On tightness and depth in superatomic Boolean algebras

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ABSTRACT: We introduce a large cardinal property which is consistent with L and show that for every superatomic Boolean algebra B and every cardinal λ with the large cardinal property, if $\text{tightness}^+(B) \geq \lambda^+$ then $\text{depth}(B) \geq \lambda$. This improves a theorem of Dow and Monk.

In [DM, Theorem C], Dow and Monk have shown that if λ is a Ramsey cardinal (see [J, p.328]) then every superatomic Boolean algebra with tightness at least λ^+ has depth at least λ. Recall that a Boolean algebra B is *superatomic* iff every homomorphic image of B is atomic. The *depth* of B is the supremum of all cardinals λ such that there is a sequence $(b_\alpha : \alpha < \lambda)$ in B with $b_\beta < b_\alpha$ for all $\alpha < \beta < \lambda$ (a *well-ordered chain* of length λ). Then depth^+ of B is the first cardinal λ such that there is no well-ordered chain of length λ in B. The *tightness* of B is the supremum of all cardinals λ such that B has a *free* sequence of length λ, where a sequence $(b_\alpha : \alpha < \lambda)$ is called *free* provided that if Γ and Δ are finite subsets of λ such that $\alpha < \beta$ for all $\alpha \in \Gamma$ and $\beta \in \Delta$, then

$$\bigcap_{\alpha\in\Gamma} -b_\alpha \cap \bigcap_{\beta\in\Delta} b_\beta \neq 0.$$

By $\text{tightness}^+(B)$ we denote the first cardinal λ for which there is no free sequence of length λ in B.

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For $b \in B$ we sometimes write b^0 for $-b$ and b^1 for b.

We improve Theorem C from [DM] in two directions. We introduce a large cardinal property which is much weaker than Ramseyness and even consistent with L (the constructible universe) and show that in Theorem C from [DM] it suffices to assume that λ has this property. Moreover we show that it suffices to assume tightness$^+(B) \geq \lambda^+$ instead of tightness$(B) \geq \lambda^+$ to conclude that depth$(B) \geq \lambda$. In particular we get:

Theorem 1. *Suppose that $0^\sharp$ exists. Let B be a superatomic Boolean algebra in the constructible universe L, and let λ be an uncountable cardinal in V. Then in L it is true that tightness$^+(B) \geq \lambda^+$ implies that depth$^+(B) \geq \lambda$.*

For the theory of $0^\sharp$ see [J, §30]. Note that λ as in Theorem 1 is a limit cardinal in L, hence it suffices to show that in L, depth$(B) \geq \kappa$ for all cardinals $\kappa < \lambda$. As was the case with the proof of Theorem C of [DM], we can't show that under the assumptions of Theorem 1, depth$(B) = \lambda$ is attained, i.e. that there is a well-ordered chain of length λ.

For the proof we consider the following large cardinal property:

Definition 2. Let λ, κ, θ be infinite cardinals, and let γ be an ordinal. The relation $R_\gamma(\lambda, \kappa, \theta)$ is defined as follows:

For every $c : [\lambda]^{<\omega} \to \theta$ there exists $A \subseteq \lambda$ of order-type γ, such that for every $u \in [A]^{<\omega}$ there exists $B \subseteq \lambda$ of order-type κ such that $\forall w \in [B]^{|u|} \quad c(w) = c(u)$.

Lemma 3. *Assume $R_\gamma(\lambda, \kappa, \theta)$, where γ is a limit ordinal. For every $c : [\lambda]^{<\omega} \to \theta$ there exists $A \subseteq \lambda$ as in the definition of $R_\gamma(\lambda, \kappa, \theta)$ such that additionally $c\restriction[A]^n$ is constant for every $n < \omega$.*

Proof: Define c' on $[\lambda]^{<\omega}$ by

$$c'\{\beta_0, \dots, \beta_{n-1}\} = \{(v, c\{\beta_i : i \in v\}) : v \subseteq n\}.$$

As θ is infinite we can easily code the values of c' as ordinals in θ and therefore apply $R_\gamma(\lambda, \kappa, \theta)$ to it. We get $A \subseteq \lambda$ of order-type γ. We shall prove that $c\restriction[A]^n$ is constant, for every $n < \omega$. Fix $w_1, w_2 \in [A]^n$. Since γ is a limit, without loss of generality we may assume that $\max(w_1) < \min(w_2)$. Let $w = w_1 \cup w_2$. By Definition 2 there exists $B \subseteq \lambda$, o.t.$B = \kappa$, such that $c'\restriction[B]^{2n}$ is constant with value $c'(w)$. Let $(\beta_\nu : \nu < \kappa)$ be the increasing enumeration of B. We have

$$c'\{\beta_0, \dots, \beta_{2n-1}\} = c'\{\beta_n, \dots, \beta_{3n-1}\}.$$

By the definition of c' we get

$$c\{\beta_0, \ldots, \beta_{n-1}\} = c\{\beta_n, \ldots, \beta_{2n-1}\} =: c_0.$$

This information is coded in $c'\{\beta_0, \ldots, \beta_{2n-1}\}$, i.e.

$$(\{0, \ldots, n-1\}, c_0),\ (\{n, \ldots, 2n-1\}, c_0) \in c'\{\beta_0, \ldots, \beta_{2n-1}\}.$$

As $c'\{\beta_0, \ldots, \beta_{2n-1}\} = c'(w)$ we conclude $c(w_1) = c(w_2) = c_0$. $\square$

Theorem 4. *Assume $R_\gamma(\lambda, \kappa, \omega)$, where γ is a limit ordinal. If B is a Boolean algebra and $(a_\nu : \nu < \lambda)$ is a sequence in B, then one of the following holds:*

(a) there exists $A \subseteq \lambda$, o.t.$(A) = \gamma$, such that $(a_\nu : \nu \in A)$ is independent;

(b) there exist $n < \omega$ and strictly increasing sequence $(\beta_\nu : \nu < \kappa)$ in λ such that, letting

$$b_\nu = \bigcup_{k<n} \bigcap_{l<n} a_{\beta_{n^2\nu+nk+l}}, \tag{$*$}$$

we have that $(b_\nu : \nu < \kappa)$ is constant;

(c) there exists a strictly decreasing sequence in B of length κ.

Corollary 5. *Assume $R_\gamma(\lambda, \kappa, \omega)$, where γ is a limit ordinal. If B is a superatomic Boolean algebra, then tightness$^+(B) > \lambda$ implies Depth$^+(B) > \kappa$.*

Proof of Corollary 5: Let $(a_\nu : \nu < \lambda)$ be a free sequence in B. As a superatomic Boolean algebra does not have an infinite independent subset, (a) is impossible. Suppose (b) were true. Define b_ν as in $(*)$. Clearly we have

$$-b_\nu \geq \bigcap_{k,l<n} a^0_{\beta_{n^2\nu+nk+l}}, \text{ and}$$

$$b_\nu \geq \bigcap_{k,l<n} a_{\beta_{n^2\nu+nk+l}}.$$

Hence if $\nu < \mu$ and $b_\nu = b_\mu$ we obtain

$$0 = -b_\nu \cap b_\mu \geq \bigcap_{k,l<n} a^0_{\beta_{n^2\nu+nk+l}} \cap \bigcap_{k,l<n} a_{\beta_{n^2\mu+nk+l}}.$$

This contradicts freeness of $(a_\nu : \nu < \kappa)$. We conclude that (c) must hold. $\square$

Proof of Theorem 4: Define $c : [\lambda]^{<\omega} \to [{}^{<\omega}2]^{<\omega}$ by

$$c\{\beta_0 < \ldots < \beta_{n-1}\} = \{\eta \in {}^n2 : \bigcap_{i<n} a_{\beta_i}^{\eta(i)} = 0\}.$$

Note that $c\{\beta_0 < \ldots < \beta_{n-1}\} = c\{\alpha_0 < \ldots < \alpha_{n-1}\}$ implies that $\{a_{\beta_0}, \ldots, a_{\beta_{n-1}}\}$ and $\{a_{\alpha_0}, \ldots, a_{\alpha_{n-1}}\}$ have the same quantifier-free diagram, i.e. for every quantifier-free formula $\phi(x_0, \ldots, x_{n-1})$ in the language of Boolean algebra,

$$B \models \phi[a_{\beta_0}, \ldots, a_{\beta_{n-1}}] \Leftrightarrow B \models \phi[a_{\alpha_0}, \ldots, a_{\alpha_{n-1}}].$$

Let $A \subseteq \lambda$ be as guaranteed for c by $R_\gamma(\lambda, \kappa, \omega)$. By Lemma 3 we may assume that $c\restriction[A]^n$ is constant, for every $n < \omega$.

If $(a_\alpha : \alpha \in A)$ is independent, we are done. Therefore we may assume that this is false. For $m < \omega$ define

$$\Gamma_m = \{\eta \in {}^m2 : \exists\{\beta_0 < \ldots < \beta_{m-1}\} \subseteq A \quad \bigcap_{i<m} a_{\beta_i}^{\eta(i)} = 0\}.$$

By assumption, in the definition of Γ_m the existential quantifier can be replace by a universal one to give the same set. There exists $m < \omega$ such that $\Gamma_m \neq \emptyset$. Define

$$\Gamma'_m = \{\eta \in \Gamma_m : \text{ no proper subsequence of } \eta \text{ belongs to } \bigcup_{k<m} \Gamma_k\}.$$

By Kruscal's Theorem [K], we have that $\bigcup_{m<\omega} \Gamma'_m$ is finite. Let n^* be minimal such that $\bigcup_{m<\omega} \Gamma'_m = \bigcup_{m<n^*} \Gamma'_m$. Then clearly we have that for every $m < \omega$ and $\eta \in \Gamma_m$, η has a subsequence in $\bigcup_{k<n^*} \Gamma'_k$. Let $m^* = (n^*)^2$, and let

$$\tau(x_0, \ldots, x_{m^*-1}) = \bigcup_{l<n^*} \bigcap_{k<n^*} x_{n^*l+k}.$$

Claim 1. *If* $\eta \in {}^{m^*}2$, $t \in \{0, 1\}$, *and in the Boolean algebra* $\{0, 1\}$, $\tau[\eta(0), \ldots, \eta(m^* - 1)] = t$, *then* $|\{i < m^* : \eta(i) = t\}| \geq n^*$. $\square$

Let $(\beta_\nu : \nu < \gamma)$ be the strictly increasing enumeration of A, and define

$$b_\nu = \tau[a_{\beta_{m^*\nu}}, a_{\beta_{m^*\nu+1}}, \ldots, a_{\beta_{m^*\nu+m^*-1}}],$$

for every $\nu < \gamma$, where the evaluation of τ takes place in B, of course. It is easy to see that the sequence $(b_\nu : \nu < \gamma)$ inherites from $(a_{\beta_\nu} : \nu < \gamma)$ the property, that any two finite subsequences of same length have the same quantifier-free diagram.

Claim 2. *If $\eta \in \Gamma_n$, then $\bigcap_{i<n} b_i^{\eta(i)} = 0$.*

Proof of Claim 2: Otherwise there exists an ultrafilter D on B such that $\bigcap_{i<n} b_i^{\eta(i)} \in D$. Define $\zeta \in {}^{nm^*}2$ by $\zeta(i) = 1$ iff $a_{\beta_i} \in D$. Then $\bigcap_{i<nm^*} a_{\beta_i}^{\zeta(i)} \in D$, and hence $\zeta \notin \Gamma_{nm^*}$. Let $h : B \to B/D = \{0, 1\}$ be the canonical homomorphism induced by D. We calculate

$$1 = h(\textstyle\bigcap_{i<n} b_i^{\eta(i)}) = \bigcap_{i<n} h(b_i)^{\eta(i)} = \bigcap_{i<n} \tau[h(a_{\beta_{m^*i}}), \dots, h(a_{\beta_{m^*(i+1)-1}})]^{\eta(i)}$$

$$= \textstyle\bigcap_{i<n} \tau[\zeta(m^*i), \dots, \zeta(m^*i + k), \dots, \zeta(m^*(i+1) - 1)]^{\eta(i)}.$$

We conclude that $\tau[\zeta(m^*i), \dots, \zeta(m^*i+k), \dots, \zeta(m^*(i+1) - 1)] = \eta(i)$, for all $i < n$, and hence by Claim 1 we can choose $j_i \in [m^*i, m^*(i + 1))$ such that $\zeta(j_i) = \eta(i)$. Clearly $i_0 < i_1$ implies that $j_{i_0} < j_{i_1}$. But this implies $\zeta \in \Gamma_{nm^*}$, a contradiction. $\square_{Claim\ 2}$

Claim 3. *If $t < \omega$, $\eta \in \Gamma_n$, $0 = k_0 < k_1 < \dots < k_t = n$, and $\eta{\restriction}[k_i, k_{i+1})$ is constant for all $i < t$, and if $\rho \in {}^t2$ is defined by $\rho(i) = \eta(k_i)$, then $\bigcap_{i<t} b_i^{\rho(i)} = 0$.*

Proof of Claim 3: Wlog we may assume that $\eta \in \Gamma'_n$ for some $n < n^*$. Indeed, otherwise we can find $m < n^*$, $\eta' \in \Gamma'_m$ and some increasing $h : m \to n$ such that $\eta'(i) = \eta(h(i))$, for all $i < m$. Then $\{h^{-1}[k_i, k_{i+1}) : i < t\}$ equals $\{[l_i, l_{i+1}) : i < s\}$ for some $l_0 = 0 < l_1 < \dots < l_{s-1} = m$. Note that $\eta'{\restriction}[l_i, l_{i+1})$ is constant, and letting $\rho' \in {}^s2$ be defined by $\rho'(i) = \eta'(l_i)$, we have $\rho'(i) = \rho(h(i))$. Hence $\bigcap_{i<s} b_i^{\rho'(i)} = 0$ implies $\bigcap_{i<t} b_i^{\rho(i)} = 0$.

Therefore we assume $\eta \in \Gamma'_n$, for some $n < n^*$. Suppose we had $\bigcap_{i<t} b_i^{\rho(i)} > 0$. Let D be an ultrafilter on B containing $\bigcap_{i<t} b_i^{\rho(i)}$. Let $h : B \to B/D$ be the canonical homomorphism. Define $\zeta \in {}^{tm^*}2$ such that $\zeta(i) = 1$ iff $a_i \in D$. Hence $\zeta \notin \Gamma_{tm^*}$. We get

$$h(\bigcap_{i<t} b_i^{\rho(i)}) = \bigcap_{i<t} \tau[\zeta(im^*), \dots, \zeta((i + 1)m^* - 1)]^{\rho(i)} = 1.$$

Hence by Claim 1,

$$\forall i < t \exists a_i \in [\{im^*, \dots, (i + 1)m^* - 1\}]^{n^*} \forall j \in a_i \quad \zeta(j) = \rho(i).$$

Define $\mu \in {}^{tn^*}2$ by $\mu(j) = \rho(i)$ iff $j \in [in^*, (i + 1)n^*)$. Then μ is a subsequence of ζ and therefore $\mu \notin \Gamma_{tn^*}$. But also η is a subsequence of μ, and hence $\eta \notin \Gamma_n$, a contradiction.

$\square_{Claim\ 3}$

Claim 4. *Suppose* $\rho \in {}^t 2$ *and* $\bigcap_{i<t} b_i^{\rho(i)} = 0$. *Let* $\zeta \in {}^{m^* t}2$ *be defined such that* $\zeta(m^* i) = \rho(i)$ *and* $\zeta \restriction [m^* i, m^*(i+1))$ *is constant for every* $i < t$. *Then* $\zeta \in \Gamma_{m^* t}$.

Proof of Claim 4: Otherwise, $\bigcap_{i<m^* t} a_i^{\zeta(i)} > 0$. Let D be an ultrafilter containing $\bigcap_{i<m^* t} a_i^{\zeta(i)}$. Let $h : B \to B/D$ be the canonical homomorphism. We have

$$h(\bigcap_{i<t} b_i^{\rho(i)}) = \bigcap_{i<t} \tau[\zeta(m^* i), \ldots, \zeta(m^*(i+1)-1)]^{\rho(i)} = \bigcap_{i<t} \tau[\rho(i), \ldots, \rho(i)]^{\rho(i)} = 1.$$

This is a contradiction. $\square_{Claim\ 4}$

Since we assume that $(a_\alpha : \alpha \in A)$ is not independent, by Claim 2 we can find $k^* < \omega$ minimal such that for some $\rho^* \in {}^{k^*}2$, $\bigcap_{i<k^*} b_i^{\rho^*(i)} = 0$. Note that $\rho^*(i+1) \neq \rho^*(i)$ for every $i < k^* - 1$. Indeed, otherwise let $\zeta \in {}^{m^* k^*}2$ be defined as in Claim 4. So $\zeta \in \Gamma_{m^* k^*}$. By Claim 3 we can find ρ' of shorter length than ρ^* such that $\bigcap_{i<|\rho'|} b_i^{\rho'(i)} = 0$, contradicting the minimal choice of k^*.

Suppose first that $k^* = 1$. We conclude that $(b_\nu : \nu < \gamma)$ either is constantly 1 or 0. The main part of the definition of $R_\gamma(\lambda, \kappa, \omega)$ then gives a sequence of length κ as desired in (b) of Theorem 4.

Secondly suppose $k^* > 1$. If $\bigcap_{i<k^*-2} b_i^{\rho^*(i)} \cap b_{k^*-2} \cap b_{k^*-1}^0 = 0$ and $\bigcap_{i<k^*-2} b_i^{\rho^*(i)} \cap b_{k^*-2}^0 \cap b_{k^*-1} = 0$, then $\bigcap_{i<k^*-2} b_i^{\rho^*(i)} \cap b_{k^*-2} = \bigcap_{i<k^*-2} b_i^{\rho^*(i)} \cap b_{k^*-1}$, and an application of the main part of the definition of $R_\gamma(\lambda, \kappa, \omega)$ gives a sequence as desired in (b).

Otherwise, if $\rho^*(k^*-2) = 1$ and $\rho^*(k^*-1) = 0$, then

$$\bigcap_{i<k^*-2} b_i^{\rho^*(i)} \cap b_{k^*-2} < \bigcap_{i<k^*-2} b_i^{\rho^*(i)} \cap b_{k^*-1}$$

, and applying the definition gives (c). Similarly if $\rho^*(k^*-2) = 0$ and $\rho^*(k^*-1) = 1$. $\square$

Theorem 6. *Assume the following:*

(1) $0^\sharp$ *exists,*

(2) $V \models \lambda$ *is an uncountable cardinal,*

(3) $\kappa, \theta < \lambda$, *and* $L \models \kappa$ *is a regular cardinal.*

Then $L \models R_\omega(\lambda, \kappa, \theta)$.

Proof: Let $c : [\lambda]^{<\omega} \to \theta$, $c \in L$, be arbitrary.

Let Y be the set of all $w \in [\lambda]^{<\omega}$ such that for every $n \leq |w|$ and $u \in [w]^n$ there exists $B \subseteq \lambda$ of order-type κ in L such that $\forall v \in [B]^n \quad c(u) = c(v)$. Clearly $Y \in L$.

Claim 1. *If in V there exists $A \in [\lambda]^\omega$ with $[A]^{<\omega} \subseteq Y$, then $L \models R_\omega(\lambda, \kappa, \theta)$.*

Proof of Claim 1: Let T be the set of all one-to-one sequences $\rho \in {}^{<\omega}\lambda$ with $\mathrm{ran}(\rho) \in Y$, ordered by extension. Then T is a tree and by assumption, T has an ω-branch in V. By absoluteness, T has an ω-branch b in L. Then $\mathrm{ran}(b)$ (or some subset) witnesses $L \models R_\omega(\lambda, \kappa, \theta)$. $\square_{Claim\ 1}$

Let $(i_\nu : \nu < \lambda^+)$ be the increasing enumeration of the club of indiscernibles of L_{λ^+}. Then $(i_\nu : \nu < \lambda)$ is the club of indiscernibles of L_λ. As $c \in L_{\lambda^+}$ there exist ordinals $\xi_0 < \ldots < \xi_{p-1} < \lambda \leq \xi_p < \ldots < \xi_{q-1} < \lambda^+$ and a Skolem term t_c such that

$$L_{\lambda^+} \models c = t_c[i_{\xi_0}, \ldots, i_{\xi_{q-1}}].$$

By indiscernibility and remarkability (see [J, p.345]) it easily follows that if $\alpha^* = \max\{\xi_{p-1}, \theta\} + 1$, then $c \restriction [\{i_\nu : \alpha^* \leq \nu < \lambda\}]^n$ is constant for every $n < \omega$, say with value c_n. Let $n < \omega$ be arbitrary. Let $\delta_0 = i_{\alpha^*+\kappa}$, $\delta_1 = i_{\alpha^*+\kappa+1}, \ldots, \delta_{n-1} = i_{\alpha^*+\kappa+n-1}$.

Claim 2. *For every $\alpha < \delta_0$ there exists a limit δ, $\alpha < \delta < \delta_0$, such that for all $\beta_0 < \ldots < \beta_{n-2} < \delta$ the following hold:*

$(*)_0$ $c\{\delta, \delta_1, \ldots, \delta_{n-1}\} = c\{\delta_0, \ldots, \delta_{n-1}\} (= c_n)$,

$(*)_1$ $c\{\beta_0, \delta, \delta_2, \ldots, \delta_{n-1}\} = c\{\beta_0, \delta_1, \ldots, \delta_{n-1}\}$,

$(*)_2$ $c\{\beta_0, \beta_1, \delta, \delta_3, \ldots, \delta_{n-1}\} = c\{\beta_0, \beta_1, \delta_2, \ldots, \delta_{n-1}\}$,

...

$(*)_{n-1}$ $c\{\beta_0, \ldots, \beta_{n-2}, \delta\} = c\{\beta_0, \ldots, \beta_{n-2}, \delta_{n-1}\}$.

Proof of Claim 2: Let $\alpha < \delta_0$ be arbitrary. Choose $\gamma < \kappa$ such that γ is a limit and $i_{\alpha^*+\gamma} > \alpha$, and let $\delta = i_{\alpha^*+\gamma}$.

Then clearly $(*)_0$ holds.

In order to prove $(*)_1$, let $\beta < \delta$ be arbitrary. There exist ordinals $\nu_0 < \ldots < \nu_{k-1} < \alpha^* + \gamma$ and a Skolem term t_β such that

$$t_\beta^{L_\lambda}[i_{\nu_0}, \ldots, i_{\nu_{k-1}}] = \beta.$$

Moreover there exist ordinals $\mu_0 < \ldots < \mu_{l-1} < \alpha^*$ and a Skolem term t such that

$$L_{\lambda^+} \models t[i_{\mu_0}, \ldots, i_{\mu_{l-1}}] = t_c[i_{\xi_0}, \ldots, i_{\xi_{q-1}}]\{t_\beta[i_{\nu_0}, \ldots, i_{\nu_{k-1}}], \delta_1, \ldots, \delta_{n-1}\}. \qquad (+)$$

Note that all indices of occurring indiscernibles, except for $\delta_1, \ldots, \delta_{n-1}$, either are at least λ or else below $\alpha^* + \gamma$. We conclude that in (+), δ_1 can be replaced by δ. The resulting statement is

$$c\{\beta, \delta_1, \ldots, \delta_{n-1}\} = c\{\beta, \delta, \delta_2, \ldots, \delta_{n-1}\},$$

as desired.

The proof of $(*)_2$—$(*)_{n-1}$ is similar. $\square_{Claim\ 2}$

It is clear that the statement of Claim 2 is absolute. Hence it is also true in L. Using this we shall prove that $[\{i_\nu : \alpha^* \leq \nu < \lambda\}]^{<\omega} \subseteq Y$. By Claim 1, this will suffice. We only have to prove that for every $n < \omega$ there exists $B \subseteq \lambda$ of order-type κ such that $B \in L$ and $\forall v \in [B]^n \quad c(v) = c_n$. Fix $n < \omega$. Working in L, we construct B inductively as $\{\gamma_\nu : \nu < \kappa\}$.

Fix $\delta_0 < \delta_1 < \ldots < \delta_{n-2} < \lambda$ as above. Apply Claim 2 in L with $\alpha = 0$ and obtain $\gamma_0 \in (0, \delta_0)$. Suppose we have gotten $(\gamma_\nu : \nu < \mu)$ for some $\mu < \kappa$. Let $\gamma^* = \sup_{\nu<\mu} \gamma_\nu + 1$. Since $\mathrm{cf}^L(\delta_0) \geq \kappa$ and $(\gamma_\nu : \nu < \mu) \in L$, we have that $\gamma^* < \delta_0$. Apply Claim 2 with $\alpha = \gamma^*$ and get $\gamma_\mu \in (\gamma^*, \delta_0)$.

We claim that $(\gamma_\nu : \nu < \kappa)$ is as desired. Indeed, let $\{\gamma_{\nu_0} < \gamma_{\nu_1} < \ldots < \gamma_{\nu_{n-1}}\}$ be arbitrary. We have

$$\begin{aligned}
& c\{\gamma_{\nu_0}, \ldots, \gamma_{\nu_{n-1}}\} =^{(*)_{n-1}} c\{\gamma_{\nu_0}, \ldots, \gamma_{\nu_{n-2}}, \delta_{n-1}\} \\
& =^{(*)_{n-2}} c\{\gamma_{\nu_0}, \ldots, \gamma_{\nu_{n-3}}, \delta_{n-2}, \delta_{n-1}\} \\
& = \ldots \\
& =^{(*)_1} c\{\gamma_{\nu_0}, \delta_1, \ldots, \delta_{n-1}\} \\
& =^{(*)_0} c_n.
\end{aligned}$$

$\square_{Theorem\ 6}$

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