On strong measure zero subsets of ${}^{\kappa}2$

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Abstract

We study the generalized Cantor space ${}^{\kappa}2$ and the generalized Baire space ${}^{\kappa}\kappa$ as analogues of the classical Cantor and Baire spaces. We equip ${}^{\kappa}\kappa$ with the topology where a basic neighborhood of a point η is the set $\{\nu : (\forall j < i)(\nu(j) = \eta(j))\}$, where $i < \kappa$.

We define the concept of a strong measure zero set of ${}^{\kappa}2$. We prove for successor $\kappa = \kappa^{<\kappa}$ that the ideal of strong measure zero sets of ${}^{\kappa}2$ is $\mathfrak{b}_\kappa$-additive, where $\mathfrak{b}_\kappa$ is the size of the smallest unbounded family in ${}^{\kappa}\kappa$, and that the generalized Borel conjecture for ${}^{\kappa}2$ is false. Moreover, for regular uncountable κ, the family of subsets of ${}^{\kappa}2$ with the property of Baire is not closed under the Souslin operation.

These results answer problems posed in [2].

1 Introduction

A systematic study of measure and category in the generalized Cantor space ${}^{\kappa}2$ and the generalized Baire space ${}^{\kappa}\kappa$ in these spaces was started in [2]; it turned out, however, that the former is quite problematic.

There are natural generalizations of the concepts of meager and strong measure zero sets from the space ${}^{\omega}\omega$ to the space ${}^{\kappa}\kappa$. Many results and their proofs concerning these concepts, e.g. the Baire Categoricity Theorem, are just straightforward generalizations of the corresponding results of ${}^{\omega}\omega$. It

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was proved in [7] that, assuming the Generalized Martin's Axiom GMA of [7], the family of meager subsets of ${}^{\omega_1}2$ is closed under unions of length $< 2^{\aleph_1}$.

In Section 2 we prove the same additivity result for the family of strong measure zero sets of ${}^{\kappa}2$.

The generalized Borel conjecture for ${}^{\kappa}2$, GBC(κ), states that every strong measure zero subset of ${}^{\kappa}2$ has the cardinality at most κ. The consistency of the Borel Conjecture for the space ${}^{\omega}2$, i.e. GBC(ω), was shown by Laver in [4]. However, in Section 3 we show that GBC(κ) fails assuming $\kappa = \kappa^{<\kappa} = \mu^+$. It is an open problem whether the statements "κ strongly inaccessible + GBC(κ)" or "κ the first (strongly) inaccessible + GBC(κ)" are consistent.

In the final section we show that the property of Baire is not preserved by the generalized Souslin operation

$$\bigcup_{f \in {}^{\kappa}\kappa} \bigcap_{i<\kappa} A_{f\restriction i}.$$

We show this by pointing out that the set CUB of characteristic functions of closed unbounded sets of κ lacks the property of Baire and yet is obtained from open sets by this Souslin operation.

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Our set theoretical notation is standard, see [3]. Ordinals are denoted by α, β, ϵ, ξ, i, j; cardinals by κ, μ and sequences by η, ν. Length of a sequence η is denoted by $\lg(\eta)$. We denote $[\alpha, \beta) = \{i \mid \alpha \leq i < \beta\}$. If η and ν are sequences, then $\eta \triangleleft \nu$ means that η is an initial segment of ν. For a cardinal κ and a set A we denote $[A]^{\kappa} = \{B \subseteq A : |B| = \kappa\}$ and $[A]^{\leq\kappa} = \{B \subseteq A : |B| \leq \kappa\}$.

2 Strong measure zero sets

Instead of the generalized Baire and Cantor spaces we study somewhat more general closed subsets of the former in this section.

Assumptions 2.1 Assume that κ is regular and uncountable. Let $T \subseteq {}^{<\kappa}\kappa$ be a normal tree with κ levels. Let T_i be the i-th level of T and $T_\kappa = \lim_\kappa(T)$. Assume that

$$i < j \leq \kappa \Rightarrow (\forall \eta \in T_i)(\exists \nu \in T_j)(\eta \triangleleft \nu).$$

We also assume that $i \leq |T_i| \leq \kappa$ for each $i < \kappa$ and $|T_\kappa| > \kappa$. Let $F_i : T_i \to |T_i|$ be one to one. We denote $\overline{F} = \langle F_i : i < \kappa \rangle$ and $\overline{F} \circ \eta = \langle F_i(\eta \restriction i) : i < \kappa \rangle$ for each $\eta \in T_\kappa$.

We introduce some notation and terminology. If $\nu \in T$ then $[\nu] = \{\eta \in T_\kappa : \nu \triangleleft \eta\}$. A set $A \subseteq T_\kappa$ is *open*, if for all $\nu \in A$ there exists $i < \kappa$ such that $[\nu \restriction i] \subseteq A$. For $X \subseteq \kappa$ and $f, g \in {}^X\kappa$,

$$f <^*_\kappa g \iff |\{i \in X : f(i) \geq g(i)\}| < \kappa.$$

The following generalization of the classical notion of a strong measure zero set was first introduced in [2].

Definition 2.2 $A \subseteq T_\kappa$ has *strong measure zero*, $A \in \mathcal{SZ}$, if for every $X \in [\kappa]^\kappa$ we can find $\langle f_\xi : \xi \in X \rangle$, $f_\xi \in T_\xi$ such that

$$A \subseteq \bigcup_{\xi \in X} [f_\xi].$$

Remark 2.3 (a) If $\kappa = \kappa^{<\kappa}$, then $T = {}^{<\kappa}2$ and $T = {}^{<\kappa}\kappa$ satisfy 2.1. So, in particular, 2.1 is true for $T = {}^{<\omega_1}\omega_1$ under CH and for $T = {}^{<\kappa}\kappa$ where κ is strongly inaccessible.

(b) If $T = {}^\kappa 2$, then T_κ does not have strong measure zero. (Choose $X = \{\xi + 1 : \xi < \kappa\}$ and $\eta(\xi) = 1 - f_\xi(\xi)$; then $\eta \in T_\kappa \setminus \cup_{\xi \in X} [f_\xi]$.)

(c) If κ is a successor and T a κ-Kurepa tree, then T_κ has strong measure zero. (If $\kappa = \mu^+$ and $X = \{x_i : i < \kappa\}$, enumerate T_{x_μ} as $\{t_\xi : \xi < \mu\}$. Choose $f_{x_\xi} = t_\xi \restriction x_\xi$.)

Next we give two characterizations of strong measure zero sets.

Lemma 2.4 *The following are equivalent for* $A \subseteq T_\kappa$

(a) A has strong measure zero

(b) if $\langle \alpha_i : i < \kappa \rangle$ *is a strictly increasing continuous sequence of ordinals* $< \kappa$ *then we can find*

$$Y_i \in [T_{\alpha_{i+1}}]^{\leq |\alpha_i|}$$

such that

$$(\forall \eta \in A)(\exists^\kappa i)(\eta \restriction \alpha_{i+1} \in Y_i).$$

Proof. (a) implies (b). Let $\langle \alpha_i : i < \kappa \rangle$ be a strictly increasing continuous sequence. For each $i < \kappa$ apply (a) to

$$X_i = \{\alpha_{j+1} : j \geq i\}$$

getting $\langle f_{i,\alpha_{j+1}} \in T_{\alpha_{j+1}} : j \geq i \rangle$. Let

$$Y_i = \{f_{\epsilon,\alpha_{i+1}} : \epsilon \leq i\}.$$

Now $|Y_i| \leq |i| \leq |\alpha_i|$ and if $\eta \in A$ then for any $i < \kappa$ there is $j \geq i$ such that $\eta \restriction \alpha_{j+1} = f_{i,\alpha_{j+1}} \in Y_j$.

(b) implies (a). Let $X \in [\kappa]^\kappa$. Choose by induction on $i < \kappa$, $\gamma_i < \kappa$ such that if i is limit then $\gamma_i = \cup\{\gamma_j : j < i\}$, and if $i = j+1$ then choose $\gamma_i > \gamma_j$ such that the set $X_j = [\gamma_j, \gamma_i) \cap X$ has cardinality $|\gamma_j|$. Apply clause (b) to $\langle \gamma_i : i < \kappa \rangle$: let

$$\langle Y_i \in [T_{\gamma_{i+1}}]^{\leq |\gamma_i|} : i < \kappa \rangle$$

be as guaranteed by clause (b). So $|Y_i| \leq |X_i|$ and we let $h_i : Y_i \to X_i$ be one to one. Let $\langle f_\xi : \xi \in X \rangle$, $f_\xi \in T_\xi$, be such that if $\xi = h_i(g)$ for $g \in Y_i$ then $f_\xi = g \restriction \xi$. As $[g] \subseteq [f_\xi]$ we are done. $\square$

Lemma 2.5 *If $\kappa = \mu^+$ and $|T_i| = \kappa$ for $i < \kappa$ large enough then the following are equivalent for $A \subseteq T_\kappa$*

(a) A has strong measure zero

(b′) like 2.4(b), but

$$Y_i \in [T_{\alpha_{i+1}}]^{\leq \mu}.$$

(c) for every $X \in [\kappa]^\kappa$, there is $f \in {}^X\kappa$ such that

$$\neg(f <^*_\kappa (\overline{F} \circ \eta) \restriction X)$$

for each $\eta \in A$.

Proof. Under the assumptions, 2.4(b) is clearly equivalent to 2.5(b′). This is where we need the assumption $\kappa = \mu^+$.

(b′) implies (c). Let $X \in [\kappa]^\kappa$. We may assume that $\alpha > \sup(\alpha \cap X)$ for each $\alpha \in X$ and if $\alpha \in [\min X, \kappa)$ then $|T_\alpha| = \kappa$. Let the closure of $X \cup \{0\}$

be enumerated in $\{\alpha_i : i < \kappa\}$ where α_i are increasing with i. Apply clause (b′) and get $\langle Y_i : i < \kappa \rangle$, $Y_i \in [T_{\alpha_{i+1}}]^{\leq \mu}$. Choose $f \in {}^X\kappa$ such that

$$f(\alpha_{i+1}) = \min\{\gamma < \kappa : F_{\alpha_{i+1}}(\eta) < \gamma \text{ for every } \eta \in Y_i\}.$$

Now let $\eta \in A$. Then $H = \{i < \kappa : \eta\restriction\alpha_{i+1} \in Y_i\}$ has cardinality κ and $F_{\alpha_{i+1}}(\eta\restriction\alpha_{i+1}) < f(\alpha_{i+1})$ for each $i \in H$. This means $\neg(f <^*_\kappa (\overline{F} \circ \eta)\restriction X)$.

(c) implies (b′). Let $\langle \alpha_i : i < \kappa \rangle$ be strictly increasing continuous sequence of ordinals $< \kappa$. We should find $\langle Y_i : i < \kappa \rangle$ as in clause (b′). Apply clause (c) for $X = \{\alpha_{i+1} : i < \kappa\}$ and get $f \in {}^X\kappa$. Let

$$Y_i = \{\eta \in T_{\alpha_{i+1}} : F_{\alpha_{i+1}}(\eta) \leq f(\alpha_{i+1})\}.$$

Let $\eta \in A$. Then $H = \{i < \kappa : F_{\alpha_{i+1}}(\eta\restriction\alpha_{i+1}) \leq f(\alpha_{i+1})\}$ has cardinality κ and $\eta\restriction\alpha_{i+1} \in Y_i$ for all $i \in H$. $\square$

A family $\mathcal{F} \subseteq {}^\kappa\kappa$ is *bounded*, if there is $g \in {}^\kappa\kappa$ such that $f <^*_\kappa g$ for all $f \in \mathcal{F}$. A family $\mathcal{F} \subseteq {}^\kappa\kappa$ is *dominating*, if for each $g \in {}^\kappa\kappa$ there is $f \in \mathcal{F}$ such that $g <^*_\kappa f$. Let $\mathfrak{d}_\kappa$ be the size of the smallest dominating family and let $\mathfrak{b}_\kappa$ be the size of the smallest unbounded family. Clearly $\kappa < \mathfrak{b}_\kappa \leq \mathfrak{d}_\kappa \leq 2^\kappa$.

For successor κ, condition (c) of Lemma 2.5 can be rephrased as follows: For each $X \in [\kappa]^\kappa$ the family $\{(\overline{F} \circ \eta)\restriction X : \eta \in A\}$ is not dominating. Hence, every $A \subseteq [T_\kappa]^{<\mathfrak{d}_\kappa}$ has strong measure zero.

In the sequel we will often abuse the terminology and say that a tree T_κ is dominating or bounded, when we actually mean that the family $\{(\overline{F} \circ \eta) : \eta \in T_\kappa\}$ is.

Remark 2.6 (a) In [1] Cummings and Shelah prove the following generalization of Easton's result on the possible behaviour of $\kappa \mapsto 2^\kappa$. Assume GCH. Then for any class function $\kappa \mapsto (\beta(\kappa), \delta(\kappa), \mu(\kappa))$ from regular cardinals to cardinal triplets satisfying

(1) the functions β, δ and μ are increasing, and

(2) $\kappa^+ \leq \beta(\kappa) = \mathrm{cf}(\beta(\kappa)) \leq \mathrm{cf}(\delta(\kappa)) \leq \delta(\kappa) \leq \mu(\kappa)$ and $\mathrm{cf}(\mu(\kappa)) > \kappa$ for all κ

there exists a class forcing preserving all cofinalities such that in the generic extension $\mathfrak{b}_\kappa = \beta(\kappa)$, $\mathfrak{d}_\kappa = \delta(\kappa)$ and $2^\kappa = \mu(\kappa)$ for all κ.

(b) If κ is strongly inaccessible, then ${}^\kappa 2$ is bounded, even though ${}^\kappa 2$ does not have strong measure zero.

Theorem 2.7 *Assume that $\kappa = \mu^+$. Then the ideal of strong measure zero sets of ${}^{\kappa}2$ is $\mathfrak{b}_\kappa$-additive.*

Proof. Assume that $\langle A_\xi : \xi < \gamma \rangle$, $\gamma < \mathfrak{b}_\kappa$, is a sequence of sets with strong measure zero. Let $A = \bigcup_{\xi<\gamma} A_\xi$. We prove that A has strong measure zero. Let $X \in [\kappa]^\kappa$. Using (c) of Lemma 2.5 for each $\xi < \gamma$ we find $f_\xi \in {}^X\kappa$ such that

$$\neg(f_\xi <^*_\kappa (\overline{F} \circ \eta){\restriction}X)$$

for all $\eta \in A_\xi$. Since the set $\{f_\xi : \xi < \gamma\}$ is bounded, there is $f \in {}^X\kappa$ such that

$$f_\xi <^*_\kappa f$$

for all $\xi < \gamma$. But then

$$\neg(f <^*_\kappa (\overline{F} \circ \eta){\restriction}X)$$

for all $\eta \in A$. Hence A is a strong measure zero set by Lemma 2.5(c). □

A version of generalized Martin's axiom for arbitrary κ, GMA(κ), is the following.

> Assume that a partial order P satisfies
>
> (a) if p and q are compatible, then they have an infimum in P.
>
> (b) if $\langle p_i : i < \gamma \rangle$ is a descending chain, where $\gamma < \kappa$, then $\inf_{i<\gamma} p_i \in P$.
>
> (c) if $\langle p_i : i < \kappa^+ \rangle$ is any sequence in P, then there are a cub set $C \subseteq \kappa^+$ and a regressive function $h : \kappa^+ \to \kappa^+$ such that for all $\alpha, \beta \in C$ we have
>
> $$\mathrm{cf}(\alpha) = \mathrm{cf}(\beta) = \kappa, h(\alpha) = h(\beta) \text{ implies } p_\alpha | p_\beta.$$
>
> Then for every family $\mathcal{D}$ of dense subsets of P such that $|\mathcal{D}| < 2^\kappa$ there is a $\mathcal{D}$-generic filter $K \subseteq P$.

It is possible to prove the relative consistency of GMA(κ) for arbitrary κ with $\kappa = \kappa^{<\kappa}$. See [7] 1.10 on page 302.

Lemma 2.8 ([6]) *Assume $\kappa = \kappa^{<\kappa}$ and GMA(κ). Then $\mathfrak{b}_\kappa = 2^\kappa$.*

Proof. Let $\{F_\alpha : \alpha < \gamma\} \subseteq {}^\kappa\kappa$, where $\gamma < 2^\kappa$. We want to construct a function F dominating all the F_α. Let P be the set of pairs (f, g) satisfying the following conditions:

(1) $f : \kappa \to \kappa$ is a partial function with $|f| < \kappa$.

(2) $g : \gamma \to \kappa$ is a partial function with $|g| < \kappa$.

(3) For all $\alpha \in \mathrm{dom}(g)$ and $j \in \mathrm{dom}(f)$ such that $j > g(\alpha)$, we have $f(j) > F_\alpha(j)$.

We define $(f_1, g_1) \leq (f_2, g_2)$ if $f_2 \subseteq f_1$ and $g_2 \subseteq g_1$.

Clearly, if (f_1, g_1) and (f_2, g_2) are compatible, then $(f_1 \cup f_2, g_1 \cup g_2)$ is their infimum. Similarly, it is easy to see that P satisfies condition (b) above.

Let then $\langle (f_\alpha, g_\alpha) : \alpha < \kappa^+\rangle$ be a sequence of conditions. We choose an arbitrary bijection $k : \kappa^+ \times \kappa \times \kappa \to \kappa^+$ and bijections $H_\beta : {}^{<\kappa}\beta \to \kappa$ for $\beta < \kappa^+$. Let

$$C_1 = \{\beta < \kappa^+ : \forall\alpha < \beta(\mathrm{dom}(g_\alpha) \subseteq \beta)\}$$

and

$$C_2 = \{\beta < \kappa^+ : k[\beta \times \kappa \times \kappa] \subseteq \beta\}.$$

For $\beta \in C_1 \cap C_2$, define $h_1(\beta) = \sup\{\alpha + 1 : \alpha \in \beta \cap \mathrm{dom}(g_\beta)\}$, and let

$$h(\beta) = k(h_1(\beta), H_\kappa(f_\beta), H_{h_1(\beta)}(g_\beta \restriction h_1(\beta))),$$

if $\mathrm{cf}(\beta) = \kappa$; otherwise, let $h(\beta) = 0$.

Claim 1 *The cub set $C = C_1 \cap C_2$ and the function h defined above witness the truth of condition (c).*

Proof: Clearly $h_1(\beta) < \beta$ whenever $\mathrm{cf}(\beta) = \kappa$. Since $C \subseteq C_2$, the function h is regressive on C. Assume $\alpha, \beta \in C$, $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta) = \kappa$, $\alpha < \beta$, and $h(\alpha) = h(\beta)$. Thus $h_1(\alpha) = h_1(\beta)$, and, by the injectivity of the H_ξ, we further have $f_\alpha = f_\beta$ and $g_\alpha \restriction h_1(\alpha) = g_\beta \restriction h_1(\beta)$. Since $\beta \in C_1$, we have $\mathrm{dom}(g_\alpha) \subseteq \beta$. Hence

$$\mathrm{dom}(g_\alpha) \cap \mathrm{dom}(g_\beta) \subseteq \beta \cap \mathrm{dom}(g_\beta) \subseteq h_1(\beta),$$

and therefore $g_\alpha \cup g_\beta$ is a function. Now $(f_\alpha, g_\alpha \cup g_\beta)$ is a common extension of (f_α, g_α) and (f_β, g_β), and the claim has been proved.

Now, let K be $\mathcal{D}$-generic, where $\mathcal{D} = \{D_i : i < \kappa\} \cup \{E_\xi : \xi < \gamma\}$ and $D_i = \{(f,g) \in P : i \in \mathrm{dom}(f)\}$, $E_\xi = \{(f,g) \in P : \xi \in \mathrm{dom}(g)\}$. Then for $F = \bigcup\{f : (f,g) \in K\}$ and $G = \bigcup\{g : (f,g) \in K\}$ we have

$$F(i) > F_\xi(i),$$

whenever $i > G(\xi)$, i.e., F dominates the family $\{F_\alpha : \alpha < \gamma\}$. □

Actually, the consistency of $\mathfrak{b}_\kappa = 2^\kappa$ can be shown by simpler means than using GMA(κ). For instance, it follows immediately from Remark 2.6(a).

Corollary 2.9 *The ideal of strong measure zero sets of* ${}^\kappa 2$ *is* 2^κ*-additive for successor* $\kappa = \kappa^{<\kappa}$*, assuming GMA*($\kappa$)*.*

Remark 2.10 Assume κ is a successor, and let $\mathcal{F}$ be a dominating family of size $\mathfrak{d}_\kappa$. Let $X \in [\kappa]^\kappa$ be such that X contains no limit ordinals. For each $f \in \mathcal{F}$ we can find $\eta_f \in T_\kappa$ such that $f <^*_\kappa (\overline{F} \circ \eta_f)\restriction X$. Now the set $A = \{\eta_f : f \in \mathcal{F}\}$ does not have strong measure zero by Lemma 2.5. Hence the ideal of strong measure zero sets is not $\mathfrak{d}^+_\kappa$-additive. So consistently $\kappa = \kappa^{<\kappa}$, the ideal is not κ^{++}-additive and $\kappa^{++} \leq 2^\kappa$.

3 The generalized Borel conjecture

The main result of this section is Lemma 3.5, which further shows the failure of the GBC under suitable assumptions.

Throughout this section we assume that κ is regular uncountable, T is a set of sequences of ordinals $< \kappa$ each of length $< \kappa$, T closed under intial segments, T_i is the set of members of T of length i, T_κ the set of sequences of length κ every initial segment of which belongs to T, and every $x \in T$ has more than one extension in T_κ. Note that in this section we do not assume $\kappa = \kappa^{<\kappa}$.

Definition 3.1 (1) $A \subseteq T_\kappa$ is *nowhere-dense*, if $T_\kappa \setminus A$ contains an open and dense set. $A \subseteq T_\kappa$ is *meager* if it is a κ-union of nowhere-dense sets. $A \subseteq {}^\kappa\kappa$ is *comeager* if ${}^\kappa\kappa \setminus A$ is meager.

(2) We say that $A \subseteq T_\kappa$ is *weakly* $(<\mu)$*-Baire* if it is not the union of $< \mu$ nowhere-dense subsets and *weakly* μ*-Baire* if it is weakly $(<\mu^+)$-Baire. A is $(< \mu)$*-Baire* (respectively, μ*-Baire*) if $A \cap U$ is weakly $(< \mu)$-Baire (weakly μ-Baire) for every set U that is open in A.

We consider the following properties of T:

Pr_0 T_κ has $> \kappa$ members.

Pr_1 T_κ is unbounded.

Pr_2 T_κ is κ-Baire.

Proposition 3.2 *(1) Pr_2 implies Pr_1, assuming that $|T_i| \geq \kappa$ for i large enough.*

(2) Pr_1 implies Pr_0.

Proof. $Pr_2 \to Pr_1$: Assume T_κ is bounded by $g \in {}^\kappa\kappa$. Let

$$A_i = \{f \in T_\kappa : (\forall j > i)(F_j(f{\restriction}j) \leq g(j))\}.$$

A_i is nowhere-dense in T_κ: Let $\nu \in T$. By our assumption we can choose a successor $\eta \in T$ of ν such that $F_j(\eta{\restriction}j) > g(j)$ for some $j > i$. Then $[\eta] \subseteq T_\kappa \setminus A_i$. Therefore $T_\kappa \subseteq \bigcup\{A_i : i < \kappa\}$ would be meager, hence not κ-Baire.

$Pr_1 \to Pr_0$: If $\eta_i \in T_\kappa$ for $i < \kappa$, then g: $g(i) = \sup\{F_i(\eta_j \restriction i) : j \leq i\}$ dominates $\overline{F} \circ \eta_i$. $\Box$

From now on, we always assume at least Pr_0.

Lemma 3.3 *Let $\kappa > \omega$ be regular. There is a sequence $\langle C_\alpha : \alpha < \mathfrak{d}_\kappa \rangle$ of cub sets of κ such that for each cub set $C \subseteq \kappa$ there is $\alpha < \mathfrak{d}_\kappa$ such that $C_\alpha \subseteq C$.*

Proof. Let $D \subseteq {}^\kappa\kappa$ be a dominating family of size $\mathfrak{d}_\kappa$. We may assume that each $g \in D$ is strictly increasing. For each $g \in D$ let

$$C_g = \{\delta < \kappa : \delta \text{ limit} \wedge \forall i (i < \delta \iff g(i) < \delta)\}.$$

C_g is a cub set: closed: Let $\delta_i \in C_g$ for all $i < j$ where $j < \kappa$ is a limit ordinal. Then for $i' < \delta = \sup\{\delta_i : i < j\}$ we have $i' < \delta_i$ for some $i < j$ and hence $g(i') < \delta_i < \delta$. unbounded: Let $\delta' < \kappa$. Let $\delta_0 = \delta'$ and $\delta_{n+1} = \sup\{g(i) : i < \delta_n\}$. Now $\delta = \{\delta_n : n < \omega\} \in C_g$.

Let $C_g = \{\gamma_{g,i} : i < \kappa\}$. Suppose $C = \{\beta_i : i < \kappa\}$ is a cub set. We assume these enumerations are strictly increasing. Then there is $g \in D$ and

$i_0 < \kappa$ such that $\beta_i < g(i)$ for all $i > i_0$. If $i_0 < j < \gamma_{\alpha,i}$ then $\beta_j < g(j) < \gamma_{g,i}$. Hence $\gamma_{g,i} = \sup\{\beta_j : j < \gamma_{g,i}\} \in C$ for each i large enough. Now we can enumerate $\{C_g \setminus i : g \in D,\ i < \kappa\}$ as $\langle C_\alpha : \alpha < \mathfrak{d}_\kappa\rangle$ and it has the required property. $\square$

Definition 3.4 *The generalized Borel's conjecture (abbreviated by* GBC*) for* T_κ is the statement $\mathcal{SZ} = [T_\kappa]^{\leq\kappa}$. GBC($\kappa$) is the statement GBC for ${}^\kappa 2$.

Theorem 3.5 *Assume that κ is regular, $|T| = \kappa$, T_κ is ($<\mathfrak{d}_\kappa$)-Baire and every $Y \in [T_\kappa]^{<\mathfrak{d}_\kappa}$ has strong measure zero. Then there is $A \in [T_\kappa]^{\mathfrak{d}_\kappa}$ which has strong measure zero.*

Proof. Let $\langle C_\alpha : \alpha < \mathfrak{d}_\kappa\rangle$ be a sequence given by Lemma 3.3. Let $\mathrm{nacc}(C_\alpha)$ be $\{i \in C_\alpha : i > \sup(C_\alpha \cap i)\}$. Let F_α be the set of functions f with domain T such that

(1) for every $x \in T$, $x <_T f(x)$ and $\lg(f(x)) \in \mathrm{nacc}(C_\alpha)$

(2) if $x \neq y$ are from T then $\lg(f(x)) \neq \lg(f(y))$

Let for $f \in F_\alpha$, $\alpha < \mathfrak{d}_\kappa$

$$X_f =^{df} \{\eta \in T_\kappa : \text{for } \kappa \text{ members } x \text{ of } T,\ f(x) <_T \eta\}.$$

Clearly X_f is comeager, since $X_f = \bigcap_{i<\kappa} X^i_f$ where $X^i_f = \{\eta : (\exists x)(\lg(x) > i \wedge f(x) <_T \eta)\}$ are open and dense: given $x \in T$, choose x' such that $x <_T x'$ and $\lg(x') > i$. Then $[f(x')] \subseteq X^i_f$.

Now we choose by induction on $\alpha < \mathfrak{d}_\kappa$, η_α and f_α such that

(1) $\eta_\alpha \in T_\kappa \setminus \{\eta_\beta : \beta < \alpha\}$

(2) $f_\alpha \in F_\alpha$

(3) $\eta_\alpha \in X_{f_\beta}$ for all $\beta < \alpha$.

If we succeed then we will show that $Z = \{\eta_\alpha : \alpha < \mathfrak{d}_\kappa\}$ is a subset of T_κ of cardinality $\mathfrak{d}_\kappa$ (by (1)) which is of strong measure zero.

Let us carry the induction. First we choose η_α to satisfy the demands, the only relevant ones are (1)+(3). But since T_κ is ($<\mathfrak{d}_\kappa$)-Baire, the set

$$(T_\kappa \setminus \{\eta_\beta : \beta < \alpha\}) \cap \bigcap_{\beta<\alpha} X_{f_\beta}$$

is non-empty. So we can find η_α which satisfies the requirements.

Next let us choose f_α, the only relevant demand is (2). Enumerate $T = \{x_i : i < \kappa\}$. We choose $f_\alpha(x_i)$ by induction on $i < \kappa$, by defining a sequence $\{\beta_i : i < \kappa\}$ such that

(1) $x_i <_T f_\alpha(x_i)$ and $f_\alpha(x_i) \in T_{\beta_i}$

(2) $\beta_i \in \mathrm{nacc}(C_\alpha) \setminus \{\beta_j : j < i\}$.

Now we show that $Z = \{\eta_\alpha : \alpha < \mathfrak{d}_\kappa\}$ has strong measure zero. Let $C = \{\alpha_i : i < \kappa\}$ be cub and choose $\alpha < \mathfrak{d}_\kappa$ such that $C_\alpha \subseteq C$. Let $C_\alpha = \{\beta_i : i < \kappa\}$. Let $Z^* = \{\eta_\beta : \beta < \mathfrak{d}_\kappa,\ \beta > \alpha\}$ and $Z' = \{\eta_\beta : \beta \leq \alpha\}$. We define $Y_i \in [T_{\alpha_{i+1}}]^{\leq |\alpha_i|}$ as follows: By our assumption Z' has strong measure zero. So there is $\langle Y'_i \in [T_{\alpha_{i+1}}]^{\leq |\alpha_i|} : i < \kappa\rangle$ such that

$$(\forall \eta \in Z')(\exists^\kappa i < \kappa)(\eta \restriction \alpha_{i+1} \in Y'_i).$$

For each $i < \kappa$ if there is $j < \kappa$ such that $\alpha_i = \beta_j$ then let

$$Y^*_i = \{f_\alpha(x) \restriction \alpha_{i+1} : \lg(f_\alpha(x)) = \beta_{j+1}\}.$$

Otherwise let $Y^*_i = \varnothing$. Let $Y_i = Y'_i \cup Y^*_i$. We claim that

$$(\forall \eta \in Z)(\exists^\kappa i < \kappa)(\eta \restriction \alpha_{i+1} \in Y_i).$$

If $\eta \in Z^*$ then $\eta \in X_{f_\alpha}$ and so $\{x \mid f_\alpha(x) <_T \eta\}$ has cardinality κ. Since $B = \{\lg(f_\alpha(x)) : f_\alpha(x) <_T \eta\} \subseteq \mathrm{nacc}(C_\alpha)$ has cardinality κ, we see that $\eta \restriction \alpha_{i+1} \in Y_i$ for each i such that $\alpha_i \in C_\alpha$. $\square$

Remark 3.6 Consistently there are such T's even if $\kappa^{<\kappa} > \kappa = cf(\kappa) > \aleph_0$.

From Theorem 3.5 we get the following corollaries.

Corollary 3.7 *Assume that T_κ is κ-Baire and $\mathfrak{d}_\kappa = \kappa^+$. Then there is a strong measure zero subset of T_κ of cardinality κ^+.*

Theorem 3.8 (Baire Category Theorem) *Assume T is a tree with κ levels, $(<\kappa)$-complete with no isolated branches. Then T_κ is κ-Baire.*

Proof. Suppose $\langle D_i : i < \kappa \rangle$ is a sequence of open dense sets. For each $i < \kappa$ let $f_i : T \to T$ be such that for every $x \in T$, $x <_T f(x)$ and $[f_i(x)] \subseteq D_i$. We will define $\eta \in \bigcap \{D_i : i < \kappa\}$ by induction. Let $x_{i+1} = f_i(x_i)$ and $x_i = \bigcup \{x_j : j < i\}$ if i is limit. Since T is $< \kappa$-complete $x_i \in T$. Now $\eta = \bigcup \{x_i : i < \kappa\} \in \bigcap_{i<\kappa} D_i$. $\square$

Corollary 3.9 *Assume $\mathfrak{d}_\kappa = \kappa^+$ and $\kappa = \kappa^{<\kappa} > \aleph_0$ and T is a tree with κ levels, $(< \kappa)$-complete with no isolated branches. Then there is a subset of T_κ of cardinality κ^+ which is of strong measure zero.*

This finally implies the failure of GBC(κ) for successor $\kappa = \kappa^{<\kappa}$.

Corollary 3.10 *If $\kappa = \kappa^{<\kappa} = \mu^+$, $|T_i| = \kappa$ for $i < \kappa$ large enough and T is closed under increasing sequences of length $< \kappa$ then there is an $A \in [T_\kappa]^{\kappa^+}$ of strong measure zero.*

Proof. Case 1. $\mathfrak{d}_\kappa > \kappa^+$ then any set of size κ^+ has strong measure zero by Lemma 2.5.

Case 2. $\mathfrak{d}_\kappa = \kappa^+$. Theorem 3.9 $\square$

Remark 3.11 The failure of the generalized Borel conjecture follows directly from this corollary by setting $T = {}^\kappa 2$.

4 The property of Baire

We show that the property of Baire is not preserved in the Souslin operation on ${}^\kappa 2$, contrary to the corresponding theorem for reals. We say that $A \subseteq {}^\kappa 2$ has *the property of Baire*, if there is an open set $O \subseteq {}^\kappa 2$ such that $(O \setminus A) \cup (A \setminus O)$ is meager.

Let

$$\text{CUB} = \{\eta \in {}^\kappa 2 : \text{ for some club } C \text{ of } \kappa \ (\forall i \in C)(\eta(i) = 1)\}.$$

Lemma 4.1 *There is a system $\langle A_\nu : \nu \in {}^{<\kappa}\kappa \rangle$ of open sets such that*

$$\text{CUB} = \bigcup_{f \in {}^\kappa\kappa} \bigcap_{i<\kappa} A_{f \restriction i}$$

Proof. For $\nu \in {}^{<\kappa}\kappa$ let

$$A_\nu = \{\eta \in {}^\kappa 2 : (\forall i \in \mathrm{dom}(\nu))(\eta(\nu(i)) = 1)\}$$

if ν is a strictly increasing continuous sequence and let A_ν be empty otherwise. Let $\eta \in \mathrm{CUB}$ and let $\langle \alpha_i : i < \kappa \rangle$ be an increasing enumeration of a club set such that $\eta(\alpha_i) = 1$ for all $i < \kappa$. Then $\eta \in A_{\langle \alpha_j : j < i \rangle}$ for all i. Conversely, if $\eta \in A_{f \restriction i}$ for all i, then clearly f is strictly increasing and continuous, hence $\eta \in \mathrm{CUB}$. $\square$

The above lemma shows that the set CUB can be obtained from open sets by means of an operation which is analogous to the Souslin operation. Thus the following result shows that the property of Baire is not preserved by this "Souslin" operation. Recall that in the space ${}^\omega 2$ the property of Baire is preserved by the ordinary Souslin operation.

Theorem 4.2 *Let $\kappa > \aleph_0$ be regular. Then* CUB *does not have the property of Baire.*

Proof. We show that for all open set O, $(O \setminus \mathrm{CUB}) \cup (\mathrm{CUB} \setminus O)$ is not meager.

Suppose first O is empty. We show that CUB is not meager. Let $R_\xi \subseteq {}^\kappa 2$ be nowhere dense for $\xi < \kappa$. We choose α_i, η_i by induction on $i \leq \kappa$ such that

(1) $\eta_i \in {}^{\alpha_i}2$

(2) if $j < i$ then $\alpha_j < \alpha_i$ and $\eta_j \lhd \eta_i$

(3) if i is limit then $\alpha_i = \bigcup_{j<i} \alpha_j$ and $\eta_i = \bigcup_{j<i} \eta_j$

(4) $\eta_{i+1}(\alpha_i) = 1$

(5) $\neg(\exists \rho)(\eta_{i+1} \lhd \rho \wedge \rho \in R_i)$.

Now $\eta_\kappa \in \mathrm{CUB} \setminus \bigcup_{\xi<\kappa} R_\xi$, whence $\mathrm{CUB} \neq \bigcup_{\xi<\kappa} R_\xi$.

If O is non-empty then we choose ν such that $[\nu] \subseteq O$. Then $O \setminus \mathrm{CUB} \supseteq [\nu] \setminus \mathrm{CUB}$. Similarly as above we show that $[\nu] \setminus \mathrm{CUB}$ is not meager. We proceed as above except $\alpha_0 = \lg(\nu)$, $\eta_0 = \nu$ and

(4′) $\eta_{i+1}(\alpha_i) = 0$.

Then $\eta_\kappa \in ([\nu] \setminus \mathrm{CUB}) \setminus \bigcup_{\xi<\kappa} R_\xi$. $\square$

Let us call a subset of ${}^\kappa 2$ *Borel* if it is a member of the smallest algebra of subsets of ${}^\kappa 2$ containing all open sets and closed under complements and unions of length $\leq \kappa$. It is proved in [2] that Borel sets have the property of Baire. Hence CUB is not Borel. This improves the result in [5] to the effect that CUB is not Π^0_3 or Σ^0_3. Assuming $\kappa = \aleph_1 = 2^{\aleph_0}$, non-Borelness of CUB follows from the stronger result that CUB and NON-STAT $= \{\eta \in {}^{\omega_1}2 :$ for some cub $C \subseteq \omega_1 (\forall i \in C)(\eta(i) = 0)\}$ cannot be separated by a Borel set [8].

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