\mathcal{F}$ is not an ultrafilter.

Now let κ be regular uncountable and assume $\langle \mathcal{F}_\alpha;\ \alpha < \kappa \rangle$ is a strictly increasing sequence of proper filters on ω with $\mathcal{F} = \bigcup_\alpha \mathcal{F}_\alpha$. Choose $A_{\alpha+1} \in \mathcal{F}_{\alpha+1} \setminus \mathcal{F}_\alpha$. Assume there are countably many $B_n \in \mathcal{F}$ such that for all α there is n with $B_n \subseteq A_{\alpha+1}$. Then for some $\alpha_0 < \kappa$, $B_n \in \mathcal{F}_{\alpha_0}$ for all n, a contradiction to the choice of A_{α_0+1}. Hence we see that $cf(\chi(\mathcal{U})) \geq \mathfrak{p}'(\mathcal{U})$ for any ultrafilter $\mathcal{U}$. □

Also notice that $\mathfrak{p}(\mathcal{U}) = \mathfrak{p}'(\mathcal{U})$ iff $\mathcal{U}$ is P–point. In particular, there are (in ZFC) ultrafilters $\mathcal{U}$ with $\mathfrak{p}'(\mathcal{U}) > \mathfrak{p}(\mathcal{U})$. Under MA this can be strengthened to

PROPOSITION 1.5. (MA) *For each regular cardinal κ with $\omega_1 \leq \kappa \leq \mathfrak{c}$, there is an ultrafilter $\mathcal{U}$ with $\mathfrak{p}(\mathcal{U}) = \omega$ and $\mathfrak{p}'(\mathcal{U}) = \kappa$.*

Proof. By Louveau's Theorem quoted above, there is an ultrafilter $\mathcal{V}$ with $\mathfrak{p}(\mathcal{V}) = \kappa$. Let $X_n := \{n\} \times \omega$ denote the vertical strips. We define an ultrafilter $\mathcal{U}$ on $\omega \times \omega$ by

$$X \in \mathcal{U} \iff \{n;\ \{m;\ \langle n, m \rangle \in X\} \in \mathcal{V}\} \in \mathcal{V}.$$

(We shall use again this type of construction in § 5.) Note that the sets $Y_n := \bigcup_{k \geq n} X_k$ witness $\mathfrak{p}(\mathcal{U}) = \omega$.

We are left with proving $\mathfrak{p}'(\mathcal{U}) = \kappa$. Given $A \in \mathcal{U}$, put $A_n = \{m;\ \langle n, m \rangle \in A\}$ and let $B_A = \{n;\ A_n \in \mathcal{V}\} \in \mathcal{V}$. Notice that if $A \subseteq^* A'$ then also $B_A \subseteq^* B_{A'}$.

First take $\lambda < \kappa$ and let $\langle A_\alpha;\ \alpha < \lambda \rangle$ be a sequence from $\mathcal{U}$. By $\mathfrak{p}(\mathcal{V}) = \kappa$, find $B \in \mathcal{V}$ with $B \subseteq^* B_{A_\alpha}$ for all α. Find $C_n \in \mathcal{V}$ such that $C_n \subseteq^* A_{\alpha,n}$ for all α with $A_{\alpha,n} \in \mathcal{V}$. Finally find $f \in \omega^\omega$ with $f(n) \geq \max(C_n \setminus A_{\alpha,n})$ for almost all n with $A_{\alpha,n} \in \mathcal{V}$, and all α. Then put $D_n = \bigcup_{k \geq n, k \in B} \{k\} \times (C_k \setminus f(k)) \in \mathcal{U}$. It is now easy to check that for each $\alpha < \lambda$ there is n with $D_n \subseteq^* A_\alpha$. Hence $\mathfrak{p}'(\mathcal{U}) \geq \kappa$.

Conversely, let $\langle B_\alpha;\ \alpha < \kappa \rangle$ witness $\mathfrak{p}(\mathcal{V}) = \kappa$, and put $A_\alpha = \bigcup_{n \in B_\alpha} X_n$. If we had $D_n \in \mathcal{U}$ such that for all α there is n with $D_n \subseteq^* A_\alpha$, then we would also get $B_{D_n} \subseteq^* B_\alpha$, a contradiction. Thus $\langle A_\alpha;\ \alpha < \kappa \rangle$ witnesses $\mathfrak{p}'(\mathcal{U}) \leq \kappa$. □

On the other hand, it is easy to see that there is always an ultrafilter $\mathcal{U}$ with $\mathfrak{p}'(\mathcal{U}) = \omega_1$ (simply take $\mathcal{A} = \{A_\alpha;\ \alpha < \omega_1\}$ strictly $\subseteq^*$–decreasing, let $\mathcal{I}$ be the ideal of pseudointersections of $\mathcal{A}$, and extend $\mathcal{A}$ to an ultrafilter $\mathcal{U}$ with $\mathcal{U} \cap \mathcal{I} = \emptyset$). This should be seen as dual to the well–known fact (see e.g. [vM, Theorem 4.4.2]) that there is always an ultrafilter $\mathcal{U}$ with $\chi(\mathcal{U}) = \mathfrak{c}$.

⋆ ⋆ ⋆

To get more restrictions on the possible values of the σ–versions of our ultrafilter characteristics, recall the following cardinal invariants.

$$\mathfrak{r}_\sigma = \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq [\omega]^\omega\ \wedge\ \forall \bar{B} \in [[\omega]^\omega]^\omega\ \exists A \in \mathcal{A}\ \forall B \in \bar{B}\ (A \subseteq^* B\ \vee\ A \subseteq^* (\omega \setminus B))\}$$

$\mathfrak{par} = \min\{|\Pi|;\ \Pi \subseteq 2^{[\omega]^2} \ \wedge\ \forall A \in [\omega]^\omega\ \exists \pi \in \Pi \ \text{ with } \pi[[A \setminus n]^2] = 2 \text{ for all } n\}$

$\mathfrak{hom} = \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq [\omega]^\omega \wedge$ for all partitions $\pi : [\omega]^2 \to 2$ there is $A \in \mathcal{A}$ such that A is homogeneous for π (that is, $|\pi[[A]^2]| = 1)\}$

The partition cardinals $\mathfrak{par}$ and $\mathfrak{hom}$ were introduced by Blass [Bl 2, section 6]. It is known that $\mathfrak{par} = \min\{\mathfrak{s}, \mathfrak{b}\}$ and that $\mathfrak{hom} = \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$ (see [Bl 2, Theorems 16 and 17], [Br, Proposition 4.2]). We see easily that $\mathfrak{c} \geq \chi_\sigma(\mathcal{U}) \geq \pi\chi_\sigma(\mathcal{U}) \geq \mathfrak{r}_\sigma$, $\chi_\sigma(\mathcal{U}) \geq \chi(\mathcal{U})$, $\pi\chi_\sigma(\mathcal{U}) \geq \pi\chi(\mathcal{U})$, $cf(\pi\chi_\sigma(\mathcal{U})) \geq \mathfrak{p}'(\mathcal{U})$, $cf(\chi_\sigma(\mathcal{U})) \geq \mathfrak{p}'(\mathcal{U})$, and that $\pi\chi_\sigma(\mathcal{U}) = \pi\chi(\mathcal{U})$ as well as $\chi_\sigma(\mathcal{U}) = \chi(\mathcal{U})$ for P–points $\mathcal{U}$. We do not know whether $\chi_\sigma(\mathcal{U}) > \chi(\mathcal{U})$ is consistent (see § 8 (1)), but we shall encounter ultrafilters $\mathcal{U}$ with $\pi\chi_\sigma(\mathcal{U}) > \pi\chi(\mathcal{U})$ in section 5. The following proposition is simply a reformulation of the well–known fact that Mathias forcing with a non–P–point adds a dominating real. We include a proof for completeness' sake.

Proposition 1.6. (Canjar, Nyikos, Ketonen, see [Ca 1, Lemma 4]) *Let $\mathcal{U}$ be an ultrafilter on ω which is not a P–point. Then:*

(a) $\pi\mathfrak{p}(\mathcal{U}) \leq \mathfrak{b}$;

(b) $\pi\chi_\sigma(\mathcal{U}) \geq \mathfrak{d}$ and $\chi(\mathcal{U}) \geq \mathfrak{d}$.

Proof. Let $\{A_n;\ n \in \omega\} \subseteq \mathcal{U}$ be decreasing with no infinite pseudointersection in $\mathcal{U}$; i.e. $A_{n+1} \subseteq A_n$ and $|A_n \setminus A_{n+1}| = \omega$ for all $n \in \omega$. Given $f \in \omega^{\uparrow\omega}$, let $A_f \in \mathcal{U}$ be such that $\min(A_f \cap (A_n \setminus A_{n+1})) \geq f(n)$ for all $n \in \omega$. Given $A \in [\omega]^\omega$, define $f_A(n) \in \omega$ by first finding the least $k \geq n$ with $A \cap (A_k \setminus A_{k+1}) \neq \emptyset$, if it exists, and then putting $f_A(n) = \min(A \cap (A_k \setminus A_{k+1}))$; otherwise let $f_A(n) = 0$.

(a) Let $\kappa < \pi\mathfrak{p}(\mathcal{U})$, $\{f_\alpha;\ \alpha < \kappa\} \subseteq \omega^{\uparrow\omega}$. Let B be a pseudointersection of the family $\{A_n;\ n \in \omega\} \cup \{A_{f_\alpha};\ \alpha < \kappa\}$. It is easy to see that f_B eventually dominates all f_α.

(b) Let $\{A_\alpha;\ \alpha < \pi\chi_\sigma(\mathcal{U})\}$ be a $\pi\sigma$–base of $\mathcal{U}$. Given $f \in \omega^{\uparrow\omega}$, let α be such that $A_\alpha \subseteq^* A_f \cap A_n$ for all n. Then f_{A_α} eventually dominates f. Thus $\{f_{A_\alpha};\ \alpha < \pi\chi_\sigma(\mathcal{U})\}$ is dominating. In case the A_α form a base, argue similarly: choose α such that $A_\alpha \subseteq^* A_f$, etc. □

We will see in 5.4 that $\pi\chi_\sigma$ and χ cannot be replaced by $\pi\chi$ in (b), in general. We notice that the above result is also true for rapid ultrafilters — with an even easier argument. However, it may fail in general (see the main results of [BlS] and [BlS 1]). The following proposition has a flavor similar to Bartoszyński's classical (and much more intricate) result [Ba] that if cov(measure) $\leq \mathfrak{b}$, then cov(measure) has uncountable cofinality.

Proposition 1.7. *If $\pi\mathfrak{p}(\mathcal{U}) \leq \mathfrak{b}$, then $cf(\pi\mathfrak{p}(\mathcal{U})) \geq \omega_1$.*

Proof. Assume λ has countable cofinality and $\pi\mathfrak{p}(\mathcal{U}) \geq \lambda$. We shall show $\pi\mathfrak{p}(\mathcal{U}) > \lambda$. Choose $\mathcal{A} \subseteq \mathcal{U}$ of size λ. Then $\mathcal{A} = \bigcup_n \mathcal{A}_n$ where $|\mathcal{A}_n| < \lambda$ and $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$. Hence we can find $X_n \in [\omega]^\omega$ with $X_n \subseteq^* A$ for all $A \in \mathcal{A}_n$. For $A \in \mathcal{A}_n$ choose a function $f_A \in \omega^\omega$ with $X_k \setminus A \subseteq f_A(k)$ for $k \geq n$. By assumption $\lambda < \mathfrak{b}$; hence there is $f \in \omega^\omega$ with $f \geq^* f_A$ for all $A \in \mathcal{A}$. Put $X := \{\min(X_k \setminus f(k));\ k \in \omega\}$. It's easy to check that $X \subseteq^* A$ for all $A \in \mathcal{A}$, and we're done. □

Proposition 1.6 and 1.7 together yield:

Corollary 1.8. *If $\mathcal{U}$ is either not a P–point or a rapid ultrafilter, then $\pi\mathfrak{p}(\mathcal{U})$ has uncountable cofinality.* □

For later use (§§ 2 and 3) we mention the following characterization of $\chi_\sigma(\mathcal{U})$.

LEMMA 1.9. $\chi_\sigma(\mathcal{U}) = \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq \mathcal{U}^\omega \ \wedge \ \forall\langle B_n;\ n \in \omega\rangle \subseteq \mathcal{U}\ \exists\langle A_n;\ n \in \omega\rangle \in \mathcal{A}\ \forall n\ (A_n \subseteq^* B_n)\}$.

Proof. Denote the cardinal on the right–hand side by $\bar{\chi}_\sigma(\mathcal{U})$. $\chi_\sigma(\mathcal{U}) \leq \bar{\chi}_\sigma(\mathcal{U})$ is trivial. To see the converse, note that for P–points $\mathcal{U}$, both cardinals coincide with the character. Hence assume $\mathcal{U}$ is not P–point; then $\mathfrak{d} \leq \chi_\sigma(\mathcal{U})$ by Proposition 1.6. Let $\{f_\beta;\ \beta < \mathfrak{d}\}$ be a dominating family which is closed under finite modifications (i.e. whenever $f \in \omega^\omega$ agrees with some f_β on all but finitely many places, then $f = f_\gamma$ for some $\gamma < \mathfrak{d}$), and let $\{\bar{A}_\alpha;\ \alpha < \chi_\sigma(\mathcal{U})\}$ be a σ–base of $\mathcal{U}$. Let $\langle A_{\alpha,n};\ n \in \omega\rangle$ enumerate $\bar{A}_\alpha$; without loss $A_{\alpha,n+1} \subseteq^* A_{\alpha,n}$. Put $A'_{\alpha,\beta,n} = A_{\alpha,f_\beta(n)}$; we leave it to the reader to verify that $\{\langle A'_{\alpha,\beta,n};\ n \in \omega\rangle;\ \alpha < \chi_\sigma(\mathcal{U}), \beta < \mathfrak{d}\}$ satisfies the defining clause of $\bar{\chi}_\sigma(\mathcal{U})$. $\square$

2. Characterizations of the coefficients of the Ramsey ideal

Let $\mathcal{I}$ be a non–trivial ideal on the Baire space ω^ω (or on one of its homeomorphic copies, $[\omega]^\omega$ or $\omega^{\uparrow\omega}$) containing all singletons. $\mathcal{F} \subseteq \mathcal{I}$ is a *base* of $\mathcal{I}$ iff given $A \in \mathcal{I}$ there is $B \in \mathcal{F}$ with $A \subseteq B$. We introduce the following four cardinal invariants associated with $\mathcal{I}$.

$$
\begin{aligned}
\mathsf{add}(\mathcal{I}) &= \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \mathcal{I} \ \wedge\ \bigcup \mathcal{F} \notin \mathcal{I}\}\\
\mathsf{cov}(\mathcal{I}) &= \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \mathcal{I} \ \wedge\ \bigcup \mathcal{F} = \omega^\omega\}\\
\mathsf{non}(\mathcal{I}) &= \min\{|F|;\ F \subseteq \omega^\omega \ \wedge\ F \notin \mathcal{I}\}\\
\mathsf{cof}(\mathcal{I}) &= \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \mathcal{I} \ \wedge\ \mathcal{F} \text{ is a base of } \mathcal{I}\}
\end{aligned}
$$

These cardinals are referred to as *additivity*, *covering*, *uniformity* and *cofinality*, respectively. They have been studied intensively in case $\mathcal{I}$ is either the ideal of Lebesgue null sets or the ideal of meager sets [BJ 1] and in some other cases as well. We note that one always has $\mathsf{add}(\mathcal{I}) \leq \mathsf{cov}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$ and $\mathsf{add}(\mathcal{I}) \leq \mathsf{non}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$; furthermore, $\mathsf{add}(\mathcal{I})$ is regular, and $cf(\mathsf{non}(\mathcal{I})) \geq \mathsf{add}(\mathcal{I})$, as well as $cf(\mathsf{cof}(\mathcal{I})) \geq \mathsf{add}(\mathcal{I})$.

Given an ultrafilter $\mathcal{U}$ on ω, we define the *Mathias forcing* associated with $\mathcal{U}$, $\mathbb{M}_\mathcal{U}$ [Ma], as follows. Conditions are pairs (r, U) with $r \in [\omega]^{<\omega}$ and $U \in \mathcal{U}$ such that $\max(r) < \min(U)$. We put $(s, V) \leq (r, U)$ iff $s \supseteq r$, $V \subseteq U$ and $s \setminus r \subseteq U$. The Mathias p.o. is σ–centered and hence *ccc*. It generically adds a real $m \in [\omega]^\omega$ which is almost included in all members of $\mathcal{U}$. For $(r, U) \in \mathbb{M}_\mathcal{U}$, we let $[r, U] = \{A \in [\omega]^\omega;\ r \subseteq A \subseteq r \cup U\}$. The ideal of *nowhere Ramsey sets* with respect to $\mathcal{U}$ (or *Ramsey null sets*) $r^0_\mathcal{U}$ consists of all $X \subseteq [\omega]^\omega$ such that given $(r, U) \in \mathbb{M}_\mathcal{U}$ there is $(r, V) \leq (r, U)$ with $X \cap [r, V] = \emptyset$. We notice that the connection between Mathias forcing and the Ramsey ideal is like the one between Cohen (random, resp.) forcing and the meager (null, resp.) ideal.

The main goal of this section is to characterize the four cardinal coefficients introduced above for the ideal $r^0_\mathcal{U}$ in terms of the cardinals in section 1. This extends a result of Louveau who had already proved that the additivity of $r^0_\mathcal{U}$ coincides with $\mathfrak{p}(\mathcal{U})$. For our characterizations we shall need

LEMMA 2.1. (Louveau, [Lo, Lemme 3.3]) *Let $\mathcal{U}$ be a P–point and $\phi : [\omega]^{<\omega} \to \mathcal{U}$. Then there is $U \in \mathcal{U}$ such that $\{s \in [\omega]^{<\omega};\ U \setminus s \subseteq \phi(s)\}$ is cofinal in $[\omega]^{<\omega}$.*

Proof. We include a proof to make the paper self–contained. Assume $\mathcal{U}$ and ϕ are as required. Since $\mathcal{U}$ is a P–point, there is $U \in \mathcal{U}$ with $U \subseteq^* \phi(s)$ for all $s \in [\omega]^{<\omega}$. Construct recursively finite sets $A_i \subseteq U$ for $i \in \omega$ by putting $A_0 := U \setminus \phi(\emptyset)$ and $A_{i+1} := U \setminus \bigcap\{\phi(s);\ \max(s) \leq \max(A_i)\}$. Then either we have $U \setminus \bigcup_i A_i \in \mathcal{U}$, and this set is as required; or $\bigcup_i A_i \in \mathcal{U}$, and one of the sets $\bigcup_i (A_{2i+1} \setminus A_{2i})$, $\bigcup_i (A_{2i} \setminus A_{2i-1})$ lies in $\mathcal{U}$ and satisfies the conclusion of the Lemma. $\square$

Theorem 1. *Let $\mathcal{U}$ be an ultrafilter on ω. Then:*
(a) (Louveau [Lo, Théorème 3.7]) $\mathsf{add}(r^0_\mathcal{U}) = \mathfrak{p}(\mathcal{U})$;
(b) $\mathsf{cov}(r^0_\mathcal{U}) = \pi\mathfrak{p}(\mathcal{U})$;
(c) $\mathsf{non}(r^0_\mathcal{U}) = \pi\chi(\mathcal{U})$;
(d) $\mathsf{cof}(r^0_\mathcal{U}) = \chi_\sigma(\mathcal{U})$.

In case $\mathcal{U}$ is a Ramsey ultrafilter, (a) through (d) were proved by Egbert Thümmel. (Note that $\chi(\mathcal{U}) = \chi_\sigma(\mathcal{U})$ in this case.)

Proof. Before plunging into the details, we describe natural ways of assigning sets in the ideal to sets in the ultrafilter, and vice–versa. Given $A \in \mathcal{U}$, let $X = X(A) := \{B \in [\omega]^\omega;\ B \not\subseteq^* A\}$ and note that $X(A) = [\omega]^\omega \setminus \bigcup_{s\in[\omega]^{<\omega}}[s, A \setminus (\max(s)+1)] \in r^0_{\mathcal{U}}$. Conversely, given $Y \in r^0_{\mathcal{U}}$, we can find a sequence $\langle B_s \in \mathcal{U};\ s \in [\omega]^{<\omega}\rangle$ such that $B_s \subseteq \omega \setminus (\max(s)+1)$, $B_s \subseteq B_t$ for $t \subseteq s$ and $Y \subseteq Y(\langle B_s;\ s \in [\omega]^{<\omega}\rangle) := [\omega]^\omega \setminus \bigcup_s [s, B_s] \in r^0_{\mathcal{U}}$. Thus sets of the form $Y(\langle B_s\rangle)$ form a base of the ideal $r^0_{\mathcal{U}}$, and it suffices to deal with such sets in order to prove the Theorem. We shall do this without further mention. Also, whenever dealing with sequences $\langle A_s \in \mathcal{U};\ s \in [\omega]^{<\omega}\rangle$ we shall tacitly assume that $A_s \subseteq \omega \setminus (\max(s)+1)$ and $A_s \subseteq A_t$ for $t \subseteq s$. We group dual results together.

(a) and (d); the inequalities $\mathsf{add}(r^0_{\mathcal{U}}) \le \mathfrak{p}(\mathcal{U})$ and $\mathsf{cof}(r^0_{\mathcal{U}}) \ge \chi_\sigma(\mathcal{U})$. Let $\{A_\alpha;\ \alpha < \mathfrak{p}(\mathcal{U})\} \subseteq \mathcal{U}$ be a witness for $\mathfrak{p}(\mathcal{U})$. Let $X_\alpha = X(A_\alpha)$. To see that $\bigcup_\alpha X_\alpha \notin r^0_{\mathcal{U}}$, fix $Y = Y(\langle B_s\rangle) \in r^0_{\mathcal{U}}$. There is $\alpha < \mathfrak{p}(\mathcal{U})$ with $y := B_\emptyset \setminus A_\alpha$ being infinite. This means $y \in X_\alpha \setminus Y$, and we're done.

For the second inequality, notice that given $\langle A_t \in \mathcal{U};\ t \in [\omega]^{<\omega}\rangle$ and $\langle B_t \in \mathcal{U};\ t \in [\omega]^{<\omega}\rangle$ with $A_\emptyset$ coinfinite and $B_t \setminus A_s$ infinite for some s and all t, we can construct $y \in Y(\langle A_t\rangle) \setminus Y(\langle B_t\rangle)$ as follows: choose $k > \max(s)$ such that $k \notin A_\emptyset$, and let $y := s \cup \{k\} \cup B_{s\cup\{k\}} \in [s \cup \{k\}, B_{s\cup\{k\}}]$; then $y \notin [t, A_t]$ for $t \subseteq s$ by the choice of k, and $y \notin [t, A_t]$ for $t \supseteq s$ by the properties of A_s. $(\star)$

Now let $\{Y_\alpha;\ \alpha < \mathsf{cof}(r^0_{\mathcal{U}})\}$ be a base of the ideal $r^0_{\mathcal{U}}$. Without loss $Y_\alpha = Y(\langle B_{\alpha,s}\rangle)$ with all $B_{\alpha,s} \in \mathcal{U}$. Fix $\bar{A} \in [\mathcal{U}]^\omega$; making its sets smaller, if necessary, we may assume that $\bar{A} = \{A_s;\ s \in [\omega]^{<\omega}\}$ with $A_s \subseteq A_t$ for $t \subseteq s$ and $A_\emptyset$ being coinfinite. Let $Y := Y(\langle A_s\rangle)$, and choose $\alpha < \mathsf{cof}(r^0_{\mathcal{U}})$ with $Y \subseteq Y_\alpha$. By $(\star)$ we get that for all $A \in \bar{A}$, there is s with $B_{\alpha,s} \subseteq^* A$, and we're done.

The inequalities $\mathsf{add}(r^0_{\mathcal{U}}) \ge \mathfrak{p}(\mathcal{U})$ and $\mathsf{cof}(r^0_{\mathcal{U}}) \le \chi_\sigma(\mathcal{U})$. We distinguish two cases. First assume $\mathcal{U}$ is not a P–point. Then the first inequality is trivial by $\mathfrak{p}(\mathcal{U}) = \omega$. Concerning the second, let $\{\{A_{\alpha,s};\ s \in [\omega]^{<\omega}\};\ \alpha < \chi_\sigma(\mathcal{U})\}$ be a σ–base of $\mathcal{U}$, recall from Proposition 1.6 that $\chi_\sigma(\mathcal{U}) \ge \mathfrak{d}$, let $\{f_\beta : [\omega]^{<\omega} \to \omega;\ \beta < \mathfrak{d}\}$ be a dominating family which is closed under finite modifications, and put $Y_{\alpha,\beta} := Y(\langle A_{\alpha,s} \setminus f_\beta(s)\rangle)$. We claim that $\{Y_{\alpha,\beta};\ \alpha < \chi_\sigma(\mathcal{U}), \beta < \mathfrak{d}\}$ is a base of $r^0_{\mathcal{U}}$. For, given $Y = Y(\langle B_s\rangle) \in r^0_{\mathcal{U}}$ with $B_s \in \mathcal{U}$ for all s, we can find first (by Lemma 1.9) an α with $A_{\alpha,s} \subseteq^* B_s$ for all s and then a β with $A_{\alpha,s} \setminus f_\beta(s) \subseteq B_s$ for all s. This easily entails $Y \subseteq Y_{\alpha,\beta}$.

Now suppose $\mathcal{U}$ is a P–point. Given $\langle B_s \in \mathcal{U};\ s \in [\omega]^{<\omega}\rangle$ satisfying additionally $B_s = B_t$ for s and t with $\max(s) = \max(t)$ (and thus $B_s \subseteq B_t$ for s,t with $\max(t) \le \max(s)$), as well as $A \in \mathcal{U}$ such that $\{s \in [\omega]^{<\omega};\ A \setminus s \subseteq B_s\}$ is cofinal in $[\omega]^{<\omega}$, we have $Y(\langle B_s\rangle) \subseteq X(A)$. $(\star\star)$ To see this, fix $s \in [\omega]^{<\omega}$, and take an arbitrary $y \in [s, A \setminus (\max(s)+1)]$. Find $t \supseteq s$ with $A \setminus t \subseteq B_t$. Letting $k := \max(t)+1$, we get $y \setminus k \subseteq A \setminus k \subseteq B_t \subseteq B_{y\cap k}$ which entails $y \in [y \cap k, B_{y\cap k}]$.

Given $\kappa < \mathfrak{p}(\mathcal{U})$ and $\{Y_\alpha;\ \alpha < \kappa\} \subseteq r^0_{\mathcal{U}}$ where $Y_\alpha = Y(\langle B_{\alpha,s}\rangle)$ with all $B_{\alpha,s} \in \mathcal{U}$, we find by Lemma 2.1 $A \in \mathcal{U}$ such that $\{s \in [\omega]^{<\omega};\ A \setminus s \subseteq B_{\alpha,s}\}$ is cofinal in $[\omega]^{<\omega}$ for all α. Thus $\bigcup_\alpha Y_\alpha \subseteq X(A) \in r^0_{\mathcal{U}}$ by $(\star\star)$. Dually, if $\{A_\alpha;\ \alpha < \chi(\mathcal{U})\}$ is a base of $\mathcal{U}$, we claim that the sets $X_\alpha = X(A_\alpha)$ form a base of our ideal. To see this, take $Y = Y(\langle B_s\rangle) \in r^0_{\mathcal{U}}$ where $B_s \in \mathcal{U}$. By Lemma 2.1 find $\alpha < \chi(\mathcal{U})$ such that $\{s \in [\omega]^{<\omega};\ A_\alpha \setminus s \subseteq B_s\}$ is cofinal in $[\omega]^{<\omega}$, and conclude by $(\star\star)$.

(b) and (c); the inequalities $\mathsf{cov}(r^0_{\mathcal{U}}) \leq \pi\mathfrak{p}(\mathcal{U})$ and $\mathsf{non}(r^0_{\mathcal{U}}) \geq \pi\chi(\mathcal{U})$. This is easy. Given a witness $\{A_\alpha \in \mathcal{U};\ \alpha < \pi\mathfrak{p}(\mathcal{U})\}$ for $\pi\mathfrak{p}(\mathcal{U})$, let $X_\alpha = X(A_\alpha)$. The X_α cover the reals, for, given $x \in [\omega]^\omega$, there is α with $x \not\subseteq^* A_\alpha$ which entails $x \in X_\alpha$. Dually, given $\{x_\alpha \in [\omega]^\omega;\ \alpha < \mathsf{non}(r^0_{\mathcal{U}})\} \notin r^0_{\mathcal{U}}$ and $A \in \mathcal{U}$, there is $\alpha < \mathsf{non}(r^0_{\mathcal{U}})$ with $x_\alpha \notin X(A)$ which means that $x_\alpha \subseteq^* A$. This shows that the x_α form a π–base of $\mathcal{U}$.

The inequalities $\mathsf{cov}(r^0_{\mathcal{U}}) \geq \pi\mathfrak{p}(\mathcal{U})$ and $\mathsf{non}(r^0_{\mathcal{U}}) \leq \pi\chi(\mathcal{U})$. We prove the second inequality first. Let $\{x_\alpha \in [\omega]^\omega;\ \alpha < \pi\chi(\mathcal{U})\}$ be a π–base of $\mathcal{U}$. Given $n \in \omega$, let $x_{\alpha,n} = x_\alpha \setminus n$. We note that $\{x_{\alpha,n};\ \alpha < \pi\chi(\mathcal{U}), n \in \omega\} \notin r^0_{\mathcal{U}}$, because, given $Y = Y(\langle B_s \rangle) \in r^0_{\mathcal{U}}$ with all $B_s \in \mathcal{U}$, we find α with $x_\alpha \subseteq^* B_\emptyset$ and thus $n \in \omega$ with $x_{\alpha,n} \subseteq B_\emptyset$, that is $x_{\alpha,n} \notin Y$.

Next, let $\kappa < \pi\mathfrak{p}(\mathcal{U})$ and $\{Y_\alpha;\ \alpha < \kappa\} \subseteq r^0_{\mathcal{U}}$; without loss $Y_\alpha = Y(\langle B_{\alpha,s} \rangle)$ with $B_{\alpha,s} \in \mathcal{U}$. We want to show that the Y_α's do not cover the reals. We distinguish two cases.

First assume $\mathcal{U}$ is not a P–point. By assumption, we find $x \in [\omega]^\omega$ with $x \subseteq^* B_{\alpha,s}$ for all α and all s. Define $g_\alpha : \omega \to x$ for $\alpha < \kappa$ recursively by:

$$g_\alpha(0) := \min\{k;\ x \setminus k \subseteq B_{\alpha,\emptyset}\}$$
$$g_\alpha(n+1) := \min\{k;\ x \setminus k \subseteq \bigcap_{s \subseteq g_\alpha(n)+1} B_{\alpha,s}\}.$$

By Proposition 1.6, we find $g : \omega \to x$ strictly increasing and eventually dominating all g_α's. Put $y := rng(g)$. To complete the argument, we shall show that $y \notin \bigcup_\alpha Y_\alpha$. Fix α. Let n_0 be minimal with $g(n) \geq g_\alpha(n)$ for all $n \geq n_0$, and put $s := \{g(i);\ i < n_0\}$. It is easy to see that $y \in [s, B_{\alpha,s}]$.

Finally assume $\mathcal{U}$ is a P–point. By Lemma 2.1 we find $A_\alpha \in \mathcal{U}$ such that $\{s \in [\omega]^{<\omega};\ A_\alpha \setminus s \subseteq B_{\alpha,s}\}$ is cofinal in $[\omega]^{<\omega}$. By assumption we find $y \in [\omega]^\omega$ with $y \subseteq^* A_\alpha$ for all α. We show again $y \notin \bigcup_\alpha Y_\alpha$. Fix α and choose $s \in [\omega]^{<\omega}$ with $y \setminus s \subseteq A_\alpha$. Find $t \supseteq s$ with $A_\alpha \setminus t \subseteq B_{\alpha,t}$. Then, letting $k := \max(t) + 1$, we have $y \setminus k \subseteq A_\alpha \setminus k \subseteq B_{\alpha,t} \subseteq B_{\alpha,y\cap k}$ which implies $y \in [y \cap k, B_{\alpha,y\cap k}]$. This completes the proof of the Theorem. $\square$

Corollary 2.2. *Let $\mathcal{U}$ be an ultrafilter on ω. Then:*
(a) $\mathsf{add}(r^0_{\mathcal{U}}) \leq \mathsf{cov}(r^0_{\mathcal{U}}) \leq \mathsf{non}(r^0_{\mathcal{U}}) \leq \mathsf{cof}(r^0_{\mathcal{U}})$.
(b) $\mathfrak{p} \leq \mathsf{cov}(r^0_{\mathcal{U}})$*; in particular, MA implies that* $\mathsf{cov}(r^0_{\mathcal{U}}) = \mathfrak{c}$.
(c) $cf(\mathsf{cof}(r^0_{\mathcal{U}})) \geq \omega_1$.

Proof. All this follows from the Theorem and the results in § 1, in particular Corollary 1.3 and Proposition 1.4. $\square$

The fact that the cardinal coefficients of *ccc*–ideals of the form $r^0_{\mathcal{U}}$ are linearly ordered distinguishes them from the *ccc*–ideals of meager and null sets (see [BJ 1]). Note, however, that the cardinal coefficients of the closely related, but non–*ccc*, ideal of *Ramsey null sets* [El] (*nowhere Ramsey sets*) r^0 are also linearly ordered. Namely, one has $\mathsf{add}(r^0) = \mathsf{cov}(r^0) = \mathfrak{h} \leq \mathsf{non}(r^0) = \mathfrak{c} < \mathsf{cof}(r^0)$ where $\mathfrak{h}$ is as usual the *distributivity number* of $\mathcal{P}(\omega)/fin$ (this is due to Plewik [Pl]).

Results with a flavor similar to our Theorem 1 were established independently by Matet [M, section 10]. He considers a situation which is both more general (filters on arbitrary regular κ instead of ultrafilters on ω) and more restricted (combinatorial properties imposed on the filters) so that our results are, to some extent, orthogonal.

For more results on the coefficients of $r^0_{\mathcal{U}}$, see, in particular, Theorem 3, Theorem 4(c) and Corollary 6.5.

3. Characterizations of the coefficients of the Louveau ideal

A tree $T \subseteq \omega^{\uparrow<\omega}$ is called *Laver tree* iff for all $\sigma \in T$ with $\sigma \supseteq stem(T)$, the set $succ_T(\sigma) := \{n \in \omega;\ \sigma\hat{\ }\langle n\rangle \in T\}$ is infinite. Given an ultrafilter $\mathcal{U}$ on ω, we define the *Laver forcing* associated with $\mathcal{U}$, $\mathbb{L}_{\mathcal{U}}$ (see [Bl 1, section 5] or [JS, section 1]), as follows. Conditions are Laver trees $T \subseteq \omega^{\uparrow<\omega}$ such that for all $\sigma \in T$ with $\sigma \supseteq stem(T)$, we have $succ_T(\sigma) \in \mathcal{U}$. We put $S \leq T$ iff $S \subseteq T$; furthermore $S \leq_0 T$ iff $S \leq T$ and $stem(S) = stem(T)$. The Laver p.o. is again a σ–centered p.o. The *Louveau ideal* $\ell^0_{\mathcal{U}}$ consists of all $X \subseteq \omega^{\uparrow\omega}$ such that given $T \in \mathbb{L}_{\mathcal{U}}$ there is $S \leq T$ with $X \cap [S] = \emptyset$ [Lo]. Louveau proved that $\ell^0_{\mathcal{U}}$ is a σ–ideal and that there is a topology $\mathcal{G}^\infty_{\mathcal{U}}$ on $\omega^{\uparrow\omega}$, finer than the usual topology, such that $\ell^0_{\mathcal{U}}$ is the ideal of the $\mathcal{G}^\infty_{\mathcal{U}}$ nowhere dense sets which coincides with the $\mathcal{G}^\infty_{\mathcal{U}}$ meager sets [Lo, 1.11 et 1.12]. (This should be compared with Ellentuck's classical results [El] on nowhere Ramsey sets.) He also showed that $\ell^0_{\mathcal{U}} = r^0_{\mathcal{U}}$ in case $\mathcal{U}$ is a Ramsey ultrafilter [Lo, Propositions 1.3 et 3.1]. In the same vein, Blass [Bl 1, pp. 238–239] and Judah–Shelah [JS, Theorem 1.20] observed that $\mathbb{L}_{\mathcal{U}}$ and $\mathbb{M}_{\mathcal{U}}$ are forcing equivalent for Ramsey $\mathcal{U}$.

We are now heading for characterizations of the cardinals add, cov, non and cof for $\ell^0_{\mathcal{U}}$ in terms of the characteristics of $\mathcal{U}$ introduced in section 1.

Theorem 2. *Let $\mathcal{U}$ be an ultrafilter on ω. Then:*
(a) $\mathsf{add}(\ell^0_{\mathcal{U}}) = \min\{\mathfrak{p}'(\mathcal{U}), \mathfrak{b}\}$;
(b) $\mathsf{cov}(\ell^0_{\mathcal{U}}) = \min\{\pi\mathfrak{p}(\mathcal{U}), \mathfrak{b}\}$;
(c) $\mathsf{non}(\ell^0_{\mathcal{U}}) = \max\{\pi\chi_\sigma(\mathcal{U}), \mathfrak{d}\}$;
(d) $\mathsf{cof}(\ell^0_{\mathcal{U}}) = \max\{\chi_\sigma(\mathcal{U}), \mathfrak{d}\}$.

Proof. As in the proof of Theorem 1, we shall stress the symmetry of the arguments, and start by fixing some notation concerning the correspondence between sets in $\mathcal{U}$ and sets in $\ell^0_{\mathcal{U}}$. For $A \in \mathcal{U}$, let $X = X(A) := \{y \in \omega^{\uparrow\omega};\ \exists^\infty n\ (y(n) \notin A)\}$; given additionally $\sigma \in \omega^{\uparrow<\omega}$, let $T = T_\sigma(A)$ be the Laver tree with stem σ and $succ_T(\tau) = A \setminus (\tau(|\tau|-1)+1)$ for $\sigma \subseteq \tau \in T$. Then we have $X(A) = \omega^{\uparrow\omega} \setminus \bigcup_{\sigma\in\omega^{\uparrow<\omega}}[T_\sigma(A)] \in \ell^0_{\mathcal{U}}$. Conversely, given $Y \in \ell^0_{\mathcal{U}}$, we can find a sequence $\langle B_\sigma \in \mathcal{U};\ \sigma \in \omega^{\uparrow<\omega}\rangle$ satisfying $B_\sigma \subseteq \omega \setminus (\sigma(|\sigma|-1)+1)$ and $B_\sigma \subseteq B_\tau$ for $\tau \subseteq \sigma$, and such that $Y \subseteq Y(\langle B_\sigma\rangle) := \omega^{\uparrow\omega} \setminus \bigcup_\sigma [T_\sigma(\langle B_\tau\rangle)]$, where $T_\sigma(\langle B_\tau\rangle)$ is the Laver tree T with stem σ and $succ_T(\tau) = B_\tau$ for $\sigma \subseteq \tau \in T$. Again, we use this convention without further comment.

(a) and (d); the inequalities $\mathsf{add}(\ell^0_{\mathcal{U}}) \leq \mathfrak{p}'(\mathcal{U})$ and $\mathsf{cof}(\ell^0_{\mathcal{U}}) \geq \chi_\sigma(\mathcal{U})$. Notice first that given $\langle A_\tau \in \mathcal{U};\ \tau \in \omega^{\uparrow<\omega}\rangle$ and $\langle B_\tau \in \mathcal{U};\ \tau \in \omega^{\uparrow<\omega}\rangle$, if $y \in \omega^{\uparrow\omega}$ satisfies $y(i) \in B_{y\restriction i} \setminus A_{y\restriction i}$ for almost all i, then

$y \in Y(\langle A_\tau \rangle) \setminus Y(\langle B_\tau \rangle)$. $\qquad (\star)$

Let $\{A_\alpha;\ \alpha < \mathfrak{p}'(\mathcal{U})\}$ be a witness for $\mathfrak{p}'(\mathcal{U})$. We show that $\bigcup_\alpha X_\alpha \notin \ell^0_{\mathcal{U}}$ where $X_\alpha = X(A_\alpha)$. This is easy, for, given $Y = Y(\langle B_\tau \rangle)$ with $B_\tau \in \mathcal{U}$, we find α with $B_\tau \not\subseteq^* A_\alpha$ for all τ, and then construct $y \in \omega^{\uparrow\omega}$ with $y(i) \in B_{y\restriction i} \setminus A_\alpha$ for all i. This gives $X_\alpha \not\subseteq Y$ by $(\star)$.

Dually, let $\{Y_\alpha;\ \alpha < \mathsf{cof}(\ell^0_{\mathcal{U}})\}$ be a base of $\ell^0_{\mathcal{U}}$; without loss $Y_\alpha = Y(\langle B_{\alpha,\sigma} \rangle)$ with $B_{\alpha,\sigma} \in \mathcal{U}$. We claim that the $\bar{B}_\alpha = \{B_{\alpha,\sigma};\ \sigma \in \omega^{\uparrow<\omega}\}$ form a witness for $\chi_\sigma(\mathcal{U})$. Let $\bar{A} = \{A_\sigma;\ \sigma \in \omega^{\uparrow<\omega}\} \in [\mathcal{U}]^\omega$ be given, and find α with $Y := Y(\langle A_\sigma \rangle) \subseteq Y_\alpha$. Assume there were $\tau \in \omega^{\uparrow<\omega}$ with $B_{\alpha,\sigma} \not\subseteq^* A_\tau$ for all σ; then we could construct $y \in \omega^{\uparrow\omega}$ with $\tau \subseteq y$ and $y(i) \in B_{\alpha,y\restriction i} \setminus A_{y\restriction i}$ for all $i \geq |\tau|$; this would contradict $Y \subseteq Y_\alpha$ by $(\star)$. Thus for all $A \in \bar{A}$ we find $B \in \bar{B}_\alpha$ with $B \subseteq^* A$, and we're done.

The inequalities $\mathsf{add}(\ell^0_{\mathcal{U}}) \geq \min\{\mathfrak{p}'(\mathcal{U}), \mathfrak{b}\}$ and $\mathsf{cof}(\ell^0_{\mathcal{U}}) \leq \max\{\chi_\sigma(\mathcal{U}), \mathfrak{d}\}$. First note that given $\langle A_\tau \in \mathcal{U};\ \tau \in \omega^{\uparrow<\omega}\rangle$ and $\langle B_\tau \in \mathcal{U};\ \tau \in \omega^{\uparrow<\omega}\rangle$ with $B_\tau \subseteq A_\tau$ for almost all τ, we have $Y(\langle A_\tau \rangle) \subseteq Y(\langle B_\tau \rangle)$. $(\star\star)$ Let $\langle \tau_n;\ n \in \omega \rangle$ enumerate $\omega^{\uparrow<\omega}$.

Let $\kappa < \min\{\mathfrak{p}'(\mathcal{U}), \mathfrak{b}\}$, and let $Y_\alpha \in \ell^0_{\mathcal{U}}$ for $\alpha < \kappa$. Assume $Y_\alpha = Y(\langle A_{\alpha,\sigma} \rangle)$ where $A_{\alpha,\sigma} \in \mathcal{U}$. By $\kappa < \mathfrak{p}'(\mathcal{U})$ find $\langle B_n \in \mathcal{U};\ n \in \omega \rangle$ such that for all α, τ, there is n with $B_n \subseteq^* A_{\alpha,\tau}$. Without loss, $B_{n+1} \subseteq B_n$ for all n. Define $g_\alpha : \omega \to \omega$ by:

$$g_\alpha(n) := \min\{m;\ B_m \subseteq^* A_{\alpha,\tau_n}\}.$$

Since $\kappa < \mathfrak{b}$, we can find $g \in \omega^\omega$ eventually dominating all g_α. Thus we have $B_{g(n)} \subseteq^* A_{\alpha,\tau_n}$ for all α and almost all n. Define a function h_α for all n with $B_{g(n)} \subseteq^* A_{\alpha,\tau_n}$ by:

$$h_\alpha(n) := \min\{m;\ B_{g(n)} \setminus m \subseteq A_{\alpha,\tau_n}\}.$$

Let h eventually dominate all h_α. Then $B_{g(n)} \setminus h(n) \subseteq A_{\alpha,\tau_n}$ for all α and almost all n. Put $B_{\tau_n} = B_{g(n)} \setminus h(n)$. By $(\star\star)$ we have $Y_\alpha \subseteq Y(\langle B_\tau \rangle)$ for all α, and $\bigcup_\alpha Y_\alpha \in \ell^0_{\mathcal{U}}$ follows.

Dually, let $\{\{B_{\alpha,\tau};\ \tau \in \omega^{\uparrow<\omega}\};\ \alpha < \chi_\sigma(\mathcal{U})\}$ be a σ–base of $\mathcal{U}$, and let $\{f_\alpha : \omega^{\uparrow<\omega} \to \omega;\ \alpha < \mathfrak{d}\}$ be a dominating family. For $\alpha < \chi_\sigma(\mathcal{U})$ and $\beta < \mathfrak{d}$, let $Y_{\alpha,\beta} := Y(\langle B_{\alpha,\tau} \setminus f_\beta(\tau) \rangle)$. Given $Y = Y(\langle A_\tau \rangle) \in \ell^0_{\mathcal{U}}$ with $A_\tau \in \mathcal{U}$, first find (by Lemma 1.9) α with $B_{\alpha,\tau} \subseteq^* A_\tau$ for all τ, and then β with $B_{\alpha,\tau} \setminus f_\beta(\tau) \subseteq A_\tau$ for almost all τ. By $(\star\star)$ we have $Y \subseteq Y_{\alpha,\beta}$, and thus the $Y_{\alpha,\beta}$ form a base of the ideal $\ell^0_{\mathcal{U}}$.

(b) and (c); the inequalities $\mathsf{cov}(\ell^0_{\mathcal{U}}) \leq \mathfrak{b}$ and $\mathsf{non}(\ell^0_{\mathcal{U}}) \geq \mathfrak{d}$. Given $f \in \omega^\omega$ and $\tau \in \omega^{\uparrow<\omega}$ let $B_{f,\tau} := \omega \setminus \max\{\tau(|\tau| - 1) + 1, f(|\tau|)\}$ and $Y_f := Y(\langle B_{f,\tau} \rangle)$. We easily see that, given an unbounded family $\{f_\alpha \in \omega^\omega;\ \alpha < \mathfrak{b}\}$, we have $\bigcup_\alpha Y_{f_\alpha} = \omega^{\uparrow\omega}$. Dually, if $\{f_\alpha \in \omega^{\uparrow\omega};\ \alpha < \mathsf{non}(\ell^0_{\mathcal{U}})\} \notin \ell^0_{\mathcal{U}}$, then for each $f \in \omega^\omega$ there is α with $f_\alpha \notin Y_f$ which means that $f \leq^* f_\alpha$; hence the f_α form a dominating family. (Notice that this is just a reformulation of the well–known fact that any Laver–like forcing adds a dominating real.)

The inequalities $\mathsf{cov}(\ell^0_{\mathcal{U}}) \leq \pi\mathfrak{p}(\mathcal{U})$ and $\mathsf{non}(\ell^0_{\mathcal{U}}) \geq \pi\chi_\sigma(\mathcal{U})$. Notice first that, given a sequence $\langle B_n \in \mathcal{U};\ n \in \omega \rangle$ with $B_{n+1} \subseteq B_n$ for all n, if we put $B_\sigma := B_{|\sigma|} \setminus (\sigma(|\sigma| - 1) + 1)$ and $Y := Y(\langle B_\sigma \rangle)$, then $y \in \omega^{\uparrow\omega} \setminus Y$ entails $rng(y) \subseteq^* B_n$ for all n. $\qquad (\dagger)$

Thus, given a witness $\{A_\alpha;\ \alpha < \pi\mathfrak{p}(\mathcal{U})\}$ for $\pi\mathfrak{p}(\mathcal{U})$, we must have $\bigcup_\alpha X(A_\alpha) = \omega^{\uparrow\omega}$ (for, if y were not in the union, $rng(y)$ would be a pseudointersection). Similarly, if $\{y_\alpha \in \omega^{\uparrow\omega};\ \alpha < \mathsf{non}(\ell^0_{\mathcal{U}})\} \notin \ell^0_{\mathcal{U}}$, then for each $\langle B_n \rangle$ as in $(\dagger)$, there is α with $y_\alpha \notin Y(\langle B_\sigma \rangle)$, and thus $y_\alpha \subseteq^* B_n$ for all n. This shows the y_α form a $\pi\sigma$–base of $\mathcal{U}$.

The inequalities $\mathsf{cov}(\ell^0_{\mathcal{U}}) \geq \min\{\pi\mathfrak{p}(\mathcal{U}), \mathfrak{b}\}$ and $\mathsf{non}(\ell^0_{\mathcal{U}}) \leq \max\{\pi\chi_\sigma(\mathcal{U}), \mathfrak{d}\}$. This is quite similar to the other two inequalities involving min and max (see above). Given $\langle A_\tau \in \mathcal{U};\ \tau \in \omega^{\uparrow<\omega}\rangle$ and $\langle B_\tau \in [\omega]^\omega;\ \tau \in \omega^{\uparrow<\omega}\rangle$ with $B_\tau \subseteq A_\tau$ for almost all τ, we have that any real $y \in \omega^{\uparrow\omega}$ with $y(i) \in B_{y\restriction i}$ for all i does not lie in $Y(\langle A_\tau\rangle)$. (‡)

Let $\kappa < \min\{\pi\mathfrak{p}(\mathcal{U}), \mathfrak{b}\}$, and let $Y_\alpha = Y(\langle A_{\alpha,\sigma}\rangle) \in \ell^0_{\mathcal{U}}$ with $A_{\alpha,\sigma} \in \mathcal{U}$ for $\alpha < \kappa$. First find $B \in [\omega]^\omega$ with $B \subseteq^* A_{\alpha,\sigma}$ for all α and σ, then define $g_\alpha : \omega^{\uparrow<\omega} \to \omega$ by:

$$g_\alpha(\tau) := \min\{m;\ B \setminus m \subseteq A_{\alpha,\tau}\}.$$

Let $g : \omega^{\uparrow<\omega} \to \omega$ eventually dominate all g_α, and put $B_\tau := B \setminus g(\tau)$. By (‡) we can construct a real not in $\bigcup_\alpha Y_\alpha$, and the family we started with is not a covering family.

Dually, let $\{B_\alpha;\ \alpha < \pi\chi_\sigma(\mathcal{U})\}$ be $\pi\sigma$–base of $\mathcal{U}$, and let $\{f_\alpha : \omega^{\uparrow<\omega} \to \omega;\ \alpha < \mathfrak{d}\}$ be a dominating family. For $\alpha < \pi\chi_\sigma(\mathcal{U})$ and $\beta < \mathfrak{d}$ choose a real $y = y_{\alpha,\beta} \in \omega^{\uparrow\omega}$ with $y(i) \in B_\alpha \setminus f_\beta(y\restriction i)$. Given $Y = Y(\langle A_\tau\rangle) \in \ell^0_{\mathcal{U}}$, first find α with $B_\alpha \subseteq^* A_\tau$ for all τ, and then β with $B_\alpha \setminus f_\beta(\tau) \subseteq A_\tau$ for almost all τ. By (‡) we know that $y_{\alpha,\beta} \notin Y$, and thus $\{y_{\alpha,\beta};\ \alpha < \pi\chi_\sigma(\mathcal{U}), \beta < \mathfrak{d}\} \notin \ell^0_{\mathcal{U}}$. This concludes the proof of the Theorem. □

Corollary 3.1. *Let $\mathcal{U}$ be an ultrafilter on ω. Then:*
(a) $\mathsf{add}(\ell^0_{\mathcal{U}}) \leq \mathsf{cov}(\ell^0_{\mathcal{U}}) \leq \mathsf{non}(\ell^0_{\mathcal{U}}) \leq \mathsf{cof}(\ell^0_{\mathcal{U}})$.
(b) $\mathfrak{p} \leq \mathsf{cov}(\ell^0_{\mathcal{U}})$*; in particular, MA implies that* $\mathsf{cov}(\ell^0_{\mathcal{U}}) = \mathfrak{c}$.
(c) $\mathsf{cov}(\ell^0_{\mathcal{U}}) \leq \mathfrak{par}$ *and* $\mathsf{non}(\ell^0_{\mathcal{U}}) \geq \mathfrak{hom}$.
(d) $cf(\mathsf{cov}(\ell^0_{\mathcal{U}})) \geq \omega_1$.

Proof. All this is immediate from the Theorem and the results concerning $\mathfrak{b}, \mathfrak{d}, \mathfrak{p}, \mathfrak{s}, \mathfrak{r}_\sigma, \mathfrak{par}$ and $\mathfrak{hom}$ mentioned in § 1, in particular Corollary 1.3 and Proposition 1.7. □

Note that, since $\ell^0_{\mathcal{U}}$ is a σ–ideal, both $\mathsf{non}(\ell^0_{\mathcal{U}})$ and $\mathsf{cof}(\ell^0_{\mathcal{U}})$ necessarily have uncountable cofinality. Again, the cardinal coefficients are linearly ordered, like those for the related non–*ccc* *Laver ideal* ℓ^0 (see [GRSS]) — or those for the *ccc*–ideal of meager sets in the *dominating topology* (see [LR]); the latter topology in fact sits strictly in between the standard topology on $\omega^{\uparrow\omega}$ and Louveau's topology $\mathcal{G}^\infty_{\mathcal{U}}$ which is relevant here.

Distinguishing the two cases whether or not $\mathcal{U}$ is a P–point, we get somewhat nicer characterizations, by 1.6 and other remarks in § 1.

Corollary 3.2. *Assume $\mathcal{U}$ is a P–point. Then:*
(a) $\mathsf{add}(\ell^0_{\mathcal{U}}) = \min\{\mathfrak{p}(\mathcal{U}), \mathfrak{b}\}$;
(b) $\mathsf{cov}(\ell^0_{\mathcal{U}}) = \min\{\pi\mathfrak{p}(\mathcal{U}), \mathfrak{b}\}$;
(c) $\mathsf{non}(\ell^0_{\mathcal{U}}) = \max\{\pi\chi(\mathcal{U}), \mathfrak{d}\}$;
(d) $\mathsf{cof}(\ell^0_{\mathcal{U}}) = \max\{\chi(\mathcal{U}), \mathfrak{d}\}$.

Assume $\mathcal{U}$ is not a P–point. Then:
(a) $\mathsf{add}(\ell^0_{\mathcal{U}}) = \mathfrak{p}'(\mathcal{U})$;
(b) $\mathsf{cov}(\ell^0_{\mathcal{U}}) = \pi\mathfrak{p}(\mathcal{U})$;
(c) $\mathsf{non}(\ell^0_{\mathcal{U}}) = \pi\chi_\sigma(\mathcal{U})$;
(d) $\mathsf{cof}(\ell^0_{\mathcal{U}}) = \chi_\sigma(\mathcal{U})$. □

For rapid P–points, the formulae get still simpler, and, in fact, the invariants for the Ramsey ideal and the Louveau ideal coincide. In view of Louveau's $r^0_{\mathcal{U}} = \ell^0_{\mathcal{U}}$ for Ramsey ultrafilters $\mathcal{U}$, Theorem 2 provides an alternative way for calculating the coeffients of $r^0_{\mathcal{U}}$.

We close this section with a diagram showing the relations between the cardinal invariants considered in this work.

$$
\begin{array}{ccccccccc}
\mathfrak{p}(\mathcal{U}) & \mathfrak{p}'(\mathcal{U}) & \pi\mathfrak{p}(\mathcal{U}) & \mathfrak{s} & \mathfrak{d} & \mathfrak{hom} & \mathsf{non}(\ell^0_{\mathcal{U}}) & \mathsf{cof}(\ell^0_{\mathcal{U}}) & \mathfrak{c} \\
 & & & & & \mathfrak{r}_\sigma & \pi\chi_\sigma(\mathcal{U}) & \chi_\sigma(\mathcal{U}) & \\
 & \mathsf{add}(\ell^0_{\mathcal{U}}) & \mathsf{cov}(\ell^0_{\mathcal{U}}) & \mathfrak{par} & \mathfrak{b} & \mathfrak{r} & \pi\chi(\mathcal{U}) & \chi(\mathcal{U}) & \\
\omega & \omega_1 & \mathfrak{p} & & & & & &
\end{array}
$$

Cardinals get larger when one moves up and to the right. Dotted lines around three cardinals say that one of them is the minimum or the maximum of the others for any ultrafilter $\mathcal{U}$ (which one of these alternatives holds being clear from the context). For ease of reading, we omitted the inequality $\pi\mathfrak{p}(\mathcal{U}) \leq \pi\chi(\mathcal{U})$.

4. Distinguishing the coefficients

Let $\mathcal{U}$ be a Ramsey ultrafilter. Since, by previous results,

$$\mathsf{cov}(r^0_{\mathcal{U}}) = \pi\mathfrak{p}(\mathcal{U}) \leq \mathfrak{b} \leq \mathfrak{d} \leq \pi\chi(\mathcal{U}) = \mathsf{non}(r^0_{\mathcal{U}}),$$

we can easily get the consistency of *there is a Ramsey ultrafilter* $\mathcal{U}$ *with* $\mathsf{cov}(r^0_{\mathcal{U}}) < \mathsf{non}(r^0_{\mathcal{U}})$. For example, this can be achieved by adjoining ω_2 Cohen reals to a model of CH. Furthermore, as mentioned in § 1, Louveau proved that MA entails the existence of a Ramsey ultrafilter $\mathcal{U}$ with

$$\mathsf{add}(r^0_{\mathcal{U}}) = \mathfrak{p}(\mathcal{U}) = \omega_1 < \mathfrak{c} = \pi\mathfrak{p}(\mathcal{U}) = \mathsf{cov}(r^0_{\mathcal{U}}).$$

We complete this cycle of results by showing the remaining consistency:

Theorem 3. *It is consistent with ZFC that there is a Ramsey ultrafilter* $\mathcal{U}$ *with* $\mathsf{non}(r^0_{\mathcal{U}}) = \pi\chi(\mathcal{U}) < \chi(\mathcal{U}) = \mathsf{cof}(r^0_{\mathcal{U}})$.

This answers half of the question in [Br, subsection 4.1]. As suggested by the referee, we note that this consistency has been well–known if one doesn't insist on $\mathcal{U}$'s Ramseyness (to see this either use the

Goldstern–Shelah model [GS] showing the consistency of $\mathfrak{r} < \mathfrak{u} := \min_{\mathcal{U}} \chi(\mathcal{U})$ and appeal to Proposition 7.1 below, or use the Bell–Kunen model [BK] (cf. Remark 4.2 below)).

Proof. We start with a model V of CH and perform a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha;\ \alpha < \omega_1 \rangle$ of *ccc* p.o.'s. We build up the Ramsey ultrafilter $\mathcal{U}$ of V_{ω_1} along the iteration as a tower of ultrafilters; in stage $2 \cdot \alpha + 1$, we shall have the Ramsey ultrafilter $\mathcal{U}_{2\cdot\alpha+1}$ in the model $V_{2\cdot\alpha+1}$. The details are as follows.

Stage α, α odd. In V_α, we let $\mathbb{Q}_\alpha = \mathbb{M}_{\mathcal{U}_\alpha}$ (Mathias forcing with the Ramsey ultrafilter $\mathcal{U}_\alpha$, see § 2 for details). Denote by $m_\alpha \in [\omega]^\omega \cap V_{\alpha+1}$ the generic Mathias real (which satisfies $m_\alpha \subseteq^* U$ for all $U \in \mathcal{U}_\alpha$).

Stage α, α even. In V_α, we let $\mathbb{Q}_\alpha = \mathbb{C}_{\omega_2}$, the p.o. for adding ω_2 Cohen reals. In $V_{\alpha+1}$, we use the ω_2 Cohen reals to produce the Ramsey ultrafilter $\mathcal{U}_{\alpha+1}$ which extends either the filter $\mathcal{F}_\alpha$ generated by $\bigcup_{\gamma<\alpha} \mathcal{U}_{2\cdot\gamma+1}$ (in case α is a limit) or the filter $\mathcal{F}_\alpha$ generated by $\mathcal{U}_{\alpha-1}$ and $\omega \setminus m_{\alpha-1}$ (in case α is successor) or the cofinite filter $\mathcal{F}_0$ (in case $\alpha = 0$). This is a standard construction (see, e.g., [Ca 2, Theorem 2], [BJ, § 3] or [St, Theorem 5.2]) which we repeat to make later arguments more transparent.

Let $V_{\alpha,\beta}$, $\beta \leq \omega_2$, denote the model obtained by adding β of the Cohen reals (so $V_{\alpha,0} = V_\alpha$ and $V_{\alpha,\omega_2} = V_{\alpha+1}$). In $V_{\alpha+1}$ enumerate the partitions of ω into infinite subsets as $\langle \langle X_{\alpha,\beta,n};\ n \in \omega \rangle;\ \beta < \omega_2 \rangle$ such that $\langle X_{\alpha,\beta,n};\ n \in \omega \rangle \in V_{\alpha,\beta}$. Let $\mathcal{U}_{\alpha,0}$ be a *careful* extension of $\mathcal{F}_\alpha$ to an ultrafilter of $V_{\alpha,0}$ (*careful* will be defined later). Fix $\beta \leq \omega_2$, and assume $\mathcal{U}_{\alpha,\gamma}$, a tower of ultrafilters in the respective models $V_{\alpha,\gamma}$, has been constructed. In case $cf(\beta) > \omega$, we let $\mathcal{U}_{\alpha,\beta} = \bigcup_{\gamma<\beta} \mathcal{U}_{\alpha,\gamma}$; and in case $cf(\beta) = \omega$, $\mathcal{U}_{\alpha,\beta}$ is a *careful* extension of $\bigcup_{\gamma<\beta} \mathcal{U}_{\alpha,\gamma}$ to the model $V_{\alpha,\beta}$. In case $\beta = \gamma + 1$ do the following.

If $\bigcup_{k<n} X_{\alpha,\gamma,k} \in \mathcal{U}_{\alpha,\gamma}$ for some n, we think of Cohen forcing $\mathbb{C}$ as adjoining a subset of ω, called $c_{\alpha,\gamma}$, in the usual way. Call such $c_{\alpha,\gamma}$ *of the first kind.* Otherwise think of Cohen forcing $\mathbb{C}$ as adjoining a subset $c_{\alpha,\gamma}$ of ω with $|c_{\alpha,\gamma} \cap X_{\alpha,\gamma,n}| = 1$ for all n. Call such $c_{\alpha,\gamma}$ *of the second kind.* Then, by genericity, $c_{\alpha,\gamma} \cap U$ is infinite for all $U \in \mathcal{U}_{\alpha,\gamma}$. In both cases, let $\mathcal{U}_{\alpha,\beta}$ be a *careful* extension of the filter generated by $\mathcal{U}_{\alpha,\gamma}$ and $c_{\alpha,\gamma}$ to an ultrafilter in $V_{\alpha,\beta}$.

This completes the construction of the Ramsey ultrafilter $\mathcal{U} = \bigcup_{\alpha<\omega_1} \mathcal{U}_{2\cdot\alpha+1}$ in the resulting model V_{ω_1}. The ω_1 Mathias reals m_α witness $\pi\chi(\mathcal{U}) = \omega_1$. Thus we are left with showing $\chi(\mathcal{U}) = \omega_2$. One inequality is clear because $\mathfrak{c} = \omega_2$. To see the other one, we have to make precise what we mean by *careful.*

Let $\mathcal{I}$ be the family of subsets I of ω for which we can find pairwise disjoint finite sets $F_\alpha \subseteq \omega_1 \times \omega_2$, $\alpha < \omega_1$, such that

$$\forall \alpha < \omega_1 \qquad\qquad I \subseteq^* \bigcup_{(\beta,\gamma)\in F_\alpha} c_{\beta,\gamma},$$

where the $c_{\beta,\gamma}$ denote the Cohen reals as explained above. Clearly, $\mathcal{I}$ is an ideal (in V_{ω_1}). Let $\mathcal{I}_{\alpha,\beta} = \mathcal{I} \cap V_{\alpha,\beta}$ for even $\alpha < \omega_1$ and $\beta \leq \omega_2$. Note that the above definition of $\mathcal{I}$ also makes sense, with the obvious adjustments, in each model $V_{\alpha,\beta}$. Call the resulting ideal $\mathcal{I}^{V_{\alpha,\beta}}$. Then one has $\mathcal{I}^{V_{\alpha,\beta}} = \mathcal{I}_{\alpha,\beta}$, and thus $\mathcal{I}_{\alpha,\beta} \in V_{\alpha,\beta}$.

(The inclusion " $\subseteq$ " is obvious. To see " $\supseteq$ ", note first that if $I \in V_{\alpha,\beta}$ and $I \subseteq^* \bigcup_{(\gamma,\delta)\in F} c_{\gamma,\delta}$, then $I \subseteq^* \bigcup_{(\gamma,\delta)\in F'} c_{\gamma,\delta}$ where $F' = F \cap ((\alpha \times \omega_2) \cup (\{\alpha\} \times \beta))$, by Cohenness of the $c_{\gamma,\delta}$. Now assume (still in $V_{\alpha,\beta}$) that $p \Vdash_{[(\alpha,\beta),\omega_1)}$ "$I \in \dot{\mathcal{I}}$ as witnessed by $\dot{F}_\zeta, \zeta < \omega_1$". Find $p_\zeta \leq p$ (in $V_{\alpha,\beta}$) such that p_ζ decides $\dot{F}_\zeta$, say $p_\zeta \Vdash_{[(\alpha,\beta),\omega_1)} \dot{F}_\zeta = F_\zeta$. By the previous remark we may assume $F_\zeta \subseteq (\alpha \times \omega_2) \cup (\{\alpha\} \times \beta)$. Since all the factors of the iteration satisfy Knaster's condition, so does the quotient $\mathbb{P}_{[(\alpha,\beta),\omega_1)}$. Thus we may assume without loss that the p_ζ are pairwise compatible. Hence the F_ζ are pairwise disjoint. Therefore $I \in \mathcal{I}^{V_{\alpha,\beta}}$ as required.)

We shall guarantee while extending the ultrafilter that

$$(\star) \qquad \mathcal{U}_{\alpha,\beta} \cap \mathcal{I}_{\alpha,\beta} = \emptyset \quad \text{for all even } \alpha < \omega_1 \text{ and all } \beta \leq \omega_2.$$

Such extensions will be called *careful.* We have various cases to consider to see that this can be done.

(1) Successor step. Assume that $\mathcal{U}_{\alpha,\beta} \cap \mathcal{I}_{\alpha,\beta} = \emptyset$ (where $\alpha < \omega_1$ is even and $\beta < \omega_2$). Since $V_{\alpha,\beta+1}$ is an extension by one Cohen real, we certainly have $\langle \mathcal{U}_{\alpha,\beta} \rangle \cap \mathcal{I}_{\alpha,\beta+1} = \emptyset$ where $\langle \mathcal{U}_{\alpha,\beta} \rangle$ denotes the filter generated by $\mathcal{U}_{\alpha,\beta}$ in $V_{\alpha,\beta+1}$. Next notice that $c_{\alpha,\beta} \cap U \notin \mathcal{I}_{\alpha,\beta+1}$ for all $U \in \mathcal{U}_{\alpha,\beta}$. (†) To see this, fix such U. Assume $c_{\alpha,\beta}$ is of the second kind. Put $U_n = U \setminus \bigcup_{k<n} X_{\alpha,\beta,k}$. All U_n lie in $\mathcal{U}_{\alpha,\beta}$ and thus not in $\mathcal{I}_{\alpha,\beta+1}$. Now note that whenever $F \subseteq (\alpha \times \omega_2) \cup (\{\alpha\} \times \beta)$ is finite with $c_{\alpha,\beta} \cap U \subseteq^* \bigcup_{(\gamma,\delta)\in F} c_{\gamma,\delta}$, then by Cohenness $U_n \subseteq^* \bigcup_{(\gamma,\delta)\in F} c_{\gamma,\delta}$ for some n. If $c_{\alpha,\beta}$ is of the first kind, the argument is even easier. This shows (†). Now we can easily extend the filter generated by $\mathcal{U}_{\alpha,\beta}$ and $c_{\alpha,\beta}$ to an ultrafilter $\mathcal{U}_{\alpha,\beta+1}$ such that $\mathcal{U}_{\alpha,\beta+1} \cap \mathcal{I}_{\alpha,\beta+1} = \emptyset$.

(2) Limit step. Let α be even, and β a limit ordinal. If $\beta > 0$, assume that $\mathcal{U}_{\alpha,\gamma} \cap \mathcal{I}_{\alpha,\gamma} = \emptyset$ for $\gamma < \beta$. In case $cf(\beta) > \omega$ we get $\mathcal{U}_{\alpha,\beta} \cap \mathcal{I}_{\alpha,\beta} = \emptyset$ because any $U \in \mathcal{U}_{\alpha,\beta}$ lies already in $V_{\alpha,\gamma}$ for some $\gamma < \beta$, and thus cannot be almost contained in the union of finitely many Cohen reals added at a later stage. If $\beta = 0$ and α non–limit, it remains to see that $\omega \setminus m_{\alpha-1} \cap U \notin \mathcal{I}_{\alpha,0}$ for all $U \in \mathcal{U}_{\alpha-1}$ which is similar to, but easier than, the argument for the $c_{\gamma,\delta}$ in (1). Whether or not α is limit and whether $cf(\beta) = \omega$ or $\beta = 0$, we can extend the given filter easily to $\mathcal{U}_{\alpha,\beta}$ such that $\mathcal{U}_{\alpha,\beta} \cap \mathcal{I}_{\alpha,\beta} = \emptyset$.

(1) and (2) clearly entail $(\star)$. If we had $\chi(\mathcal{U}) \leq \omega_1$, we could find $U \in \mathcal{U}$ which is almost included in ω_2 of the Cohen reals which we added to $\mathcal{U}$ in the course of the construction, and this would contradict $(\star)$. Thus $\chi(\mathcal{U}) \geq \omega_2$, and the proof is complete. □

Remark 4.1. ω_1 and ω_2 in the above proof can be replaced by arbitrary regular $\kappa < \lambda$. The argument is the same: $\chi(\mathcal{U}) \leq \lambda$ by $\mathfrak{c} = \lambda$, $\chi(\mathcal{U}) \geq \lambda$ by $(\star)$, $\pi\chi(\mathcal{U}) \leq \kappa$ by the κ Mathias reals and $\pi\chi(\mathcal{U}) \geq \kappa$ by the fact that the iteration has length κ which implies $\pi\chi(\mathcal{U}) \geq \mathfrak{d} \geq \mathsf{cov}(\text{meager}) \geq \kappa$ (where the first two inequalitites are in ZFC). Here, $\mathsf{cov}(\text{meager})$ denotes the smallest size of a family of meager sets covering the reals. It is well–known (and easy to see) that $\mathfrak{d} \geq \mathsf{cov}(\text{meager})$.

Remark 4.2. The method of the proof of Theorem 3 can also be used to show there is a Ramsey ultrafilter $\mathcal{U}$ with $\chi(\mathcal{U}) = \mathfrak{c}$ in the Bell–Kunen model (see [BK]; this model is gotten by a finite support iteration of *ccc* p.o.'s of length ω_{ω_1} over a model of CH, forcing MA for small p.o.'s at limit steps of the form $\omega_{\alpha+1}$; it satisfies $\mathfrak{c} = \omega_{\omega_1}$ and $\pi\chi(\mathcal{V}) = \omega_1$ for all ultrafilters $\mathcal{V}$). Hence it is consistent there is a Ramsey ultrafilter $\mathcal{U}$ with $\pi\chi(\mathcal{U}) = \omega_1 < \omega_{\omega_1} = \chi(\mathcal{U})$.

5. A plethora of π–characters

This section is devoted to understanding the spectrum of possible values for the π–character and clearing up the relationship between π–character and $\pi\sigma$–character. For this, we need to discuss two ultrafilter constructions. First, let $\mathcal{V}$ and $\mathcal{V}_n$ be ultrafilters on ω, and define an ultrafilter $\mathcal{U}$ on $\omega \times \omega$ by

$$X \in \mathcal{U} \Longleftrightarrow \{n;\ \{m;\ \langle n,m\rangle \in X\} \in \mathcal{V}_n\} \in \mathcal{V}.$$

(Note that we used this construction already once in the proof of Proposition 1.5.) We call $\mathcal{U}$ the $\mathcal{V}$*–sum* of the $\mathcal{V}_n$, $\mathcal{U} = \sum_{\mathcal{V}} \mathcal{V}_n$. In case all $\mathcal{V}_n$ are the same ultrafilter $\mathcal{W}$, we write $\mathcal{U} = \mathcal{V} \times \mathcal{W}$ and call it the *product* of $\mathcal{V}$ and $\mathcal{W}$. Then we have:

PROPOSITION 5.1. *(a)* $\min\{\pi\chi(\mathcal{V}), \sum_{\mathcal{V}} \pi\chi(\mathcal{V}_n)\} \leq \pi\chi(\mathcal{U}) \leq \sum_{\mathcal{V}} \pi\chi(\mathcal{V}_n)$.

(b) If $\mathcal{U} = \mathcal{V} \times \mathcal{W}$*, we have* $\pi\chi(\mathcal{U}) = \pi\chi(\mathcal{W})$.

(c) $\pi\chi_\sigma(\mathcal{U}) \geq \pi\chi_\sigma(\mathcal{V})$.

(d) If $\mathcal{U} = \mathcal{V} \times \mathcal{W}$*, we have* $\pi\chi_\sigma(\mathcal{U}) = \max\{\pi\chi_\sigma(\mathcal{V}), \pi\chi_\sigma(\mathcal{W}), \mathfrak{d}\}$.

Here, given cardinals λ_α, $\alpha \in R$, and an ultrafilter $\mathcal{D}$ on R, $\sum_{\mathcal{D}} \lambda_\alpha$ denotes the $\mathcal{D}$*–limit* of the λ_α, that is the least cardinal κ such that $\{\alpha;\ \lambda_\alpha \leq \kappa\} \in \mathcal{D}$.

Proof. For the purposes of this proof, let $X_n = \{n\} \times \omega$ denote the vertical strips.

(a) The second inequality is easy, for we can take the union of π–bases of the appropriate $\mathcal{V}_n$'s (considered as ultrafilters on the X_n's) as a π–base for $\mathcal{U}$. For the first inequality, let $\kappa <$ min and $\mathcal{A} = \{A_\alpha;\ \alpha < \kappa\} \subseteq [\omega \times \omega]^\omega$. We want to show $\mathcal{A}$ is not a π–base of $\mathcal{U}$. Without loss, all A_α are either contained in one X_n or intersect each X_n at most once. For the second kind of A_α, let $B_\alpha = \{n;\ A_\alpha \cap X_n \neq \emptyset\}$. There is $C \in \mathcal{V}$ such that the A_α of the first kind are not a π–base of $\mathcal{V}_n$ for $n \in C$; let $D_n \in \mathcal{V}_n$ witness this. Since the B_α don't form a π–base of $\mathcal{V}$, choose $E \subseteq C$ witnessing this. We now see easily that $A_\alpha \not\subseteq^* \bigcup_{n \in E} \{n\} \times D_n \in \mathcal{U}$, as required.

(b) By (a) it suffices to prove that $\pi\chi(\mathcal{U}) \geq \pi\chi(\mathcal{W})$. Let $\kappa < \pi\chi(\mathcal{W})$, and $\mathcal{A} = \{A_\alpha;\ \alpha < \kappa\} \subseteq [\omega \times \omega]^\omega$. We want to show $\mathcal{A}$ is not a π–base of $\mathcal{U}$. For this simply let $B_\alpha = \{m;\ \langle n,m\rangle \in A_\alpha$ for some $n\}$, find $C \in \mathcal{W}$ which does not almost contain any of the B_α which are infinite, and note that $A_\alpha \not\subseteq^* \{\langle n,m\rangle;\ m > n$ and $m \in C\} \in \mathcal{U}$ (for any α), as required. (Note that the same argument shows that $\pi\chi(\mathcal{U}) \geq \pi\chi(f(\mathcal{U}))$ for all ultrafilters $\mathcal{U}$ and all finite–to–one functions f.)

(c) This is easy, for given a $\pi\sigma$–base $\mathcal{A}$ of $\mathcal{U}$, the family $\{B_A;\ A \in \mathcal{A}\} \cap [\omega]^\omega$ where $B_A = \{n;\ A \cap X_n \neq \emptyset\}$ is a $\pi\sigma$–base of $\mathcal{V}$.

(d) By (c) we know $\pi\chi_\sigma(\mathcal{U}) \geq \pi\chi_\sigma(\mathcal{V})$; $\pi\chi_\sigma(\mathcal{U}) \geq \pi\chi_\sigma(\mathcal{W})$ is proved as in (b); finally, $\pi\chi_\sigma(\mathcal{U}) \geq \mathfrak{d}$ follows from Proposition 1.6 because $\mathcal{U}$ is not a P–point. So we are left with showing that $\pi\chi_\sigma(\mathcal{U}) \leq$ max.

For this choose $\pi\sigma$–bases $\{A_\alpha;\ \alpha <$ max$\}$ of $\mathcal{W}$ and $\{B_\beta;\ \beta <$ max$\}$ of $\mathcal{V}$, as well as a dominating family $\{f_\gamma;\ \gamma <$ max$\}$. Put $C_{\alpha,\beta,\gamma} = \{\langle n,m\rangle;\ n \in B_\beta$ and $m = \min(A_\alpha \setminus f_\gamma(n)\}$. To see that the $C_{\alpha,\beta,\gamma}$ form a $\pi\sigma$–base of $\mathcal{U}$, take $D_k \in \mathcal{U}$, put $E_k = \{n;\ \{m;\ \langle n,m\rangle \in D_k\} \in \mathcal{W}\} \in \mathcal{V}$, and find β with $B_\beta \subseteq^* E_k$ for all k. Also let $F_{k,n} = \{m;\ \langle n,m\rangle \in D_k\} \in \mathcal{W}$ for $n \in E_k$, and find α with $A_\alpha \subseteq^* F_{k,n}$ for all k,n. Define $g_k \in \omega^\omega$ such that $A_\alpha \subseteq F_{k,n} \setminus g_k(n)$ for all n, and find γ such that $f_\gamma \geq^* g_k$ for all k. It is now easy to see that $C_{\alpha,\beta,\gamma} \subseteq^* D_k$ for all k. □

Another ultrafilter construction goes as follows: let λ be a regular uncountable cardinal, let $\mathcal{T} = \langle T_\alpha;\ \alpha < \lambda\rangle$ be a tower, let $\mathcal{V}_\alpha$, $\alpha < \lambda$, be ultrafilters on ω with $T_\alpha \in \mathcal{V}_\alpha$, and let $\mathcal{D}$ be a uniform ultrafilter on λ. Define $\mathcal{U}$ as the *$\mathcal{D}$–limit* of the $\mathcal{V}_\alpha$, i.e.

$$U \in \mathcal{U} \Longleftrightarrow \{\alpha;\ U \in \mathcal{V}_\alpha\} \in \mathcal{D}.$$

Then:

Proposition 5.2. $\lambda \leq \pi\chi(\mathcal{U}) \leq \lambda \cdot \sum_{\mathcal{D}} \pi\chi(\mathcal{V}_\alpha)$.

Proof. Note that $\mathcal{T} \subseteq \mathcal{U}$. Since $\mathcal{T}$ has no pseudointersection, $\pi\chi(\mathcal{U}) \geq \lambda$ is immediate. To see the other inequality, let $\mathcal{A}_\alpha$ be π–bases of the $\mathcal{V}_\alpha$ for appropriate α's. Then $\bigcup_\alpha \mathcal{A}_\alpha$ is a π–base of $\mathcal{U}$. This shows $\pi\chi(\mathcal{U}) \leq \lambda \cdot \sum_{\mathcal{D}} \pi\chi(\mathcal{V}_\alpha)$. □

As an immediate consequence we see

Corollary 5.3. *Let $\kappa < \lambda$ be regular uncountable cardinals such that there is an ultrafilter $\mathcal{V}$ with $\pi\chi(\mathcal{V}) = \kappa$ and a tower of height λ. Then there is an ultrafilter $\mathcal{U}$ with $\pi\chi(\mathcal{U}) = \lambda$.* □

Theorem 4. *(a) Let R be a set of regular uncountable cardinals in $V \models GCH$. Then there is a forcing notion $\mathbb{P}$ such that*

$$V^{\mathbb{P}} \models \text{“for all } \lambda \in R \text{ there is an ultrafilter } \mathcal{U} \text{ such that } \pi\chi(\mathcal{U}) = \lambda\text{”}.$$

(b) It is consistent there is an ultrafilter $\mathcal{U}$ with $\pi\chi(\mathcal{U}) < \pi\chi_\sigma(\mathcal{U})$. More explicitly, given $\kappa < \lambda$ regular uncountable, we can force $\pi\chi(\mathcal{U}) = \kappa$ and $\pi\chi_\sigma(\mathcal{U}) = \lambda$ for some ultrafilter $\mathcal{U}$.

(c) It is consistent there is an ultrafilter $\mathcal{U}$ with $\pi\chi(\mathcal{U}) = \mathsf{non}(r^0_{\mathcal{U}}) = \omega_\omega$. In particular $\pi\chi(\mathcal{U}) = \mathsf{non}(r^0_{\mathcal{U}})$ consistently has countable cofinality.

Proof. (a) We plan to adjoin an ultrafilter $\mathcal{U}$ with $\pi\chi(\mathcal{U}) = \omega_1$ and towers $\mathcal{T}_\lambda$ of height λ for each $\lambda \in R$. Then the result will follow by 5.3. Note that the consistency of the existence of towers of different heights was proved by Dordal [Do 1, section 2] with essentially the same argument.

Let $\mu > \sup(R)^+$ be a regular cardinal. We shall have $\mathbb{P} = \mathbb{P}^0 \times \mathbb{P}^1$ where $\mathbb{P}^0$ is the Easton product which adds μ subsets to λ for each $\lambda \in R$ and $\mathbb{P}^1$ is a *ccc* forcing notion. Since $\mathbb{P}^1$ is still *ccc* in $V^{\mathbb{P}^0}$, cofinalities and cardinals are preserved.

$\mathbb{P}^1$ is an iteration $\mathbb{P}^2 \star \dot{\mathbb{P}}^3$ where $\mathbb{P}^2$ is the finite support product of the forcings $\mathbb{Q}^\lambda$, $\lambda \in R$, which add families $\{C^\lambda_\eta;\ \eta \in 2^{<\lambda}\}$ of infinite subsets of ω such that

- $\eta \subseteq \theta$ implies $C^\lambda_\theta \subseteq^* C^\lambda_\eta$ and
- $C^\lambda_{\eta\hat{}\langle 0\rangle} \cap C^\lambda_{\eta\hat{}\langle 1\rangle}$ is finite,

with finite conditions (see the proof of Theorem 5 for a similar, but more complicated, forcing). In $V^{\mathbb{P}^2}$, $\mathbb{P}^3$ is a finite support iteration of length ω_1 of Mathias forcings with an ultrafilter (see §§ 2 and 4) which adds an ultrafilter $\mathcal{U}$ all of whose cardinal coefficients are equal to ω_1. (Alternatively, we could define $\mathbb{P}^3$ in $V^{\mathbb{P}^0 \times \mathbb{P}^2}$.)

Since $\mathbb{P}^0$ is still ω_1–distributive over $V^{\mathbb{P}^1}$, it doesn't add reals, and $\mathcal{U}$ still is an ultrafilter with $\pi\chi(\mathcal{U}) = \omega_1$ in $V^{\mathbb{P}}$ $(\star)$. Also, in $V^{\mathbb{P}}$ we have $\mathfrak{c} < \mu$, but $2^\lambda = \mu$ for all $\lambda \in R$. Given $f \in 2^\lambda$ with $f{\restriction}\alpha \in V$ for all $\alpha < \lambda$,

$$\mathcal{T}_f = \{C^\lambda_{f\restriction\alpha};\ \alpha < \lambda\}$$

forms a $\subseteq^*$–decreasing chain. Because of the Easton product, we have μ such $\mathcal{T}_f$'s for each λ. Since $\mathfrak{c} < \mu$, not all of them can have a pseudointersection. Hence, for each $\lambda \in R$, there is a tower $\mathcal{T}_\lambda$ of height λ $(\star\star)$. (In fact, a density argument shows none of them has a pseudointersection, see the proof of Theorem 5.) By Corollary 5.3 as well as $(\star)$ and $(\star\star)$, we have, for each $\lambda \in R$, an ultrafilter $\mathcal{V}$ with $\pi\chi(\mathcal{V}) = \lambda$.

(b) By [BlS] we know it is consistent there is a P–point $\mathcal{V}$ with $\kappa = \pi\chi(\mathcal{V}) = \pi\chi_\sigma(\mathcal{V}) < \mathfrak{d} = \lambda$. Put $\mathcal{U} = \mathcal{V} \times \mathcal{V}$. Then $\pi\chi(\mathcal{U}) = \pi\chi(\mathcal{V}) = \kappa$ by 5.1 (b) and $\pi\chi_\sigma(\mathcal{U}) = \mathfrak{d} = \lambda$ by 5.1 (d). (Instead of [BlS], the ultrafilters gotten in the construction in Theorem 5 could be used for this consistency.)

(c) By (a) we can force ultrafilters $\mathcal{V}_\alpha$ for all regular α with $\omega_1 \leq \alpha \leq \omega_{\omega+1}$. Then $\mathcal{U} = \sum_{\mathcal{V}_{\omega_{\omega+1}}} \mathcal{V}_n$ satisfies $\pi\chi(\mathcal{U}) = \omega_\omega$, by 5.1 (a). $\square$

We conclude this section with the discussion of several refinements of Theorem 4. The construction in part (b) of the above proof also shows that the result in Lemma 1.6 is sharp and cannot be improved.

Corollary 5.4. *It is consistent there is an ultrafilter $\mathcal{U}$ which is not a P–point such that $\pi\chi(\mathcal{U}) < \mathfrak{d}$.* $\square$

The result in Theorem 4 (a) will be superseded by Theorem 5 in the next section. We still gave its proof because it is much simpler and also because of the following two consequences of the construction which we cannot get from Theorem 5.

Corollary 5.5. *In the statement of Theorem 4 (a), we can delete the word "regular".*

Proof. Assume without loss that whenever $\lambda \in R$ is singular, then $R \cap \lambda$ is cofinal in λ. We show the construction in the proof of Theorem 4 (a) produces an ultrafilter $\mathcal{U}$ with $\pi\chi(\mathcal{U}) = \lambda$.

For $\mu \in R\cap\lambda$ we added towers $\mathcal{T}_\mu = \langle T_{\alpha,\mu};\ \alpha < \mu\rangle$ such that $\bigcup_{\mu\in R\cap\lambda} \mathcal{T}_\mu$ is a filter base (this is immediate from the definition of the forcing $\mathbb{P}^2$). Put $S = \{F;\ F$ is a finite subset of $\{\langle\alpha,\mu\rangle;\ \alpha < \mu$ and $\mu \in R \cap \lambda\}\}$. Clearly $|S| = \lambda$. For $F \in S$ let $\mathcal{V}_F$ be an ultrafilter on $\bigcap_{\langle\alpha,\mu\rangle\in F} T_{\alpha,\mu}$ with $\pi\chi(\mathcal{V}_F) = \omega_1$. Let $\mathcal{D}$ be a uniform ultrafilter on S such that for any $F \in S$, $\{G \in S;\ G \supseteq F\} \in \mathcal{D}$. Put $\mathcal{U} = \lim_{\mathcal{D}} \mathcal{V}_F = \{X \subseteq \omega;\ \{F \in S;\ X \in \mathcal{V}_F\} \in \mathcal{D}\}$. We have to show $\pi\chi(\mathcal{U}) = \lambda$. This is no more but an elaboration of the argument in Proposition 5.2.

$\pi\chi(\mathcal{U}) \leq \lambda$ is immediate since the union of the π–bases of the $\mathcal{V}_F$ is a π–base of $\mathcal{U}$. To see $\pi\chi(\mathcal{U}) \geq \lambda$, it suffices to show that $\mathcal{T}_\mu \subseteq \mathcal{U}$ for each $\mu \in R \cap \lambda$. Fix $\alpha < \mu$. As $\{G \in S;\ G \ni \langle\alpha,\mu\rangle\} \in \mathcal{D}$ and $T_{\alpha,\mu} \in \mathcal{V}_G$ for any $G \in S$ with $\langle\alpha,\mu\rangle \in G$, it follows that $T_{\alpha,\mu} \in \mathcal{U}$, as required. $\square$

Corollary 5.6. *In Theorem 4 (a), we can additionally demand that the dominating number $\mathfrak{d}$ is an arbitrary regular uncountable cardinal. In particular, there may be many different π–characters below $\mathfrak{d}$.*

Proof. Simply replace the forcing $\mathbb{P}^3$ in the proof of Theorem 4 (a) by the forcing from [BlS] which adds an ultrafilter $\mathcal{U}$ with $\chi(\mathcal{U}) = \omega_1$ while forcing $\mathfrak{d} = \kappa$ where κ is an arbitrary regular cardinal. $\square$

6. The spectral problem

By Louveau's Theorem mentioned in § 1, we know it is consistent that there are simultaneously ultrafilters with many different values for $\mathfrak{p}$. The same is true for $\pi\chi$, as proved in the preceding section. We now complete this cycle of results by showing how to get the consistency of the simultaneous existence of many ultrafilter characters and, dually, of many values for $\pi\mathfrak{p}$.

Theorem 5. *Let R be a set of regular uncountable cardinals in $V \models GCH$. Then there is a forcing notion $\mathbb{P}$ such that*

$$V^{\mathbb{P}} \models \text{“for all } \lambda \in R \text{ there is an ultrafilter } \mathcal{U} \text{ such that } \chi(\mathcal{U}) = \pi\chi(\mathcal{U}) = \lambda\text{”.}$$

In fact, the ultrafilters we construct in the proof are all P–points.

Proof. We plan to adjoin, for each $\lambda \in R$, a matrix $\langle E^{\alpha}_{\lambda,\gamma};\ \alpha < \omega_1, \gamma < \lambda\rangle$ of subsets of ω such that the following conditions are met:

(i) $\langle E^{\alpha}_{\lambda,\gamma};\ \gamma < \lambda\rangle$ forms a tower;

(ii) $\alpha < \beta < \omega_1$ entails $E^{\beta}_{\lambda,\gamma} \subseteq^* E^{\alpha}_{\lambda,\gamma}$;

(iii) for each $X \subseteq \omega$ we find a pair $\langle \alpha, \gamma\rangle$ such that either $E^{\alpha}_{\lambda,\gamma} \subseteq^* X$ or $E^{\alpha}_{\lambda,\gamma} \subseteq^* \omega \setminus X$.

Clearly, this is enough: all three conditions imply the matrix generates an ultrafilter, we get $\chi(\mathcal{U}) \le \lambda$ by the size of the matrix, and (i) entails $\pi\chi(\mathcal{U}) \ge \lambda$.

We now describe the forcing we use. We shall have $\mathbb{P} = \mathbb{P}^0 \times \mathbb{P}^1$ where $\mathbb{P}^1$ is *ccc* and $\mathbb{P}^0$ is the Easton product of the forcing notions adding one subset of λ with conditions of size $< \lambda$ for $\lambda \in R$. Since $\mathbb{P}^0$ is ω_1–closed, it preserves the *ccc* of $\mathbb{P}^1$, and thus $\mathbb{P}$ preserves cofinalities and cardinals. However, we shall look at $\mathbb{P}$ as first forcing with $\mathbb{P}^1$ and then with $\mathbb{P}^0$. In $V^{\mathbb{P}^1}$, the closure property of $\mathbb{P}^0$ is lost, but it is still ω_1–distributive.

To define $\mathbb{P}^1$, put

$$\mu = \begin{cases} \sup(R) & \text{if this has uncountable cofinality} \\ \sup(R)^+ & \text{otherwise} \end{cases}$$

Let $\zeta = \mu \cdot \omega_1$, and let $\{A_\lambda;\ \lambda \in R\}$ be a partition of ζ such that $|A_\lambda \cap [\mu \cdot \beta, \mu(\beta + 1))| = \mu$ for each $\beta < \omega_1$. $\mathbb{P}^1$ shall be a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha;\ \alpha < \zeta\rangle$ of *ccc* p.o.'s such that

$$\Vdash_\alpha \text{“}|\dot{\mathbb{Q}}_\alpha| \le \mu\text{”}$$

for all $\alpha < \zeta$. Since we have GCH in the ground model, this implies $V^{\mathbb{P}^1} \models \mathfrak{c} \le \mu$ so that we can enumerate the names of subsets of ω arising in the extension in order type ζ. More explicitly, we shall have a sequence $\langle \dot{X}_\alpha;\ \alpha < \zeta\rangle$ such that

- $\Vdash_\alpha$ “$\dot{X}_\alpha \subseteq \omega$” and
- whenever $\dot{X}$ is a $\mathbb{P}_\alpha$–name for a subset of ω and $\lambda \in R$, then there is $\beta \ge \alpha$, $\beta \in A_\lambda$, such that

$$\Vdash_\beta \text{“}\dot{X} = \dot{X}_\beta\text{”}.$$

Clearly this can be done.

Along the iteration, we want to add, for $\lambda \in R$ and $\alpha \in A_\lambda$, a system $\langle C^\alpha_\eta;\ \eta \in 2^{<\lambda} \cap V \rangle$ of infinite subsets of ω lying in $V^{\mathbb{P}_{\alpha+1}}$ such that

(1) $\eta \subseteq \theta$ implies $C^\alpha_\theta \subseteq^* C^\alpha_\eta$;

(2) $C^\alpha_{\eta\hat{}\langle 0\rangle} \cap C^\alpha_{\eta\hat{}\langle 1\rangle}$ is finite;

(3) if $\alpha < \beta$ both belong to A_λ, then $C^\beta_\eta \subseteq^* C^\alpha_\eta$;

(4) in $V^{\mathbb{P}_{\alpha+1}}$, the set $\{\eta \in 2^{<\lambda} \cap V;\ C^\alpha_\eta \subseteq^* X_\alpha$ or $C^\alpha_\eta \cap X_\alpha$ is finite$\}$ is dense (and open) in $2^{<\lambda} \cap V$.

In (4), X_α denotes, of course, the interpretation of $\dot{X}_\alpha$ in $V^{\mathbb{P}_{\alpha+1}}$.

We are ready to describe the factors $\mathbb{Q}_\alpha$ of the iteration. Fix α and work in $V^{\mathbb{P}_\alpha}$. We distinguish two cases:

Case 1. $\alpha = \min(A_\lambda)$ or $cf(A_\lambda \cap \alpha) \geq \omega$.

$\mathbb{Q}_\alpha$ consists of pairs $\langle s, F\rangle$ where $F \subseteq A_\lambda \cap \alpha$ is finite (the second part of the condition is missing in case $\alpha = \min(A_\lambda)$) and s is a finite partial function with $dom(s) \subseteq 2^{<\lambda} \cap V$ and such that $s(\eta) \subseteq \omega$ is finite for all $\eta \in dom(s)$. We stipulate $\langle s, F\rangle \leq \langle t, G\rangle$ iff $G \subseteq F$, $dom(t) \subseteq dom(s)$ and $t(\eta) \subseteq s(\eta)$ for all $\eta \in dom(t)$ as well as

- $\bullet_1$ if $\eta \subseteq \theta$ belong to $dom(t)$, then $s(\theta) \setminus t(\theta) \subseteq s(\eta) \setminus t(\eta)$;
- $\bullet_2$ if $\eta\hat{}\langle 0\rangle, \eta\hat{}\langle 1\rangle \in dom(t)$, then $s(\eta\hat{}\langle 0\rangle) \setminus t(\eta\hat{}\langle 0\rangle)$ and $s(\eta\hat{}\langle 1\rangle) \setminus t(\eta\hat{}\langle 1\rangle)$ are disjoint;
- $\bullet_3$ if $\alpha \in G$ and $\eta \in dom(t)$, then $s(\eta) \setminus t(\eta) \subseteq C^\alpha_\eta$.

This forcing is easily seen to be *ccc*, and it adds a system $\langle D^\alpha_\eta;\ \eta \in 2^{<\lambda} \cap V\rangle$ of subsets of ω which satisfies (1) through (3) above by $\bullet_1$ through $\bullet_3$.

Case 2. $\beta = \max(\alpha \cap A_\lambda)$.

Let $\mathbb{Q}_\alpha$ be the trivial ordering, and define $D^\alpha_\eta := C^\beta_\eta$ for all $\eta \in 2^{<\lambda} \cap V$.

This completes the construction of the $\mathbb{Q}_\alpha$, and, hence, of the forcing $\mathbb{P}^1$. We still have to explain how to get the C^α_η from the D^α_η in the model $V^{\mathbb{P}_{\alpha+1}}$. For this, note that the set

$$\{\eta \in 2^{<\lambda} \cap V;\ D^\alpha_\eta \subseteq^* X_\alpha \text{ or } D^\alpha_\eta \cap X_\alpha \text{ is finite or}$$
$$\text{for all } \theta \supseteq \eta, \text{ both } D^\alpha_\theta \cap X_\alpha \text{ and } D^\alpha_\theta \cap (\omega \setminus X_\alpha) \text{ are infinite}\}$$

is dense and open in $2^{<\lambda} \cap V$. Let

$$C^\alpha_\eta = \begin{cases} D^\alpha_\eta \cap X_\alpha & \text{if } \eta \text{ enjoys the third property} \\ D^\alpha_\eta & \text{otherwise} \end{cases}$$

Then (2) and (3) are trivially true, and it is easy to check that (1) and (4) are satisfied as well. Thus we are done with the construction of the required system.

Next, let $f_\lambda \in 2^\lambda$, $\lambda \in R$, be the generic Easton functions. Also let $B_\lambda \subseteq A_\lambda$ be a cofinal subset of order type ω_1. For $\alpha < \omega_1$, set $E^\alpha_{\lambda,\gamma} = C^{B_\lambda(\alpha)}_{f_\lambda \restriction \gamma}$ where $B_\lambda(\alpha)$ denotes the α–th element of B_λ. We claim the $E^\alpha_{\lambda,\gamma}$ satisfy (i) through (iii) above.

Now, (ii) is immediate from (3). To see (i), first note that the $E^\alpha_{\lambda,\gamma}$ for fixed α form a decreasing chain, by (1). Next use a genericity argument to see that this chain has no pseudointersection, as follows. Work in

$V^{\mathbb{P}^1}$. By distributivity (see above), $\mathbb{P}^0$ adds no new reals over $V^{\mathbb{P}^1}$. Hence it suffices to check that given any $X \subseteq \omega$ in $V^{\mathbb{P}^1}$, the set $\{\eta \in 2^{<\lambda};\ X \not\subseteq^* C^\alpha_\eta\}$ is dense in $2^{<\lambda} \cap V$. But this is trivial by (2). Finally, (iii) is taken care of by an exactly analogous density argument involving (4). Hence we're done. $\square$

In the remainder of this section, we discuss several improvements of, and variations on, the above result which are corollaries to the construction.

Corollary 6.1. *If we care only about characters, we can relax the assumption about R in Theorem 5 to: "R is a set of cardinals of uncountable cofinality."*

Proof. We confine ourselves to describing the changes we need to make in the proof of Theorem 5. We additionally adjoin, for $\lambda \in R$ singular, a matrix $\langle E^\alpha_{\lambda,\Gamma};\ \alpha < \omega_1, \Gamma \in [\lambda]^\omega \cap V\rangle$ of subsets of ω such that, in addition to (ii) and (iii) (with γ replaced by Γ), the following conditions are met:

(i_a) $\Gamma \subseteq \Delta$ implies $E^\alpha_{\lambda,\Delta} \subseteq^* E^\alpha_{\lambda,\Gamma}$;

(i_b) $\Gamma \not\subseteq \Delta$ implies $E^\alpha_{\lambda,\Delta} \not\subseteq^* E^\beta_{\lambda,\Gamma}$ for all α, β.

Then (i_a), (ii) and (iii) imply the matrix generates an ultrafilter $\mathcal{U}$ with $\chi(\mathcal{U}) \leq \lambda$, while ($\mathrm{i}_b$) gives us that $\chi(\mathcal{U}) \geq \lambda$.

With $\mathbb{P}^0$ we also adjoin a function f_λ from λ to 2 with countable conditions for each singular $\lambda \in R$. In the partition of ζ include the A_λ for singular λ. $\mathbb{P}^1$ is as before except that we still have to define $\mathbb{Q}_\alpha$ for $\alpha \in A_\lambda$ where λ is singular.

For such λ and $\alpha \in A_\lambda$, we add a system $\langle C^\alpha_\eta;\ \eta \in V, \eta : \lambda \to 2$ is a partial function with countable domain$\rangle$ of subsets of ω such that, in addition to (3) and (4), we have

(1_a) $\eta \subseteq \theta$ implies $C^\alpha_\theta \subseteq^* C^\alpha_\eta$;

(1_b) $\eta \not\subseteq \theta$ implies $C^\alpha_\theta \not\subseteq^* C^\beta_\eta$ for all α, β;

(2) if η and θ are incompatible, then $C^\alpha_\eta \cap C^\alpha_\theta$ is finite.

The corresponding D^α_η are produced as before and satisfy (1_a), (1_b), (2) and (3). Note that (1_b) for the D^α_η is easily preserved in Case 1 by a genericity argument. C^α_η is defined from D^α_η as previously and satisfies (1_a), (2), (3) and (4). To see that is also satisfies (1_b) suppose that $C^\alpha_\theta \subseteq^* C^\beta_\eta$ for some $\alpha \geq \beta$ and $\eta \not\subseteq \theta$. Then θ has an extension $\bar\theta$ such that η and $\bar\theta$ are incompatible and thus $|C^\beta_\eta \cap C^\beta_{\bar\theta}| < \omega$ by (2). By (1_a), $C^\alpha_{\bar\theta} \subseteq^* C^\alpha_\theta$ which means that $|C^\alpha_{\bar\theta} \cap C^\beta_{\bar\theta}| < \omega$, contradicting (3). (This is the only place where (2) is needed.)

Finally put $E^\alpha_{\lambda,\Gamma} = C^{B_\lambda(\alpha)}_{f_\lambda \restriction \Gamma}$ and use (1_a), (1_b), (3) and (4) to check that (i_a), (i_b), (ii) and (iii) are satisfied. $\square$

Corollary 6.2. *In Theorem 5, we can additionally demand that all the ultrafilters produced are Ramsey ultrafilters.*

Proof. We replace the $\mathbb{Q}_\alpha$ in the iteration by $\mathbb{Q}_\alpha \star \mathbb{C}$ where $\mathbb{C}$ denotes Cohen forcing (so $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha \star \dot{\mathbb{Q}}_\alpha \star \mathbb{C}$). Apart from that, it suffices to change the way the C^α_η are defined from the D^α_η. Instead of listing names for subsets of ω, we list names for partitions of ω, as $\langle \dot X_{\alpha,n};\ n \in \omega\rangle$. Assume we are at step α.

Look at

$$\{\eta \in 2^{<\lambda} \cap V;\ D^\alpha_\eta \subseteq^* \bigcup_{n<N} X_{\alpha,n} \text{ for some } N \text{ or}$$
$$D^\alpha_\theta \text{ meets infinitely many } X_{\alpha,n} \text{ for all } \theta \supseteq \eta\}$$

This is again dense and open in $2^{<\lambda} \cap V$. Now let Y be a Cohen real over $V^{\mathbb{P}_\alpha \star \dot{\mathbb{Q}}_\alpha}$, and think of Y as a subset of ω which meets each $X_{\alpha,n}$ once. Then let

$$C^\alpha_\eta = \begin{cases} D^\alpha_\eta \cap Y & \text{if } \eta \text{ enjoys the second property above} \\ D^\alpha_\eta & \text{otherwise} \end{cases}$$

We now see that the C^α_η satisfy

(4') in $V^{\mathbb{P}_{\alpha+1}}$, the set $\{\eta \in 2^{<\lambda} \cap V;\ C^\alpha_\eta \subseteq^* \bigcup_{n<N} X_{\alpha,n}$ for some N or $|C^\alpha_\eta \cap X_{\alpha,n}| \leq 1$ for all $n\}$ is dense (and open) in $2^{<\lambda} \cap V$.

And thus we get by genericity

(iii') for each partition $\langle X_n;\ n \in \omega\rangle$ of ω, we find a pair $\langle \alpha, \gamma\rangle$ such that either $E^\alpha_{\lambda,\gamma} \subseteq^* \bigcup_{n<N} X_n$ for some N or $|E^\alpha_{\lambda,\gamma} \cap X_n| \leq 1$ for all n,

which guarantees Ramseyness. $\square$

⋆ ⋆ ⋆

By Theorem 5, we can get a plethora of ultrafilter characters. This suggests it might be interesting to know whether an arbitrary set of regular cardinals can be realized as the set of possible ultrafilter characters in some model of ZFC. To this end we define

- $\mathsf{Spec}(\chi) = \{\lambda;\ \chi(\mathcal{U}) = \lambda$ for some ultrafilter $\mathcal{U}$ on $\omega\}$, the *character spectrum*;
- $\mathsf{Spec}(\pi\chi) = \{\lambda;\ \pi\chi(\mathcal{U}) = \lambda$ for some ultrafilter $\mathcal{U}$ on $\omega\}$, the π*–character spectrum*;
- $\mathsf{Spec}^\star(\chi) = \{\lambda;\ \chi(\mathcal{U}) = \pi\chi(\mathcal{U}) = \lambda$ for some ultrafilter $\mathcal{U}$ on $\omega\}$.

Unfortunately, we have no limitative results on $\mathsf{Spec}(\chi)$ and on $\mathsf{Spec}(\pi\chi)$ (see section 8 for some questions on this; in particular, question (5)), but we can prove the following which answers the spectral question for $\mathsf{Spec}^\star(\chi)$ in many cases.

Theorem 6. *Let R be a non–empty set of uncountable regular cardinals in $V \models GCH$. Then there is a forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$, for all regular λ which are neither successors of singular limits of R nor inaccessible limits of R, we have*

$$\lambda \in R \Longleftrightarrow \lambda \in \mathsf{Spec}^\star(\chi).$$

Proof. We use a modification of the partial order in Theorem 5, and confine ourselves to describing the differences of the two proofs. Let $\nu = \min(R)$. Put $\zeta = \mu \cdot \nu$. $\mathbb{P}^0$ is defined exactly as before, and $\mathbb{P}^1$ is a finite support iteration of length ζ which

(a) takes care of all the p.o.'s described in the proof of Theorem 5; and

(b) forces MA for all p.o.'s of size $< \mu$ at each limit step of the form $\mu \cdot \beta$ where $\beta < \nu$ is a successor ordinal.

It's clear that this can be done. By Theorem 5, we know that $R \subseteq \mathsf{Spec}^\star(\chi)$. We proceed to show the other direction.

Let $\lambda \notin R$ be a regular cardinal which is neither a successor of a singular limit of R nor an inaccessible limit of R. Assume there is, in $V^{\mathbb{P}}$, an ultrafilter $\mathcal{U}$ with $\chi(\mathcal{U}) = \lambda$. Since $\mathfrak{c} = \mu$, and μ doesn't qualify as λ (because either μ is a limit of R (and thus either inaccessible or not regular) or μ is a successor of a singular limit of R or $\mu = \sup(R) = \max(R) \in R$), we know $\lambda < \mu$. Since the cofinality of the iteration is $\nu \in R$, we also see $\lambda > \nu$. We shall show that $\pi\chi(\mathcal{U}) < \lambda$. Let $\mathcal{F} \subseteq \mathcal{U}$ be a base of $\mathcal{U}$ of size λ. Work in $V^{\mathbb{P}^1}$. The forcing $\mathbb{P}^0$ decomposes as a product $\mathbb{P}^{<\lambda} \times \mathbb{P}^{>\lambda}$ because $\lambda \notin R$. The first part has size $< \lambda$. This follows from the Easton support in case λ is a successor of an inaccessible, and is trivial in the other cases. The second part is λ^+–distributive, and thus adds no new sets of size λ. Hence $\mathcal{F} \in V^{\mathbb{P}^{<\lambda} \times \mathbb{P}^1}$.

For each $p \in \mathbb{P}^{<\lambda}$, let (in $V^{\mathbb{P}^1}$) $\mathcal{F}_p = \{F \subseteq \omega;\ p \Vdash_{\mathbb{P}^{>\lambda}} F \in \dot{\mathcal{F}}\}$. Clearly, $\mathcal{F} \subseteq \bigcup_p \mathcal{F}_p$. Next, for each $\beta < \nu$, let (in $V^{\mathbb{P}_{\mu \cdot \beta}}$) $\mathcal{F}_{p,\beta} = \{F \subseteq \omega;\ F \in V^{\mathbb{P}_{\mu\cdot\beta}}$ and $\Vdash_{\mathbb{P}^1 / \mathbb{P}_{\mu \cdot \beta}} F \in \dot{\mathcal{F}}_p\}$. By *ccc*–ness of the iteration, we have $\mathcal{F}_p = \bigcup_\beta \mathcal{F}_{p,\beta}$. Since we forced MA along the way, each $\mathcal{F}_{p,\beta}$ has a pseudointersection which we call $G_{p,\beta}$. By construction, the $G_{p,\beta}$ form a π–base of $\mathcal{U}$ which has size $< \lambda$, as required. $\square$

Note the similarity between this result and Dordal's result [Do 1, Corollary 2.6] on the spectrum of tower heights. The latter is easier to prove because a tower is an easier combinatorial object than an ultrafilter. — Of course, the restrictions on λ in the above theorem come from the size of the set of Easton conditions, and the present proof does not work in the other cases. In the other direction, we can show that certain cardinals may arise as characters even if they don't belong to R:

Proposition 6.3. *In the model constructed in Theorem 6: if R contains cofinally many ω_n, then $\omega_{\omega+1} \in \mathsf{Spec}(\chi)$.*

Proof. Let $S = R \cap \omega_\omega$, let $\nu = \min(R)$ as before, and put $S' = \{n;\ \omega_n \in S\}$. Assume we have, for $n \in S'$, an ultrafilter $\mathcal{V}_n$ on ω which is generated by a matrix $\langle E^\alpha_{\omega_n,\gamma};\ \alpha < \nu, \gamma < \omega_n\rangle$ satisfying (i) through (iii) in the proof of Theorem 5. Put $\mathcal{U} := \sum_{\mathcal{V}} \mathcal{V}_n$ where $\mathcal{V}$ is any ultrafilter with $\chi(\mathcal{V}) = \nu < \omega_\omega$ and $S' \in \mathcal{V}$. We shall show that $\chi(\mathcal{U}) = \omega_{\omega+1}$.

To see $\chi(\mathcal{U}) \geq \omega_{\omega+1}$, it suffices to show that $\chi(\mathcal{U}) \geq \omega_n$ for all n, by Proposition 1.4. This is easy: fix n and $\mathcal{F} \subseteq \mathcal{U}$ with $|\mathcal{F}| = \omega_n$; for $m > n$ with $m \in S'$ find $A_m \in \mathcal{V}_m$ such that $F \cap (\{m\} \times \omega) \not\subseteq^* \{m\} \times A_m$ for any $F \in \mathcal{F}$ with $F \cap (\{m\} \times \omega) \in \mathcal{V}_m$, and put $A = \bigcup_{m > n, m \in S'} \{m\} \times A_m \in \mathcal{U}$; then $F \not\subseteq^* A$ for any $F \in \mathcal{F}$, as required.

To see $\chi(\mathcal{U}) \leq \omega_{\omega+1}$, note first that $\mathfrak{d} = \nu$ by construction. Now, let $\{g_\delta;\ \delta < \nu\}$ be a dominating family, and let $\{V_\zeta;\ \zeta < \nu\}$ be a base of $\mathcal{V}$; without loss, each $V \in \mathcal{V}$ strictly contains at least one V_ζ. Also let $\{f_\eta;\ \eta < \omega_{\omega+1}\} \subseteq \prod S := \{f : S' \to \omega_\omega;\ f(n) < \omega_n\}$ be a dominating family; i.e. given $f \in \prod S$ there is $\eta < \omega_{\omega+1}$ such that $f(n) < f_\eta(n)$ for all $n \in S'$ (such a family clearly exists in the ground model; it also exists in the generic extension because *pcf* is left unchanged by the forcing). For $\alpha, \delta, \zeta < \nu$ and $\eta < \omega_{\omega+1}$, put $A_{\alpha,\delta,\zeta,\eta} = \bigcup_{n \in V_\zeta} \{n\} \times (E^\alpha_{\omega_n, f_\eta(n)} \setminus g_\delta(n))$ and check that the $A_{\alpha,\delta,\zeta,\eta}$ form a base of $\mathcal{U}$. $\square$

Note that for $\chi(\mathcal{U}) \geq \omega_{\omega+1}$ we used no extra assumptions while the proof of $\chi(\mathcal{U}) \leq \omega_{\omega+1}$ involved the special shape of the ultrafilters $\mathcal{V}_n$ as well as $\mathfrak{d} < \omega_\omega$ (which is necessary by 1.6). We don't know whether a similar result can be proved without these assumptions (see § 8 (6)). Also, contrary to the situation for $\pi\chi$

(see Corollary 5.6), we don't know whether we can have many characters below $\mathfrak{d}$.

⋆⋆⋆

We finally come to the result dual to Theorem 5.

Theorem 7. *Let R be a set of regular uncountable cardinals in $V \models GCH$. Then there is a forcing notion $\mathbb{P}$ such that*

$$V^{\mathbb{P}} \models \text{“} \textit{for all } \lambda \in R \textit{ there is an ultrafilter } \mathcal{U} \textit{ such that } \mathfrak{p}(\mathcal{U}) = \pi\mathfrak{p}(\mathcal{U}) = \lambda \text{”}.$$

Proof. Again, let

$$\mu = \begin{cases} \sup(R) & \text{if this has uncountable cofinality} \\ \sup(R)^+ & \text{otherwise} \end{cases}$$

and adjoin, for all $\lambda \in R$, matrices $\langle E^\alpha_{\lambda,\gamma};\ \alpha < \mu, \gamma < \lambda \rangle$ of subsets of ω such that (i) thru (iii) in the proof of Theorem 5 are satisfied with ω_1 replaced by μ. It is immediate that the matrices will generate ultrafilters of the required sort. The rest of the proof of Theorem 5 carries over with very minor changes which we leave to the reader. □

As in Corollary 6.2 we get

Corollary 6.4. *In Theorem 7, we can additionally demand that all the ultrafilters produced are Ramsey ultrafilters.* □

One can again define $\mathsf{Spec}(\mathfrak{p})$, $\mathsf{Spec}(\pi\mathfrak{p})$ and $\mathsf{Spec}^\star(\mathfrak{p})$ in the obvious fashion, but we do not know of any restrictive results (like, e.g., Theorem 6) concerning these spectra. — The second part of the following corollary — which is immediate from Theorems 5 and 7, Corollaries 6.2 and 6.4, and results mentioned in § 1 — answers the other half of the question in [Br, subsection 4.1].

Corollary 6.5. *(a) It is consistent with ZFC that for some Ramsey ultrafilter $\mathcal{U}$, $\mathsf{cov}(r^0_{\mathcal{U}}) = \pi\mathfrak{p}(\mathcal{U}) < \mathfrak{par}$.*

(b) It is consistent with ZFC that for some Ramsey ultrafilter $\mathcal{U}$, $\mathsf{non}(r^0_{\mathcal{U}}) = \pi\chi(\mathcal{U}) > \mathfrak{hom}$. □

7. Connection with reaping and splitting

As we remarked in § 1, for any ultrafilter $\mathcal{U}$ on ω we have $\pi\mathfrak{p}(\mathcal{U}) \leq \mathfrak{s}$ and $\pi\chi(\mathcal{U}) \geq \mathfrak{r}$. Furthermore, it follows from the results in §§ 5 and 6 that it is consistent to have an ultrafilter $\mathcal{U}$ with $\pi\mathfrak{p}(\mathcal{U}) < \mathfrak{s}$, as well as to have one with $\pi\chi(\mathcal{U}) > \mathfrak{r}$. Still there is a close connection between the $\pi\chi(\mathcal{U})$ and $\mathfrak{r}$, as shown by the following well–known result whose proof we repeat for completeness' sake.

Proposition 7.1. (Balcar–Simon, [BS, Theorem 1.7]) $\mathfrak{r} = \min_{\mathcal{U}} \pi\chi(\mathcal{U})$.

Proof. Let $\mathcal{A}$ be a reaping family of size $\mathfrak{r}$. Without loss, $\mathcal{A}$ is downward closed, that is, whenever $A \in \mathcal{A}$, then $\{B \in \mathcal{A};\ B \subseteq A\}$ is a reaping family inside A. This entails it can be shown by induction that given pairwise disjoint X_i, $i \in n$, with $\bigcup_{i\in n} X_i = \omega$, there are $i \in n$ and $A \in \mathcal{A}$ with $A \subseteq X_i$ $(\star)$. Let $\mathcal{I}$ be the ideal generated by sets $X \subseteq \omega$ with $A \not\subseteq^* X$ for all $A \in \mathcal{A}$. By $(\star)$, $\mathcal{I}$ is a proper ideal. Hence it can be extended to a maximal ideal whose dual ultrafilter $\mathcal{U}$ has $\mathcal{A}$ as a π–base, and thus witnesses $\pi\chi(\mathcal{U}) = \mathfrak{r}$. □

Let us note that Balcar and Simon proved a much more general result: the analogue of 7.1 holds in fact for a large class of Boolean algebras.

We shall now see that there is no dual form of this proposition.

Theorem 8. *It is consistent with ZFC that $\pi\mathfrak{p}(\mathcal{U}) = \omega_1$ for all ultrafilters $\mathcal{U}$ on ω, yet $\mathfrak{s} = \mathfrak{c} = \omega_2$.*

For the proof of this Theorem we need to introduce several notions and prove a few preliminary Lemmata. Given a limit ordinal $\delta < \omega_2$, let $\langle \delta_\zeta;\ \zeta \in cf(\delta)\rangle$ be a fixed continuously increasing sequence with $\delta = \bigcup_\zeta \delta_\zeta$. We define sequences $\bar{A}^\alpha = \langle A^\alpha_\beta \subseteq \alpha;\ \beta < \omega_1\rangle$ for $\alpha < \omega_2$ recursively as follows.

- $A^0_\beta = \emptyset$
- $A^{\alpha+1}_\beta = A^\alpha_\beta \cup \{\alpha\}$
- $A^\delta_\beta = \{\gamma < \delta;\ \gamma \in A^{\delta_\zeta}_\beta$ for all ζ with $\delta_\zeta > \gamma\}$ in case $cf(\delta) = \omega$
- $A^\delta_\beta = \{\gamma < \delta;\ \gamma < \delta_\zeta$ for some $\zeta < \beta$, and $\gamma \in A^{\delta_\zeta}_\beta$ for all $\zeta < \beta$ with $\delta_\zeta > \gamma\}$ in case $cf(\delta) = \omega_1$

We leave it to the reader to verify that all A^α_β are at most countable and that for all $\gamma < \alpha$, the set $\{\beta < \omega_1;\ \gamma \in A^\alpha_\beta\}$ contains a club.

Now, suppose $\mathfrak{F} = \{\mathcal{F}^\gamma = \{F^\gamma_\beta;\ \beta < \omega_1\};\ \gamma < \alpha\}$ is a family of filter bases on ω. We call $\mathfrak{F}$ *α–nice* (or simply *nice* if the α in question is clear from the context) iff, given $X \in [\omega]^\omega$ and a set $\{f_j;\ j \in \omega\}$ of one–to–one functions in ω^ω, there is a club $C = C(X, \langle f_j\rangle_j) \subseteq \omega_1$ such that $|X \setminus \bigcup_{j<k}\bigcup_{\gamma\in\Gamma} f_j^{-1}(F^\gamma_\beta)| = \omega$ for all $\beta \in C$, all $k \in \omega$ and all finite $\Gamma \subseteq A^\alpha_\beta$. Furthermore, if $\mathcal{U}$ is a Ramsey ultrafilter, then $\mathfrak{F}$ is called *$\mathcal{U} - \alpha$–nice* (or simply *$\mathcal{U}$–nice*) iff given $f \in \omega^\omega$ one–to–one, there is a club $D = D(f) \subseteq \omega_1$ such that for all $\beta \in D$ there exists $U \in \mathcal{U}$ with $f[U] \cap F^\gamma_\beta$ being finite for all $\gamma \in A^\alpha_\beta$. There is a two–way interplay between niceness and $\mathcal{U}$–niceness (see 7.3 and 7.4): given $\mathfrak{F}$ nice, we can, in certain circumstances, construct $\mathcal{U}$ such that $\mathfrak{F}$ is $\mathcal{U}$–nice; on the other hand, after forcing with $\mathbb{M}_\mathcal{U}$ where $\mathfrak{F}$ is $\mathcal{U}$–nice, $\mathfrak{F}$ is still nice in the generic extension. This is the core of our arguments, and guarantees the preservation of niceness in finite support iterations with forcings of the form $\mathbb{M}_\mathcal{U}$ as a factor.

Lemma 7.2. (CH) *If $\mathfrak{F} = \{\mathcal{F}^\gamma = \{F^\gamma_\beta;\ \beta < \omega_1\};\ \gamma < \alpha\}$ is α–nice and $\mathcal{U}$ is an ultrafilter, then there is $\mathcal{F}^\alpha = \{F^\alpha_\beta;\ \beta < \omega_1\} \subseteq \mathcal{U}$ such that $\mathfrak{F} \cup \{\mathcal{F}^\alpha\}$ is $(\alpha+1)$–nice.*

Lemma 7.3. (CH) *If $\mathfrak{F}$ is nice, then there is a Ramsey ultrafilter $\mathcal{U}$ such that $\mathfrak{F}$ is $\mathcal{U}$–nice.*

Lemma 7.4. *Assume $\mathcal{U}$ is a Ramsey ultrafilter, and $\mathfrak{F}$ is $\mathcal{U}$–nice. Then $\Vdash_{\mathbb{M}_\mathcal{U}}$ "$\mathfrak{F}$ is nice".*

Lemma 7.5. *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha;\ \alpha < \delta\rangle$, δ a limit ordinal, be a finite support iteration of ccc p.o.'s, and let $\dot{\mathcal{F}}^\alpha = \{\dot{F}^\alpha_\beta;\ \beta < \omega_1\}$ be $\mathbb{P}_\alpha$–names for filter bases on ω such that $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathfrak{F}}^{\alpha+1} := \{\dot{\mathcal{F}}^\gamma;\ \gamma < \alpha+1\}$ is $(\alpha+1)$–nice" for $\alpha < \delta$. Then $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathfrak{F}}^\delta = \bigcup_{\alpha<\delta} \dot{\mathfrak{F}}^{\alpha+1}$ is δ–nice".*

Proof of Lemma 7.2. Let $\{X_\beta;\ \beta < \omega_1\}$ enumerate $[\omega]^\omega$ so that each $X \in [\omega]^\omega$ occurs uncountably often. Also let $\{f_\beta;\ \beta < \omega_1\}$ enumerate the one–to–one functions of ω^ω. For X_η and $\langle f_\zeta \rangle_{\zeta<\eta}$, let $C_\eta = C(X_\eta, \langle f_\zeta \rangle_{\zeta<\eta})$ witness the niceness of $\mathfrak{F}$. Without loss $\min(C_\eta) > \eta$. It suffices to construct sets $F^\alpha_\beta \in \mathcal{U}$ such that for all $\eta < \beta$ with $\beta \in C_\eta$ and all finite $\Gamma \subseteq A^\alpha_\beta \cup \{\alpha\}$ and finite $Z \subseteq \eta$, we have $|X_\eta \setminus \bigcup_{\zeta\in Z} \bigcup_{\gamma\in\Gamma} f^{-1}_\zeta(F^\gamma_\beta)| = \omega$. F^α_0 is any member of $\mathcal{U}$.

Assume the $F^\alpha_{\beta'}$ have been constructed for $\beta' < \beta$. Let $\{\eta_k;\ k \in \omega\}$ enumerate the $\eta < \beta$ with $\beta \in C_\eta$. Also, let $\{\zeta_i;\ i \in \omega\}$ enumerate β. By the properties of the C_η with $\beta \in C_\eta$ and standard thinning arguments, we can find sets $X'_k \subseteq X_{\eta_k}$ such that

(i) for all $\zeta \in \eta_k$ and $\gamma \in A^\alpha_\beta$ we have that $X'_k \cap f^{-1}_\zeta(F^\gamma_\beta)$ is finite;

(ii) the X'_k are pairwise disjoint;

(iii) given k and i, either f_{ζ_i} is almost equal to some f_{ζ_j} with $j < i$ on the set X'_k or $f_{\zeta_i}[X'_k]$ is almost disjoint from $f_{\zeta_j}[\bigcup_\ell X'_\ell]$ for all $j < i$.

Now choose infinite sets X^0_k and X^1_k such that $X^0_k \cup X^1_k = X'_k$ and $X^0_k \cap X^1_k = \emptyset$. Put $X^0 = \bigcup_k X^0_k$ and $X^1 = \bigcup_k X^1_k$. Build disjoint sets $Y^0 = \bigcup_k Y^0_k$ and $Y^1 = \bigcup_k Y^1_k$ as follows. $Y^0_0 = f_{\zeta_0}[X^0], Y^1_0 = f_{\zeta_0}[X^1], ..., Y^0_k = f_{\zeta_k}[X^0] \setminus \bigcup_{j<k} Y^1_j, Y^1_k = f_{\zeta_k}[X^1] \setminus \bigcup_{j<k} Y^0_j, ...$ We have either $Y^0 \notin \mathcal{U}$ or $Y^1 \notin \mathcal{U}$; without loss the former holds, and we let $F = F^\alpha_\beta = \omega \setminus Y^0 \in \mathcal{U}$. F is as required because we can now show by induction on j that $f^{-1}_{\zeta_j}(F) \cap X^0_k$ is finite for all j and k. This completes the proof of 7.2. $\square$

Proof of Lemma 7.3. Let $\{f_\beta;\ \beta < \omega_1\}$ enumerate the one–to–one functions in ω^ω, and build a tower $\{U_\beta;\ \beta < \omega_1\}$ which generates a Ramsey ultrafilter $\mathcal{U}$. Guarantee that $\mathcal{U}$ will be Ramsey in the successor steps of the construction. If β is a limit such that $\beta \in \bigcap_{\theta<\beta} C(U_\theta, \langle f_\zeta \rangle_{\zeta<\theta})$, we can choose U_β such that $U_\beta \subseteq^* U_\theta$ for all $\theta < \beta$ and such that $U_\beta \cap f^{-1}_\theta(F^\gamma_\beta)$ is finite for all $\theta < \beta$ and $\gamma \in A^\alpha_\beta$; otherwise let U_β be any set almost included in all U_θ's where $\theta < \beta$. Thus, if $f = f_\eta$, $D(f) = \{\beta > \eta;\ \beta \in \bigcap_{\theta<\beta} C(U_\theta, \langle f_\zeta \rangle_{\zeta<\theta})\}$ is a diagonal intersection of clubs, and thus a club, and witnesses $\mathcal{U}$–niceness. $\square$

Proof of Lemma 7.4. Let $\dot{X}$ and $\dot{f}^j$, $j \in \omega$, be $\mathbb{M}_\mathcal{U}$–names for objects in $[\omega]^\omega$ and for one–to–one functions in ω^ω, respectively. Let $\dot{x}$ be the name for the increasing enumeration of $\dot{X}$. Replacing $\dot{X}$ by a name for a subset of $\dot{X}$, if necessary, we may assume that

$$\Vdash_{\mathbb{M}_\mathcal{U}} \text{“}\dot{f}^j(\dot{x}(n)) \geq \dot{m}(n)\text{” for all } j \text{ and all } n \geq j, \qquad (\star)$$

where $\dot{m}$ denotes the canonical name for the Mathias–generic. Since $\mathbb{M}_\mathcal{U}$ and $\mathbb{L}_\mathcal{U}$ are forcing equivalent (see § 3), we can think of $\mathbb{M}_\mathcal{U}$ as forcing with Laver trees T such that the set of successors lies in $\mathcal{U}$ for every node above the stem. Fix $n \in \omega$ and $j \leq n$. Set

$$A(n,j) = \{\sigma \in \omega^{<\omega};\ \text{ some } T^j_{\sigma,n} = T \in \mathbb{L}_\mathcal{U} \text{ with } \mathit{stem}(T) = \sigma \text{ decides the value } \dot{f}^j(\dot{x}(n))\}.$$

Furthermore put

$$B(n,j) = \{\sigma \in \omega^{<\omega};\ \sigma \notin A(n,j) \text{ and } U^j_{\sigma,n} := \{k \in \omega;\ \sigma\hat{\ }\langle k \rangle \in A(n,j)\} \in \mathcal{U}\}.$$

Call a triple (σ, n, j) *relevant* iff $\sigma \in B(n,j)$. For relevant triples (σ, n, j) define $f = f^j_{\sigma,n} : U^j_{\sigma,n} \to \omega$ by

$$f(k) = \text{the value forced to } \dot{f}^j(\dot{x}(n)) \text{ by } T^j_{\sigma\hat{\ }\langle k \rangle,n}.$$

Using that $\mathcal{U}$ is Ramsey and that f cannot be constant on a set from $\mathcal{U}$ by the definition of $B(n,j)$, we may assume that f is one–to–one on $U^j_{\sigma,n}$, by pruning that set if necessary. By $\mathcal{U}$–niceness find a club $D(f)$ such that for all $\beta \in D$ there is $U \in \mathcal{U}$ with $f[U] \cap F^\gamma_\beta$ being finite for all $\gamma \in A^\alpha_\beta$. Let C be the intersection of all $D(f^j_{\sigma,n})$ where (σ,n,j) is relevant. We claim that

$$\Vdash_{\mathbb{L}_\mathcal{U}} \text{“}|\dot f^j[\dot X] \cap F^\gamma_\beta| < \omega\text{” for all } j \in \omega, \beta \in C \text{ and } \gamma \in A^\alpha_\beta. \qquad (\star\star)$$

Clearly this suffices to complete the proof of the Lemma.

To see $(\star\star)$, fix $j \in \omega, \beta \in C, \gamma \in A^\alpha_\beta$ and $T \in \mathbb{L}_\mathcal{U}$. Put $\ell := \max\{j, |stem(T)|\}$. We shall recursively construct $T' \leq T$ such that

$$T' \Vdash_{\mathbb{L}_\mathcal{U}} \text{“}\dot f^j(\dot x(n)) \notin F^\gamma_\beta\text{” for all } n \geq \ell. \qquad (\star\star\star)$$

Along the construction we shall guarantee that if $\sigma \in T' \cap A(n,j)$ for some $n \geq \ell$, then $T'_\sigma := \{\tau \in T';\ \tau \subseteq \sigma$ or $\sigma \subseteq \tau\} \leq T^j_{\sigma,n}$ and that the value forced to $\dot f^j(\dot x(n))$ by $T^j_{\sigma,n}$ does not belong to F^γ_β. By $(\star)$ we see that $stem(T) \notin A(n,j)$ for all $n \geq \ell$. Hence we can put $stem(T)$ into T'. To do the recursion step, assume we put $\sigma \supseteq stem(T)$ into T'. Again by $(\star)$, the set N of all n such that $\sigma \in A(n,j)$ is finite. The same holds for the set M of all n such that $\sigma \in B(n,j)$. By definition of C, we can find $U \in \mathcal{U}$ such that $U \subseteq U^j_{\sigma,n}$ and $f^j_{\sigma,n}[U] \cap F^\gamma_\beta = \emptyset$ for all $n \in M$. Now put $\sigma\hat{}\langle k\rangle$ into T' iff $k \in U$ and $\sigma\hat{}\langle k\rangle \in T^j_{\sigma,n}$ for all $n \in N$ and $\sigma\hat{}\langle k\rangle \notin A(n,j)$ for all $n \in \omega \setminus (N \cup M)$. Using again $(\star)$, it is easily seen that the set of all k satisfying these three clauses belongs to $\mathcal{U}$. This completes the recursive construction of T'. It is now easy to see that T' indeed satisfies $(\star\star\star)$. $\square$

Proof of Lemma 7.5. Let $\dot X$ and $\dot f^j$, $j \in \omega$, be $\mathbb{P}_\delta$–names for objects in $[\omega]^\omega$ and one–to–one functions in ω^ω, respectively. First assume that $cf(\delta) = \omega$, and that $\delta = \bigcup_n \delta_n$ (where the δ_n form the sequence fixed before the definition of the $\bar A^\alpha$). Now construct $\mathbb{P}_{\delta_n}$–names $\dot X_n$ and $\dot f^j_n$ which can be thought of as approximations to our objects as follows. Step into V_{δ_n}. Find a decreasing sequence of conditions $\langle p_{n,m};\ m \in \omega\rangle \in \mathbb{P}_{[\delta_n,\delta)}$ such that $p_{n,m}$ decides the m–th element of $\dot X$ as well as $\dot f^j(k)$ for $j,k \leq m$. Let X_n be the set of elements forced into $\dot X$ by this sequence, and let f^j_n be the function whose values are forced to $\dot f^j$ by this sequence. The niceness of the $\mathfrak{F}^{\delta_n+1}$ in the models V_{δ_n} provides us with clubs $C_n = C(X_n, \langle f^j_n\rangle_j)$ as witnesses.

Back in V, we have names $\dot C_n$ for these witnesses. By *ccc*–ness find a club C in the ground model which is forced to be contained in all $\dot C_n$ by the trivial condition of $\mathbb{P}_\delta$. We claim that C witnesses the niceness of $\mathfrak{F}^\delta$ in V_δ. To see this, take $\beta \in C, k \in \omega$ and $\Gamma \subseteq A^\delta_\beta$ finite. Also fix $\ell \in \omega$ and $p \in \mathbb{P}_\delta$. Find n such that $p \in \mathbb{P}_{\delta_n}$ and $\Gamma \subseteq \delta_n$. Step into $V_{\delta_n} = V[G_{\delta_n}]$ where $p \in G_{\delta_n}$. Since $\Gamma \subseteq A^{\delta_n}_\beta$ by construction of the A^δ_β, we know that $|X_n \setminus \bigcup_{j<k}\bigcup_{\gamma\in\Gamma}(f^j_n)^{-1}(F^\gamma_\beta)| = \omega$. Hence we can find $i > \ell$ in this set and m large enough so that

$$p_{n,m} \Vdash_{\mathbb{P}_{[\delta_n,\delta)}} \text{“}i \in \dot X \setminus \textstyle\bigcup_{j<k}\bigcup_{\gamma\in\Gamma}(\dot f^j)^{-1}(F^\gamma_\beta)\text{”}.$$

Thus we see that

$$\Vdash_{\mathbb{P}_\delta} \text{“}|\dot X \setminus \textstyle\bigcup_{j<k}\bigcup_{\gamma\in\Gamma}(\dot f^j)^{-1}(\dot F^\gamma_\beta)| = \omega\text{”},$$

as required.

Next assume that $cf(\delta) = \omega_1$, and that $\delta = \bigcup_\zeta \delta_\zeta$. Find $\alpha < \delta$ such that $\dot X$ and $\dot f^j$ are $\mathbb{P}_\alpha$–names, step into V_α, and let $X = \dot X[G_\alpha]$, $f^j = \dot f^j[G_\alpha]$. By niceness of the $\mathfrak{F}^{\delta_\zeta+1}$ for $\delta_\zeta > \alpha$, we get $\mathbb{P}_{\delta_\zeta}$–names for clubs, $\dot{C}_\zeta = \dot{C}(X, \langle f^j \rangle_j)$. Without loss, we can assume that $C_\zeta = \dot{C}_\zeta[G_{\delta_\zeta}] \in V_\alpha$ (by *ccc*–ness). Then $C = \{\beta < \omega_1;\ \beta \in \bigcap_{\zeta<\beta} C_\zeta\}$ is a diagonal intersection of clubs, and thus club, and is easily seen to witness the niceness of $\dot{\mathfrak{F}}^\kappa[G_\kappa]$ (use again the definition of $\bar{A}^\delta$). This completes the proof of the Lemma. $\square$

Proof of Theorem 8. Let $E = \{\alpha < \omega_2;\ cf(\alpha) = \omega_1\}$. We shall start with a model V which satisfies GCH and additionally $\diamondsuit_{\omega_2}(E)$. The latter is used as a guessing principle to ensure that we took care of every ultrafilter along the iteration. For example we could take $V = L$ (see [De, chapter IV, Theorem 2.2]). Then we perform a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha;\ \alpha < \omega_2 \rangle$ of *ccc* p.o.'s over V. We think of the diamond sequence as acting on the product $\omega_2 \times \mathbb{P}_{\omega_2}$; more explicitly, we use a sequence $\langle S_\alpha \subseteq \alpha \times \mathbb{P}_\alpha;\ \alpha \in E \rangle$ such that for all $T \subseteq \omega_2 \times \mathbb{P}_{\omega_2}$, the set $\{\alpha \in E;\ T \cap (\alpha \times \mathbb{P}_\alpha) = S_\alpha\}$ is stationary. This we can do since the initial segments $\mathbb{P}_\alpha$ of the iteration will have size ω_1. Furthermore we shall have a $\mathbb{P}_{\omega_2}$–name $\dot{f}$ for a bijection between ω_2 and $[\omega]^\omega$ such that for all $\alpha \in E$, we have

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{f} \restriction \alpha \text{ is a bijection between } \alpha \text{ and } [\omega]^\omega \cap V[\dot{G}_\alpha]\text{”}.$$

The existence of such a name is, again, straightforward.

The details of the construction are as follows. In V_α, we shall have

(a) a Ramsey ultrafilter $\mathcal{U}_\alpha$ such that $\dot{\mathbb{Q}}_\alpha[G_\alpha] = \mathbb{M}_{\mathcal{U}_\alpha}$;

(b) a filter base $\mathcal{F}^\alpha = \{F^\alpha_\beta;\ \beta < \omega_1\}$ such that $\mathfrak{F}^{\alpha+1} := \{\mathcal{F}^\gamma;\ \gamma \leq \alpha\}$ is both nice and $\mathcal{U}_\alpha$–nice.

Let α be arbitrary. By either Lemma 7.4 or 7.5 and induction, $\mathfrak{F}^\alpha = \{\mathcal{F}^\gamma;\ \gamma < \alpha\}$ is nice in V_α. In case S_α is a $\mathbb{P}_\alpha$–name for a subset of α and

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{f}[S_\alpha] \text{ is an ultrafilter”},$$

we let $\mathcal{V} = \dot{f}[S_\alpha][G_\alpha] \in V_\alpha$; otherwise $\mathcal{V}$ is an arbitrary ultrafilter of V_α. By Lemma 7.2 find $\mathcal{F}^\alpha \subseteq \mathcal{V}$ such that $\mathfrak{F}^{\alpha+1}$ is nice. Then apply Lemma 7.3 to get $\mathcal{U}_\alpha$ such that $\mathfrak{F}^{\alpha+1}$ is $\mathcal{U}_\alpha$–nice. This completes the construction.

It remains to see that V_{ω_2} is as required. $\mathfrak{c} = \mathfrak{s} = \omega_2$ is immediate because all the factors of the iteration are of the form $\mathbb{M}_\mathcal{U}$ for some Ramsey ultrafilter. To see $\pi\mathfrak{p}(\mathcal{V}) = \omega_1$ for every ultrafilter $\mathcal{V}$, take a $\mathbb{P}_{\omega_2}$–name $T \subseteq \omega_2 \times \mathbb{P}_{\omega_2}$ such that $\dot{f}[T][G_{\omega_2}] = \mathcal{V}$; without loss $\Vdash_{\mathbb{P}_{\omega_2}}$ “$\dot{f}[T]$ is an ultrafilter”. We easily get a club $C \subseteq \omega_2$ such that for all $\alpha \in C \cap E$, we have that $T \cap (\alpha \times \mathbb{P}_\alpha)$ is a name and

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{f}[T \cap (\alpha \times \mathbb{P}_\alpha)] \text{ is an ultrafilter in } V[\dot{G}_\alpha]\text{”}.$$

Hence we find $\alpha \in C \cap E$ with $T \cap (\alpha \times \mathbb{P}_\alpha) = S_\alpha$. This means that in V_α, we chose $\mathcal{F}^\alpha \subseteq \dot{f}[S_\alpha][G_\alpha]$ such that $\mathcal{F}^\alpha$ had no pseudointersection in V_{ω_2}. Since $\mathcal{F}^\alpha \subseteq \dot{f}[T][G_{\omega_2}] = \mathcal{V}$ has size ω_1, $\pi\mathfrak{p}(\mathcal{V}) = \omega_1$ follows. $\square$

Remark 7.6. If one cares only about P–points $\mathcal{U}$, then the conclusion of Theorem 8 is much easier to prove because *niceness* can be replaced by a simpler notion. Also, A. Dow has remarked that $\mathfrak{s} = \omega_2$ and $\pi\mathfrak{p}(\mathcal{U}) = \omega_1$ for all P–points $\mathcal{U}$ is true in Dordal's factored Mathias real model [Do], and the referee has pointed out that one of the models of [BlS 1] even satisfies $\mathfrak{s} = \omega_2$ and $\chi(\mathcal{U}) = \omega_1$ for all P–points $\mathcal{U}$. The latter is so because the forcing construction increases $\mathfrak{s}$ and is P–point–preserving [BlS 1, Theorems 3.3 and 5.2]. The former holds because in Dordal's model all ω_1–towers are preserved along the iteration (for the successor step, one uses a result of Baumgartner [Do, Theorem 2.2], saying that Mathias forcing does not destroy any towers; the limit step is taken care of by the type of iteration used [Do, Lemma 4.2]). An easy reflection argument shows each P–point $\mathcal{U}$ contains such an ω_1–tower, and $\pi\mathfrak{p}(\mathcal{U}) = \omega_1$ follows. We do not know whether Dordal's model even satisfies $\pi\mathfrak{p}(\mathcal{U}) = \omega_1$ for all ultrafilters $\mathcal{U}$. For this, one would have to extend Baumgartner's result quoted above to filter bases. However, our construction is more general, for slight modifications in the proof show the consistency of the statement in Theorem 8 with large continuum; more explicitly:

Remark 7.7. Let $\kappa \geq \mathfrak{c}$ be a regular cardinal in $V \models \diamondsuit_{\kappa^+}(E)$. Then there is a generic extension of V which satisfies $\pi\mathfrak{p}(\mathcal{U}) = \kappa$ for all ultrafilters $\mathcal{U}$ and $\mathfrak{s} = \mathfrak{c} = \kappa^+$. To see this, simply replace ω, ω_1 and ω_2 by $< \kappa$, κ and κ^+ (respectively) in the above proof, and change the definitions of $\bar{A}^\alpha$ and niceness accordingly. Then Lemmata 7.4 and 7.5 still hold (with a modified proof, of course) and Lemmata 7.2 and 7.3 are true if the assumption is changed to $MA + \mathfrak{c} = \kappa$. This means that along the iteration we also have to force MA cofinally often with *ccc* p.o.'s of size $< \kappa$. This is no problem since it can be shown (with an argument similar to the modified proof of Lemma 7.5) that such p.o.'s preserve (the modified) niceness. We leave details to the reader.

8. Questions with comments

There are numerous interesting questions connected with the cardinals we have studied which are still open.

(1) *Does* $\chi_\sigma(\mathcal{U}) = \chi(\mathcal{U})$ *for all ultrafilters* $\mathcal{U}$*?*

We note that $\chi(\mathcal{U}) = \chi_\sigma(\mathcal{U})$ as long as $\chi(\mathcal{U}) < \omega_\omega$; furthermore, $\chi(\mathcal{U}) = \chi_\sigma(\mathcal{U})$ in the absence of $0^\sharp$ (these remarks are due to W. Just).

(2) (Vojtáš, cf. [Va, Problem 1.4]) *Does* $\mathfrak{r} = \mathfrak{r}_\sigma$*?*

This problem is connected with Miller's question whether $cf(\mathfrak{r}) = \omega$ is consistent (see [Mi, p. 502] and [Mi 1, Problem 3.4]).

(3) *Does* $\mathfrak{r}_\sigma = \min_{\mathcal{U}} \pi\chi_\sigma(\mathcal{U})$*?*

A negative answer would provide us with a dual form of Theorem 8, and rescue some of the symmetry lost in § 7.

(4) *Can* $\pi\mathfrak{p}(\mathcal{U})$ *be consistently singular?*

Let us recall (§ 1) that $\mathfrak{p}(\mathcal{U})$ is regular and notice that $\pi\chi(\mathcal{U})$ and $\chi(\mathcal{U})$ are consistently singular — simply add ω_{ω_1} Cohen reals or see §§ 5 and 6! So $\pi\mathfrak{p}(\mathcal{U})$ is the only cardinal for which this question is of interest. Furthermore, we may ask whether $cf(\pi\mathfrak{p}(\mathcal{U})) \geq \mathfrak{p}(\mathcal{U})$. (Note this is true for $\pi\chi(\mathcal{U})$ and $\chi(\mathcal{U})$, see § 1.) The only information we have about $cf(\pi\mathfrak{p}(\mathcal{U}))$ is given in 1.7 and 1.8. The problem seems connected with Vaughan's problem concerning the possible singularity of $\mathfrak{s}$ (cf. [Va, Problem 1.2]).

(5) (Spectral problem at regulars) *Assume* $\mathfrak{c} = \omega_3$ *and there is an ultrafilter* $\mathcal{U}$ *with* $\chi(\mathcal{U}) = \omega_1$*. Does this imply there is an ultrafilter* $\mathcal{V}$ *with* $\chi(\mathcal{V}) = \omega_2$*? With* $\pi\chi(\mathcal{V}) = \omega_2$*?*

(Of course, this is just the smallest interesting case of a much more general problem.) Note that the assumptions imply that there are ultrafilters $\mathcal{U}$ and $\mathcal{V}$ with $\pi\chi(\mathcal{U}) = \chi(\mathcal{U}) = \omega_1$ and $\pi\chi(\mathcal{V}) = \chi(\mathcal{V}) = \omega_3$ (Bell–Kunen [BK], see also [vM, Theorem 4.4.3]). By Theorem 6, we know there is not necessarily an ultrafilter $\mathcal{W}$ with $\pi\chi(\mathcal{W}) = \chi(\mathcal{W}) = \omega_2$. Of course, there is a corresponding problem on $\mathfrak{p}$ and $\pi\mathfrak{p}$. Finally, similar questions can be asked about special classes of ultrafilters. For example, it would be interesting to know what can be said about the spectrum of possible characters of P–points.

(6) (Spectral problem at singulars) *Let* λ *be singular (of uncountable cofinality). Assume that* $\mathsf{Spec}(\chi)$ *is cofinal in* λ*. Does* $\lambda \in \mathsf{Spec}(\chi)$*? Similar question for* $\mathsf{Spec}(\pi\chi)$*. What about* λ^+*?*

The only (partial) results we have in this direction are Proposition 5.1 (a), Theorem 4 (c) and Proposition 6.3.

(7) *Let* R *be a set of cardinals of uncountable cofinality in* $V \models GCH$*. Show there is a generic extension of* V *which has ultrafilters* $\mathcal{U}$ *with* $\chi(\mathcal{U}) = \pi\chi(\mathcal{U}) = \lambda$ *for each* $\lambda \in R$*.*

For regulars, this was done in Theorem 5. For singulars it was done *separately* for $\pi\chi$ and χ in Corollaries 5.5 and 6.1. We don't know how to do it simultaneously. Note, however, that given a single singular cardinal λ of uncountable cofinality in $V \models CH$, we can always force an ultrafilter with $\chi(\mathcal{U}) = \pi\chi(\mathcal{U}) = \lambda$: simply add λ Cohen reals; then, in fact, all ultrafilters $\mathcal{U}$ satisfy $\chi(\mathcal{U}) = \pi\chi(\mathcal{U}) = \lambda$.

(8) *Is there, in ZFC, an ultrafilter $\mathcal{U}$ with $\pi\chi(\mathcal{U}) = \chi(\mathcal{U})$?*

By the result of Bell and Kunen ([BK], [vM, Theorem 4.4.3]), this is true if $\mathfrak{c}$ is regular. The Bell–Kunen model [BK] which has no ultrafilter $\mathcal{U}$ with $\pi\chi(\mathcal{U}) = \mathfrak{c}$ has one with $\pi\chi(\mathcal{U}) = \chi(\mathcal{U}) = \omega_1$ instead. The dual question, whether there is an ultrafilter $\mathcal{U}$ with $\mathfrak{p}(\mathcal{U}) = \pi\mathfrak{p}(\mathcal{U})$, is independent, by the second author's P–point independence Theorem [Sh]. However, we may still ask whether one always has an ultrafilter $\mathcal{U}$ with $\mathfrak{p}'(\mathcal{U}) = \pi\mathfrak{p}(\mathcal{U})$.

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