SUCCESSOR OF SINGULARS: COMBINATORICS AND NOT COLLAPSING CARDINALS $\leq \kappa$ IN $(< \kappa)$-SUPPORT ITERATIONS SH667

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Abstract. On the one hand we deal with $(< \kappa)$-supported iterated forcing notions which are $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$-complete, have in mind problems on Whitehead groups, uniformizations and the general problem. We deal mainly with the caes of a successor of the singular cardinal. This continues [Sh 587]. On the other hand we deal with complimentary ZFC combinatorial results.

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Annotated Content

§1 GCH implies for successor of singular no stationary S has unformization

We show that a major improvement in [Sh 587] over [Sh 186] for inaccessible (every ladder on S has uniformization rather than some ladder on S) cannot be done for successor of singulars. This is continued in §4.
1.1 Fact: Assume

(a) λ is strong limit singular with $2^\lambda = \lambda^+$, let $\mathrm{cf}(\lambda) = \sigma$
(b) $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \sigma\}$ is stationary.

Then we can find $\langle < \gamma^\delta_i : i < \sigma >: \delta \in S\rangle$ such that

(α) γ^δ_i is increasing (with i) with limit δ
(β) if $\mu < \lambda$ and $f : \lambda^+ \to \mu$ then the following set is stationary: $\{\delta \in S : f(\gamma^\delta_{2i}) = f(\gamma^\delta_{2i+1})$ for every $i < \sigma\}$.
Moreover
$(\beta)^+$ if $f_i : \lambda^+ \to \mu_i, \mu_i < \lambda$ for $i < \sigma$ then the following set is stationary: $\{\delta \in S : f_i(\gamma^\delta_{2i}) = f_i(\gamma^\delta_{2i+1})$ for every $i < \sigma\}$.

Proof. This will prove 1.2, too. We first concentrate on $(\alpha) + (\beta)$ only.
Let $\lambda = \sum\limits_{i<\sigma} \lambda_i, \lambda_i$ a cardinal increasing continuous with i, $\lambda_{i+1} > 2^{\lambda_i}, \lambda_0 > 2^\sigma$. For $\alpha < \lambda^+$, let $\alpha = \bigcup\limits_{i<\sigma} a_{\alpha,i}$ such that $|a_{\alpha,i}| \le \lambda_i$. Without loss of generality $\delta \in S \Rightarrow \delta$ divisible by λ^ω (ordinal exponentiation). For $\delta \in S$ let $\langle \beta^\delta_i : i < \sigma\rangle$ be increasing continuous with limit δ, β^δ_i divisible by λ and > 0. For $\delta \in S$ let $\langle b^\delta_i : i < \sigma\rangle$ be such that: $b^\delta_i \subseteq \beta^\delta_i, |b^\delta_i| \le \lambda_i, b^\delta_i$ is increasing continuous with i and $\delta = \bigcup\limits_{i<\sigma} b^\delta_i$ (e.g. we can let $b^\delta_i = \bigcup\limits_{j_1,j_2<i} a_{\beta^\delta_{j_1},j_2} \cup \lambda_i)$. We further demand $\lambda_i \subseteq b^\delta_i \cap \lambda$. Let $\langle f^*_\alpha : \alpha < \lambda^+\rangle$ list the two-place functions with domain an ordinal $< \lambda^+$ and range $\subseteq \lambda^+$. Let $S = \bigcup\limits_{\mu<\lambda} S_\mu$, with each S_μ stationary and $\langle S_\mu : \mu < \lambda\rangle$ pairwise disjoint. We now fix $\mu < \lambda$ and will choose $\bar\gamma^\delta = \langle \gamma^\delta_i : i < \sigma\rangle$ for $\delta \in S_\mu$ such that clause (α) holds and clause (β) holds (that is for every $f : \lambda^+ \to \mu$ for stationary many $\delta \in S_\mu$ the conclusion of clause (β) holds), this clearly suffices.
Now for $\delta \in S_\mu$ and $i < j < \sigma$ we can choose $\zeta^\delta_{i,j,\varepsilon}$ (for $\varepsilon < \lambda_j$) (really here we use just $\varepsilon = 0, 1$) such that:

(A) $\langle \zeta^\delta_{i,j,\varepsilon} : \varepsilon < \lambda_j \rangle$ is a strictly increasing sequence of ordinals

(B) $\beta^\delta_i < \zeta^\delta_{i,j,\varepsilon} < \beta^\delta_{i+1}$, (can even demand $\zeta^\delta_{i,j,\varepsilon} < \beta^\delta_i + \lambda$)

(C) $\zeta^\delta_{i,j,\varepsilon} \notin \{\zeta^\delta_{i_1,j_1,\varepsilon_1} : j_1 < j, \varepsilon_1 < \lambda_{j_1}$ (and $i_1 < \sigma$, really only $i_1 = i$ matters)$\}$

(D) for every $\alpha_1, \alpha_2 \in b^\delta_j$, the sequence $\langle \mathrm{Min}\{\lambda_j, f^*_{\alpha_1}(\alpha_2, \zeta^\delta_{i,j,\varepsilon})\} : \varepsilon < \lambda_j \rangle$ is constant i.e.: one of the following occurs:

  (α) $\varepsilon < \lambda_j \Rightarrow (\alpha_2, \zeta^\delta_{i,j,\varepsilon}) \notin \mathrm{Dom}(f^*_{\alpha_1})$

  (β) $\varepsilon < \lambda_j \Rightarrow f^*_{\alpha_1}(\alpha_2, \zeta^\delta_{i,j,\varepsilon}) = f^*_{\alpha_1}(\alpha_2, \zeta^\delta_{i,j,0})$, well defined

  (γ) $\varepsilon < \lambda_j \Rightarrow f^*_{\alpha_1}(\alpha_2, \zeta^\delta_{i,j,\varepsilon}) \geq \lambda_j$, well defined.

For each $i < j < \sigma$ we use "λ is strong limit $> \lambda_j \geq \sum\limits_{j_1 < j} \lambda_{j_1} + \sigma$".
Let $G = \{g : g$ a function from σ to σ such that $(\forall i < \sigma)(i < g(i)\}$.
For each function $g \in G$ we try $\bar\gamma^{g,\delta} = \langle \zeta^\delta_{i,g(i),0}, \zeta^\delta_{i,g(i),1} : i < \sigma \rangle$ i.e. $\langle \zeta^{g,\delta}_{2i}, \zeta^{g,\delta}_{2i+1} \rangle = \langle \gamma^\delta_{i,g(i),0}, \gamma^\delta_{i,g(i),1} \rangle$.
Now we ask for each $g \in G$:

<u>Question</u>$^\mu_g$: Does $\langle \bar\gamma^{g,\delta} : \delta \in S_\mu \rangle$ satisfy
$(\forall f \in {}^{\lambda^+}\mu)(\exists^{\mathrm{stat}} \delta \in S_\mu)(\bigwedge\limits_{i<\sigma} f(\gamma^{g,\delta}_{2i}) = f(\gamma^{g,\delta}_{2i+1}))$?.

If for some $g \in G$ the answer is yes, we are done. Assume not, so for each $g \in G$ we can find $f_g : \lambda^+ \to \mu$ and a club E_g of λ^+ such that:

$$\delta \in S_\mu \cap E_g \Rightarrow (\exists i < \sigma)(f_g(\gamma^{g,\delta}_{2i}) \neq f_g(\gamma^{g,\delta}_{2i+1}))$$

which means

$$\delta \in S_\mu \cap E_g \Rightarrow (\exists i < \sigma)[f_g(\zeta^\delta_{i,g(i),0}) \neq f_g(\zeta^\delta_{i,g(i),1})].$$

Let $G = \{g_\varepsilon : \varepsilon < 2^\sigma\}$, so we can find a 2-place function f^* from λ^+ to μ satisfying $f^*(\varepsilon, \alpha) = f_{g_\varepsilon}(\alpha)$ when $\varepsilon < 2^\sigma, \alpha < \lambda^+$. Hence for each $\alpha < \lambda^+$ there is $\gamma[\alpha] < \lambda^+$ such that $f^* \restriction \alpha = f^*_{\gamma[\alpha]}$.

Let $E^* = \bigcap\limits_{\varepsilon < 2^\sigma} E_{g_\varepsilon} \cap \{\delta < \lambda^+ :$ for every $\alpha < \delta$ we have $\gamma[\alpha] < \delta\}$. Clearly it is a club of λ^+, hence we can find $\delta \in S_\mu \cap E^*$. Now $\beta^\delta_{i+1} < \delta$ hence $\gamma[\beta^\delta_{i+1}] < \delta$ (as $\delta \in E^*$) but $\delta = \bigcup\limits_{i<\sigma} b^\delta_i$ hence for some $j < \sigma, \gamma[\beta^\delta_{i+1}] \in b^\delta_j$; as b^δ_j increases with

j we can define a function $h : \sigma \to \sigma$ by $h(i) =$ Min$\{j : j > i+1$ and $\mu < \lambda_j$ and $\gamma[\beta^\delta_{i+1}] \in b^\delta_j\}$. So $h \in G$ hence for some $\varepsilon(*) < 2^\sigma$ we have $h = g_{\varepsilon(*)}$. Now looking at the choice of $\zeta^\delta_{i,h(i),0}, \zeta^\delta_{i,h(i),1}$ we know (remember $2^\sigma < \lambda_0 \subseteq b^\delta_j$ and $\mu < \lambda_{h(i)}$)

$$(\forall \varepsilon < 2^\sigma)(\forall \alpha \in b^\delta_{h(i)})[\mathrm{Rang}(f^*_\alpha) \subseteq \mu \ \ \& \ \ \mathrm{Dom}(f^*_\alpha) \supseteq \beta^\delta_{i+1} \to f^*_\alpha(\varepsilon, \zeta^\delta_{i,h(i),0}) = f^*_\alpha(\varepsilon, \zeta^\delta_{i,h(i),1})].$$

In particular this holds for $\varepsilon = \varepsilon(*), \alpha = \gamma[\beta^\delta_{i+1}]$, so we get

$$f^*_{\gamma[\beta^\delta_{i+1}]}(\varepsilon(*), \zeta^\delta_{i,h(i),0}) = f^*_{\gamma[\beta^\delta_{i+1}]}(\varepsilon(*), \zeta^\delta_{i,h(i),1}).$$

By the choice of f^* and of $\gamma[\beta^\delta_{i+1}]$ this means

$$f_{g_{\varepsilon(*)}}(\zeta^\delta_{i,h(i),0}) = f_{g_{\varepsilon(*)}}(\zeta^\delta_{i,h(i),1})$$

but $h = g_{\varepsilon(*)}$ and the above equality means $f^*_{g_{\varepsilon(*)}}(\gamma^{g_{\varepsilon(*)},\delta}_{2i}) = f^*_{g_{\varepsilon(*)}}(\gamma^{g_{\varepsilon(*)},\delta}_{2i+1})$, and this holds for every $i < \sigma$, and $\delta \in E^* \Rightarrow \delta \in E_{g_{\varepsilon(*)}}$ so we get a contradiction to the choice of $(f_{g_{\varepsilon(*)}}, E_{\varepsilon(*)})$.
So we have finished proving $(\alpha) + (\beta)$.

How do we get $(\beta)^+$ of 1.1, too?

The first difference is in phrasing the question, now it is, for $g \in G$:

<u>Question</u>$^\mu_g$: Does $\langle \bar\gamma^{g,\delta} : \delta \in S_\mu \rangle$ satisfy:

$$\Big((\forall f_0 \in {}^{\lambda^+}\mu_0)(\forall f_1 \in {}^{\lambda^+}\mu_1) \ldots (\forall f_i \in {}^{\lambda^+}\mu_i) \ldots\Big)_{i<\sigma} (\exists^{\mathrm{stat}} \delta \in S_\mu)(\bigwedge_{i<\sigma} f_i(\gamma^{g,\delta}_{2i}) = f_i(\gamma^{g,\delta}_{2i+1})).$$

If for some g the answer is yes, we are done, so assume not so we have $f_{g,i} \in {}^{\lambda^+}(\mu_i)$ for $g \in G, i < \sigma$ and club E_g of λ^+ such that

$$\delta \in S_\mu \cap E_g \Rightarrow (\exists i < \sigma)(f_{g,i}(\gamma^{g,\delta}_{2i}) \neq f_{g,i}(\gamma^{g,\delta}_{2i+1})).$$

A second difference is the choice of f^* as $f^*(\sigma\varepsilon + i, \alpha) = f_{g_\varepsilon,i}(\alpha)$ for $\varepsilon < 2^\sigma$, $i < \sigma, \alpha < \lambda^+$.
Lastly, the equations later change slightly. $\square_{1.1}$

1.2 Fact: 1) Under the assumptions (a) + (b) of 1.1 letting $\bar{\lambda} = \langle \lambda_i : i < \sigma \rangle$ be increasingly continuous with limit λ such that $2^\sigma < \lambda_0, 2^{\lambda_i} < \lambda_{i+1}$ we have $(*)_1 + (*)_2$ where

$(*)_1$ we can find $\langle < \gamma^\delta_\zeta : \zeta < \lambda >: \delta \in S \rangle$ such that

- (α) γ^δ_ζ is increasing in ζ with limit δ
- $(\beta)^+$ if $f_i : \lambda^+ \to \lambda_{i+1}$, for $i < \sigma$, then the following set is stationary $\{\delta \in S : f_i(\gamma^\delta_\zeta) = f_i(\gamma^\delta_\xi)$ when $\zeta, \xi \in [\lambda_i, \lambda_{i+1})$ for every $i < \sigma\}$

$(*)_2$ moreover if $F_i : [\lambda^+]^{<\lambda} \to [\lambda^+]^{\lambda^+}$ for $i < \sigma$ (or just $F_i : [\lambda^+]^{<\lambda} \to [\lambda^+]^{\lambda}$) and $\sup(w) < \min(F_i(w))$ for $w \in [\lambda^+]^{<\lambda}$, for each $i < \sigma$, then in addition we can demand

- (i) $\{\gamma^\delta_\zeta : \zeta \in [\lambda_i, \lambda_{i+1}]\} \subseteq F_i(\{\gamma^\delta_\zeta : \zeta < \lambda_i\})$,
- (ii) $|\{\langle \gamma^\delta_\zeta : \zeta < \zeta^* \rangle : \gamma^\delta_{\zeta^*} = \gamma\}| \leq \lambda$ for each $\gamma < \lambda^+$ and $\zeta^* < \sigma$

.

2) Assume $\lambda, \langle \lambda_i : i < \sigma \rangle$ are as in part (1) and $\langle C_\delta : \delta \in S \rangle$ is given, it guess clubs (for λ^+, which mean that for every club E of λ^+ the set $\{\delta \in S : C_\delta \subseteq E\}$ is a stationary subset of λ^+) and $C_\delta = \{\alpha[\delta, i] : i < \sigma\}, \alpha[\delta, i]$ divisible by λ^ω increasing in i with limit $\delta, \langle \mathrm{cf}(\alpha[\delta, i+1]) : i < \sigma \rangle$ is increasing with limit λ and let $\beta(\delta, i) = \sum\limits_{j<i} \lambda_j \times \mathrm{cf}(\alpha[\delta, j])$. Then

$(*)$ we can find $\langle < \gamma^\delta_\zeta : \zeta < \lambda >: \delta \in S \rangle$ such that

- (α) $\langle \gamma^\delta_\zeta : \zeta < \lambda \rangle$ is increasing with limit δ, (for $\delta \in S$)
- (β) $\sup\{\gamma^\delta_\zeta : \gamma^\delta_\zeta < \beta[\delta, j+1]\} = \alpha[\delta, j]$
- (γ) for every $f_i \in {}^{(\lambda^+)}(\mu_i)$ for $i < \sigma$ where $\mu_i < \lambda$ and club E of λ^+, for stationarily many $\delta \in S$ we have $\{\gamma^\delta_i : i < \lambda\} \subseteq E$ and $f_i(\gamma^\delta_\zeta) = f_i(\gamma^\delta_\varepsilon)$, when $\zeta, \varepsilon \in [\beta[\delta, i] + \lambda_i \xi, \beta[\delta, i] + \lambda_i \xi + \lambda_i)$ and $\xi < \mathrm{cf}(\alpha[\delta, i])$).

Proof. 1) The same proof as in 1.1 for $(*)_1$, but see a proof after the proof of 4.2.
2) Should be clear, too. $\square_{1.2}$

§2 Case C: Forcing for successor of singular

We continue [Sh 587].

2.1 Hypothesis. 1) λ strong limit singular $\sigma = \mathrm{cf}(\lambda) < \lambda, \kappa = \lambda^+, \mu^* \geq \kappa, 2^\lambda = \lambda^+$.

2.2 Definition. 1) Let $\mathfrak{C}_{<\kappa}(\mu^*)$ be the family of $\hat{\mathscr{E}}_0 \subseteq \{\bar{a} : \bar{a} = \langle a_i : i \leq \alpha\rangle$ where $\alpha < \kappa, a_i \in [\mu^*]^{<\kappa}$ increasing continuous, and $a_i \cap \kappa \in \kappa\}$ such that: for every $\theta = \mathrm{cf}(\theta) < \lambda, \chi$ large enough and $x \in \mathscr{H}(\chi)$ we can find $\langle N_i : i \leq \theta\rangle$ obeying $\bar{a} \in \hat{\mathscr{E}}_0$ (with error some n see [Sh 587, B.5.1(1)]) and such that $x \in N_0$; this repeats [Sh 587, B.5.1(2)]; formally we should say that $\bar{N}$ obeys $\bar{a}$ for μ^*.
2) $\mathfrak{C}^1_{<\kappa}(\mu^*)$ is the family of $\hat{\mathscr{E}}_1 \subseteq \{\bar{a} : \bar{a} = \langle a_i : i \leq \sigma\rangle, a_i$ increasing continuous, $i < \sigma \Rightarrow |a_i| < \lambda$ and $\lambda + 1 \subseteq \bigcup\limits_{i<\sigma} a_i\}$.

2.3 Definition. 1) We say $\bar{M} = \langle M_i : i \leq \sigma\rangle$ is ruled by $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$ if, for some $\chi > \mu^*$:

- (a) $\hat{\mathscr{E}}_0 \in \mathfrak{C}_{<\kappa}(\mu^*), \hat{\mathscr{E}}_1 \in \mathfrak{C}^1_{<\kappa}(\mu^*)$
- (b) for[1] some $\langle \bar{M}^i : -1 \leq i < \sigma\rangle$ and $\langle \bar{N}^i : -1 \leq i < \sigma\rangle$ we have:
  - (α) $M_i \prec (\mathscr{H}(\chi), \in, <^*_\chi)$
  - (β) $\bar{M}$ obeys some $\bar{a} \in \hat{\mathscr{E}}_1$ for some finite error (so for some n, for every $i, a_i \subseteq M_i \cap \mu^* \subseteq a_{i+n}$) and $\bar{M} \restriction (i+1) \in M_{i+1}$ and $j < i \Rightarrow M_j \prec M_j$ and M_i is increasing continuous
  - (γ) $[M_{i+1}]^{2^{\|M_i\|}} \subseteq M_{i+1}$ for i a limit ordinal $< \sigma$
  - (δ) $\bar{M}^i = \langle M^i_\alpha : \alpha \leq \delta_i\rangle, \bar{N}^i = \langle N^i_\alpha : \alpha \leq \delta_i\rangle$ and $M^i_\alpha \prec N^i_\alpha \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ and $\lambda + 1 \subseteq N^i_\alpha$ and $\|M^i_\alpha\| = \|M^i_\alpha\|^{\|M_i\|}$ for $\alpha < \delta_i$ non limit, $[M^i_\beta]^{\|M_i\|} \subseteq M^i_{\beta+1}, \beta < \delta_i$
  - (ε) $\langle N^i_\alpha : \alpha \leq \delta_i\rangle = \bar{N}^i$ obeys some $\bar{b}_i \in \hat{\mathscr{E}}_0$ for some finite error and $\bar{M}^i, \bar{N}^i$ are increasing continuous
  - (ζ) $M_{i+1} = M^i_{\delta_i} \subseteq N^i_{\delta_i}$ and $\langle(\bar{M}^j, \bar{N}^j) : j < i\rangle \in M^i_0$
  - (η) $\delta_i \subseteq M_{i+1}$ (hence $\delta_i < \lambda$) and $\lambda \subseteq N^i_\alpha$,
  - (θ) $\mathrm{cf}(\delta_i) > 2^{\|M_i\|}$ for i limit,
  - (ι) $\bar{N}^i \restriction (\alpha+1), \bar{M}^i \restriction (\alpha+1) \in M^i_{\alpha+1}$ for $\alpha < \delta_i, i < \sigma$ hence $N^i_\beta = \mathrm{Sk}_{(\mathscr{H}(\chi),\in,<^*_\chi)}(M^i_\beta \cup \lambda)$ when $i < \omega\sigma$ and $\beta \leq \delta_i$ is a limit ordinal

[1] we may later ignore the $i = -1$ in our notation

(κ) $N^i_{\delta_i} \prec N^j_0$ for $i < j$

(λ) $M_i \prec M^i_0, M_i \in M^i_0$.

2) We say above that $(\langle \bar M^i : i < \sigma\rangle, \langle \bar N^i : i < \sigma\rangle)$ is an $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$-approximation to $\bar M$.
3) Let $\mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ be the family of $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$ such that:

(a) $\hat{\mathscr{E}}_0 \in \mathfrak{C}_{<\kappa}(\mu^*)$ and $\hat{\mathscr{E}}_1 \in \mathfrak{C}^1_{<\kappa}(\mu^*)$
(b) for χ large enough and $x \in \mathscr{H}(\chi)$ we can find $\bar M$ which is ruled by $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$ and $x \in M_0$
(c) $\hat{\mathscr{E}}_0$ is closed (see below).

4) $\hat{\mathscr{E}}_0$ is closed if $\langle a_i : i \le \alpha\rangle \in \hat{\mathscr{E}}_0, \gamma \le \beta \le \alpha$ implies $\langle a_i : i \in [\beta, \gamma]\rangle \in \hat{\mathscr{E}}_0$.

Remark. 1) In Definition 2.3(1), letting $\bar N = \bar N^0 \hat{\ } \bar N^1 \ldots$ i.e. $\bar N = \langle N_i : i < \lambda\rangle, N_\varepsilon =: N^i_\alpha$ if $\varepsilon = \sum\limits_{j<i} \delta_j + \alpha$; so $\ell g(\bar N) = \lambda$ and $\bar N \restriction (i_0 + 1) \in N_{i_0+1}$ so $\bar N$ is $\prec$-increasingly continuous, and $\gamma < \lambda \Rightarrow \bar N \restriction \gamma \in N_{\gamma+1}$.

2.4 Claim. *1) Assume $\hat{\mathscr{E}}_0 \in \mathfrak{C}_{<\kappa}(\mu^*)$ and $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_i : i < \gamma\rangle$ is a $(<\kappa)$-support iteration such that $\Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ is strongly $\hat{\mathscr{E}}_0$-complete" for each $i < \gamma$, see [Sh 587, B.5.3(3)].*
Then $\mathbb{P}_\gamma$ is strongly $\hat{\mathscr{E}}_0$-complete (hence $\mathbb{P}_\gamma/\mathbb{P}_\beta$).
2) If $\mathbb{Q}$ is $\hat{\mathscr{E}}_0$-complete, then $\mathbf{V}^{\mathbb{Q}} \models \hat{\mathscr{E}}_0$ non-trivial.

Proof. By [Sh 587, B.5.6] (here the choice "for any regular cardinal $\theta < \kappa$" rather than "for any cardinal $\theta < \kappa$" in [Sh 587, B.5.1(2)] is important). $\square_{2.4}$

2.5 Definition. Let $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1) \in \mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ and let $\mathbb{Q}$ be a forcing notion.
1) For a sequence $\bar M = \langle M_i : i \le \sigma\rangle$ ruled by $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$ with an $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$-approximation $(\langle \bar M^i : i < \sigma\rangle, \langle \bar N^i : i < \sigma\rangle)$ and a condition $r \in \mathbb{Q}$ we define a game $\mathfrak{G}^{\spadesuit}_{\bar M, \langle \bar M^i : i<\sigma\rangle, \langle \bar N^i : i<\sigma\rangle}(\mathbb{Q}, r)$ between two players COM and INC.

The play lasts σ moves during which the players construct a sequence $\langle i_0, p, \langle p_i, \bar q_i : i_0 - 1 \le i < \sigma\rangle\rangle$ such that $i_0 < \sigma$ is non-limit, $p \in M_{i_0} \cap \mathbb{Q}, p_i \in M_{i+1} \cap \mathbb{Q}, \bar q_i = \langle q_{i,\varepsilon} : \varepsilon < \delta_i\rangle \subseteq \mathbb{Q}$ (where $\delta_i + 1 = \ell g(\bar N^i)$).

The player INC first decides what is $i_0 < \delta$ and then it chooses a condition $p \in \mathbb{Q} \cap M_{i_0}$ stronger than r. Next, at the stage $i \in [i_0 - 1, \delta)$ of the game, COM chooses $p_i \in \hat{\mathbb{Q}} \cap M_{i+1}$ such that:

(*i*) $p \leq_{\mathbb{Q}} p_i$
(*ii*) $(\forall j < i)(\forall \varepsilon < \delta_j)(q_{j,\varepsilon \leq_{\mathbb{Q}}} p_i)$,
(*iii*) if i is a non-limit ordinal, <u>then</u> $p_i \in \hat{\mathbb{Q}}$ is minimal satisfying (i) + (ii)
(*iv*) if i is a limit ordinal, <u>then</u> $p_i \in \mathbb{Q}$.

Now the player INC answers choosing an increasing sequence $\bar{q}_i = \langle q_{i,\varepsilon} : \varepsilon < \delta_i \rangle$ such that $p_i \leq_{\mathbb{Q}} q_{i,0}$ and $\bar{q}_i$ is $(\bar{N}^i \restriction [\alpha, \delta_i], \mathbb{Q})^*$-generic for some $\alpha < \delta_i$ (see [Sh 587, B.5.3.1]) and $\beta < \delta_i \Rightarrow \bar{q}_i \restriction (\beta + 1) \in M_{i,\beta+1}$.
The player COM wins if it has always legal moves and the sequence $\langle p_i : i < \omega\sigma \rangle$ has an upper bound in $\mathbb{Q}$.
2) We say that the forcing notion $\mathbb{Q}$ is complete for $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$ or $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$-complete if

(*a*) $\mathbb{Q}$ is strongly complete for $\hat{\mathscr{E}}_0$ and
(*b*) for a large enough regular χ, for some $x \in \mathscr{H}(\chi)$, for every sequence $\bar{M}$ ruled by $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$ with an $\hat{\mathscr{E}}_0$-approximation $(\langle \bar{M}^i : i < \sigma \rangle, \langle \bar{N}^i : i < \sigma \rangle)$ and such that $x \in M_0$ and for any condition $r \in \mathbb{Q} \cap M_0$, the player INC does not have a winning strategy in the game $\mathfrak{G}^{\spadesuit}_{\bar{M}, \langle \bar{M}^i : i < \sigma \rangle, \langle \bar{N}^i : i < \sigma \rangle}(\mathbb{Q}, r)$.

2.6 Proposition. Assume

(*a*) $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1) \in \mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$,
(*b*) $\mathbb{Q}$ is a forcing notion for $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$.

Then $\Vdash_{\mathbb{Q}}$ "$(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1) \in \mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$".

Proof. Straightforward (and not used in this paper).

2.7 Proposition. Assume that $\hat{\mathscr{E}} \in \mathfrak{C}_{<\kappa}(\mu^*)$ is closed and $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ is a $(< \kappa)$-support iteration of forcing notions which are strongly complete for $\hat{\mathscr{E}}$. Let $\mathscr{T} = (T, <^+ M, \mathrm{rk})$ be a standard $(w, \alpha_0)^\gamma$-tree (see [Sh 587, A.3.3]), $\|T\| < \lambda, w \subseteq \gamma, \alpha_0$ an ordinal, and let $\bar{p} = \langle p_t : t \in T \rangle \in FTr'(\bar{\mathbb{Q}})$, see [Sh 587, A.3.2]. Suppose that $\mathscr{I}$ is an open dense subset of $\mathbb{P}_\gamma$. <u>Then</u> there is $\bar{q} = \langle q_t : t \in T \rangle \in FTr'(\bar{\mathbb{Q}})$ such that $\bar{p} \leq \bar{q}$ and for each $t \in T$

(a) $q_t \in \{q \restriction \mathrm{rk}(t) : q \in \mathscr{I}\}$, and
(b) for each $\alpha \in \mathrm{Dom}(q_t)$, one of the following occurs:
  (i) $q_t(\alpha) = p_t(\alpha)$
  (ii) $\Vdash_{\mathbb{P}_\alpha}$ "$q_t(\alpha) \in \underset{\sim}{\mathbb{Q}}_\alpha$" (not just in the completion $\hat{\underset{\sim}{\mathbb{Q}}}_\alpha$)
  (iii) $\Vdash_{\mathbb{P}_\alpha}$ "there is $r \in \underset{\sim}{\mathbb{Q}}_\alpha$ such that $\hat{\underset{\sim}{\mathbb{Q}}}_\alpha \models p_t(\alpha) \leq r \leq q_t(\alpha)$" (not really needed).

Proof. Just like the proof of [Sh 587, B.7.1].

Our next proposition corresponds to [Sh 587, B.7.2] which corresponds to [Sh 587, A.3.6]. The difference with [Sh 587, B.7.2] is the appearance of the $\bar{M}, \bar{M}^i$.

2.8 Proposition. Assume that $\hat{\mathscr{E}} \in \mathfrak{C}_{<\kappa}(\mu^*)$ is closed and $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ is a $(< \kappa)$-support iteration and $x = \langle \underset{\sim}{x}_\alpha : \alpha < \gamma \rangle$ is such that

$$\Vdash_{\mathbb{P}_\alpha} \text{"}\underset{\sim}{\mathbb{Q}}_\alpha \text{ is strongly complete for } \hat{\mathscr{E}} \text{ with witness } \underset{\sim}{x}_\alpha\text{"}$$

(for $\alpha < \gamma$). Further suppose that

(α) $(\bar{N}, \bar{a})$ is an $\hat{\mathscr{E}}$-complementary pair (see [Sh 587, B.5.1]), $\bar{N} = \langle N_i : i \leq \delta \rangle$ and $x, \hat{\mathscr{E}}, \bar{\mathbb{Q}} \in N_0$,
(β) $\mathscr{T} = (T, <^+, \mathrm{rk}) \in N_0$ is a standard $(w, \alpha_0)^\gamma$-tree, $w \subseteq \gamma \cap N_0, \|w\| < \mathrm{cf}(\delta), \alpha_0$ is an ordinal, $\alpha_1 = \alpha_0 + 1$ and $0 \in w$
(γ) $\bar{p} = \langle p_t : t \in T \rangle \in FTr'(\bar{\mathbb{Q}}) \cap N_0, w \in N_0$, (of course $\alpha_0 \in N_0$, on FTr' see [Sh 587, A.3.2]),
(δ) $\bar{M} = \langle M_i : i \leq \delta \rangle, M_i \prec (\mathscr{H}(\chi), \in, <^*_\chi), M_i$ is increasing continuous, $[M_i]^{\|w\| + |\mathscr{T}|} \subseteq M_{i+1}$ and the pair $(\bar{M} \restriction (i+1), \bar{N} \restriction (i+1))$ belongs to $M_{i+1}, M_i \prec N_i$ and $w \cup \{x, \hat{\mathscr{E}}_0, \mathbb{Q}\} \in M_0$
(ε) for $i \leq \delta$, $\mathscr{T}_i = (T_i, <_i, \mathrm{rk}_i)$ is such that T_i consists of all sequences $t = \langle t_\zeta : \zeta \in \mathrm{dom}(t) \rangle$ such that $\mathrm{dom}(t)$ is an initial segment of w, and
  (i) each t_ζ is a sequence of length α_1
  (ii) $\langle t_\zeta \restriction \alpha_0 : \zeta \in \mathrm{dom}(t) \rangle \in T$
  (iii) for each $\zeta \in \mathrm{dom}(t)$, either $t_\zeta(\alpha_0) = *$ or $t_\zeta(\alpha_0) \in M_i$ is a $\mathbb{P}_\zeta$-name for an element of $\underset{\sim}{\mathbb{Q}}_\zeta$ and

  if $t_\zeta(\alpha) \neq *$ for some $\alpha < \alpha_0$, then $t_\zeta(\alpha_0) \neq *$,

(iv) $\text{rk}_i(t) = \min(w \cup \{\zeta\} \setminus \text{dom}(t))$ and $<_i$ is the extension relation.

Then

(a) each $\mathscr{T}_i$ is a standard $(w, \alpha_1)^\gamma$-tree, $\|T_i\| \le \|T\| \cdot \|M_i\|^{\|w\|}$ and if $i < \delta$ then $T_i \in N_{i+1}$

(b) $\mathscr{T}$ is the projection of each $\mathscr{T}_i$ onto (w, α_0) and $\mathscr{T}_i$ is increasing with i

(c) there is $\bar{q} = \langle q_t : t \in T_\delta \rangle \in FTr'(\bar{\mathbb{Q}})$ such that

(i) $\bar{p} \le_{\text{proj}_T^{T_\delta}} \bar{q}$

(ii) if $t \in T_\delta \setminus \{<>\}$ then the condition $q_t \in \mathbb{P}'_{\text{rk}_\delta(t)}$ is an upper bound of an $(\bar{N} \restriction [i_0, \delta], \mathbb{P}_{\text{rk}_\delta(t)})^*$-generic sequence (where $i_0 < \delta$ is such that $t \in T_{i_0}$) and for every $\beta \in \text{dom}(q_t) = N_\delta \cap \text{rk}(t), q_t(\beta)$ is a name for the least upper bound in $\hat{\underset{\sim}{\mathbb{Q}}}_\beta$ of an $(\bar{N}[\underset{\sim}{G}_\beta] \restriction [\xi, \delta), \underset{\sim}{\mathbb{Q}}_\beta)^*$-generic sequence (for some $\xi < \delta$).
[Note that by [Sh 587, B.5.5], the first part of the demand on q_t implies that if $i_0 \le \xi$ then $q_t \restriction \beta$ forces that $(\bar{N}[\underset{\sim}{G}_\beta] \restriction [\xi, \delta], \bar{a} \restriction [\xi, \delta])$ is an $\hat{\mathscr{S}}$-complementary pair.]

(iii) if $t \in T_\delta, t' = \text{proj}_T^{T_\delta}(t) \in T, \zeta \in \text{dom}(t)$ and $t_\zeta(\alpha_0) \ne *$, then $q_t \restriction \zeta \Vdash_{\mathbb{P}_\zeta}$ "$p_{t'}(\zeta) \le_{\hat{\underset{\sim}{\mathbb{Q}}}_\zeta} t_\zeta(\alpha_0) \Rightarrow t_\zeta(\alpha_0) \le_{\hat{\underset{\sim}{\mathbb{Q}}}_\zeta} q_t(\zeta)$",

(iv) $q_{<>} = p_{<>}$.

Proof. Clauses (a) and (b) should be clear. Clause (c) is proved as in [Sh 587, B.7.2]. $\square_{2.8}$

Remark. In 2.9 below is proved as in the inaccessible case i.e. the proofs of ([Sh 587, B.7.3]) with $\bar{M}, \langle \bar{N}^i : i < \sigma \rangle$ as in Definition 2.5. We define the trees point: in stage i using trees $\mathscr{T}_i$ with set of levels $w_i = M_i \cap \gamma$ and looking at all possible moves of COM, i.e. $p_i \in M_{i+1} \cap \mathbb{P}_\gamma$, so constructing this tree of conditions in δ_i stages, in stage $\varepsilon < \delta_i$, has $|N^i_\varepsilon \cap M_{i+1}|^{2^{\|M_i\|}}$ nodes.

Now

$$p \in \mathbb{P}_\gamma \cap M_{i+1} \not\Rightarrow \text{ Dom}(p) \subseteq M_{i+1} \text{ but}$$
$$p \in \mathbb{P}_\gamma \cap M_{i+1} \Rightarrow \text{ Dom}(p) \subseteq M_\sigma = \bigcup_{i < \omega\sigma} N^i_{\delta_i}$$
$$p \in \mathbb{P}_\gamma \cap N^i_\varepsilon \Rightarrow \text{ Dom}(p) \subseteq N^i_\varepsilon.$$

So in limit cases $i < \sigma$: the existence of limit is by the clause (μ) of Definition 2.3. In the end we use the winning of the play and then need to find a branch in the tree of conditions of level σ: like Case A using $\hat{\mathscr{E}}_0$. $\square_{2.9}$

2.9 Theorem. *Suppose that* $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1) \in \mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$ *(so* $\hat{\mathscr{E}}_0 \in \mathfrak{C}_{<\kappa}(\mu^*)$*) and* $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ *is a* $(<\kappa)$*-support iteration such that for each* $\alpha < \kappa$

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\underset{\sim}{\mathbb{Q}}_\alpha \textit{ is complete for } (\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)\text{”}.$$

Then

(a) $\Vdash_{\mathbb{P}_\gamma} (\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1) \in \mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$, *moreover*

(b) $\mathbb{P}_\gamma$ *is complete for* $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$.

Proof. We need only part (a) of the conclusion, so we concentrate on it. Let χ be a regular large enough regular cardinal, $\underset{\sim}{x}$ be a name for an element of $\mathscr{H}(\chi)$ and $p \in \mathbb{P}_\gamma$. Let $\underset{\sim}{x}_\alpha \in \mathscr{H}(\chi)$ be a $\mathbb{P}_\alpha$-name for the witness that $\underset{\sim}{\mathbb{Q}}_\alpha$ is (forced to be) complete for $\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$ and let $\bar{x} = \langle \underset{\sim}{x}_\alpha : \alpha < \gamma \rangle$. Since $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1) \in \mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$, we find $\bar{M} = \langle M_i : i \le \sigma \rangle$ which is ruled by $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$ with an $\hat{\mathscr{E}}_0$-approximation $\langle \bar{M}^i, \bar{N}^i : -1 \le i < \sigma \rangle$ and such that $p, \bar{\mathbb{Q}}, \underset{\sim}{x}, \bar{x}, \hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1 \in M_0$ (see 2.3). Let $\bar{N}^i = \langle N^i_\varepsilon : \varepsilon \le \delta_i \rangle$ and let $\bar{a}^i \in \hat{\mathscr{E}}_0$ be such that $(\bar{N}^i, \bar{a}^i)$ is an $\hat{\mathscr{E}}_0$-complementary pair and let $\bar{M}^i = \langle M^i_\varepsilon : \varepsilon \le \delta_i \rangle$. Let $w_i = \{0\} \cup \bigcup\limits_{\omega j \le i} (\gamma \cap M_{\omega j})$ (for $i \le \delta$). By the demands of 2.3 we know that $\|w_i\| <$ cf$(\delta_i), w_i \in M^i_0$.

By induction on $i \le \sigma$ we define standard $(w_i, i)^\gamma$-trees $\mathscr{T}_i \in M_{i+1}$ and $\bar{p}^i = \langle p^i_t : t \in T_i \rangle \in FTr'(\bar{\mathbb{Q}}) \cap M_{i+1}$ such that $\|T_i\| \le \|M_i\|^{\|w_i\|} \le \|M_{i+1}\|$ if i is limit or $0, w_{i+1} = w_i$ hence $\mathscr{T}_{i+1} = \mathscr{T}_i$, and if $j < i \le \delta$ then $\mathscr{T}_j =$ proj$^{(w_i,i+1)}_{(w_j,j+1)}(\mathscr{T}_i)$ and $\bar{p}^j \le_{\text{proj}^{\mathscr{T}_i}_{\mathscr{T}_j}} \bar{p}^i$.

<u>CASE 1</u>: $i = 0$.

Lt T^*_0 consist of all sequences $\langle t_\zeta : \zeta \in$ dom$(t) \rangle$ such that dom(t) is an initial segment of w_0 and $t_\zeta = <>$ for $\zeta \in$ dom(t). Thus T^*_0 is a standard $(w_0, 0)^\gamma$-tree, $\|T^*_0\| = \|w_0\| + 1$. For $t \in T^*_0$ let $p^{*0}_t = p \restriction$ rk$^*_0(t)$. Clearly the sequence $\bar{p}^{*0} = \langle p^{*0}_t : t \in T^*_0 \rangle$ is in $FTr'(\bar{\mathbb{Q}}) \cap N^{-1}_0$. Apply 2.8 to $\hat{\mathscr{E}}_0, \bar{\mathbb{Q}}, \bar{N}^{-1}, \mathscr{T}^*_0, w_0$ and $\bar{p}^{*0}$ (note that $\|M^{-1}_\varepsilon\|^{\|w_0\|} \subseteq M^{-1}_\varepsilon$ for $\varepsilon < \delta_0$). As a result we get a $(w_0, 1)^\gamma$-tree $\mathscr{T}_0$ (the one called $\mathscr{T}_{\delta_0}$ there) and $\bar{p}^0 = \langle p^0_t : t \in T_0 \rangle \in FTr'(\bar{\mathbb{Q}}) \cap M_1$ (the one called $\bar{q}$ there) satisfying clauses (ε),(c)(i)-(iv) of 2.8 and such that $\|T_0\| \leq \|N^{-1}_{\delta_0}\|^{\|w_0\|} = \|M_0\|^{\|w_0\|} = \|M_0\|$ (remember $\mathrm{cf}(\delta_0) > 2^{\|M_0\|}$). So, in particular, if $t \in T_0, \zeta \in \mathrm{dom}(t)$ then $t_\zeta(0) \in M_1$ is either $*$ of a $\mathbb{P}_\zeta$-name for an element of $\underset{\sim}{\mathbb{Q}}_\zeta$.

Moreover, we additionally require that $(\mathscr{T}_0, \bar{p}^0)$ is the $<^*_\chi$-first with all these properties, so $\mathscr{T}_0, \bar{p}^0 \in M_1$.

CASE 2: $i = i_0 + 1$.

We proceed similarly to the previous case. Suppose we have defined $\mathscr{T}_{i_0}$ and $\bar{p}^{i_0}$ such that $\mathscr{T}_{i_0}, \bar{p}^{i_0} \in M_{i_0+1}, \|T_{i_0}\| \leq \|M_{i_0+1}\|$. Let $\mathscr{T}^*_i$ be a standard $(w_i, i_0)^\gamma$-tree such that

> T^*_i consists of all sequences $\langle t_\zeta : \zeta \in \mathrm{dom}(t)\rangle$ such that $\mathrm{dom}(t)$ is an initial segment of w_i and

$$\langle t_\zeta : \zeta \in \mathrm{dom}(t) \cap w_{i_0}\rangle \in T_{i_0} \text{ and } (\forall \zeta \in \mathrm{dom}(t) \setminus w_{i_0})(\forall j < i_0)(t_\zeta(j) = *).$$

Thus, $\mathscr{T}_{i_0} = \mathrm{proj}^{(w_i,i)}_{(w_{i_0},i_0)}(\mathscr{T}^*_i)$ and $\|T^*_i\| \leq \|M_i\|$. Let $p^{*i}_t = p^{i_0}_{t'} \restriction \mathrm{rk}^*_i(t)$ for $t \in T^*_i, t' = \mathrm{proj}^{\mathscr{T}_i}_{\mathscr{T}_{i_0}}(t)$. Now apply 2.8 to $\hat{\mathscr{E}}_0, \bar{\mathbb{Q}}, \bar{N}^{i_0}, \mathscr{T}^*_i, w_i$ and $\bar{p}^{*i}$ (check that the assumptions are satisfied). So we get a standard $(w_i, i_0+1)^\gamma$-tree $\mathscr{T}_i$ and a sequence $\bar{p}^i$ satisfying $(\varepsilon), (c)(i)-(iv)$ of 2.8, and we take the $<^*_\chi$-pair $(\mathscr{T}_i, \bar{p}^i)$ with these properties. In particular, we will have $\|T_i\| \leq \|M_{i_0}\| \cdot \|N^{i_0}_{\delta_i}\|^{\|M_{i_0}\|} = \|M_{i_0+1}\|$ and $\bar{p}^i, \mathscr{T}_i \in M_{i+1}$.

CASE 3: i is a limit ordinal.

Suppose we have defined $\mathscr{T}_j, \bar{p}^j$ for $j < i$ and we know that $\langle(\mathscr{T}_j, \bar{p}^j) : j < i\rangle \in M_{i+1}$ (this is the consequence of taking "the $<^*_\chi$-first such that ..."). let $\mathscr{T}^*_i = \lim(\langle \mathscr{T}_j : j < i\rangle)$. Now, for $t \in T^*_i$ we would like to define p^{*i}_t as the limit of $p^j_{\mathrm{proj}^{\mathscr{T}^*_i}_{\mathscr{T}_j}(t)}$. However, our problem is that we do not know if the limit exists. Therefore, we restrict ourselves to these t for which the respective sequence has an upper bound. To be more precise, for $t \in \mathscr{T}^*_i$ we apply the following procedure.

$\bigotimes$ Let $t^j = \mathrm{proj}^{\mathscr{T}^*_i}_{\mathscr{T}_j}(t)$ for $j < i$. Try to define inductively a condition $p^{*i}_t \in \mathbb{P}_{\mathrm{rk}^*_i(t)}$ such that $\mathrm{dom}(p^{*i}_t) = \cup\{\mathrm{dom}(p^j_{t^j}) \cap \mathrm{rk}^*_i(t) : j < i\}$. Suppose we have successfully defined $p^{*i}_t \restriction \alpha$ for $\alpha \in \mathrm{dom}(p^{*i}_t)$, in such a way that $p^{*i}_t \restriction \alpha \geq p^j_{t^j} \restriction \alpha$ for all $j < i$. We know that

$$p^{*i}_t \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \text{" the sequence } \langle p^j_{t^j}(\alpha) : j < i\rangle \text{ is } \leq_{\underset{\sim}{\hat{\mathbb{Q}}}_\alpha} \text{-increasing".}$$

So now, if there is a $\mathbb{P}_\alpha$-name $\underset{\sim}{\tau}$ for an element of $\underset{\sim}{\mathbb{Q}}_\alpha$ such that

$$p^{*i}_t \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \text{“}(\forall j < i)(p^j_{t^j}(\alpha) \leq_{\hat{\underset{\sim}{\mathbb{Q}}}_\alpha} \underset{\sim}{\tau})\text{”},$$

then we take the $\mathbb{P}_\alpha$-name of the lub of $\langle p^j_{t^j}(\alpha) : j < i, p^j_{t^j}(\alpha) \neq *\rangle$ in $\hat{\mathbb{Q}}$, and we continue. If there is no such $\underset{\sim}{\tau}$ then we decide that $t \notin \mathscr{T}^+_i$ and we stop the procedure[2].

Now, let $\mathscr{T}^+_i$ consist of those $t \in T^*_i$ for which the above procedure resulted in a successful definition of $p^{*i}_t \in \mathbb{P}_{\mathrm{rk}^*_i(t)}$. It might not be clear at the moment if T^+_i containss anything more than $<>$, but we will see that this is the case. Note that

$$\|T^+_i\| \leq \|T^*_i\| \leq \prod_{j<i} \|T_j\| \leq \prod_{j<i} \|M_j\| \leq 2^{\|M_i\|} \leq \|M^i_0\|.$$

Moreover, for nonlimit $\varepsilon > 2$ we have $\|M^i_\varepsilon\|^{\|w_i\|+\|T^+_i\|} \leq \|M^i_\varepsilon\|^{\|M_i\|} \subseteq M^i_{\varepsilon+1}$ and $\mathscr{T}^+_i, \bar{p}^{*i} \in M_{i+1}$. Let $\mathscr{T}_i = \mathscr{T}^*_i, \bar{p}^i = \bar{p}^{*i}$ (this time there is no need to take the $<^*_\chi$-first pair as the process leaves no freedom). So we have finished Case 3.

After the construction is carried out we continue in a similar manner as in [Sh 587, A.3.7] (but note slightly different meaning of the $*$'s here).

So we let $\mathscr{T}_\sigma = \lim(\langle \mathscr{T}_i : i < \sigma\rangle)$. It is a standard $(\sigma,\sigma)^\gamma$-tree. By induction on $\alpha \in w_\sigma \cup \{\gamma\}$ we choose $q_\alpha \in \mathbb{P}'_\alpha$ and a $\mathbb{P}_\alpha$-name $\underset{\sim}{t}_\alpha$ such that:

(a) $\Vdash_{\mathbb{P}_\alpha}$ “$\underset{\sim}{t}_\alpha \in T_{\omega\sigma}$ & $\mathrm{rk}_\delta(\underset{\sim}{t}_\alpha) = \alpha$” and let $i^\alpha_0 = \min\{i < \delta : \alpha \in M_i\} < \sigma$,

(b) $\Vdash_{\mathbb{P}_\alpha}$ “$\underset{\sim}{t}_\beta = \underset{\sim}{t}_\alpha \restriction \beta$” for $\beta < \alpha$,

(c) $\mathrm{dom}(q_\alpha) = w_\delta \cap \alpha$,

(d) if $\beta < \alpha$ then $q_\beta = q_\alpha \restriction \beta$,

(e) $p^i_{\mathrm{proj}^{\mathscr{T}_\delta}_{\mathscr{T}_i}(\underset{\sim}{t}_\alpha)}$ is well defined and $p^i_{\mathrm{proj}^{\mathscr{T}_\delta}_{\mathscr{T}_i}(\underset{\sim}{t}_\alpha)} \restriction \alpha \leq q_\alpha$ for each $i < \omega\sigma$,

(f) for each $\beta < \alpha$

[2] Generally in such situation we can act as in 2.7 to get a real decision, i.e. if $p^{*i}_t \restriction (\alpha+1)$ is not well defined while $p^{*i}_t \restriction \alpha$ is well defined then $p^{*i}_t \restriction \alpha \Vdash$ “the sequence $\langle p^j_{t^j}(\alpha) : j < i\rangle$ has no $\leq_{\hat{\underset{\sim}{\mathbb{Q}}}_\alpha}$-upper bound. But the need has not arisen here.

$q_\alpha \Vdash_{\mathbb{P}_\alpha}$ "$(\forall i < \delta)((\underset{\sim}{t}_{\beta+1})_\beta(i) = * \Leftrightarrow i < i_0^\beta)$ and the sequence

$$\langle i_0^\beta, p^{i_0^\beta}_{\mathrm{proj}^{\mathscr{T}_\delta}_{\mathscr{T}_{i_0^\beta}}(\underset{\sim}{t}_{\beta+1})}(\beta), \langle (\underset{\sim}{t}_{\beta+1})_\beta(i), p^i_{\mathrm{proj}^{\mathscr{T}_\delta}_{\mathscr{T}_i}(\underset{\sim}{t}_{\beta+1})}(\beta) : i_0^\beta \le i < \delta\rangle\rangle$$

is a result of a play of the game $\mathfrak{G}^\spadesuit_{\bar M[\underset{\sim}{G}_\beta], \langle \bar N^i[\underset{\sim}{G}_\beta]: i<\delta\rangle}(\underset{\sim}{\mathbb{Q}}_\beta, 0_{\underset{\sim}{\mathbb{Q}}_\beta})$,

won by player COM",

(g) the condition q_α forces (in $\mathbb{P}_\alpha$) that
"the sequence $\bar M[\underset{\sim}{G}_{\mathbb{P}_\alpha}] \restriction [i_\alpha, \delta]$ is ruled by $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$ and
$\langle \bar N^i[\underset{\sim}{G}_{\mathbb{P}_\alpha}] : i_0^\alpha \le i < \sigma\rangle$ is its $\hat{\mathscr{E}}_0$-approximation".

(Remember: $\hat{\mathscr{E}}_1$ is closed under end segments). This is done completely parallely to the last part of the proof of [Sh 587, A.3.7].

Finally, look at the condition q_γ and the clause (g) above. $\square_{2.9}$

<u>2.10 Generalization</u> 1) $\hat{\mathscr{E}}_1$ is a set of triples $\langle \bar a, \langle \bar b^i, \bar a^i : i < \sigma\rangle, \bar\lambda\rangle, \bar a = \langle a_i : i \le \sigma\rangle, \bar a^i = \langle a^i_\alpha : \alpha \le \delta_i\rangle, \bar b^i = \langle b^i_\alpha : \alpha \le \delta_i\rangle \in \hat{\mathscr{E}}_0, a^i_{\delta_i} = a_{i+1}, a_i \subseteq b^i_0, \lambda = \langle \lambda_i : i < \sigma\rangle$ an increasing sequence of cardinals $< \lambda, \sum \lambda_i = \lambda$.
2) We say $(\bar M, \langle \bar M^i : i < \sigma\rangle, \langle \bar N^i : i < \sigma\rangle)$ obeys $(\bar a, \langle \bar b^i : i < \bar\lambda\rangle$ <u>if</u>: $M_i \cap \mu^* = a_i, \bar N^i$ obeys $\bar b^i$ all things in 2.3 but $\lambda_i \ge \|M_i\|, \lambda_i \ge \prod_{j\le i} \|M_j\|, [M^i_\alpha]^{\lambda_i} \subseteq M^i_{\alpha+1}$ for $\alpha < \delta_i$ (so earlier $\lambda_i = 2^{\|M_i\|}$).

<u>2.11 Conclusion</u> 1) Assume

(a) $S \subseteq \{\delta < \kappa : \mathrm{cf}(\delta) = \sigma\}$ is stationary not reflecting
(b) $\bar{\mathbf{a}} = \langle \bar a_\delta : \delta \in S\rangle, \bar a_\delta = \langle a_{\delta,i} : i \le \sigma\rangle, \delta = a_{\delta,\sigma}$ and $a_{\delta,i}$ increasing with i and $i < \sigma \Rightarrow |a_{\delta,i}| < \lambda$ and $\sup(a_{\delta,i}) < \delta$
[variant: $\bar\lambda^\delta = \langle \lambda^\delta_i : i < \sigma\rangle$ increasing with limit λ]
(c) we let $\mu^* = \kappa, \hat{\mathscr{E}}_0 = \hat{\mathscr{E}}_0[S] = \{\bar a : \bar a = \langle a_i : i \le \alpha\rangle, \alpha < \kappa, a_i \in \kappa \backslash S$ increasing continuous$\}$
(d) $\hat{\mathscr{E}}_1 = \{\bar a_\delta : \delta \in S\}$
(or $\{\langle \bar a_\delta, \langle \bar a^{\delta,i}, \bar b^{i,\delta} : i < \sigma\rangle, \bar\lambda^\delta\rangle : \delta \in S\}$ appropriate for (2.10)
(e) we assume the pair $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1) \in \mathfrak{C}^\spadesuit_{<\kappa}(\mu^*)$
(f) $\mu = \mu^\kappa, \kappa < \tau = \mathrm{cf}(\tau) < \mu$.

Then for some $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$-complete forcing notion $\mathbb{P}$ of cardinality μ we have

$$\Vdash_{\mathbb{P}} \text{“forcing axiom for } (\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)\text{-complete forcing notion of cardinality } \leq \kappa \text{ and } < \tau \text{ of open dense sets”}$$

and in $\mathbf{V}^{\mathbb{P}}$ the set S is still stationary (by preservation of $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$-nontrivial).

2) If clauses (a),(c) holds and $\diamondsuit_S$, then for some $\bar{\mathbf{a}}$, if we define $\hat{\mathscr{E}}_1$ as in clause (d) then clause (b),(d),(e) holds.

Proof. 1) See more in the end of §3.
2) Easy. $\square_{2.11}$

2.12 Application: In $\mathbf{V}^{\mathbb{P}}$ of 2.11:

(a) if
  (i) $\theta < \lambda, A_\delta \subseteq \delta = \sup(A_\delta)$ for $\delta \in S$,
  (ii) $|A_\delta| < \theta$
  (iii) $\bar{h} = \langle h_\delta : \delta \in S\rangle, h_\delta : A \to \theta$
  (iv) $A_\delta \subseteq \bigcup\{a_{\delta,i+1} \backslash a_{\delta,i} : i < \sigma\}$,
  then for some $h : \kappa \to \theta$ and club E of κ we have $(\forall \delta \in S \cap E)[h_\delta \subseteq^* h]$ where $h' \subseteq^* h''$ means that $\sup(\mathrm{Dom}(h')) > \sup\{\alpha : \alpha \in \mathrm{Dom}(h')$ and $\alpha \notin \mathrm{Dom}(h'')$ or $\alpha \in \mathrm{Dom}(h'')$ & $h'(\alpha) \neq h''(\alpha)\}$

(b) if we add: “h_δ constant”, then we can omit the assumption (iii)

(c) we can weaken $|A_\delta| < \theta$ to $|A_\delta \cap a_{\delta,i+1}| \leq |a_{\delta,i}|$

(d) in (c) we can weaken $|A_\delta| \leq \theta \vee |A_\delta \cap a_{\delta,i+1}| \leq |a_{\delta,i}|$ to $h_\delta \restriction a_{\delta,i+1}$ belongs to $M_{i+1} \cap N^i_\alpha$ for some $\alpha < \delta_i$
(remember $\mathrm{cf}(\sup a_{\delta,i+1}) > \lambda^\delta_i$).

2.13 Remark. 1) Compared to [Sh 186] the new point in the application is (b).
2) You may complain why not having the best of (a) + (b), i.e. combine their good points. The reason is that this is impossible by §1, §4; the situation is different in the inaccessible case.

Proof. Should be clear. Still we say something in case h_δ constant, that is (b).
Let

$$\mathbb{Q} = \{(h, C) : h \text{ is a function with domain an ordinal } \alpha < \kappa = \lambda^+, C \text{ a closed subset of } \alpha + 1, \alpha \in C \text{ and } (\forall \delta \in C \cap S \cap (\alpha + 1))(h_\delta \subseteq^* h)\}.$$

with the partial order being inclusion.

For $p \in \mathbb{Q}$ let $p = (h^p, C^p)$.

So clearly if $(h, C) \in \mathbb{Q}$ and $\alpha = \text{Dom}(h) < \beta \in \kappa$ <u>then</u> for some h_1 we have $h \subseteq h_1 \in \mathbb{Q}_1$, $\text{Dom}(h_1) = \beta$; moreover, if $\gamma < \theta$ & $\beta \notin S$ then $(h, C) \le (h \cup \gamma_{[\alpha,\beta]}, C \cup \{\beta\}) \in \mathbb{Q}$.

The main point is proving $\mathbb{Q}$ is complete for $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$. Now "$\mathbb{Q}$ is strongly complete for $\hat{\mathscr{E}}_0$" is proved as in [Sh 587, B.6.5.1,B.6.5.2] (or 3.14 below which is somewhat less similar). The main point is clause (b) of 2.5(2); that is, let $\bar{M}, \langle \bar{M}^i : i < \omega\sigma\rangle, \langle \bar{N}^i : i < \omega\sigma\rangle$ be as there. In the game $\mathfrak{G}_{\bar{M},\langle N_i : i < \omega\sigma\rangle}(r, \mathbb{Q})$ from 2.5(1), we can even prove that the player COM has a winning strategy: in stage i (non-trivial): if h_δ is constantly $\gamma < \theta$ or just $h_\delta \restriction (A_\delta \cap a_{\delta,i+1} \backslash a_{\delta,i})$ is constantly $\gamma < \theta$ then we let

$$p_i = \Big(\cup \{h^{q^j_\zeta} : j < i \text{ and } \zeta < \delta_i\} \cup \gamma_{[N^i_{\delta_i} \cap \kappa, \beta_i)}, \text{closure}\big(\cup\{C^{q^j_\zeta} : j < i \text{ and } \zeta < \delta_i\} \cup \{\beta_i\}\big)\Big)$$

for some $\beta_i \in M_{i+1} \cap \kappa \backslash M_i$ large enough such that $A_\delta \cap M_{i+1} \cap \kappa \subseteq \beta_i$. $\square$**?**

—> scite{2.10} undefined

Remark. In the example of uniformizing (see [Sh 587]) if we use this forcing, the density is less problematic.

2.14 Claim. *1) In* **?***'s conclusion we can omit the club E that is let $E = \kappa$ and*

—> *scite{2.10} undefined*

demand $(\forall \delta \in S)(h_\delta \subseteq^* h)$ provided that we add in **?***, recalling $S \subseteq \kappa$ does not*

—> *scite{2.10} undefined*

reflect is a set of limit ordinals and

$$\bar{A} = \langle A_\delta : \delta \in S\rangle, A_\delta \subseteq \delta = \sup(A_\delta)$$

satisfies

(∗) $\delta_1 \neq \delta_2$ *in* $S \Rightarrow \sup(A_{\delta_1} \cap A_{\delta_2}) < \delta_1 \cap \delta_2$.

2) If $(\forall \delta \in S)(otp(A_\delta) = \theta$ *this always holds.*

Proof. We define $\mathbb{Q} = \{h :$ Dom(h) is an ordinal $< \kappa$ and $h(\beta) \neq 0 \wedge \beta \in$ Dom$(h) \rightarrow (\exists \delta \in S)[h_\delta(\beta) = h(\beta)]$ and $\delta \in ($Dom$(h) + 1) \cap S$ implies $h_\delta \subseteq^* h\}$ ordered by $\subseteq$. Now we should prove the parallel of the fact:

⊠′ if $p \in \mathbb{Q}, \alpha =$ Dom$(p) < \beta < \kappa$ then there is q such that $p \leq q \in \mathbb{Q}$ and Dom$(q) = \beta$.

Why this holds? We can find $\langle A'_\delta : \delta \in S \cap (\beta+1)\rangle$ such that $A'_\delta \subseteq A_\delta, \sup(A_\delta \backslash A'_\delta) < \delta$ and $\bar{A}' = \langle A'_\delta : \delta \in S \cap (\beta + 1)\rangle$ is pairwise disjoint.

Now choose q as follows

$$\mathrm{Dom}(q) = \beta$$

$$q(j) = \begin{cases} p(j) & \underline{\text{if}} \quad j < \alpha \\ h_\delta(j) & \underline{\text{if}} \quad j \in A'_\delta \backslash \alpha \text{ and } \delta \in S \cap (\beta + 1) \backslash (\alpha + 1) \\ 0 & \underline{\text{if}} \text{ otherwise.} \end{cases}$$

Why does $\bar{A}'$ exist? Prove by induction on β that for any $\bar{A}^1, \langle A'_\delta : \delta \in S \cap (\alpha + 1)\rangle$ as above and β satisfying $\alpha < \beta < \kappa$, we can end extend $\bar{A}^1$ to $\langle A'_\delta : \delta \in S \cap (\beta + 1)\rangle$ which is as above. $\square_{2.14}$

2.15 Remark. Note: concerning κ inaccessible we could immitate what is here: having $M_{i+1} \underset{\neq}{\prec} N^i_{\delta_i}, \bigcup\limits_{i<\delta} M_i = \bigcup\limits_{i<\delta} N^i_{\delta_i}$.

As long as we are looking for a proof that no sequence of length $< \kappa$ are added, the gain is meagre (restricting the $\bar{q}$'s by $\bar{q} \restriction \alpha \in N'_{\alpha+1}$). Still if you want to make the uniformization and some diamond we may consider this.

<u>2.16 Comment</u>: We can weaken further the demand, by letting COM have more influence. E.g. we have (in 2.3) $\delta_i = \lambda_i =$ cf$(\lambda_i) = \|M_{i+1}\|, D_i$ a $|a_i|^+$-complete filter on λ_i, the choice of $\bar{q}^i$ in the result of a game in which INC should have chose a set of player $\in D_i$ and $\diamondsuit_{D_i}$ holds (as in the treatment of case E^* here).

The changes are obvious, but I do not see an application at the moment.

§3 κ^+-c.c. and κ^+-pic

We intend to generalize pic of [Sh:f, Ch.VIII,§1]. The intended use is for iteration with each forcing $> \kappa$ - see use in [Sh:f]. In [Sh 587, B.7.4] we assume each $\mathbb{Q}_i$ of cardinality $\leq \kappa$. Usually $\mu = \kappa^+$.
Note: $\hat{\mathscr{E}}_0$ is as in the accessible case, in [Sh 587] but this part works in the other cases. In particular, in Cases A,B (in [Sh 587]'s context) if the length of $\bar{a} \in \hat{\mathscr{E}}_0$ is $< \lambda$ (remember $\kappa = \lambda^+$), then we have $(< \lambda)$-completeness implies $\hat{\mathscr{E}}_0$-completeness AND in 3.7 even $\bar{a} \in \hat{\mathscr{E}}_0 \Rightarrow \ell g(\bar{a}) = \omega$ is O.K.

In Case A on the $S_0 \subseteq S^\kappa_\lambda$ if $\ell g(\bar{a}) = \lambda, a_\lambda \in S_0$ is O.K., too. STILL can start with other variants of completeness which is preserved.

3.1 Context: We continue [Sh 587, B.5.1-B.5.7(1)] (except the remark [Sh 587, B.5.2(3)]) under the weaker assumption $\kappa = \kappa^{<\kappa} > \aleph_0$, so κ is not necessarily strongly inaccessible; also in our $\hat{\mathscr{E}}$'s we allow $\bar{a}$ such that $|a_\delta| = |\delta|$ is strongly inaccessible.

3.2 Definition. Assume:

$\boxtimes(a)$ $\mu = \text{cf}(\mu) > |\alpha|^{<\kappa}$ for $\alpha < \mu$

(b) the triple $(\kappa, \mu^*, \hat{\mathscr{E}}_0)$ satisfies: $\kappa = \text{cf}(\kappa) > \aleph_0, \mu^* \geq \kappa, \hat{\mathscr{E}}_0 \subseteq \{\bar{a} : \bar{a}$ an increasing continuous sequence of members of $[\mu^*]^{<\kappa}$ of limit length $< \kappa$ with $a_i \cap \kappa \in \kappa\}$ and

(c) $S^\square \subseteq \{\delta < \mu : \text{cf}(\delta) \geq \kappa\}$ stationary.

For $\ell = 1,2$ we say $\mathbb{Q}$ satisfies $(\mu, S^\square, \hat{\mathscr{E}}_0)$-pic$_\ell$ if: for some $x \in \mathscr{H}(\chi)$ (can be omitted, essentially, i.e. replaced by $\mathbb{Q}$) we have

$(*)$ if

(α) $S \subseteq S^\square$ is stationary and $\langle \mu, S, \hat{\mathscr{E}}_0, x\rangle \in N^\alpha_0$

(β) for $\alpha \in S, \delta_\alpha < \kappa$, and

(i) if $\ell = 1, \bar{N}^\alpha = \langle N^\alpha_i : i \leq \delta_\alpha\rangle$ and $c_\alpha = \delta_\alpha$ and $\bar{N}^{\alpha,*} = \bar{N}^\alpha$

(ii) if $\ell = 2$ then $\bar{N}^{\alpha,*} = \langle N^\alpha_i : i \leq \delta_\alpha\rangle, \bar{N}^\alpha = \langle N^\alpha_i : i \in c^+_\alpha\rangle$ where $c_\alpha \subseteq \delta_\alpha = \sup(c_\alpha), c^+_\alpha = c_\alpha \cup \{\delta_\alpha\}, c_\alpha$ is closed, $\gamma < \beta \in c_\alpha \Rightarrow c_\alpha \cap \gamma \in N^\alpha_\beta$

(γ) $(\bar{N}^\alpha, \bar{a}^\alpha)$ is $\hat{\mathscr{E}}_0$-complementary (see [Sh 587, B.5.3]); so $\bar{N}^\alpha$ obeys $\bar{a}^\alpha \in \hat{\mathscr{E}}_0$ (with some error n_α) (so here we have $\|N^\alpha_{\delta_\alpha}\| < \kappa, \delta_\alpha < \kappa$)

(δ) $\bar{p}^\alpha$ is $(\bar{N}^\alpha, \mathbb{Q})^1$-generic (see [Sh 587, Definition B.5.3.1])

(ε) $\alpha \in N^\alpha_0$ and

(i) if $\ell = 1$, then for some club C of μ for every $\alpha \in S$ we have $\langle (\bar N^\beta, \bar p^\beta) : \beta \in S \cap C \cap \alpha \rangle$ belong to N^α_0

(ii) if $\ell = 2$, <u>then</u> for some club C of μ for every $\alpha \in S \cap C$ and $i < \delta_\alpha$ we have $\langle (\bar N^{\beta,*} \restriction (i+1), \bar p^\beta \restriction (i+1)) : \beta \in S \cap C$ belongs to N^α_{i+1}

(ε) we define a function g with domain S as follows: $g(\alpha) = (g_0(\alpha), g_1(\alpha))$ where $g_0(\alpha) = N^\alpha_{\delta_\alpha} \cap (\bigcup\limits_{\beta<\alpha} N^\beta_{\delta_\beta})$ and $g_1(\alpha) = (N^\alpha_{\delta_\alpha}, N^\alpha_i, c)_{i<\delta_1, c \in g_0(\alpha)} / \cong$,

<u>then</u> we can find a club C of μ such that:
if $\alpha < \beta$ & $g(\alpha) = g(\beta)$ & $\alpha \in C \cap S$ & $\beta \in C \cap S$ <u>then</u> $\delta_\alpha = \delta_\beta, g(\alpha) = g(\beta)$, for some $h, N^\alpha_{\delta_\alpha} \underset{h}{\cong} N^\beta_{\delta_\beta}$ (really unique), and for each $i < \delta_\alpha$ the function h maps N^α_i to N^β_i, p^α_i to p^β_i and $\{p^\alpha_i : i < \delta_\alpha\} \cup \{p^\beta_i : i < \delta_\beta\}$ has an upper bound.

3.3 Claim. *Assume $\boxtimes$, i.e. $(a), (b), (c)$ of 3.2 and*

(d) *$\hat{\mathscr E}_0$ is non-trivial, which means: for every χ large enough and $x \in \mathscr H(\chi)$ there is $\bar N = \langle N_i : i \le \delta \rangle$ increasingly continuous, $N_i \prec (\mathscr H(\chi), \in), x \in N_i, \|N_i\| < \kappa, \bar N \restriction (i+1) \in N_{i+1}$ and $\bar N$ obeys some $\bar a \in \hat{\mathscr E}_0$ with some finite error n)*

(e) *$\mathbb Q$ is a strongly $c\ell(\hat{\mathscr E}_0)$-complete forcing notion (hence adding no new bounded subsets of κ) where $c\ell(\hat{\mathscr E}_0) =: \{\bar a \restriction [\alpha, \beta] : \bar a \in \hat{\mathscr E}_0$ and $\alpha \le \beta \le \ell g(\bar a)\}$*

(f) *$\mathbb Q$ satisfies $(\mu, S^\square, \hat{\mathscr E}_0)$-pic$_\ell$ where $\ell \in \{1,2\}$.*

<u>Then</u> $\mathbb Q$ satisfies the μ-c.c. provided that

$(*)$ *$\ell = 1$ <u>or</u> $\ell = 2$ and $\hat{\mathscr E}_0$ is fat, see below.*

3.4 Definition. We say $\hat{\mathscr E}_0 \in \mathfrak C^-_{<\kappa}(\mu^*)$ is fat, if in the following game $\partial_{\kappa,\mu^*}(\hat{\mathscr E}_0)$ between fat and lean, the fat player has a winning strategy.

A play last κ moves; in the α-th move:

<u>Case 1</u>: α nonlimit.

The player lean chooses a club $Y_\alpha \subseteq [\mu^*]^{<\kappa}$, the fat player chooses $a_\alpha \in Y_\alpha$ and $\mathscr P_\alpha \subseteq \{c : c \subseteq \alpha$ is closed$\}$ of cardinality $< \kappa$.

<u>Case 2</u>: α limit.

We let $Y_\alpha = [\mu_0]^{<\kappa}$ and $a_\alpha = \cup\{a_\beta : \beta < \alpha\}$ and the player fat chooses $\mathscr{P}_\alpha \subseteq \{C : C \subseteq \alpha \text{ is closed}\}$ of cardinality $< \kappa\}$.

In a play, fat wins iff for some limit ordinal α and $c \in \mathscr{P}_\alpha$ we have:

$(*)(i)$ $\beta \in c \Rightarrow c \cap \beta \in \mathscr{P}_\beta$

(ii) $\alpha = \sup(c)$

(iii) $\langle a_\beta : \beta \in c \cup \{\alpha\}\rangle \in \hat{\mathscr{E}}_0$.

3.5 Remark. 0) With more care in the game Definition 3.10 we incorporate choosing the $\bar{p}^\alpha$'s. In 3.7$(*)(\varepsilon)(ii)$ we can add $\langle N^\beta_{i+1} : \beta \in \alpha \cap c\rangle$ belongs to N^α_{i+1}.
1) In the Definition 3.4, without loss of generality $c \in \mathscr{P}_\alpha \ \& \ \beta \in c \Rightarrow c \cap \beta \in \mathscr{P}_\beta$.
2) If κ is strongly inaccessible without loss of generality we have $\mathscr{P}_\alpha = \mathscr{P}(\alpha)$, so fat has a winning strategy.
3) In general being fat is a weak demand, e.g. if $\hat{\mathscr{E}}_0 \supseteq \{\bar{a} : \bar{a} = \langle a_i : i \leq \omega\rangle, a_\omega = \bigcup\limits_n a_n, a_i \in [\mu^*]^{<\kappa}$ is increasing.

Proof of 3.9. Case 1: $\ell = 1$.

Assume $p_\alpha \in \mathbb{Q}$ for $\alpha < \mu$ and let χ be large enough and x as in Definition 3.2. We choose $(\bar{N}^\alpha, \bar{p}^\alpha)$ by induction on $\alpha < \mu$ as follows. If $\langle(\bar{N}^\beta, \bar{p}^\beta) : \beta < \alpha\rangle$ is already defined, as $\hat{\mathscr{E}}_0$ is non-trivial there is a pair $(\bar{N}^\alpha, \bar{a}^\beta)$ which is $\hat{\mathscr{E}}_0$-complementary and $\langle(\bar{N}^\beta, \bar{p}^\beta) : \beta < \alpha\rangle, \mathbb{Q}, \langle p_\beta : \beta < \mu\rangle, p_\alpha, \alpha, x$ belong to N^α_0 and let $\bar{N}^\alpha = \langle N^\alpha_i : i \leq \delta_i\rangle$. So $p_\alpha \in N^\alpha_0$ and we can choose $p_{\alpha,i} \in N^\alpha_{i+1}$ such that $p_\alpha = p_{\alpha,0}$ and $\langle p_{\alpha,i} : i < \delta_\alpha\rangle$ is $(\bar{N}^\alpha, \mathbb{Q})^1$-generic.
[Why? By the proof of [Sh 587, B.5.6.4].] Now by "$\mathbb{Q}$ is $(\mu, S^\square, \hat{\mathscr{E}}_0)$-pic$_\ell$", for some $\alpha < \beta$ in $S^\square, \{p^\alpha_i : i < \delta_\alpha\} \cup \{p^\beta_i : i < \delta_\beta\}$ has a common upper bound hence in particular, p_α, p_β are compatible.

Case 2: $\ell = 2$.

Assume $p_\alpha \in \mathbb{Q}$ for $\alpha < \mu$ and let χ be large enough. Let **St** be a winning strategy for the player fat in the game $\partial_{\kappa,\mu^*}(\hat{\mathscr{E}}_0)$. Now we choose by induction on $i < \kappa$. The tuple $(N^\alpha_i, \mathscr{P}^\alpha_i, Y^\alpha_i, \bar{p}^\alpha_i)$ where $\bar{p}^\alpha_i = \langle p^\alpha_{i,c} : c \in \mathscr{P}^\alpha_i\rangle$ for $\alpha < \mu$ such that:

$\boxtimes(a)$ $M^\alpha_i \prec (\mathscr{H}(\chi), \in, <^*_\chi)$

(b) M^α_i increasing continuous in i

(c) $\|M^\alpha_i\| < \kappa$ and $\langle M^\alpha_j : j \leq i\rangle \in M^\alpha_{i+1}$ and $M^\alpha_i \cap \kappa \in \kappa$ and $p_\alpha \in M^\alpha_i$,

(d) $\langle Y^\alpha_j, M^\alpha_j \cap \mu^*, \mathscr{P}^\alpha_j : j \leq i\rangle$ is an initial segment of a play of $\partial_{\kappa,\mu^*}(\hat{\mathscr{E}}_0)$ in which the player fat uses his winning strategy **St**

(e) $\langle (M^\beta_j, \mathscr{P}^\beta_j, Y^\beta_j, \bar p^\beta_i) : j \le i, \beta \in S\rangle$ belongs to N^α_{i+1} (hence $\mathscr{P}^\alpha_j \subseteq M^\alpha_{j+1}$, etc.)

(f) $p^\alpha_{i,c} \in \mathbb{Q} \cap N^\alpha_{i+1}$

(g) if $c \in \mathscr{P}^\alpha_i$ and $\langle p^\alpha_{j,c\cap j} : j \in c\rangle$ has an upper bound then $p^\alpha_{i,c}$ is such a bound

(h) $p^\alpha_{i,c} \in \cap\{\mathscr{I} : \mathscr{I} \in M^\alpha_i$ is a dense open subset of $\mathbb{Q}\}$.

Can we carry the induction?

For i limit let $M^\alpha_i = \cup\{M^\alpha_j : j < i\}$ and choose $Y^\alpha_i, \mathscr{P}^\alpha_i$ by clause (d) i.e. by the rules of the game $\partial_{\kappa,\mu^*}(\hat{\mathscr{E}}_0)$ and p^α_i by clause (g) + (h) (possible as forcing by $\mathbb{Q}$ adds no new sequences of length $< \kappa$ of members of $\mathbf{V}$). For i non-limit, let $x_i = \langle (M^\beta_j, \mathscr{P}^\beta_j, Y^\beta_j, \bar p^\beta_j) : j \le i, \beta \in S\rangle$ let $Y^\alpha_i = \{a : a \in [\mu^*]^{<\kappa}$ and $\alpha \in a$ and $a = \mu^* \cap \mathrm{Sk}^{<\kappa}_{(\mathscr{H}(\chi),\in,<^*_\chi)}(\{x_i \times \mathbb{Q}, \mathbf{St}, \alpha\})\}$ ($Sk^{<\kappa}$ means $a \in Y^\alpha_i \Rightarrow a \cap \kappa \in \kappa$) and let $(a^\alpha_i, \mathscr{P}^\alpha_i)$ be the move which the strategy $\mathbf{St}$ dictate to the player fat if the i-th move of lean is Y^α_i (and the play so far is $\langle (Y^\alpha_j, M^\alpha_j \cap \mu^*, \mathscr{P}_{\alpha,j}) : j < i)\rangle$. Now we choose $M^\alpha_i = \mathrm{Sk}^{<\kappa}_{(\mathscr{H}(\chi),\in,<^*_\chi)}(\{x_i, \mathbb{Q}, \mathbf{St}, \alpha\})$ and $\mathscr{P}^\alpha_i$ has already been chosen and $\bar p^\alpha_i = \langle p^\alpha_{i,c} : c \in \mathscr{P}^\alpha_i\rangle$ as in the limit case.

Having carried the induction, for each $\alpha \in S$ in the play $\langle (Y^\alpha_i, M^\alpha_i \cap \mu^*, \mathscr{P}^\alpha_i) : i < \kappa\rangle$ the player fat wins the game having used the strategy $\mathbf{St}$, hence there are a limit ordinal $i_\alpha < \kappa$ and closed $c_\alpha \in \mathscr{P}_{i_\alpha}$ and $i_\alpha = \sup(c_\alpha)$ and $\langle M^\alpha_j : j \in c_\alpha \cup \{i_\alpha\}\rangle$ obeys some member $\bar a_\alpha$ of $\hat{\mathscr{E}}_0$. As $\mathbb{Q}$ is $c\ell(\hat{\mathscr{E}}_0)$-complete we can prove by induction on $j \in c_\alpha \cup \{i_\alpha\}$ that $\varepsilon < j$ & $\varepsilon \in C_\alpha \Rightarrow \mathbb{Q} \models p^\alpha_{\varepsilon,c_\alpha\cap\varepsilon} \le p^\alpha_{j,c_\alpha\cap j}$.

Let $\delta_\alpha = i_\alpha, N^\alpha_i = M^\alpha_i$ for $i \le \delta_\alpha$ and $\bar p^\alpha = \langle p^\alpha_i : i \in c_\alpha\rangle$. Now continue as in Case 1. $\square_{3.3}$

3.6 Claim. *If $(*)$ of Definition 3.2, we can allow $Dom(g)$ to be a subset of ScapC, $\langle A_i : i < \mu\rangle$ be an increasingly continuous sequence of sets, $|A_i| < \mu, N^\alpha_{\delta_\alpha} \subseteq A_{\alpha+1}$ replacing the definition of g, g_0 and by $g_0(\alpha) = N^\alpha_{\delta_\alpha} \cap A_\alpha$ and g_1 by $g_1(\alpha) = (N^\alpha_{\delta_\alpha}, N^\alpha_i, c)_{i<\delta_\alpha, c\in g_0(c)}/\cong$ (and get equivalent definition).*

Remark. If $\mathrm{Dom}(g) \cap S^\square$ is not stationary, the definition says nothing.

Proof. Straight.

3.7 Claim. *Assume clauses $\boxtimes$, i.e. (a), (b), (c) of 3.2 and (d) of 3.3.*

For $(< \kappa)$-support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha\rangle$, if we have $\Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ is $(\mu, S^\square, \hat{\mathscr{E}}_0)$-pic$_\ell$" for each $i < \alpha$ and forcing with $Lim(\bar{\mathbb{Q}})$ add no bounded subsets of κ, <u>then</u> $\mathbb{P}_\gamma$ and $\mathbb{P}_\gamma/\mathbb{P}_\beta$, for $\beta \le \gamma \le \ell g(\bar{\mathbb{Q}})$ are $\hat{\mathscr{E}}_0$-complete $(\mu, S^\square, \hat{\mathscr{E}}_0)$-pic$_\ell$.

3.8 Remark. We can omit the assumption "Lim($\bar{\mathbb{Q}}$) add no bounded subsets of κ" if we add the assumption $c\ell(\hat{\mathscr{E}}_0) \in \mathfrak{C}_{<\kappa}(\mu^*)$, see [Sh 587, Def.B.5.1(2)], because with the later assumption the former follows by [Sh 587, B.5.6].

Proof. Similar to [Sh:f, Ch.VIII]. We first concentrate on

Case 1: $\ell = 1$.

It is enough to prove for $\mathbb{P}_\alpha$.
We prove this by induction on α. Let $\Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ is $(\mu, S^\square, \hat{\mathscr{E}}_0)$-pic$_\ell$ as witnessed by $\underset{\sim}{x}_i$ and let $\underset{\sim}{\chi}_i =$ Min$\{\chi : \underset{\sim}{x}_i \in \mathscr{H}(\chi)\}$".

Let $x = (\mu^*, \kappa, \mu, S^\square, \hat{\mathscr{E}}_0, \langle(\underset{\sim}{\chi}_i, \underset{\sim}{x}_i) : i < \ell g(\bar{\mathbb{Q}})\rangle)$ and assume χ is large enough such that $x \in \mathscr{H}(\chi)$ and let $\langle(\bar{N}^\alpha, \bar{p}^\alpha) : \alpha \in S\rangle$ be as in Definition 3.2, so $S \subseteq S^\square$ is stationary and $\bar{N}^\alpha = \langle N^\alpha_i : i \le \delta_\alpha\rangle$. We define a g by

$\boxtimes_1$ g is a function with domain S

$\boxtimes_2$ $g(\alpha) = \langle g_\ell(\alpha) : \ell < 2\rangle$ where
$g_0(\alpha) = (N^\alpha_{\delta_\alpha}) \cap (\bigcup\limits_{\beta<\alpha} N^\beta_{\delta_\beta})$
$g_1(\alpha) =$ the isomorphic type of $(N^\alpha_{\delta_\alpha}, N^\alpha_i, p^\alpha_i, c)_{c \in g_0(\alpha)}$.

Let C be a club of μ such that $\alpha \in S \cap C \Rightarrow \langle(\bar{N}^\beta, \bar{p}^\beta) : \beta < \alpha\rangle \in N^\alpha_0$, (recall $\ell = 1$).
Fix y such that $S_y = \{\alpha \in S : g(\alpha) = y$ and $\alpha \in C\}$ is stationary.

Let $w_\alpha = \bigcup\limits_{i<\delta_\alpha}$ Dom(p^α_i), $w^*_y = w_\alpha \cap g_0(\alpha)$ for $\alpha \in S_y$ (as $\alpha \in S_y$, clearly the set does not depend on the α). For each $\zeta \in w^*_y$ we define a $\mathbb{P}_\zeta$-name, $\underset{\sim}{S}_{y,\zeta}$ as follows:

$$\underset{\sim}{S}_{y,\zeta} = \{\alpha \in S_y : (\forall i < \delta_\alpha)(p^\alpha_i \restriction \zeta \in \underset{\sim}{G}_{\mathbb{P}_\zeta})\}.$$

Now we try to apply Definition 3.2 in $\mathbf{V}^{\mathbb{P}_\zeta}$ to
$\Big\langle(\langle N^\alpha_i[\underset{\sim}{G}_{\mathbb{P}_\zeta}] : i \le \delta_\alpha\rangle, \langle p^\alpha_i(\zeta)[\underset{\sim}{G}_{\mathbb{P}_\zeta}] : i < \delta_\alpha\rangle) : \alpha \in \underset{\sim}{S}_{y,\zeta}[\underset{\sim}{G}_{\mathbb{P}_\zeta}]\Big\rangle$. Clearly, if $\underset{\sim}{S}_{y,\zeta}[\underset{\sim}{G}_{\mathbb{P}_\zeta}]$ is a stationary subset of μ, we can apply it and $\underset{\sim}{g}_{y,\zeta}$ be the $\mathbb{P}_\zeta$-name of a function with domain $\underset{\sim}{S}_{y,\zeta}$ defined like g in $(*)$ of Definition 3.2. Now $\underset{\sim}{g}_{y,\zeta}$ is well defined, and actually can be computed if we use $A_\beta = \cup\{N^\alpha_{\delta_\alpha}[\underset{\sim}{G}_{\mathbb{P}_\zeta}] : \alpha < \beta\}$. So by an induction hypothesis on α there is a suitable $\mathbb{P}_\zeta$-name $\underset{\sim}{C}_\zeta$ of a club of μ such that in addition, if $\underset{\sim}{S}_{y,\zeta}[G_{\mathbb{P}_\zeta}]$ is not a stationary subset of μ, let $\underset{\sim}{C}_\zeta[G_{\mathbb{P}_\zeta}]$ be a club of μ

disjoint to it. But as $\mathbb{P}_\zeta$ satisfies the μ-c.c. without loss of generality $\underset{\sim}{C}_\zeta = C_\zeta$ so $C' = C \cap \bigcap\limits_{\zeta \in w^*_y} C_\zeta$ is a club of μ. Now choose $\alpha_1 < \alpha_2$ from $S_y \cap C'$ and we choose by induction on $\varepsilon \in w' = w^*_y \cup \{0, \ell g(\bar Q)\}$ a condition $q_\varepsilon \in \mathbb{P}_\varepsilon$ such that:

$\boxtimes_3(i)$ $\varepsilon_1 < \varepsilon \Rightarrow q_{\varepsilon_1} = q_\varepsilon \restriction \varepsilon_1$

(ii) q_ε is a bound to $\{p^{\alpha_1}_u \restriction \varepsilon : i < \delta_{\alpha_1}\} \cup \{p^{\alpha_2}_i \restriction \varepsilon : i < \delta_{\alpha_2}\}$.

For $\varepsilon = 0$ let $q_0 = \emptyset$. We have nothing to do really if ε is with no immediate predecessor in w, we let q_ε be $\cup\{q_{\varepsilon_1} : \varepsilon_1 < \varepsilon, \varepsilon_1 \in w'\}$. So let $\varepsilon = \varepsilon_1 + 1, \varepsilon_1 \in w'$; now if $q_\varepsilon \in G \subseteq \mathbb{P}_{\varepsilon_1,2}, G$ generic over V, <u>then</u> $\alpha_1, \alpha_2 \in \underset{\sim}{S}_{y,\varepsilon_1}[G]$, hence $\underset{\sim}{S}_{y,\zeta}[G] \cap C_{\varepsilon_1}$ is non-empty, hence is stationary, and we use Definition 3.2.

<u>Case 2</u>: $p = 2$.

Similar proof. $\square_{3.7}$

3.9 Claim. *Assume $\mu = cf(\mu) > \kappa, (\forall \alpha < \mu)(|\alpha|^{<\kappa} < \mu), S \subseteq \{\delta < \mu : cf(\delta) \geq \kappa\}$ is stationary. If $|\mathbb{Q}| \leq \kappa$ or just $< \mu, \mathscr{E}_0 \in \mathfrak{C}^-_{<\kappa}(\mu^*)$, that is $\subseteq \{\bar a : \bar a$ increasingly continuous of length $< \kappa, a_i \in [\mu^*]^{<\kappa}$ and $a_i \cap \kappa \in \kappa\}$ non-trivial, possibly just for one cofinality say $\aleph_0$, <u>then</u> $\mathbb{Q}$ satisfies κ^+-pic$_\ell$.*

Proof. Trivial, we get same sequence of condition or just see the proof of [Sh 587, B.7.4]. $\square_{3.9}$

<u>3.10 Discussion</u>: 1) What is the use of pic?

In the forcing axioms instead "$|\mathbb{Q}| \leq \kappa$" we can write "$\mathbb{Q}$ satisfies the $(\mu, S^\square, \hat{\mathscr{E}}_0)$-pic". This strengthens the axioms.

In [Sh:f] in some cases the length of the forcing is bounded (there ω_2) but here no need (as in [Sh:f, Ch.VII,§1]).

This section applies to all cases in [Sh 587] and its branches.

2) Note that we can demand that the p^α_i satisfies some additional requirements (in Definition 3.2) say $p^\alpha_{2i} = F_{\mathbb{Q}}(\bar N \restriction (2i+1), \bar p^\alpha \restriction (2i+1))$.

Let us see how this gives some improvement of the results of [Sh 576, B.8] on $\mathfrak{C}^\spadesuit_{<\kappa}(\mu^*)$, see [Sh 587, B.5.7.3].

3.11 Definition. Assume

⊛ $\kappa > \aleph_0$ is strongly inaccessible and $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1) \in \mathfrak{C}^\spadesuit_{<\kappa}(\mu^*)$ and θ_0, θ_1 are regular cardinals $> \kappa, \theta_2$ a cardinal $> \kappa$ (let $\bar\theta = (\theta_0, \theta_1, \theta_2)$, the usual case is $\theta_0 = \kappa^+$) and $\hat{\mathscr{E}} \subseteq \hat{\mathscr{E}}_1$ is nontrivial (see in Definition 3.3, clause (d)) and $\ell \in \{1, 2\}$.

Let $Ax^{\kappa}_{\theta_1,\theta_2}(\hat{\mathscr{E}}_0,\hat{\mathscr{E}}_1,\mathscr{E})$, the forcing axiom for $(\hat{\mathscr{E}}_0,\hat{\mathscr{E}}_1,\mathscr{E})$, and $\bar\theta = (\theta_0,\theta_1,\theta_2)$ be the following statement:

⊠ <u>if</u>

- (i) $\mathbb{Q}$ is a focing notion of cardinality $< \theta_1$
- (ii) $\mathbb{Q}$ is complete for $(\hat{\mathscr{E}}_0,\hat{\mathscr{E}}_1)$, see Definition [Sh 587, B.5.9(3)]
- (iii) $\mathbb{Q}$ satisfies $(\theta_0, S^{\square}, \hat{\mathscr{E}})$-pic$_\ell$
- (iv) $\mathscr{I}_i$ is a dense subset of $\mathbb{Q}$ for $i < i^* < \theta_2$,

<u>then</u> there is a directed $H \subseteq \mathbb{Q}$ such that $(\forall i < i^*)(H \cap \mathscr{I}_i \neq \emptyset)$.

3.12 Theorem. *Assume ⊛ of Definition 3.11 and $\mu = \mu^{<\theta_1} = \mu^{<\theta_0} \geq \theta_0 + \theta_2$. <u>Then</u> there is a forcing notion $\mathbb{P}$ such that:*

- (α) *$\mathbb{P}$ is complete for $\hat{\mathscr{E}}_0$*
- (β) *$\mathbb{P}$ has cardinality μ*
- (γ) *$\mathbb{P}$ satisfies the θ_0-c.c. and even the $(\kappa,\theta_0,\hat{\mathscr{E}})$-pic$_\ell$*
- (δ) *$\mathbb{P}$ is complete for $(\hat{\mathscr{E}}_0,\hat{\mathscr{E}}_1)$, hence $\Vdash_{\mathbb{P}}$ "$(\hat{\mathscr{E}}_0,\hat{\mathscr{E}}_1) \in \mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$" and more*
- (ε) *$\Vdash_{\mathbb{P}}$ "$Ax^{\kappa}_{\bar\theta}(\hat{\mathscr{E}}_0,\hat{\mathscr{E}}_1,\mathscr{E})$.*

Proof. Like the proof of [Sh 587, B.8.2], using 3.7 instead of [Sh 587, B.7.4]. $\square_{3.12}$

We may wonder how large can a stationary $S \subseteq \kappa$ be?

3.13 Claim. *1) Assume*

- ⊛(a) *κ is strongly inaccessible $> \aleph_0$*
- (b) *$S \subseteq \kappa$ is stationary*
- (c) *for letting $\mu^* = \kappa$ and $\hat{\mathscr{E}}_0 = \hat{\mathscr{E}}_0[S] = \{\bar a \in \mathfrak{C}_{<\kappa}(\mu^*)$: for every $i \leq \ell g(\bar a)$ we have $a_i \notin S\}$ we have $\hat{\mathscr{E}}_0 \in \mathfrak{C}_{<\kappa}(\mu^*)$*
- (d) *we let $\hat{\mathscr{E}}_1 = \hat{\mathscr{E}}_1[S] = \{\bar a \in \mathfrak{C}_{<\kappa}(\mu^*)$: for every nonlimit $i \leq ellg(\bar a)$ we have $a_i \notin S\}$.*

<u>Then</u>

- (α) *$(\hat{\mathscr{E}}_0,\hat{\mathscr{E}}_1) \in \mathfrak{C}^{\spadesuit}_{<\kappa}(\mu^*)$, see [Sh 587, B.5.7(3)].*

2) The parallel of 2.11.

We now deal with forcing the failure of diamond on the set of inaccessibles.

3.14 Claim. *Assume*

(a) $\kappa, S, \hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1$ *are as in 3.13*

(b) *if* $S_{bd} =: \{\theta < \kappa : \theta$ *strongly inaccessible,* $S \cap \theta$ *is stationary in* θ *and* $\diamondsuit_{S\cap\theta}\}$ *is not a stationary subset of* κ

(c) $\bar{A} = \langle A_\alpha : \alpha \in S \rangle, A_\alpha \subseteq \alpha$

(d) $\mathbb{Q} = \mathbb{Q}_{\bar{A}_1}$ *is as in Definition 3.15 below*

(e) $\hat{\mathscr{E}} \subseteq \hat{\mathscr{E}}_0$ *is nontrivial.*
<u>*Then*</u>

(α) $\mathbb{Q}$ *is complete for* $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$

(β) $\mathbb{Q}$ *satisfies the* $(\kappa, \kappa^+, \hat{\mathscr{E}})$*-pic*$_\ell$

(γ) $\mathbb{Q}$ *satisfies the* κ^+*-c.c.*

3.15 Definition. For $\kappa = \text{cf}(\kappa), S \subseteq \kappa = \sup(S), \bar{A} = \langle A_\alpha : \alpha \in S \rangle$, with $A_\alpha \subseteq \alpha$ we define the forcing notions $\mathbb{Q} = \mathbb{Q}^{ad}_{\bar{A}}$ as follows:

(a) $p \in \mathbb{Q}$ iff

(i) $p = (c, A) = (c^p, A^p)$

(ii) c is $\emptyset$ or a closed bounded subset of κ hence has a last element

(iii) $A \subseteq \sup(c)$ such that

(iv) if $\alpha \in C \cap S$ then $A \cap \alpha \neq A_\alpha$

(b) $p \leq q$ iff

(i) c^p is an initial segment of c^q

(ii) $A^p = A^q \cap \sup(c^p)$.

Proof of 3.14. We concentrate on part (1), part (2)'s proof is similar. Now

$(*)_1$ for every $\alpha < \kappa, \mathscr{I}_\alpha = \{p \in \mathbb{Q} : \alpha < \sup(c^p)\}$ is dense open.
[Why? If $p \in \mathbb{Q}$, let $\beta = \sup(c^p) + 1 + \alpha$ and $q = (c^p \cup \{\beta\}, A^p)$, so $p \leq q \in \mathscr{I}_\alpha$.]

$(*)_2$ If $\delta < \kappa$ is a limit ordinal, $\langle p_i : i < \delta \rangle$ is $\leq_{\mathbb{Q}}$-increasing and $\sup(c^{p_i}) \leq \alpha_{i+1} < \sup(c^{p_{i+1}})$ for $i < \delta$, and for limit $i, \alpha_i = \cup\{\alpha_j : j < i\}$ and $\{\alpha_{1+i} : i < \delta\}$

is disjoint to S, <u>then</u> $p = (\bigcup_{i<\delta} c_i^{p_i}, \bigcup_{i<\delta} A^{p_i})$ is a $\le_{\mathbb{Q}}$-lub of $\langle p_i : i < \delta\rangle$.
[Why? Just think.]

$(*)_3$ forcing with $\mathbb{Q}$ add no new sequences of length $< \kappa$ of ordinals (or members of $\mathbf{V}$).
[Why? By $(*)_2$+ the assumption $\circledast$, clause (c) of Claim 3.13 as in [Sh 587, B.6].]

$(*)_4$ $\mathbb{Q}$ is complete for $\hat{\mathscr{E}}_0$
[Why? Just think.]

$(*)_5$ $\mathbb{Q}$ is complete for $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$, see [Sh 587, Def.B.5.9(3)].
[Why? Let χ be large enough and let $\langle M_i : i < \delta\rangle$ be ruled by $(\hat{\mathscr{E}}_0, \hat{\mathscr{E}}_1)$, with $\hat{\mathscr{E}}_0$-approximation $\langle(\bar N^i, \bar a^i) : i < \delta\rangle$, see [Sh 587, Def.B.5.9(1)] and $r \in \mathbb{Q} \cap M_0$ and $S, \kappa, \bar A \in M_0$ and we have to prove that the player COM has a winning strategy in the game $\Game_{\bar M, \langle \bar N^i : i<\delta\rangle}(\mathbb{Q}, r)$.]
For this we proved by induction on $\delta < \kappa$ (a limit ordinal) the statement

$\boxtimes_\delta$ if $\langle M_i : i \le \delta\rangle, \langle \bar N^i : i < \delta\rangle, r$ are as above (but α may be a nonlimit ordinal) $\bar b = \langle b_i : i < \delta\rangle, b_i \in [M_{i+1} \cap \kappa \backslash M_i]^{\le \|M_i\|}$ and $B \subseteq M_\delta \cap \kappa$ (or just $B \subseteq \cup\{b_i : i < \delta\}$, <u>then</u> we can find p such that $r \le p \in \mathbb{Q}$ and $A^p \cap b_i = B \cap b_i$ for every $i < \delta$ and $\sup(c^p) = M_\delta \cap \kappa$.

<u>Case 1</u>: α nonlimit. Trivial.

<u>Case 2</u>: α limit and for some $i < \alpha$ we have $\mathrm{cf}(\delta) \le \|M_i\|$.

Let $\theta =$ cf(θ) and let $\langle \delta_\varepsilon : \varepsilon \le \theta\rangle$ be increasing continuous, $\delta_0 = 0, \|M_{\delta_1}\| > \theta$ and $\delta_\theta = \delta$.

Choose $b \subseteq M_{\delta_1+1} \cap \kappa \backslash M_{\delta_1} \backslash b_{\delta_1}$ of cardinality θ and choose $b' \subseteq b$ such that $\zeta \in (\varepsilon, \delta] \Rightarrow A_{M_{\delta_\zeta} \cap \kappa} \cap b \ne b'$. By the induction hypothesis, we can find $r_{\delta_1} \in M_{\delta_1+1}$ such that $\sup(c^{r_1}) = M_{\delta_1} \cap \kappa, r \le r_{\delta_0}, \beta < \delta_1 \Rightarrow A^{r_1} \cap b_\beta = B \cap b_\beta$ and r_1 is $(M_\beta, \mathbb{Q})$-generic for every $\beta \le \delta_1$. Let r_1^+ be such that $r_{\delta_1} \le r^+_{\delta_1} \in \mathbb{Q} \cap M_{\delta_1+1}$ and $\sup(b_{\delta_1} \cup b) < \sup(r^+_{\delta_1})$ and $A^{r^+_{\delta_1}} \cap b_{\delta_1} = B \cap b_{\delta_1}$ and $A^{r_1^+} \cap b = b'$. Now we choose by induction on $\varepsilon \in [2, \delta]$, a condition r_ε such that $r_\varepsilon \in M_{\delta_\varepsilon+1}$, $\sup(c^{r_\varepsilon}) = M_{\delta_\varepsilon} \cap \kappa, r_1^+ \le r_\varepsilon, [\zeta \in [2, \varepsilon) \Rightarrow r_\zeta \le r_\varepsilon]$ and $\beta < \delta_\varepsilon \Rightarrow A^{r_\varepsilon} \cap b_\beta = B \cap b_\varepsilon$ and r_ε is $(M_\gamma, \mathbb{Q})$-generic for $\gamma \le \delta_\varepsilon$. For limit $\varepsilon, r_\varepsilon$ is uniquely determined and it $\in \mathbb{Q}$ by the choice of r_1^+. For ε nonlimit use the induction hypothesis for $\langle M_\beta : \beta \in [\delta_\varepsilon + 1, \delta_{\varepsilon+1}]\rangle$.

<u>Case 3</u>: Neither Case 1 nor Case 2.

So α is strongly inaccessible, call it θ and $\theta = M_\theta \cap \kappa$; so as $\{\kappa, S\} \in M_\theta \prec (\mathscr{H}(\chi), \in, <^*_\chi)$, necessarily $\delta = \sup(S), \delta \in S_{\mathrm{bd}}$ and $\neg \diamondsuit_{\theta \cap S}$ (e.g. $\theta \cap S$ is not stationary in S). Choose for each $\beta < \theta$, an ordinal $\gamma_\beta \in M_{\beta+1} \cap \kappa \backslash M_\beta \backslash b_\beta$ and let $A'_i = \{j < i : \gamma_j \in A_{M_\beta \cap \kappa}\}$ for $i \in S \cap \theta$.

Now $\langle A'_i : i \in S \cap \theta \rangle$ cannot be a diamond sequence for θ hence we can find $X \subseteq \theta$ and club C^- of θ such that $\delta \in X \cap S \Rightarrow A^-_\delta \neq X \cap \delta$. Let $C = \{i < \theta : i$ limit, $(\forall j < i)(\gamma_j < i)$ and $i \in C^-$ and $M_i \cap \kappa = i\}$, clearly C is a club of θ. Let $b^+_\beta = a_\beta \cup \{\gamma_\beta\}, B^+ = B \cup \{\gamma_\beta : \beta \in X\}$, and proceed naturally. $\square_{3.14}$

3.16 Remark. So we can iterate and get that (G.C.H. and) diamond fail for "most" stationary subsets of any strongly inaccessibles. We shall return to this elsewhere.

§4 Existence of non-free Whitehead (and $\mathrm{Ext}(G,\mathbb{Z}) = \{0\}$) abelian groups in successor of singulars

In [Sh 587], the consistency with GCH of the following is proved for some regular uncountable κ: there is a κ-free nonfree abelian group of cardinality κ, and all such groups are Whitehead. We use κ inaccessible, here we ask: is this assumption necessary for the first such κ?
The following claim seems to support the hope for a positive answer.

4.1 Claim. *Assume*

- (a) *λ is strong limit singular,* $\sigma = cf(\lambda) < \lambda, \kappa = \lambda^+ = 2^\lambda$
- (b) $S \subseteq \{\delta < \kappa : cf(\delta) = \sigma\}$ *is stationary*
- (c) *S does not reflect or at least*
- $(c)^-$ $\bar{A} = \langle A_\delta : \delta \in S\rangle, otp(A_\delta) = \sigma, \sup(A_\delta) = \delta$ *and* $\bar{A}$ *is λ-free, i.e., for every* $\alpha^* < \kappa$ *we can find* $\langle \alpha_\delta : \delta \in \alpha^* \cap S\rangle, \alpha_\delta < \delta$ *such that* $\langle A_\delta \backslash \alpha_\delta : \delta \in S \cap \alpha^*\rangle$ *is a sequence of pairwise disjoint sets*
- (d) $\langle G_i : i \leq \sigma\rangle$ *is a sequence of abelian groups such that:*
  - (α) $\delta < \sigma$ *limit* $\Rightarrow G_\delta = \bigcup\limits_{i<\delta} G_i$
  - (β) $i < j \leq \sigma \Rightarrow G_j/G_i$ *free and* $G_i \subseteq G_j$
  - (γ) $G_\sigma/\bigcup\limits_{i<\sigma} G_i$ *is not Whitehead*
  - (δ) $|G_\sigma| < \lambda$
  - (ε) $G_0 = \{0\}$.

Then
1) There is a strongly κ-free abelian group of cardinality κ which is not Whitehead, in fact $\Gamma(G) \subseteq S$.
2) There is a strongly κ-free abelian group G^* of cardinality κ satisfying $HOM(G^*,\mathbb{Z}) = \{0\}$, *in fact* $\Gamma(G^*) \subseteq S$ *(in fact the same abelian group can serve).*
3) We can rephrase clause $(d)(\gamma)$ of the assumption, i.e. "$G_\sigma/\bigcup\limits_{i<\sigma} G_i$ is not Whitehead" by:

$(d)(\gamma)^-$ *some* $f^* \in HOM(\bigcup\limits_{i<\sigma} G_i, \mathbb{Z})$ *cannot be extended to* $f' \in HOM(G_\sigma,\mathbb{Z})$.

We first note:

4.2 Claim. *Assume*

(a) λ *strong limit singular,* $\sigma = cf(\lambda) < \lambda, \kappa = 2^\lambda = \lambda^+$

(b) $S \subseteq \{\delta < \kappa : cf(\delta) = \sigma$ *and* λ^ω *divides* δ *for simplicity*$\}$ *is stationary*

(c) $A_\delta \subseteq \delta = \sup(A_\delta)$, $otp(A_\delta) = \sigma, A_\delta = \{\alpha_{\delta,\zeta} : \zeta < \sigma\}$ *increasing with* ζ

(d) *let* $h_0 : \kappa \to \kappa$ *and* $h_1 : \kappa \to \sigma$ *be such that* $(\forall \alpha < \kappa)(\forall \zeta < \sigma)(\forall \gamma \in (\alpha, \kappa))(\exists^\lambda \beta \in [\gamma, \gamma + \lambda])(h_0(\beta) = \alpha$ *and* $h_1(\beta) = \zeta)$, *and* $(\forall \alpha < \kappa) h_0(\alpha) \leq \alpha$

(e) *Let* $\bar{\lambda} = \langle \lambda_\zeta : \zeta < \sigma \rangle$ *be increasing continuous with limit* λ *such that* $\lambda_0 = 0$ *and* $\zeta < \sigma \Rightarrow \lambda_{\zeta+1} = cf(\lambda_{\zeta+1}) > \sigma$.

Then we can choose $\langle (g_\delta, \langle \gamma^\delta_\zeta : \zeta < \lambda \rangle) : \delta \in S \rangle$ *such that*

$\bigodot_1$(i) $\langle \gamma^\delta_\zeta : \zeta < \lambda \rangle$ *is strictly increasing with limit* δ

(ii) *if* $\lambda_\zeta \leq \xi < \lambda_{\zeta+1}$ *then* $h_0(\gamma^\delta_\xi) = h_0(\gamma^\delta_{\lambda_\zeta}) = \alpha_{\delta,\zeta}$ *and* $h_1(\gamma^\delta_\xi) = h_1(\gamma^\delta_{\lambda_\zeta}) = \zeta$

(iii) h^*_δ *a partial function from* κ *to* κ, $sup(Dom(h^*_\delta)) < \gamma^\delta_\zeta$ *for* $\delta \in S$

$\bigodot_2$ *for every* $f : \kappa \to \kappa, B \in [\kappa]^{<\lambda}$ *and* $g^2_\zeta : \kappa \to \lambda_{\zeta+1}$ *for* $\zeta < \sigma$ *there are stationarily many* $\delta \in S$ *such that:*

(i) $h^*_\delta = f \restriction B$

(ii) *if* $\lambda_\zeta \leq \xi < \lambda_{\xi+1}$ *then* $g^2_\zeta(\gamma^\delta_\xi) = g^2_\zeta(\gamma^\delta_{\lambda_\zeta})$.

Remark. Note that when subtraction or division[3] is meaningful, $\bigodot_2$ is quite strong.

Proof. By the proofs of 1.1, 1.2 (can use guessing clubs by $\alpha_{\delta,\zeta}$'s, can demand that $\beta^\delta_{2\zeta}, \beta^\delta_{2\zeta+1} \in [\alpha_{\delta,\zeta}, \alpha_{\delta,\zeta} + \lambda)$.

But to help the reader we give a proof.

Let $\lambda = \sum\limits_{i<\sigma} \lambda_i, \lambda_i$ increasing continuous, $\lambda_{i+1} > 2^{\lambda_i}, \lambda_0 = 0, \lambda_1 > 2^\sigma$. Let $M_i \prec (\mathscr{H}((2^\kappa)^+), \in, <^*)$ be increasing continuous, $\|M_i\| = \lambda, \langle M_j : j \leq i \rangle \in M_{i+1}, \lambda + 1 \subseteq M_i$ and $\{\bar{A}, h_0, h_1, \bar{\lambda}\} \in M_0$. For $\alpha < \lambda^+$, let $\alpha = \bigcup\limits_{i<\sigma} a_{\alpha,i}$ such that $|a_{\alpha,i}| \leq \lambda_i$ and $a_{\alpha,i} \in M_{\alpha+1}$ and even $\langle < a_{\beta,i} : i < \sigma >: \beta \leq \alpha \rangle \in M_{\alpha+1}$. Without loss of generality $\delta \in S \Rightarrow \delta$ divisible by λ^ω (ordinal exponentiation). For $\delta \in S$

[3] i.e. x_β belongs to some additive group G^* for $\beta < \kappa, \hat{g} \in$ Hom$(G^*, H^*), g(\beta) = \hat{g}(x_\beta)$ then for some δ as in $\bigodot_2$, we have $g(x^0_{\beta^\delta_\xi} - x_{\beta^\delta_{\lambda_\zeta}})$ is 0_{H^*}; similarly for multiplicative groups

let $\bar\beta^\delta = \langle \beta^\delta_i : i < \sigma\rangle$ be increasing continuous with limit δ, β^δ_i divisible by λ and > 0. For $\delta \in S$ let $\langle b^\delta_i : i < \sigma\rangle$ be such that: $b^\delta_i \subseteq \beta^\delta_i, |b^\delta_i| \le \lambda_i, b^\delta_i$ is increasingly continuous in i and $\delta = \bigcup\limits_{i<\sigma} b^\delta_i$ (e.g. $b^\delta_i = \bigcup\limits_{j_1,j_2<i} a_{\beta^\delta_{j_1,j_2}} \cup \lambda_i$). We further demand $\lambda_i \subseteq b^\delta_i \cap \lambda$. Let $\langle f^*_\alpha : \alpha < \lambda^+\rangle$ list the two-place functions with domain an ordinal $< \lambda^+$ and range $\subseteq \lambda^+$. Let H be the set of functions h, $\mathrm{Dom}(h) \in [\kappa]^{<\lambda}$, $\mathrm{Rang}(h) \subseteq \kappa$, so $|H| = \kappa$. Let $S = \cup\{S_h : h \in H\}$, with each S_h stationary and $\langle S_h : h \in H\rangle$ pairwise disjoint. Without loss of generality $\delta \in S_h \Rightarrow \sup(\mathrm{Dom}(h)) < \beta^\delta_0$. Let h^*_δ be h when $\delta \in S_h$. We now fixed $h \in H$ and will choose $\bar\gamma^\delta = \langle \gamma^\delta_i : i < \lambda\rangle$ for $\delta \in S_h$ such that clauses $\bigodot_1 + \bigodot_2$ for our fixed h (and $\delta \in S_h$ ignoring h in $\bigodot_2$) hold, this clearly suffices.
Now for $\delta \in S_h$ and $i < \sigma$ and $g \in {}^\sigma\sigma$ we can choose $\zeta^\delta_{i,g,\varepsilon}$ (for $\varepsilon < \lambda_{i+1}$) such that:

- (A) $\langle \zeta^\delta_{i,g,\varepsilon} : \varepsilon < \lambda_{i+1}\rangle$ is a strictly increasing sequence of ordinals
- (B) $\beta^\delta_i < \zeta^\delta_{i,g,\varepsilon} < \beta^\delta_{i+1}$, (can even demand $\zeta^\delta_{i,j,\varepsilon} < \beta^\delta_i + \lambda$)
- (C) $h_0(\zeta^\delta_{i,g,\varepsilon}) = \alpha_{\delta,i}$ and $h_1(\zeta^\delta_{i,g,\varepsilon}) = i$
- (D) for[4] every $\alpha_1, \alpha_2 \in b^\delta_{g(i)}$, the sequence $\langle \mathrm{Min}\{\lambda_{g(i)}, f^*_{\alpha_1}(\alpha_2, \zeta^\delta_{i,g,\varepsilon}) : \varepsilon < \lambda_{i+1}\}\rangle$ is constant i.e. one of the following occurs:
  - (α) $\varepsilon < \lambda_{i+1} \Rightarrow (\alpha_2, \zeta^\delta_{i,g,\varepsilon}) \notin \mathrm{Dom}(f^*_{\alpha_1})$
  - (β) $\varepsilon < \lambda_{i+1} \Rightarrow f^*_{\alpha_1}(\alpha_2, \zeta^\delta_{i,g,\varepsilon}) = f^*_{\alpha_1}(\alpha_2, \zeta^\delta_{i,j,0})$ well defined
  - (γ) $\varepsilon < \lambda_j, f^*_{\alpha_1}(\alpha_2, \zeta^\delta_{i,g,\varepsilon}) \ge \lambda_j$, well defined. We can add $\langle f^*_{\alpha_1}(\alpha_2, \zeta^\delta_{i,g,\varepsilon}) : \varepsilon < \lambda_i\rangle$ is constant or strictly increasing.
- (E) for some $j < \sigma$, we have $(\forall \varepsilon < \lambda_{i+1})[\zeta^\delta_{i,g,\varepsilon} \in a_{\alpha,j}]$ where $\alpha = \sup\{\zeta^\delta_{i,g,\varepsilon} : \varepsilon < \lambda_{i+1}\}$, (remember $\sigma \ne \lambda_{i+1}$ are regular).

For each function $g \in {}^\sigma\sigma$ we try $\bar\gamma^{g,\delta} = \langle \gamma^{\delta,g}_\varepsilon : \varepsilon < \lambda\rangle$ be: if $\lambda_i \le \varepsilon < \lambda_{i+1}$ then $\gamma^{\delta,g}_\alpha = \zeta^\delta_{i,g,\varepsilon}$.
Now for some g it works. $\square_{4.2}$

Proof of 1.2(1). Let $M = \cup\{M_\alpha : \alpha < \kappa\}, M_\alpha \prec (\mathscr{H}(2^\kappa)^+), \in)$ has cardinality λ, M_α is increasing continuous, $\langle M_\beta : \beta \le \alpha\rangle \in M_\alpha$ and $\langle F_i : i < \sigma\rangle$ belongs to M_0. Let $E_0 = \{\delta < \kappa : M_\delta \cap \kappa = \delta\}$ and $E =$ acc(E). The proof is like the proof of 4.2 with the following changes:

- (i) $\beta^\delta_i \in E_0$ for $\delta \in S \cap E$

[4] we can use a colouring which uses e.g. $\langle \zeta^\delta_{j,g,\varepsilon} : j < i, \varepsilon < \lambda_{j+1}\rangle$ as a parameter

(*ii*) in clause (A) we demand $\langle \zeta^\delta_{i,g,\varepsilon} : g \in G, \varepsilon < \lambda_{i+1}\rangle$ belongs to $M_{\beta^\delta_{i+1}}$ (hence also $\langle \zeta^\delta_{j,g,\varepsilon} : g \in G, \varepsilon < \lambda_{j+1} : j \le i\rangle$ belongs to $M_{\beta^\delta_{i+1}}$)

(*iii*) clause (c) is replaced by: $\zeta^\delta_{i,g,\varepsilon} \in F_i(\{\zeta^\delta_{j,g\restriction(j+1),\varepsilon} : \varepsilon < \lambda_{j+1}$ and $j < i\})$.

$\square_{1.2}$

Proof of 4.1. 1) We apply 4.2 to the $\langle A_\delta : \delta \in S\rangle$ from 4.1, and any h_0, h_1 as in clause (d) of 4.2.
Let $\{t^{i,j}_\gamma + G_i : \gamma < \theta^{i,j}\}$ be a free basis of G^j/G^i for $i < j \le \sigma$. If $i = 0, j = \sigma$ we may omit the i,j, i.e. $t_\zeta = t^{0,\sigma}_\zeta$ and $\theta = \theta^{0,\sigma}$. Let $\theta + \aleph_0 = |G_\sigma| < \lambda$; actually $\theta^{\zeta,\zeta+1} < \lambda_\zeta$ is enough; without loss of generality $\theta < \lambda_1$ in 4.2. Let $\beta^\delta_{\zeta,i} = \gamma^\delta_{\xi(\zeta,i)}$ where $\xi(\zeta,i) = \bigcup\limits_{\varepsilon<\zeta} \lambda_\varepsilon + 1 + i$ for $\delta \in S, \zeta < \sigma, i < \theta$.
Let $\beta_\delta(*) = \text{Min}\{\beta : \beta \in \text{Dom}(h^*_\delta), h^*_\delta(\beta) \ne 0\}$, if well defined where h^*_δ is from 4.2.
Clearly (see $\bigodot_1(iii)$ of 4.2) we have $\beta_\delta(*) \notin \{\beta^\delta_{\zeta,i} : \zeta < \sigma, i < \theta\}$ (or omit $\lambda_\zeta, \beta^\delta_{\zeta,i}$ for ζ too small). We define an abelian group G^*: it is generated by $\{x_\alpha : \alpha < \kappa\} \cup \{y^\delta_\gamma : \gamma < \theta$ and $\delta \in S\}$ freely except for the relations:

$(*)_1$ $\sum\limits_{\gamma<\theta} a_\gamma y^\delta_\gamma = \sum\{b_{\zeta,\gamma}(x_{\beta^\delta_{\zeta,\gamma}} - x_{\gamma^\delta_{\lambda_\zeta}}) : \zeta < \sigma \text{ and } \gamma < \theta^{\zeta,\zeta+1}\}$
when $G_\sigma \models \sum\limits_{\gamma<\theta^{0,\sigma}} a_\gamma t_\gamma = \sum\{b_{\zeta,\gamma} t^{\zeta,\zeta+1}_\gamma : \zeta < \sigma \text{ and } \gamma < \theta^{\zeta,\zeta+1}\}$ where
$a_\gamma, b_{\zeta,\gamma} \in \mathbb{Z}$ but all except finitely many are zero.

There is a (unique) homomorphism $\mathbf{g}_\delta$ from G_σ into G^* induced by $\mathbf{g}_\delta(t_\gamma) = y^\delta_\gamma$. As usual it is an embedding. Let $\text{Rang}(\mathbf{g}_\delta) = G^{<\delta>}$.
For $\beta < \kappa$ let G^*_β be the subgroup of G^* generated by $\{x_\alpha : \alpha < \beta\} \cup \{y^\delta_\gamma : \gamma < \theta^{0,\sigma}$ and $\delta \in \beta \cap S\}$. It can be described similarly to G^*.

<u>Fact A</u>: G^* is strongly λ-free.

Proof. For $\alpha^* < \beta^* < \kappa$, we can find $\langle \alpha_\delta : \delta \in S \cap (\alpha^*, \beta^*]\rangle$ such that $\langle A_\delta \backslash \alpha_\delta : \delta \in S \cap (\alpha^*, \beta^*]\rangle$ are pairwise disjoint and disjoint to α^* hence the sequence $\langle \{\beta^\delta_{\zeta,i} : i < \theta, \zeta \in (\text{Min}\{\xi < \sigma : \beta^\delta_{\zeta,0} > \alpha_\delta\}, \sigma)\} : \delta \in S \cap (\alpha^*, \beta^*]\rangle$ is a sequence of pairwise disjoint sets.
For $\delta \in S \cap (\alpha^*, \beta^*]$, let $\zeta_\delta = \text{Min}\{\zeta : \beta^\delta_{\zeta,0} > \alpha_\delta\} < \sigma$. Now easily $G^*_{\beta^*+1}$ is generated as an extension of $G^*_{\alpha^*+1}$ by $\{\mathbf{g}_\delta(t^{\zeta_\delta,\sigma}_\gamma) : \gamma < \theta^{\zeta_\delta,\sigma}$ and $\delta \in S \cap (\alpha^*, \beta^*]\} \cup \{x_\alpha :$

$\alpha \in (\alpha^*, \beta^*]$ and for no $\delta \in S \cap (\alpha^*, \beta^*]$ do we have $\alpha \in \{\beta^\delta_{\zeta,i} : i < \theta^{\zeta,\sigma}$ and $\zeta < \zeta_\delta\}\}$; moreover $G^*_{\beta^*+1}$ is freely generated (as an extension of $G^*_{\alpha^*+1}$). So $G^*_{\beta^*+1}/G^*_{\alpha^*+1}$ is free, as also G^*_1 is free we have shown Fact A.

Fact B: G^* is not Whitehead.

Proof. We choose by induction on $\alpha \le \kappa$, an abelian group H_α and a homomorphism $\mathbf{h}_\alpha : H_\alpha \to G^*_\alpha = \langle \{x_\beta : \beta < \alpha\} \cup \{y^\delta_\gamma : \gamma < \theta, \delta \in S \cap \alpha\}\rangle_{G^*}$ increasing continuous in α, with kernel $\mathbb{Z}, \mathbf{h}_0 =$ zero and $\mathbf{k}_\alpha : G^*_\alpha \to H_\alpha$ is a not necessarily linear mapping such that $\mathbf{h}_\alpha \circ \mathbf{k}_\alpha = \mathrm{id}_{G^*_\alpha}$. We identify the set of members of $H_\alpha, G_\alpha, \mathbb{Z}$ with subsets of $\lambda \times (1+\alpha)$ such that $O_{H_\alpha} = O_{\mathbb{Z}} = 0$.
Usually we have no freedom or no interesting freedom. But we have for $\alpha = \delta + 1$, $\delta \in S$. What we demand is ($G^{\langle \delta \rangle}$ - see before Fact A):

$(*)_2$ letting $H^{<\delta>} = \{x \in H_{\delta+1} : \mathbf{h}_{\delta+1}(x) \in G^{<\delta>}\}$, if $s^* = g_\delta(x_{\beta_\delta(*)}) \in \mathbb{Z} \backslash \{0\}$ (g_δ from 4.2), then there is no homomorphism $f_\delta : G^{<\delta>} \to H^{<\delta>}$ such that

- (α) $f_\delta(x_{\beta^\delta_{\zeta,i}}) - \mathbf{k}_\delta(x_{\beta^\delta_{\zeta,i}}) \in \mathbb{Z}$ is the same for all $i \in (\bigcup\limits_{\varepsilon<\zeta} \lambda_\varepsilon, \lambda_\zeta]$
- (β) $\mathbf{h}_{\delta+1} \circ f_\delta = \mathrm{id}_{G^{<\delta>}}$.

[Why is this possible? By non-Whiteheadness of $G^\sigma / \bigcup\limits_{i<\sigma} G^i$ that is see $(d)(\gamma)^-$ in 4.1.]
The rest should be clear.

Proof of 4.1(2). Of course, similar to that of 4.1(1) but with some changes.

Step A: Without loss of generality there is a homomorphism f^* from $\bigcup\limits_{i<\sigma} G^i$ to $\mathbb{Z}$ which cannot be extended to a homormopshim from G_σ to $\mathbb{Z}$.
[Why? Standard, see [Fu].]

Step B: During the construction of G^*, we choose G^*_α by induction on $\alpha \le \kappa$, but if $h^*_\delta(0)$ from 4.2 is a member of G^*_δ in $(*)_1$ we replace $(x_{\beta^\delta_{\zeta,\gamma}} - x_{\gamma^\delta_{\lambda_\zeta}})$ by $\big(x_{\beta^\delta_{\zeta,\gamma}} - x_{\beta^\delta_{\lambda_\zeta}} + f^*(t^{\zeta,\zeta+1}_\gamma) g_\delta(0)\big)$, note that $f^*(t^{\zeta,\zeta+1}_\gamma) \in \mathbb{Z}$ and $h^*_\delta(0) \in G^*_\delta$.
So if in the end $f : G^* \to \mathbb{Z}$ is a non-zero homomorphism, let $x^* \in G^*$ be such that

$f(x^*) \neq 0$ and[5] $|f^*(x^*)|$ is minimal under this, so without loss of generality it is 1. Hence for some $\delta \in S$ we have:

$(*)_3$ $f(g_\delta(0)) = 1_{\mathbb{Z}}$

$(*)_4$ $f(x_{\gamma^\delta_{\lambda_\zeta+1+1+\gamma}}) = f(x_{\gamma^\delta_{\lambda_\zeta}})$ for $\gamma \in \lambda_{\zeta+1} \backslash \lambda_\zeta$
that is $f(x_{\beta^\delta_{\zeta,\gamma}}) = f(x_{\gamma^\delta_{\lambda_zeta}})$

(in fact this holds for stationarily many ordinals $\delta \in S$).

So we get an easy contradiction.

3) The proof is included in the proof of part (2). $\square_{4.1}$

We also note the following consequence of a conclusion of an instance of GCH.

4.3 Claim. *Assume*

(a) $\lambda = \mu^+$ *and* $\mu > \sigma = cf(\mu)$
(b) $\lambda = \lambda^\theta$ *where* $\theta = 2^\sigma$
(equivalently $\mu^\theta = \mu^+ > 2^\theta$*)*
(c) $S \subseteq \{\delta < \lambda : cf(\delta) = \sigma\}$ *is stationary*
(d) $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ *with* η_δ *an increasing sequence of length* σ *with limit* δ.

Then we can find $\langle \bar{A}^\delta : \delta \in S \rangle$ *such that:*

(α) $\bar{A}^\delta = \langle A^\delta_i : i < \sigma \rangle$
(β) $A^\delta_i \in [\delta]^{<\mu}$ *and* $sup(A^\delta_i) < \delta$
$(\beta)^+$ *for some* $\langle \lambda^*_i : i < \sigma \rangle$ *increasing with limit* λ, $|A^\delta_i| < \lambda^*_i$,
(γ) *for every* $h : \lambda \to \lambda$*, for stationarily many* $\delta \in S$ *we have* $(\forall i < \sigma)[h(\eta_\delta(i)) \in A^\delta_i]$.

4.4 Remark. 1) We can restrict ourselves to $h : \lambda \to \mu$ in clause (γ), and then, of course, can use $\langle < A^\delta_i : i < \sigma >: \delta \in S \rangle$ with $A^\delta_i \subseteq \mu$.
2) We can add to the conclusion "$A^\delta_i \subseteq \eta_\delta(i+1)$" if $\bar{\eta}$ guess clubs.

Proof. Let $\langle \lambda_i : i < \sigma \rangle$ be increasing continuous with limit μ. Let $\langle \bar{\alpha}_\gamma : \gamma < \lambda \rangle$ list ${}^\theta\lambda$, so $\bar{\alpha}_\gamma = \langle \alpha_{\gamma,\varepsilon} : \varepsilon < \theta \rangle$ and without loss of generality $\alpha_{\gamma,\varepsilon} \leq \gamma$. For each $\delta \in S$ let $\langle b^\delta_i : i < \sigma \rangle$ be an increasing continuous sequence of subsets of δ with union δ such that $|b^\delta_i| < \mu$ and $\sup(b^\delta_i) < \delta$; for $(\beta)^+$, moreover $|b^\delta_i| \leq \lambda_i$;

[5] What does this mean? $f^*(x^*)$ is an integer so its absolute value is well defined

this is possible as $\mathrm{cf}(\delta) = \sigma = \mathrm{cf}(\mu) < \mu$. Let $\langle g_\varepsilon : \varepsilon < \theta\rangle$ list ${}^\sigma\sigma$ and define $A_i^{\varepsilon,\delta} =: \{\alpha_{\gamma,\varepsilon} : \gamma \in b^\delta_{g_\varepsilon(i)}\}$. Now $A_i^{\varepsilon,\delta}$ is a set of cardinality $\le |b^\delta_{g_\varepsilon(i)}| < \mu$ and $\sup(A_i^{\varepsilon,\delta}) \le \sup(b^\delta_{g_\varepsilon(i)})$ (as we have demanded that $\alpha_{\gamma,\varepsilon} \le \gamma$) but $\sup(b^\delta_{g_\varepsilon(i)}) < \delta$ by the choice of the b^δ_j's hence $\sup(A_i^{\varepsilon,\delta}) < \delta$. So for each $\varepsilon < \theta$ the sequence $\bar{\mathbf{A}}^\varepsilon =: \langle \bar{A}^{\varepsilon,\delta} : \delta \in S\rangle$, where $\bar{A}^{\varepsilon,\delta} = \langle A_i^{\varepsilon,\delta} : i < \sigma\rangle$ satisfies clauses $(\alpha) + (\beta)$ and $(\beta)^+$ when relevant. Hence it suffices to prove that for some $\varepsilon < \theta$ the sequence $\bar{\mathbf{A}}^\varepsilon$ satisfy clause (γ), too. Assume toward contradiction that for every $\varepsilon < \theta$ the sequence $\bar{\mathbf{A}}^\varepsilon$ fails clause (γ) hence there is $h_\varepsilon : \lambda \to \lambda$ which exemplifies this, that is for some club E_ε of $\lambda, \delta \in E_\varepsilon \cap S \Rightarrow (\exists i < \sigma)[h_\varepsilon(\eta_\delta(i)) \notin A_i^{\varepsilon,\delta}]$. So for every $\beta < \lambda$ the sequence $\langle h_\varepsilon(\beta) : \varepsilon < \theta\rangle$ belongs to ${}^\theta\lambda$, hence is equal to $\bar{\alpha}_{h(\beta)}$ for some $h(\beta) < \lambda$. Clearly $E = \{\delta < \lambda : \delta$ a limit ordinal and $(\forall\beta < \delta)h(\beta) < \delta\}$ is a club of λ (recall $\theta < \lambda$) hence we can find $\delta(*) \in E \cap S$. We define $g^* : \sigma \to \sigma$ by $g^*(i) = \mathrm{Min}\{j < \sigma : h(\eta_{\delta(*)}(j)) \in b^\delta_j\}$, now g^* is well defined as, for $i < \sigma$ the ordinal $h(\eta_{\delta(*)}(i))$ is $< \delta(*)$ (as $\delta(*) \in E$) and $\eta_{\delta(*)}(i) < \delta(*)$) and $\delta = \bigcup\limits_{j<\sigma} b^\delta_j$. As $g^* \in {}^\sigma\sigma$ clearly for some $\varepsilon(*) < \theta$ we have $g_{\varepsilon(*)} = g^*$.

So, for any $i < \sigma$ let $\gamma_i = h(\eta_{\delta(*)}(i))$, now $h(\eta_{\delta(*)}(i)) \in b^\delta_{g^*(i)}$ (by the choice of g^*) and $g^*(i) = g_{\varepsilon(*)}(i)$ by the choice of $\varepsilon(*)$, together $\gamma_i \in b^\delta_{g_{\varepsilon(*)}(i)}$. But $A_i^{\varepsilon(*),\delta(*)} = \{\alpha_{\gamma,\varepsilon(*)} : \gamma \in b^\delta_{g_{\varepsilon(*)}(i)}\}$ by the choice of $A_i^{\varepsilon(*),\delta(*)}$ hence $\alpha_{\gamma_i,\varepsilon(*)} \in A_i^{\varepsilon(*),\delta(*)}$, but as $\gamma_i = h(\eta_{\delta(*)}(i))$, by the choice of h we have $h_{\varepsilon(*)}(\eta_{\delta(*)}(i)) = \alpha_{\gamma_i,\varepsilon(*)} \in A_i^{\varepsilon(*),\delta(*)}$.

So $(\forall i < \sigma)(h_\varepsilon(\eta_{\delta(*)}(i)) \in A_i^{\varepsilon(*),\delta(*)})$, which by the choice of h_ε implies $\delta(*) \notin E_{\varepsilon(*)}$ but $\delta(*) \in E \subseteq \bigcap\limits_{\varepsilon<\sigma} E_\varepsilon$, contradiction. $\square_{4.3}$

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