ON CIESIELSKI'S PROBLEMS

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Abstract. In the present paper we discuss some problems formulated in Ciesielski [3].

0. Introduction. I was asked to read and comment on Ciesielski's survey paper [3]. I have found it very exciting and illuminating. Quite naturally I was not able to resist the temptation to look mainly at the open problems formulated in this nice paper. Some of them are related to my research *in progress* and may be solved soon. This is in particular the case with Problems 5, 1 and 6, for which relevant information should be given by Ciesielski Shelah [4] and Rosłanowski Shelah [14]. For some other problems (like [3, Problem 3]) I have ideas that could work and this may materialize in a continuation of the present paper.

Here we would like to present answers to three problems and address a fourth one. In the first section we solve [3, Problem 8] and we show that, consistently, $\mathrm{d}_{\mathfrak{c}}$ is a singular cardinal and $\mathfrak{e}_{\mathfrak{c}} < \mathrm{d}_{\mathfrak{c}}$ (in 1.9; see 1.1 for the definitions of $\mathfrak{e}_{\mathfrak{c}}, \mathrm{d}_{\mathfrak{c}}$). In the next section we present some results relevant for [3, Problem 9]. We do not solve the problem, but it was formulated in a very general way (*When does* $\mathrm{d}_{\mathfrak{c}} = \mathrm{d}^*_{\mathfrak{c}}$ *or* $\mathfrak{e}_{\mathfrak{c}} = \mathfrak{e}^*_{\mathfrak{c}}$ *hold?*) making the full answer rather difficult. The third section answers [3, Problem 7]. We show there that the Martin Axiom for σ–centered forcing notions implies that for every function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ there are functions $g_n, h_n : \mathbb{R} \longrightarrow \mathbb{R}$, $n < \omega$, such that $f(x,y) = \sum\limits_{n=0}^{\infty} g_n(x) \cdot h_n(y)$. Finally, in the next section we deal

Date: September 15, 2020.
1991 *Mathematics Subject Classification.* Primary: 26Axx, 03E05;
Secondary 03E35, 03E50.
The research was partially supported by "Basic Research Foundation" of the Israel Academy of Sciences and Humanities. Publication 675.

with countably continuous functions and we show (in 4.2) that in the Cohen model they are exactly the functions f with the property that

$$(\forall U\in[\mathbb{R}]^{\aleph_1})(\exists U^*\in[U]^{\aleph_1})(f\restriction U^* \text{ is continuous}).$$

This answers negatively [3, Problem 4].

Notation: Our notation is rather standard and compatible with that of classical textbooks on Set Theory (like Jech [11] or Bartoszyński Judah [1]). However in forcing we keep the convention that *a stronger condition is the larger one.*

Notation 0.1. We will keep the following rules for our notation:

(1) $\alpha,\beta,\gamma,\delta,\xi,\zeta,i,j\ldots$ will denote ordinals.
(2) $\kappa,\lambda,\mu,\theta\ldots$ will stand for cardinal numbers, $\mathfrak{c}$ is the cardinality of the continuum.
(3) A bar above a name indicates that the object is a sequence, usually $\bar{X}$ will be $\langle X_i : i<\ell g(\bar{X})\rangle$, where $\ell g(\bar{X})$ denotes the length of $\bar{X}$.
(4) A tilde indicates that we are dealing with a name for an object in forcing extension (like $\underset{\sim}{x}$). The canonical $\mathbb{P}$–name for a generic filter is called $\underset{\sim}{G}_{\mathbb{P}}$.
(5) For a cardinal κ, the quantifiers $(\exists^\kappa i)$ and $(\forall^\kappa i)$ are abbreviations for "there is κ many i such that…" and "for all but less than κ many $i\ldots$", respectively.
(6) otp stands for "order type". When using elements of the pcf–theory we will follow the notation and terminology of [16]. In particular, tcf will stand for "true cofinality" and $J^{\rm bd}_\theta$ will denote the ideal of bounded subsets of θ.

1. Around d_κ and $\mathfrak{e}_\kappa$.

Definition 1.1. Let $\theta\le\kappa$ be cardinals.

(1) Let $\mathcal{S}^\theta_\kappa \stackrel{\rm def}{=} \prod\limits_{i<\kappa}[\kappa]^{<\theta}$.
(2) We define the following cardinal coefficients of the space ${}^\kappa\kappa$:

$$\begin{aligned}
\mathrm{d}_\kappa &= \min\{|F| : F\subseteq {}^\kappa\kappa \ \&\ (\forall g\in{}^\kappa\kappa)(\exists f\in F)(\exists^\kappa i<\kappa)(f(i)=g(i))\},\\
\mathfrak{e}_\kappa &= \min\{|F| : F\subseteq {}^\kappa\kappa \ \&\ (\forall g\in{}^\kappa\kappa)(\exists f\in F)(\forall^\kappa i<\kappa)(f(i)\neq g(i))\},\\
\mathfrak{d}_\kappa &= \min\{|F| : F\subseteq {}^\kappa\kappa \ \&\ (\forall g\in{}^\kappa\kappa)(\exists f\in F)(\forall^\kappa i<\kappa)(g(i)<f(i))\},\\
\mathfrak{b}_\kappa &= \min\{|F| : F\subseteq {}^\kappa\kappa \ \&\ (\forall g\in{}^\kappa\kappa)(\exists f\in F)(\exists^\kappa i<\kappa)(g(i)<f(i))\},\\
\mathrm{c}(\kappa,\theta) &= \min\{|G| : G\subseteq \mathcal{S}^\theta_\kappa \ \&\ (\forall g\in{}^\kappa\kappa)(\exists\bar{S}\in G)(\forall^\kappa i<\kappa)(g(i)\in S_i)\},\\
\mathrm{c}^-(\kappa,\theta) &= \min\{|G| : G\subseteq \mathcal{S}^\theta_\kappa \ \&\ (\forall g\in{}^\kappa\kappa)(\exists\bar{S}\in G)(\exists^\kappa i<\kappa)(g(i)\in S_i)\},\\
\mathrm{b}(\kappa,\theta) &= \min\{|F| : F\subseteq {}^\kappa\kappa \ \&\ (\forall\bar{S}\in\mathcal{S}^\theta_\kappa)(\exists f\in F)(\forall^\kappa i<\kappa)(f(i)\notin S_i)\}.\\
\mathrm{b}^-(\kappa,\theta) &= \min\{|F| : F\subseteq {}^\kappa\kappa \ \&\ (\forall\bar{S}\in\mathcal{S}^\theta_\kappa)(\exists f\in F)(\exists^\kappa i<\kappa)(f(i)\notin S_i)\}.
\end{aligned}$$

(3) For functions $f,g\in {}^\kappa\kappa$ we say that f *dominates* g (in short: $g<^*_\kappa f$) if $(\forall^\kappa i<\kappa)(g(i)<f(i))$.
[Thus $\mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa$ are the unbounded number and the dominating number, respectively, of the partial order $({}^\kappa\kappa,<^*_\kappa)$.]

Remark 1.2. (1) The cardinal invariants introduced in 1.1 are natural generalizations of those studied in Set Theory of the Reals; see e.g. Bartoszyński Judah [1] or Goldstern Shelah [10].
(2) Using 1.1, we may reformulate [3, Problem 8] as follows:
(a) Is it consistent that $\mathrm{d}_\mathfrak{c}>\mathfrak{e}_\mathfrak{c}$?
(b) Can $\mathrm{d}_\mathfrak{c}$ be a singular cardinal?
(see [3, 4.7, 4.12]).

Proposition 1.3. (1) *The partial order* $({}^\kappa\kappa,<^*_\kappa)$ *is* $\mathfrak{b}_\kappa$*–directed. The cardinal* $\mathfrak{b}_\kappa$ *is regular. If* κ *is regular then* $\mathfrak{b}_\kappa=\mathrm{b}^-(\kappa,\kappa)$.
(2) $\mathfrak{b}_\kappa\leq\mathrm{d}_\kappa$. *If* κ *is a successor then* $\mathrm{d}_\kappa=\mathfrak{b}_\kappa$.
(3) $\mathrm{cf}(\kappa)<\mathrm{c}^-(\kappa,\kappa)$ *and* $\theta<\kappa\ \Rightarrow\ \kappa<c^-(\kappa,\theta^+)$.
(4) *Assume that either* $\theta<\mathrm{cf}(\kappa)$ *or* $\theta=\mathrm{cf}(\kappa)$ *is a successor cardinal. Then* $\mathrm{d}_\kappa=\mathrm{c}^-(\kappa,\theta)$.

Proof 1) and 3) Should be clear.

2) For a function $f\in{}^\kappa\kappa$ let $f^+\in{}^\kappa\kappa$ be defined by $f^+(i)=f(i)+1$. Clearly, if $F\subseteq{}^\kappa\kappa$ is a family witnessing the minimum in the definition of d_κ then $\{f^+:f\in F\}$ is a $<^*_\kappa$–unbounded family. Hence $\mathfrak{b}_\kappa\leq\mathrm{d}_\kappa$.

Assume now that $\kappa=\mu^+$ and let $F\subseteq{}^\kappa\kappa$ be $<^*_\kappa$–unbounded, $|F|=\mathfrak{b}_\kappa$. Note that necessarily $\mathfrak{b}_\kappa>\kappa$. For each $\alpha<\kappa$ fix a sequence $\langle\beta_{\alpha,\xi}:\xi<\mu\rangle$ such that $\{\beta_{\alpha,\xi}:\xi<\mu\}=\alpha+1$. For $f\in F$ and $\xi<\mu$ let $h^f_\xi\in{}^\kappa\kappa$ be such that $(\forall i<\kappa)(h^f_\xi(i)=\beta_{f(i),\xi})$. Let

$$F^*\stackrel{\mathrm{def}}{=}\{h^f_\xi: f\in F\ \&\ \xi<\mu\}.$$

Then $|F^*|\leq|F|+\mu=\mathfrak{b}_\kappa$. Suppose $g\in{}^\kappa\kappa$. By the choice of F, we find $f\in F$ such that the set $A\stackrel{\mathrm{def}}{=}\{i<\kappa: g(i)<f(i)\}$ is of cardinality κ. For $i\in A$ let $\xi_i<\mu$ be such that $g(i)=\beta_{f(i),\xi_i}$. Then for some $\xi<\mu$ the set $A_\xi=\{i\in A:\xi_i=\xi\}$ is of size κ. Look at the function $h^f_\xi\in F^*$: for every $i\in A_\xi$ we have $g(i)=h^f_\xi(i)$.

4) First note that plainly $\mathrm{c}^-(\kappa,\theta)\leq\mathrm{d}_\kappa$, so we have to show the converse inequality (under our assumptions).

Assume $\theta<\mathrm{cf}(\kappa)$. Let $G\subseteq\mathcal{S}^\theta_\kappa$ be such that $|G|=\mathrm{c}^-(\kappa,\theta)$ and

$$(\forall g\in{}^\kappa\kappa)(\exists\bar{S}\in G)(\exists^\kappa i<\kappa)(g(i)\in S_i).$$

For $\bar{S} \in G$ and $i < \kappa$ fix an enumeration $\{\beta^{\bar{S},i}_\varepsilon : \varepsilon < \varepsilon^{\bar{S},i}\}$ of S_i (so $\varepsilon^{\bar{S},i} < \theta$). Next define functions $h^{\bar{S}}_\varepsilon \in \kappa$ (for $\bar{S} \in G$ and $\varepsilon < \theta$) by

$$h^{\bar{S}}_\varepsilon(i) = \begin{cases} \beta^{\bar{S},i}_\varepsilon & \text{if } \varepsilon < \varepsilon^{\bar{S},i}_\varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $g \in {}^\kappa\kappa$. Take $\bar{S} \in G$ such that $(\exists^\kappa i<\kappa)(g(i) \in S_i)$. Then for some $\varepsilon < \theta$ we have

$$(\exists^\kappa i<\kappa)(g(i) = \beta^{\bar{S},i}_\varepsilon = h^{\bar{S}}_\varepsilon(i)),$$

and hence we may conclude that $\mathrm{d}_\kappa \leq \mathrm{c}^-(\kappa,\theta)+\theta$ is witnessed by the family $\{h^{\bar{S}}_\varepsilon : \bar{S} \in G \ \&\ \varepsilon < \theta\}$. Finally we note that $\mathrm{c}^-(\kappa,\theta)+\theta = \mathrm{c}^-(\kappa,\theta)$ (by (3); remember $\mathrm{c}^-(\kappa,\theta) \geq \mathrm{c}^-(\kappa,\kappa)$).

If $\theta = \mathrm{cf}(\kappa)$ is a successor cardinal, say $\theta = \mu^+$, then we proceed similarly: we may assume that for each $\bar{S} \in G$ and $i < \kappa$ we have $|S_i| = \mu = \varepsilon^{\bar{S},i}$ and we finish as above (as $\mu < \mathrm{cf}(\kappa)$). ■

Proposition 1.4. *Assume that κ is a strong limit singular cardinal,* $\mathrm{cf}(\kappa) = \theta > \aleph_0$. *Then*

$$\mathrm{c}^-(\kappa,\theta) = \mathrm{c}(\kappa,\theta) = \mathrm{d}_\kappa = 2^\kappa.$$

Proof Clearly $\kappa < \mathrm{c}^-(\kappa,\theta) \leq \mathrm{c}(\kappa,\theta) \leq 2^\kappa$ (remember 1.3(3)) and $\mathrm{c}^-(\kappa,\theta) \leq \mathrm{d}_\kappa \leq 2^\kappa$, so it suffices to show that $\mathrm{c}^-(\kappa,\theta) \geq 2^\kappa$.

Suppose that $G \subseteq \mathcal{S}^\theta_\kappa$, $|G| = \mu$, $\kappa < \mu < \mu^+ \leq 2^\kappa$.

Choose an increasing continuous sequence $\langle \kappa_i : i < \theta\rangle$ such that

$$\theta < \kappa_0 \quad \text{and} \quad \sup_{i<\theta} \kappa_i = \kappa \quad \text{and} \quad (\forall i < \theta)(2^{\sum\limits_{j<i} \kappa_j + \aleph_0} < \kappa_i).$$

Next, using [16, Ch. VIII, §1], pick $\bar{\chi} = \langle \chi_i : i < \theta\rangle$ such that

(i) $\bar{\chi}$ is a strictly increasing sequence of regular cardinals,
(ii) $\kappa_i < \chi_i < \kappa$ for each $i < \theta$,
(iii) $\mathrm{tcf}(\prod\limits_{i<\theta} \chi_i/J^{\mathrm{bd}}_\theta) = \mu^+$.

Now, for every $\bar{S} \in G$ define a function $h^{\bar{S}} \in \prod\limits_{i<\theta} \chi_i$ by

$$h^{\bar{S}}(i) = \sup\{\alpha < \chi_i : \alpha \in S_\gamma \ \text{ and } \ \gamma < \kappa_i\}.$$

Note that $|\{\alpha < \chi_i : \alpha \in S_\gamma,\ \gamma < \kappa_i\}| \leq \kappa_i \cdot \theta < \chi_i$, so (as χ_i is regular) $h^{\bar{S}}(i) < \chi_i$. It follows from (iii) that there a function $h \in \prod\limits_{i<\theta} \chi_i$ such that

$$(\forall \bar{S} \in G)(h^{\bar{S}} <_{J^{\mathrm{bd}}_\theta} h).$$

Finally define a function $g \in {}^\kappa\kappa$ by:

$$\text{if} \quad \sup_{j<i} \kappa_j \leq \gamma < \kappa_i \quad \text{then} \quad g(\gamma) = h(i).$$

Note that for each $\bar{S} \in G$ we have

$$\{\gamma < \kappa : g(\gamma) \in S_\gamma\} \subseteq \bigcup \{[\sup_{j<i} \kappa_j, \kappa_i) : i < \theta,\ h^{\bar{S}}(i) \geq h(i)\} \subseteq \kappa_{j(\bar{S})},$$

where $j(\bar{S}) = \min\{j < \theta : \{i < \theta : h^{\bar{S}}(i) \geq h(i)\} \subseteq j\}$. Consequently the function g shows that the family G cannot witness the minimum in the definition of $\mathrm{c}^-(\kappa, \theta)$ and we are done. ■

Remark 1.5. Actually much weaker assumptions are sufficient to get the conclusion of 1.4. For example, almost always we may allow $\theta = \aleph_0$ (see [17]).

Proposition 1.6. *If κ is a singular cardinal, $\theta < \kappa$ then $\mathfrak{e}_\kappa = \mathrm{b}(\kappa, \theta) = \kappa^+$.*

Proof First note that $\kappa < \mathfrak{e}_\kappa \leq \mathrm{b}(\kappa, \theta)$, so it is enough to show that $\mathrm{b}(\kappa, \theta) \leq \kappa^+$.

By [16, Ch. II, 1.5], we may find an increasing sequence $\langle \chi_i : i < \mathrm{cf}(\kappa) \rangle$ of regular cardinals cofinal in κ and such that

$$\theta < \chi_0, \quad \mathrm{tcf}(\prod_{i<\mathrm{cf}(\kappa)} \chi_i / J^{\mathrm{bd}}_{\mathrm{cf}(\kappa)}) = \kappa^+ \quad \text{and} \quad (\forall i < \mathrm{cf}(\kappa))(\sup_{j<i} \chi_j < \chi_i < \kappa).$$

Let $\langle h_\alpha : \alpha < \kappa^+ \rangle \subseteq \prod\limits_{i<\mathrm{cf}(\kappa)} \chi_i$ be a $<_{J^{\mathrm{bd}}_{\mathrm{cf}(\kappa)}}$–increasing sequence cofinal in $(\prod\limits_{i<\mathrm{cf}(\kappa)} \chi_i, <_{J^{\mathrm{bd}}_{\mathrm{cf}(\kappa)}})$. For $i < \mathrm{cf}(\kappa)$ put $\mu_i \stackrel{\mathrm{def}}{=} \sup\limits_{j<i} \chi_j$. Then the sequence $\langle \mu_i : i < \mathrm{cf}(\kappa) \rangle$ is increasing continuous with limit κ. Now we define functions $f_\alpha \in {}^\kappa\kappa$ (for $\alpha < \kappa^+$) by:

$$\mu_i \leq \xi < \mu_{i+1} \ \&\ i < \mathrm{cf}(\kappa) \quad \Rightarrow \quad f_\alpha(\xi) = h_\alpha(i+1).$$

We claim that

$$(\forall \bar{S} \in \mathcal{S}^\theta_\kappa)(\exists \alpha < \kappa^+)(\forall^\kappa \xi < \kappa)(f_\alpha(\xi) \notin S_\xi).$$

So suppose $\bar{S} \in \mathcal{S}^\theta_\kappa$. Define a function $h^{\bar{S}} \in \prod\limits_{i<\mathrm{cf}(\kappa)} \chi_i$ by

$$h^{\bar{S}}(i) = \sup\{\alpha < \chi_i : \alpha \in S_\xi \ \text{ and } \ \xi < \mu_i\}$$

(note that the set on the right-hand side of the formula above is of size $< \chi_i$ so the supremum is below χ_i). Take $\alpha < \kappa^+$ and $j^* < \mathrm{cf}(\kappa)$ such that

$$j^* \leq i < \mathrm{cf}(\kappa) \quad \Rightarrow \quad h^{\bar{S}}(i) < h_\alpha(i),$$

and note that then

$$\{\xi<\kappa: f_\alpha(\xi)\in S_\xi\}\subseteq \mu_{j^*}.$$

So we are done. ■

Proposition 1.7. *If κ is singular and $\theta<\kappa$ then*

(a) $\mathrm{d}_\kappa\geq \mathrm{c}^-(\kappa,\theta)\geq \mathrm{pp}_{J^{\rm bd}_{\mathrm{cf}(\kappa)}}(\kappa)$,

(b) $\mathrm{d}_\kappa\geq \mathfrak{e}_\kappa$ *and if* $\mathrm{d}_\kappa=\mathfrak{e}_\kappa$ *then*

$$\mathrm{cf}(\kappa)\leq\theta<\kappa\quad\Rightarrow\quad \mathrm{pp}_\theta(\kappa)=\kappa^+.$$

Proof To show clause (a) take any $\mu<\mathrm{pp}_{J^{\rm bd}_{\mathrm{cf}(\kappa)}}(\kappa)$ and essentially repeat the proof of 1.6 for μ^+ (remember [16, Ch. II, 2.3]). The assertion (b) follows from (a) and 1.6. ■

Proposition 1.8. *Assume that $\mathbb{P}$ is a $\mathrm{cf}(\theta)$–cc forcing notion.*

(1) $\Vdash_{\mathbb{P}}$ " $(\forall\bar S\in\mathcal{S}^\theta_\kappa)(\exists\bar S^*\in\mathcal{S}^\theta_\kappa\cap\mathbf{V})(\forall i<\kappa)(S_i\subseteq S^*_i)$ ".

(2) $\Vdash_{\mathbb{P}}$ " $\mathrm{c}(\kappa,\theta)\geq(\mathrm{c}(\kappa,\theta))^{\mathbf{V}}$ *and* $\mathrm{c}^-(\kappa,\theta)\geq(\mathrm{c}^-(\kappa,\theta))^{\mathbf{V}}$ ".

(3) *If* $\kappa^{<\mathrm{cf}(\theta)}=\kappa$ *then*

$$\Vdash_{\mathbb{P}}\text{ `` }\mathrm{c}(\kappa,\theta)=(\mathrm{c}(\kappa,\theta))^{\mathbf{V}}\textit{ and }\mathrm{c}^-(\kappa,\theta)=(\mathrm{c}^-(\kappa,\theta))^{\mathbf{V}}\text{ ''.}$$

(4) *If $\theta=\mathrm{cf}(\theta)<\kappa$ and either $\theta<\mathrm{cf}(\kappa)$ or $\theta=\mathrm{cf}(\kappa)$ is a successor cardinal then* $\Vdash_{\mathbb{P}}$ " $\mathrm{d}_\kappa=(\mathrm{d}_\kappa)^{\mathbf{V}}$ ".

Proof 1) Suppose that $\underset{\sim}{A}$ is a $\mathbb{P}$–name for a set of ordinals, $\Vdash_{\mathbb{P}}|\underset{\sim}{A}|<\theta$. Since $\mathbb{P}$ satisfies the $\mathrm{cf}(\theta)$–cc, we find a cardinal $\mu<\theta$ and a $\mathbb{P}$–name $\underset{\sim}{h}$ such that $\Vdash_{\mathbb{P}}$" $\underset{\sim}{h}:\mu\xrightarrow{\rm onto}\underset{\sim}{A}$ ". By the $\mathrm{cf}(\theta)$–cc again, we find sets B_i (for $i<\mu$) such that $|B_i|<\mathrm{cf}(\theta)$ and $\Vdash_{\mathbb{P}}\underset{\sim}{h}(i)\in B_i$. Let $A=\bigcup\limits_{i<\mu}B_i$. Then $\Vdash_{\mathbb{P}}\underset{\sim}{A}\subseteq A$ and: if $\mathrm{cf}(\theta)<\theta$ then $|A|\leq\mu\cdot\mathrm{cf}(\theta)<\theta$ and if $\mathrm{cf}(\theta)=\theta$ then $|A|<\theta$ as $\mu<\mathrm{cf}(\theta)$. The rest should be clear.

4) Let $F\subseteq{}^\kappa\kappa$, $F\in\mathbf{V}$ be a family witnessing the minimum in the definition of d_κ. We are going to show that

$$\Vdash_{\mathbb{P}}\text{ `` }(\forall g\in{}^\kappa\kappa)(\exists f\in F)(\exists^\kappa i<\kappa)(g(i)=f(i))\text{ ''.}$$

So suppose that $p\in\mathbb{P}$ and $\underset{\sim}{g}$ are such that $p\Vdash$"$\underset{\sim}{g}\in{}^\kappa\kappa$". Choose a sequence $\langle p_i:i<\kappa\rangle$ of conditions and a function $g\in{}^\kappa\kappa$ such that

$$(\forall i<\kappa)(p\leq p_i\ \ \&\ \ p_i\Vdash_{\mathbb{P}}\underset{\sim}{g}(i)=g(i)).$$

By the choice of F we find $f\in F$ such that the set $A\stackrel{\rm def}{=}\{i<\kappa:g(i)=f(i)\}$ is of size κ. Next choose a condition $q\geq p$ such that

$$q\Vdash_{\mathbb{P}}\text{ `` }|\{i\in A:p_i\in\underset{\sim}{G}_{\mathbb{P}}\}|=\kappa\text{ ''.}$$

[Possible, as otherwise $p \Vdash$"$|\{i \in A : p_i \in \underset{\sim}{G}_{\mathbb{P}}\}| \leq \mu$" for some $\mu < \kappa$ (remember that $\theta \leq \mathrm{cf}(\kappa)$). So we have a $\mathbb{P}$–name $\underset{\sim}{h}$ for a function from μ into A such that $p \Vdash (\forall i \in A)(p_i \in \underset{\sim}{G}_{\mathbb{P}} \Rightarrow i \in \mathrm{rng}(\underset{\sim}{h}))$. For each $\zeta \in \mu$ the set $B_\zeta = \{i \in A : (\exists p' \geq p)(p' \Vdash \underset{\sim}{h}(\zeta) = i)\}$ is of size $< \theta$ and hence $|\bigcup\limits_{\zeta<\mu} B_\zeta| \leq \theta \cdot \mu < \kappa$. Take any $i \in A \setminus \bigcup\limits_{\zeta<\mu} B_\zeta$ and look at the condition p_i.]
Now note that the condition q forces " $(\exists^\kappa i \in A)(\underset{\sim}{g}(i) = g(i))$ ".

Thus we have proved that $\Vdash_{\mathbb{P}}$"$\mathrm{d}_\kappa \leq (\mathrm{d}_\kappa)^{\mathbf{V}}$". For the converse inequality we use 1.3(4) and 1.8(2). Thus we get

$$\Vdash_{\mathbb{P}} \text{" } \mathrm{d}_\kappa = \mathrm{c}^-(\kappa,\theta) \geq (\mathrm{c}^-(\kappa,\theta))^{\mathbf{V}} = (\mathrm{d}_\kappa)^{\mathbf{V}} \text{ ",}$$

finishing the proof. ■

Now may get the affirmative answer to [3, Problem 8] (see 1.2(2)):

Conclusion 1.9. It is consistent that $\mathrm{d}_{\mathfrak{c}}$ is a singular cardinal and $\mathfrak{e}_{\mathfrak{c}} < \mathrm{d}_{\mathfrak{c}}$ (modulo existence of high enough measurables).

PROOF First we force that there is κ satisfying the assumptions of 1.4 and such that 2^κ singular. How? Start with a supercompact Laver indestructible κ and make 2^κ to have cofinality κ^+, κ still supercompact. Next force κ to have cofinality $\aleph_1$, say as in Magidor [13]. (By [15] we can make κ to be the ω_1–th fix point among the alephs.) So now we have $\mathrm{d}_\kappa = 2^\kappa$, $\mathrm{cf}(2^\kappa) = \kappa^+ < 2^\kappa$. Next add κ Cohen reals. Since this forcing satisfies the $\aleph_1$–cc and is of cardinality κ we conclude that, by 1.8(4), in the final universe $\mathrm{d}_\kappa = \mathrm{d}_{\mathfrak{c}}$ remains the same (so it is singular). Finally, by 1.6, we know that in the resulting model $\mathfrak{e}_{\mathfrak{c}} = \kappa^+ < \mathrm{d}_{\mathfrak{c}}$. ■

Remark 1.10. (1) In fact, if we waive the requirement "$\mathrm{d}_{\mathfrak{c}}$ is singular" then $2^\kappa = \kappa^{++}$ is enough for the proof, so we can get even $\kappa = \aleph_{\omega_1}$, $\mathrm{d}_\kappa = \aleph_{\omega_1+2}$ and $\mathfrak{e}_\kappa = \aleph_{\omega_1+1}$.

(2) What is the consistency strength? By Gitik [8] the consistency strength of
(⊕) κ is measurable and $2^\kappa > \kappa^+$
is that of the existence of a measurable cardinal κ of Mitchell order κ^{++}. By Gitik [7], the consistency strength of
(⊗) κ is measurable and $2^\kappa = \aleph_{\kappa^+}$
is that of the existence of a hypermeasurable cardinal with sequence of measures of length $\aleph_{\kappa^+}$.

2. Around d^*_κ and $\mathfrak{e}^*_\kappa$. In this section we address [3, Problem 9]. The problem reads

When does $\mathrm{d}_{\mathfrak{c}} = \mathrm{d}^*_{\mathfrak{c}}$ or $\mathfrak{e}_{\mathfrak{c}} = \mathfrak{e}^*_{\mathfrak{c}}$ hold?

(see 2.1 for the definitions of $\mathrm{d}^*_{\mathfrak{c}}$, $\mathfrak{e}^*_{\mathfrak{c}}$). Though we do not answer the question fully, we are able to give examples of situations in which the equalities hold. The results here should be combined with those from the previous section, of course.

Definition 2.1. We define the following cardinal coefficients of the space ${}^\kappa\kappa$:

$$\mathrm{d}^*_\kappa = \min\{|F| : F \subseteq {}^\kappa\kappa \ \&\ (\forall G\in[{}^\kappa\kappa]^\kappa)(\exists f\in F)(\forall g\in G)(\exists^\kappa i<\kappa)(f(i)=g(i))\},$$

$$\mathfrak{e}^*_\kappa = \min\{|F| : F \subseteq {}^\kappa\kappa \ \&\ (\forall G\in[{}^\kappa\kappa]^\kappa)(\exists f\in F)(\forall g\in G)(\forall^\kappa i<\kappa)(f(i)\neq g(i))\},$$

Proposition 2.2. (1) [Ciesielski and Jordan; see Jordan [12]] *If* $\kappa=\kappa^{<\kappa}$ *then* $\mathrm{d}_\kappa=\mathrm{d}^*_\kappa$ *and* $\mathfrak{e}_\kappa=\mathfrak{e}^*_\kappa$.
(2) *If κ is a successor cardinal then* $\mathrm{d}^*_\kappa\leq \mathrm{cf}([\mathrm{d}_\kappa]^\kappa,\subseteq)$.
(3) *Suppose that $\bar\lambda$ is an increasing sequence cofinal in κ, $\ell g(\bar\lambda)=\delta\leq\kappa$ such that* $\mathrm{tcf}(\prod\limits_{i<\delta}\lambda_i/J^{\mathrm{bd}}_\delta)=\theta$. *Then* $\mathfrak{e}_\kappa\leq\mathfrak{e}^*_\kappa\leq\theta\leq\mathrm{d}_\kappa\leq\mathrm{d}^*_\kappa$.
(4) *If κ is singular then* $\mathfrak{e}_\kappa=\mathfrak{e}^*_\kappa=\kappa^+$.

Proof 3) Repeat the proof of 1.6, 1.7 with suitable (minor) changes, see below too.

4) A minor modification of the proof of 1.6 shows it. Proceed like there, but instead of functions $h^{\bar S}$ (for $\bar S\in\mathcal{S}^\theta_\kappa$) consider functions $h_{\bar g}\in\prod\limits_{i<\mathrm{cf}(\kappa)}\chi_i$ (for $\bar g=\langle g_\xi:\xi<\kappa\rangle\subseteq{}^\kappa\kappa$) defined by

$$h_{\bar g}(i)=\sup\{\alpha<\chi_i:\alpha=g_\xi(\zeta),\ \xi,\zeta<\mu_i\}.$$

The rest should be clear. ■

Remark 2.3. (1) Concerning the assumptions of 2.4 below, note that by Gitik Shelah [9] there may be such ultrafilters in various cases. Necessarily $\theta\geq\kappa^+$; it can be κ^+, which is the interesting case, and can have 2^κ singular. See more in Džamonja Shelah [5].
(2) Concerning 2.4(c), note that *if* κ is just strongly inaccessible, $\kappa<\mu$ and $f\in{}^\kappa\kappa$ satisfies $(\forall i<\kappa)(\mathrm{cf}(f(i))>|i|)$ (more if we want to preserve being a large cardinal) *then* there is a κ–strategically closed κ^+–cc forcing notion $\mathbb{Q}$ such that $\Vdash_{\mathbb{Q}}$“ $\prod\limits_{i<\kappa}f(i)/J^{\mathrm{bd}}_\kappa$ is μ–directed ”. (Just iterate the forcing adding $g\in\prod\limits_{i<\kappa}f(i)$ dominating all members

of $\prod\limits_{i<\kappa} f(i)$ from the ground model; so a condition fixes $g \restriction \alpha$ (for some $\alpha < \kappa$) and promises $g \geq^* g_0 \in \prod\limits_{i<\kappa} f(i)$.)

(3) So under the assumption of 2.4 (all parts), $\mathfrak{e}^*_\kappa \leq \mathrm{d}_\kappa$.

Proposition 2.4. *Suppose that cardinals θ, κ are such that there is a normal ultrafilter $\mathcal{D}$ on κ generated by θ sets. Then*

(a) $\mathfrak{e}^*_\kappa \leq \theta + \kappa^+$,
(b) if *for every family $\mathcal{A} \subseteq \mathcal{D}$ of size $< \mu$ there is $B \in \mathcal{D}$ such that* $(\forall A \in \mathcal{A})(B \subseteq^* A)$ then $\mu \leq \mathrm{d}_\kappa$,
(c) if *there is a function $f \in {}^\kappa\kappa$ such that $\prod\limits_{i<\kappa} f(i)/J^{\mathrm{bd}}_\kappa$ is μ–directed* then $\mu \leq \mathrm{d}_\kappa$.

Proof (a) Let $\mu = \kappa^+$ and let us assume that $\mathrm{otp}({}^\kappa\kappa/\mathcal{D}) > \kappa^+$ (the other case is handled similarly). Pick up a function $f \in {}^\kappa\kappa$ such that

$$(\forall i < j < \kappa)(i < f(i) < f(j) \text{ and } f(i) \text{ is a regular cardinal})$$

and $\mathrm{otp}(\prod\limits_{i<\kappa} f(i)/\mathcal{D}) = \mu$ (remember that $\mathcal{D}$ is a normal ultrafilter on κ). Let $E \subseteq \kappa$ be a club of κ such that

$$(\forall \delta \in E)(\forall i < \delta)(f(i) < \delta),$$

and let $\{A_\alpha : \alpha < \theta\} \subseteq \mathcal{D}$ be a family generating $\mathcal{D}$ and such that $A_\alpha \subseteq E$ (for all $\alpha < \theta$). Choose a sequence $\bar{g} \subseteq \prod\limits_{i<\kappa} f(i)$ such that $\ell g(\bar{g}) = \mu$ and $\langle g_\zeta/\mathcal{D} : \zeta < \mu\rangle$ is $<_\mathcal{D}$–increasing and cofinal in $\prod\limits_{i<\kappa} f(i)/\mathcal{D}$, and $g_\zeta(i) > i$ (for $\zeta < \mu$ and $i < \kappa$).

For $\alpha < \theta$ and $\zeta < \mu$ we choose a function $h_{\alpha,\zeta} \in {}^\kappa\kappa$ such that $h_{\alpha,\zeta}(i) = g_\zeta(\min(A_\alpha \setminus i))$ (for $i < \kappa$). Let $F = \{h_{\alpha,\zeta} : \alpha < \theta,\ \zeta < \mu\}$, so $|F| \leq \theta + \mu$.

Next, for a function $h \in {}^\kappa\kappa$ define $h^f \in \prod\limits_{i<\kappa} f(i)$ by:

$$h^f(i) = \begin{cases} h(i) & \text{if } h(i) < f(i), \\ 0 & \text{if } h(i) \geq f(i). \end{cases}$$

and choose an ordinal $\zeta(h) < \mu$ such that $h^f <_\mathcal{D} g_{\zeta(h)}$. Then the set $A^h \stackrel{\mathrm{def}}{=} \{i < \kappa : h^f(i) < g_{\zeta(h)}(i)\}$ is in $\mathcal{D}$. Since the set

$$A_h \stackrel{\mathrm{def}}{=} \{i < \kappa : i \text{ is limit and } (\forall j < i)(h(j) < i)\}$$

is a club of κ (so in $\mathcal{D}$) we may choose $\alpha(h) < \theta$ such that $A_{\alpha(h)} \subseteq A^h \cap A_h$.

Suppose now that $G \in [{}^{\kappa}\kappa]^{\kappa}$, say $G = \{h_\xi : \xi < \kappa\}$. Take $\zeta < \mu$ such that $\sup\limits_{\xi<\mu} \zeta(h_\xi) < \zeta$ and let $\alpha < \theta$ be such that

$$A_\alpha \subseteq \mathop{\triangle}_{\xi<\kappa} A_{\alpha(h_\xi)} \cap \mathop{\triangle}_{\xi<\kappa} \{i < \kappa : g_{\zeta(h_\xi)}(i) < g_\zeta(i)\}.$$

Claim 2.4.1. *If $\xi < i < \kappa$ then $h_\xi(i) \neq h_{\alpha,\zeta}(i)$.*

Proof of the claim: First assume that $\xi < i$, $i \in A_\alpha$. Then, by the choice of α, we have $g_{\zeta(h_\xi)}(i) < g_\zeta(i)$ and $i \in A_{\alpha(h_\xi)} \subseteq A^{h_\xi}$. Consequently,

either $h_\xi(i) \geq f(i)$ or $h_\xi(i) = (h_\xi)^f(i) < g_{\zeta(h_\xi)}(i) < g_\zeta(i) = h_{\alpha,\zeta}(i) < f(i)$

(and so $h_\xi(i) \neq h_{\alpha,\zeta}(i)$). So suppose now that $\xi < i$, $i \notin A_\alpha$. Let $j = \min(A_\alpha \setminus i)$. Then $j \in A_{\alpha(h_\xi)} \subseteq A_{h_\xi}$ and $i < j$, so $h_\xi(i) < j$ and $h_{\alpha,\zeta}(i) = g_\zeta(j) > j$. Hence $h_\xi(i) \neq h_{\alpha,\zeta}(i)$.

It follows from 2.4.1 that the family F exemplifies $\mathfrak{e}^*_\kappa \leq \theta + \mu$.

(b) It is similar to (a). Using the assumptions we choose $f, E, A_\alpha, \bar{g}$ as there and we define $h^f, \zeta(h), \alpha(h)$ (for $h \in {}^{\kappa}\kappa$) in the same manner. Exactly as in 2.4.1 we show that for each $h \in {}^{\kappa}\kappa$ and $i \in {}^{\kappa}\kappa$ we have $h_{\alpha(h),\zeta(h)}(i) \neq h(i)$ (just consider two cases: $i \in A_{\alpha(h)}$ and $i \notin A_{\alpha(h)}$).

(c) Similarly. ■

3. Representing functions on the plane. In this section we answer [3, Problem 7] showing that it is consistent that $\mathfrak{c} > \aleph_1$ but for every function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ there exist functions $g_n, h_n : \mathbb{R} \longrightarrow \mathbb{R}$, $n < \omega$, such that

$$f(x, y) = \sum_{n=0}^{\infty} g_n(x) \cdot h_n(y).$$

Let us start with the following technical lemma.

Lemma 3.1. *Assume* **MA**(σ*-centered*). *Suppose that B is an infinite subset of ω, X is a set of size $< \mathfrak{c}$ and $f, g_n : X \longrightarrow \mathbb{R}$ (for $n \in B$) are such that*

($\otimes$) *the sets*

$$A_x \stackrel{\text{def}}{=} \{n \in B : g_n(x) \neq 0\}$$

for $x \in X$ are infinite almost disjoint.

Then *there is a sequence $\langle b_n : n \in B \rangle$ of rational numbers such that*

(1) $b_n \neq \frac{1}{(n+1)^2}$ *for all $n \in B$, and*
(2) $f(x) = \sum\limits_{n \in B} g_n(x) \cdot b_n$ *for each $x \in X$.*

Proof Let $\mathbb{Q} = \mathbb{Q}(X, f, B, \langle g_n : n \in B \rangle)$ be the following forcing notion: **a condition in** $\mathbb{Q}$ is a triple $p = (\bar{b}^p, m^p, \sigma^p) = (\bar{b}, m, \sigma)$ such that

- $m \in \omega$, σ is a finite function such that $\mathrm{dom}(\sigma) \subseteq X$ and $\mathrm{rng}(\sigma) \subseteq B \cap (m+1)$,
- $\bar{b} = \langle b_n : n \in B \cap m \rangle$ is a sequence of rational numbers, $b_n \neq \frac{1}{(n+1)^2}$ for $n \in B \cap m$,
- for each $x \in \mathrm{dom}(\sigma)$, the sequence

$$\langle |f(x) - \sum_{n \in B \cap k} g_n(x) \cdot b_n| : k \in B \cap [\sigma(x), m] \rangle$$

is non-increasing;

the order of $\mathbb{Q}$ is the natural one: $\quad p \leq q \quad$ if and only if

$$\bar{b}^p \trianglelefteq \bar{b}^q, \quad m^p \leq m^q \quad \text{and} \quad \sigma^p \subseteq \sigma^q.$$

Claim 3.1.1. *$\mathbb{Q}$ is a non-trivial σ–centered forcing notion.*

Proof of the claim: Consider the space ${}^X\omega$ equipped with the product topology of discrete copies of ω. By Engelking Karłowicz [6], this space is separable (as $|X| \leq \mathfrak{c}$). So let $\{\eta_k : k < \omega\} \subseteq {}^X\omega$ be a dense subset of ${}^X\omega$. For $m, k \in \omega$ and a sequence $\bar{b} = \langle b_n : n \in B \cap m \rangle$ of rationals let

$$Q_k^{m,\bar{b}} \stackrel{\mathrm{def}}{=} \{p \in \mathbb{Q} : m^p = m \ \&\ \bar{b}^p = \bar{b} \ \&\ \sigma^p \subseteq \eta_k\}.$$

Since there are countably many possibilities for $\langle m, \bar{b}, k \rangle$ as above and each member of $\mathbb{Q}$ belongs to some $Q_k^{m,\bar{b}}$ (remember the choice of η_k's), it is enough to show that the sets $Q_k^{m,\bar{b}}$ are directed. So let $p_0, \ldots, p_{\ell-1} \in Q_k^{m,\bar{b}}$. Then $\bar{b}^{p_i} = \bar{b}$, $m^{p_i} = m$ and $\sigma^{p_i} \subseteq \eta_k$ (for $i < \ell$). Put $q = (\bar{b}, m, \bigcup\limits_{i<\ell} \sigma^{p_i})$. It should be clear that $q \in Q_k^{m,\bar{b}}$ is a condition stronger than all $p_0, \ldots, p_{\ell-1}$.

Now, for $x \in X$ and a positive rational number ε let

$$\mathcal{I}_x^\varepsilon \stackrel{\mathrm{def}}{=} \Big\{p \in \mathbb{Q} : x \in \mathrm{dom}(\sigma^p) \ \&\ |f(x) - \sum_{n \in B \cap m^p} g_n(x) \cdot b_n| < \varepsilon\Big\}.$$

Claim 3.1.2. *For every $x \in X$ and a rational $\varepsilon > 0$ the set $\mathcal{I}_x^\varepsilon$ is an open dense subset of $\mathbb{Q}$.*

Proof of the claim: Let $q \in \mathbb{Q}$ and let $r \in \mathbb{Q}$ be defined as follows. If $x \in \mathrm{dom}(\sigma^q)$ then $r = q$, otherwise

$$\bar{b}^r = \bar{b}^q, \quad m^r = m^q \quad \text{and} \quad \sigma^r = \sigma^q \cup \{(x, m^q)\}.$$

(So r is a condition stronger than q and $x \in \mathrm{dom}(\sigma^r)$.) Use the assumption $(\otimes)$ to choose $m^* > m^r$ such that

$$m^* \in B \cap A_x \cap \bigcap \{B \setminus A_y : y \in \mathrm{dom}(\sigma^r) \setminus \{x\}\},$$

remember $A_y = \{n \in B : g_n(y) \neq 0\}$. Let

$$\varepsilon^* = \frac{1}{2} \min\{\varepsilon, |f(x) - \sum_{n \in B \cap m^r} g_n(x) \cdot b_n|\}$$

(so $\varepsilon > \varepsilon^* \geq 0$) and let

- $m^p = \min(B \setminus (m^* + 1))$, $\sigma^p = \sigma^r$,
- $b^p_n = b^r_n$ if $n \in B \cap m^r$, $b^p_n = 0$ if $n \in B \cap [m^r, m^*)$ and $b^p_{m^*} \neq \frac{1}{(m^*+1)^2}$ be a rational number such that

$$f(x) - \sum_{n \in B \cap m^*} g_n(x) \cdot b^p_n - \varepsilon^* \leq g_{m^*}(x) \cdot b^p_{m^*} \leq f(x) - \sum_{n \in B \cap m^*} g_n(x) \cdot b^p_n + \varepsilon^*$$

(clearly the choice is possible as $g_{m^*}(x) \neq 0$; if $\varepsilon^* = 0$ then $b^p_{m^*} = 0$).

One easily checks now that the above choice defines a condition $p \in \mathcal{I}^\varepsilon_x$ stronger than r.

It follows from 3.1.1, 3.1.2 that we may use **MA**(σ-centered) to find a directed set $G \subseteq \mathbb{Q}$ such that $G \cap \mathcal{I}^\varepsilon_x \neq \emptyset$ for each $x \in X$ and a positive rational ε. Let $\bar{b} = \bigcup\{\bar{b}^p : p \in G\}$. It should be clear that the sequence $\bar{b}$ is as required. ■

Definition 3.2. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$. An *f–approximation* is a tuple $p = (X^p, \bar{g}^p_0, \bar{g}^p_1, \mathcal{D}^p) = (X, \bar{g}_0, \bar{g}_1, \mathcal{D})$ such that

(a) $X \subseteq \mathbb{R}$,

(b) $\bar{g}_\ell = \langle g_{\ell,n} : n < \omega \rangle$, $\quad g_{\ell,n} : X \longrightarrow \mathbb{R}$ (for $\ell < 2$, $n < \omega$), for $\ell < 2$, $x \in X$ let $\bar{a}_{\ell,x} = \langle g_{\ell,n}(x) : n < \omega \rangle$,

(c) $(\forall x, y \in X)(f(x,y) = \sum\limits_{n=0}^{\infty} g_{0,n}(x) \cdot g_{1,n}(x))$,

(d) $\mathcal{D}$ is a filter on ω including all co-finite subsets of ω and generated by $\leq |X| + \aleph_0$ sets,

(e) if $x \in X$, $\ell < 2$ then

$$\{n < \omega : g_{\ell,n}(x) \in \{0, \tfrac{1}{(n+1)^2}\}\} \in \mathcal{D}, \qquad \text{and}$$

$$A_{\bar{a}_{\ell,x}} = A^p_{\bar{a}_{\ell,x}} \stackrel{\text{def}}{=} \{n < \omega : g_{\ell,n}(x) = \tfrac{1}{(n+1)^2}\} \neq \emptyset \mod \mathcal{D},$$

(f) no finite union of sets $A_{\bar{a}_{\ell,x}}$ (for $\ell < 2$, $x \in X$) is in $\mathcal{D}$,

(g) if $(\ell_1, x_1) \neq (\ell_2, x_2)$, $\ell_1, \ell_2 < 2$, $x_1, x_2 \in X$ then $\bar{a}_{\ell_1,x_1} \neq \bar{a}_{\ell_2,x_2}$ and the intersection $A^p_{\bar{a}_{\ell_1,x_1}} \cap A^p_{\bar{a}_{\ell_2,x_2}}$ is finite.

The set $\mathcal{AP}^f$ of all approximations carries a natural partial order: for f–approximations p, q we let $\quad p \leq q \quad$ if and only if

$$X^p \subseteq X^q, \quad \mathcal{D}^p \subseteq \mathcal{D}^q \quad \text{and} \quad g^p_{\ell,n} \subseteq g^q_{\ell,n} \quad (\text{for } \ell < 2 \text{ and } n < \omega).$$

Theorem 3.3. *Assume* $\mathbf{MA}$(σ-*centered*). *Let* $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$. *Suppose that* $p \in \mathcal{AP}^f$ *is such that* $|X^p| < \mathfrak{c}$ *and* $r^* \in \mathbb{R} \setminus X^p$. Then *there is* $q \in \mathcal{AP}^f$ *such that*

$$p \le q \quad \text{and} \quad X^q = X^p \cup \{r^*\}.$$

Proof First choose pairwise disjoint infinite subsets B_0, B_1, B_2 of $\omega \setminus \{0\}$ such that for $m < 3$:

(α) $(\forall B \in \mathcal{D}^p)(B_m \subseteq^* B)$,
(β) $(\forall \ell < 2)(\forall x \in X^p)(|B_m \cap A_{\bar{a}^p_{\ell,x}}| = \aleph_0)$,
(γ) no finite union of sets $A_{\bar{a}_{\ell,x}}$ (for $\ell < 2$, $x \in X$) almost includes B_m.

(There are such sets by $\mathbf{MA}$(σ-centered); remember 3.2(e),(f).) Next choose disjoint infinite subsets B^0_0, B^1_0, B^2_0 of B_0 such that for $k < 3$

(δ) $(\forall \ell < 2)(\forall x \in X^p)(|B^k_0 \cap A_{\bar{a}^p_{\ell,x}}| < \aleph_0)$.

(Again, easily possible by our assumptions and 3.2(g) and (γ) above.)

Now we start defining an f–approximation q. We let

- $X^q = X^p \cup \{r^*\}$,
- $\mathcal{D}^q$ be the filter generated by $\mathcal{D}^p \cup \{B_0\}$,
- $g^q_{\ell,n}(x) = g^p_{\ell,n}(x)$ for $x \in X^p$, $\ell < 2$ and $n < \omega$,
- if $n \in \omega \setminus B_1$ then

$$g^q_{0,n}(r^*) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{1}{(n+1)^2} & \text{if } n \in B^0_0, \\ 0 & \text{if } n \in \omega \setminus (B_1 \cup B^0_0 \cup \{0\}); \end{cases}$$

and if $n \in \omega \setminus B_2$ then

$$g^q_{1,n}(r^*) = \begin{cases} f(r^*, r^*) & \text{if } n = 0, \\ \frac{1}{(n+1)^2} & \text{if } n \in B^1_0, \\ 0 & \text{if } n \in \omega \setminus (B_2 \cup B^1_0 \cup \{0\}). \end{cases}$$

Now we want to define $g^q_{0,n}(r^*)$, $g^q_{1,n}(r^*)$ for other n, but we have to be careful with that to ensure that the clause 3.2(c) is satisfied. It should be clear at the moment that we do not have to worry anymore about that clause if $x, y \in X^p$ or $x = y = r^*$ (for the last case inspect the definition above and the choice of B_m, B^0_0, B^1_0). So now we use 3.1 to finish the definition. First note that for each $x \in X$ the sets

$$\{n \in \omega \setminus B_1 : g^q_{0,n}(r^*) \cdot g^p_{1,n}(x) \neq 0\} \quad \text{and} \quad \{n \in \omega \setminus B_2 : g^p_{0,n}(x) \cdot g^q_{1,n}(r^*) \neq 0\}$$

are finite (remember clauses (α) and (δ)). Apply 3.1 to the set B_1, functions $g^p_{1,n}$ (for $n \in B_1$) and the mapping

$$x \mapsto f(r^*, x) - \sum_{n \in \omega \setminus B_1} g^q_{0,n}(r^*) \cdot g^p_{1,n}(x)$$

(note that the sum is actually finite) to find $g^q_{0,n}(r^*)$ (for $n \in B_1$) such that $g^q_{0,n}(r^*) \neq \frac{1}{(n+1)^2}$ and for each $x \in X$

$$f(r^*,x) - \sum_{n\in\omega\setminus B_1} g^q_{0,n}(r^*)\cdot g^p_{1,n}(x) = \sum_{n\in B_1} g^q_{0,n}(r^*)\cdot g^p_{1,n}(x).$$

Next use 3.1 for B_2, $g^p_{0,n}$ (for $n \in B_2$) and the mapping

$$x \mapsto f(x,r^*) - \sum_{n\in\omega\setminus B_2} g^p_{0,n}(x)\cdot g^q_{1,n}(r^*)$$

to choose $g^q_{1,n}(r^*)$ (for $n \in B_2$) such that $g^q_{1,n}(r^*) \neq \frac{1}{(n+1)^2}$ and for $x \in X$

$$f(x,r^*) - \sum_{n\in\omega\setminus B_2} g^p_{0,n}(x)\cdot g^q_{1,n}(r^*) = \sum_{n\in B_2} g^p_{0,n}(x)\cdot g^q_{1,n}(r^*).$$

This finishes the definition of $g^q_{\ell,n}(x)$ for $\ell < 2$, $x \in X^q$ and $n < \omega$. Checking that $(X^q, \bar{g}^q_0, \bar{g}^q_1, \mathcal{D}^q) \in \mathcal{AP}^f$ is as required is straightforward. ■

Since $\leq$–increasing sequences of f–approximations have (natural) upper bounds we may use 3.3 to prove inductively the following.

Conclusion 3.4. Assume **MA**(σ-centered). Then for every function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ there are functions $g_n, h_n : \mathbb{R} \longrightarrow \mathbb{R}$, $n < \omega$, such that

$$f(x,y) = \sum_{n=0}^{\infty} g_n(x)\cdot h_n(y).$$

Remark 3.5. Regarding the assumptions of 3.4, remember that by Bell [2] **MA**(σ-centered) is equivalent to $\mathfrak{p} = \mathfrak{c}$.

Let us finish this section with the following "negative" result.

Proposition 3.6. *Let $\mathbb{P}$ be the forcing notion for adding $\aleph_2$ Cohen reals. Then, in $\mathbf{V}^{\mathbb{P}}$, there is a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that there are* **no** *functions $g_n, h_n : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying*

$$(\forall x,y \in \mathbb{R})(f(x,y) = \sum_{n=0}^{\infty} g_n(x)\cdot h_n(y)).$$

Proof Since we may break $\mathbb{P}$ into two steps each adding $\aleph_2$ Cohen reals, we may assume that $\mathbf{V} \models \neg\mathbf{CH}$. So fix a sequence $\langle \eta_i : i < \aleph_2\rangle$ of pairwise distinct real numbers. Then $\mathbb{P}$ may be interpreted as the partial order of all finite functions p such that $\mathrm{dom}(p) \subseteq \{(\eta_i,\eta_j) : i,j < \aleph_2\}$ and $\mathrm{rng}(p) \subseteq 2$ ordered by the inclusion. For a set $A \subseteq \{(\eta_i,\eta_j) : i,j < \aleph_2\}$ let $\mathbb{P}_A = \{p \in \mathbb{P} : \mathrm{dom}(p) \subseteq A\}$ (so $\mathbb{P}_A \lessdot \mathbb{P}$).

Let $\underset{\sim}{f}$ be a $\mathbb{P}$–name for a function from $\mathbb{R}^2$ to $\mathbb{R}$ such that $\Vdash_{\mathbb{P}} \bigcup\{p : p \in \underset{\sim}{G}_{\mathbb{P}}\} \subseteq \underset{\sim}{f}$. Suppose that $\underset{\sim}{g}_n, \underset{\sim}{h}_n$ (for $n < \omega$) are $\mathbb{P}$–names for functions from $\mathbb{R}$ to $\mathbb{R}$.

Claim 3.6.1.

$$\Vdash_{\mathbb{P}} \text{“ } (\exists i < j < \aleph_2)(\underset{\sim}{f}(\eta_i, \eta_j) \neq \sum_{n=0}^{\infty} \underset{\sim}{g}_n(\eta_i) \cdot \underset{\sim}{h}_n(\eta_j)) \text{ ”.}$$

Proof of the claim: Let $q \in \mathbb{P}$. For each $i < \aleph_2$ fix a countable subset A_i of $\{(\eta_\xi, \eta_\zeta) : \xi, \zeta < \aleph_2\}$ such that $\mathrm{dom}(q) \subseteq A_i$ and for some $\mathbb{P}_{A_i}$–names $\underset{\sim}{r}_n, \underset{\sim}{s}_n$ (for $n < \omega$) we have $\Vdash_{\mathbb{P}}$“ $\underset{\sim}{g}_n(\eta_i) = \underset{\sim}{r}_n$ and $\underset{\sim}{h}_n(\eta_i) = \underset{\sim}{s}_n$ ”. Let

$$B_i \stackrel{\text{def}}{=} \{\xi : (\exists \zeta < \aleph_2)((\eta_\xi, \eta_\zeta) \in A_i \ \text{ or } \ (\eta_\zeta, \eta_\xi) \in A_i)\}$$

(clearly each B_i is countable). Plainly, for $i \in S^2_1 \stackrel{\text{def}}{=} \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}$ we have $\sup(B_i \cap i) < i$ and hence for some $j < \aleph_2$ the set $S = \{i \in S^2_1 : \sup(B_i \cap i) = j\}$ is stationary. Choose $i_0 < i_1$ from S such that $\sup(B_{i_0}) < i_1$. Let

$$Y = \{(\eta_\xi, \eta_\zeta) : \{\xi, \zeta\} \subseteq B_{i_0} \ \text{ or } \ \{\xi, \zeta\} \subseteq B_{i_1}\}.$$

Note that $(\eta_{i_0}, \eta_{i_1}) \notin Y$. Since $\underset{\sim}{g}_n(\eta_{i_0}), \underset{\sim}{h}_n(\eta_{i_1})$ are (essentially) $\mathbb{P}_Y$–names and $q \in \mathbb{P}_Y$, we find a condition $p \in \mathbb{P}_Y$ stronger than q and deciding the statement “$\sum\limits_{n=0}^{\infty} \underset{\sim}{g}_n(\eta_{i_0}) \cdot \underset{\sim}{h}_n(\eta_{i_1}) \leq \frac{1}{2}$”. Let $r \in \mathbb{P}$ be a condition stronger than p such that $(\eta_{i_0}, \eta_{i_1}) \in \mathrm{dom}(r)$ and

$$r(\eta_{i_0}, \eta_{i_1}) = \begin{cases} 1 & \text{if } p \Vdash_{\mathbb{P}_Y} \text{“ } \sum\limits_{n=0}^{\infty} \underset{\sim}{g}_n(\eta_{i_0}) \cdot \underset{\sim}{h}_n(\eta_{i_1}) \leq \frac{1}{2} \text{ ”,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$r \Vdash_{\mathbb{P}} \underset{\sim}{f}(\eta_{i_0}, \eta_{i_1}) \neq \sum_{n=0}^{\infty} \underset{\sim}{g}_n(\eta_{i_0}) \cdot \underset{\sim}{h}_n(\eta_{i_1}),$$

finishing the proof. ■

4. Countably continuous functions.

Our aim here is to show that, consistently, **CH** fails but every $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$(\forall U \in [\mathbb{R}]^{\aleph_1})(\exists U^* \in [U]^{\aleph_1})(f \restriction U^* \text{ is continuous})$$

is countably continuous (see Definition 4.1 below). This answers negatively [3, Problem 4].

Definition 4.1. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is *countably continuous* if there is a partition $\langle X_n : n < \omega \rangle$ of $\mathbb{R}$ such that the restriction of f to any X_n is continuous.

Theorem 4.2. *It is consistent with* $\neg$**CH** *that every function* $f:\mathbb{R}\longrightarrow\mathbb{R}$ *such that*

$(\oplus)_f$ $\quad (\forall U\in[\mathbb{R}]^{\aleph_1})(\exists U^*\in[U]^{\aleph_1})(f\restriction U^*$ *is continuous*)

is countably continuous.

Proof Start with $\mathbf{V}\models$ **CH** and let $\lambda>\aleph_1$ be a cardinal such that $\lambda^{\aleph_0}=\lambda$. Let $\mathbb{P}_\lambda$ be a forcing notion for adding λ many Cohen reals. So $\mathbb{P}_\lambda$ can be represented as the set of all finite partial functions $p:\mathrm{dom}(p)\longrightarrow 2$, $\mathrm{dom}(p)\subseteq\lambda$, ordered by the inclusion.

For a set $A\subseteq\lambda$ let $\mathbb{P}_A=\{p\in\mathbb{P}_\lambda:\mathrm{dom}(p)\subseteq A\}$. Then $\mathbb{P}_A\lessdot\mathbb{P}_\lambda$. Plainly, $\mathbb{P}_\lambda$ is a ccc forcing notion and $\Vdash_{\mathbb{P}_\lambda}\mathfrak{c}=\lambda>\aleph_1$.

We are going to show that in $\mathbf{V}^{\mathbb{P}_\lambda}$, every real function f satisfying $(\oplus)_f$ is countably continuous. To this end suppose that $\underset{\sim}{f}$ is a $\mathbb{P}_\lambda$–name for a function from $\mathbb{R}$ into $\mathbb{R}$ such that

$$\Vdash_{\mathbb{P}_\lambda}\text{“ }\underset{\sim}{f}\text{ is not countably continuous ”.}$$

By induction on $\alpha<\omega_1$ choose an increasing continuous sequence $\langle A_\alpha:\alpha<\omega_1\rangle$ such that for each $\alpha<\omega_1$:

(1) $A_\alpha\in[\lambda]^{\aleph_1}$;
(2) if $\underset{\sim}{\eta}$ is a $\mathbb{P}_{A_\alpha}$–name for a real then $\underset{\sim}{f}(\underset{\sim}{\eta})$ is a $\mathbb{P}_{A_{\alpha+1}}$–name;
(3) if $\bar{\underset{\sim}{h}}=\langle\underset{\sim}{h}_n:n<\omega\rangle$ is a $\mathbb{P}_{A_\alpha}$–name for an ω–sequence of partial real functions such that $\mathrm{dom}(\underset{\sim}{h}_n)$ is a Borel set and $\underset{\sim}{h}_n$ is continuous on its domain (for $n<\omega$) then each $\underset{\sim}{h}_n$ is a $\mathbb{P}_{A_{\alpha+1}}$–name and there is a $\mathbb{P}_{A_{\alpha+1}}$–name $\underset{\sim}{\eta}$ for a real such that

$$\Vdash_{\mathbb{P}_{A_{\alpha+1}}}\text{“ }(\forall n<\omega)(\underset{\sim}{f}(\underset{\sim}{\eta})\neq\underset{\sim}{h}_n(\underset{\sim}{\eta}))\text{ ”.}$$

(There are no problems with carrying out the construction.) Let $A=\bigcup\limits_{\alpha<\omega_1}A_\alpha$, so $A\in[\lambda]^{\aleph_1}$ and $\Vdash_{\mathbb{P}_A}$ **CH**. It should be clear that, by (2) above, we have a $\mathbb{P}_A$–name $\underset{\sim}{f}^A$ such that $\Vdash_{\mathbb{P}_\lambda}$“ $\underset{\sim}{f}^A=\underset{\sim}{f}\restriction\mathbb{R}\cap\mathbf{V}^{\mathbb{P}_A}$ ”. Moreover, by (3), we know that

$$\Vdash_{\mathbb{P}_A}\text{“ }\underset{\sim}{f}^A\text{ is not countably continuous ”.}$$

Now, using [3, 3.11], we conclude that (remember that we have **CH** in $\mathbf{V}^{\mathbb{P}_A}$)

$$\Vdash_{\mathbb{P}_A}\text{“ }(\exists U\in[\mathbb{R}]^{\aleph_1})(\forall U^*\in[U]^{\aleph_1})(\underset{\sim}{f}^A\restriction U^*\text{ is not continuous })\text{ ”.}$$

Let $G\subseteq\mathbb{P}_A$ be a generic filter over $\mathbf{V}$. Work in $\mathbf{V}[G]$. Let $U\in[\mathbb{R}]^{\aleph_1}$ be such that $\underset{\sim}{f}^A[G]\restriction U^*$ is not continuous for any uncountable $U^*\subseteq U$. We want to show that this property of the function $\underset{\sim}{f}^A[G]$ and the set U is preserved by the quotient forcing $\mathbb{P}_\lambda/\mathbb{P}_A$ (which is isomorphic to $\mathbb{P}_{\lambda\setminus A}$, of course). So suppose that $p \in \mathbb{P}_\lambda/\mathbb{P}_A$ is such that

$$p \Vdash_{\mathbb{P}_\lambda/\mathbb{P}_A} \text{“ } (\exists U^* \in [U]^{\aleph_1})(\underset{\sim}{f}^A[G]\restriction U^* \text{ is continuous) ”}.$$

Every continuous function on a set $U^* \subseteq \mathbb{R}$ can be extended to a continuous function on a Π^0_2–set. Now, both Π^0_2–sets and continuous functions on them are coded by reals. Consequently we find a countable set $B \subseteq \lambda \setminus A$ such that $\mathrm{dom}(p) \subseteq B$ and for some $\mathbb{P}_B$–names $\underset{\sim}{W}, \underset{\sim}{h}$ we have

$$p \Vdash_{\mathbb{P}_\lambda/\mathbb{P}_A} \text{“ } \begin{array}{l} \underset{\sim}{W} \text{ is a } \Pi^0_2\text{–subset of } \mathbb{R} \text{ , } \underset{\sim}{h} : \underset{\sim}{W} \longrightarrow \mathbb{R} \text{ is continuous and} \\ (\exists^{\aleph_1}\eta \in U)(\eta \in \underset{\sim}{W} \ \ \& \ \ \underset{\sim}{f}^A[G](\eta) = \underset{\sim}{h}(\eta)) \text{ ”}. \end{array}$$

The property stated above is absolute from $\mathbf{V}^{\mathbb{P}_\lambda}$ to $\mathbf{V}^{\mathbb{P}_{A\cup B}}$, so the condition p forces the respective sentence in $\mathbb{P}_B$. Now, the forcing notion $\mathbb{P}_B$ is countable so it has the property that every uncountable set of ordinals in the extension contains an uncountable subset from the ground model. Consequently, we find (still in $\mathbf{V}[G]$) an uncountable set $U_0 \subseteq U$ such that

$$p \Vdash_{\mathbb{P}_B} \text{“ } \begin{array}{l} \underset{\sim}{W} \text{ is a } \Pi^0_2\text{–subset of } \mathbb{R}, \ \underset{\sim}{h} : \underset{\sim}{W} \longrightarrow \mathbb{R} \text{ is continuous and} \\ (\forall \eta \in U_0)(\eta \in \underset{\sim}{W} \ \ \& \ \ \underset{\sim}{f}^A[G](\eta) = \underset{\sim}{h}(\eta)) \text{ ”}. \end{array}$$

Thus $p \Vdash_{\mathbb{P}_B}$ “$\underset{\sim}{f}^A[G]\restriction U_0$ is continuous ”, and hence easily this statement has to hold in $\mathbf{V}[\tilde{G}]$ already, a contradiction. ■

References

[1] Bartoszyński, T. and Judah, H., *Set Theory: On the Structure of the Real Line.* A K Peters, Wellesley, Massachusetts (1995).

[2] Bell, M, *On the combinatorial principle $P(\mathfrak{c})$* Fundamenta Mathematicae, 114 (1981), 149–157.

[3] Ciesielski, K., *Set Theoretic Real Analysis.* Journal of Applied Analysis, to appear.

[4] Ciesielski, K. and Shelah, S., *F288: an application of oracle ccc.* In progress.

[5] Džamonja, M. and Shelah, S., *Universal graphs at successors of singular strong limit cardinals.* Preprint.

[6] Engelking, R. and Karłowicz, M., *Some theorems of set theory and their topological consequences.* Fundamenta Mathematicae, 57 (1965), 275–285.

[7] Gitik, M., *On measurable cardinals violating the continuum hypothesis.* Annals of Pure and Applied Logic, 63 (1993), 227–240.

[8] Gitik, M., *The negation of the singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$.* Annals of Pure and Applied Logic, 43 (1989), 209–234.

[9] Gitik, M. and Shelah, S., *On densities of box products.* Topology and its Applications, **accepted**.

[10] Goldstern, M. and Shelah, S., *Many simple cardinal invariants.* Archive for Mathematical Logic, 32 (1993), 203–221.

[11] Jech, T., *Set theory.* Academic Press, New York (1978).

[12] Jordan, F., *Cardinal invarants connected with adding real functions*. Real Anal. Exchange, 22 (1996/97), 696–713.

[13] Magidor, M., *Changing cofinality of cardinals*. Fundamenta Mathematicae, XCIX (1978), 61–71.

[14] Rosłanowski, A. and Shelah, S., *Norms on possibilities III: strange subsets of the real line*. In progress.

[15] Shelah, S., *The singular cardinals problem: independence results*. In *Surveys in set theory*, volume 87 of London Math. Soc. Lecture Note Ser., pages 116–134. Cambridge Univ. Press, Cambridge-New York, 1983.

[16] Shelah, S., *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides*. Oxford University Press, 1994.

[17] Shelah, S., *Analytical Guide and Corrections to [Sh:g]*. Preprint, available from `http://www.math.rutgers.edu/~shelah`.

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