On versions of ♣ on cardinals larger than $\aleph_1$

Mirna Džamonja
School of Mathematics
University of East Anglia
Norwich, NR47TJ, UK
M.Dzamonja@uea.ac.uk
Saharon Shelah
Mathematics Department
Hebrew University of Jerusalem
91904 Givat Ram, Israel
and
Rutgers University
New Brunswick, New Jersey
USA
shelah@sunset.huji.ac.il

[1]

Abstract

We give two results on guessing unbounded subsets of λ^+. The first is a positive result and applies to the situation of λ regular and at least equal to $\aleph_3$, while the second is a negative consistency result which applies to the situation of λ a singular strong limit with $2^\lambda > \lambda^+$. The first result shows that in ZFC there is a guessing of unbounded subsets of $S^{\lambda^+}_\lambda$. The second result is a consistency result (assuming a supercompact cardinal exists) showing that a natural guessing fails.

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A result of Shelah in [Sh 667] shows that if $2^\lambda = \lambda^+$ and λ is a strong limit singular, then the corresponding guessing holds.

Both results are also connected to an earlier result of Džamonja-Shelah in which they showed that a certain version of ♣ holds at a successor of singular just in ZFC. The first result here shows that the result of Fact 0.2 can to a certain extent be extended to the successor of a regular. The negative result here gives limitations to the extent to which one can hope to extend the mentioned Džamonja-Shelah result. [2]

0 Introduction and background

The combinatorial principle ♣ is a weakening of $\diamondsuit$ which, at $\aleph_1$, means that there is a sequence $\langle A_\delta : \delta \text{ limit } < \omega_1\rangle$ such that every A_δ is an unbounded subset of δ, and for every unbounded subset A of ω_1, there are stationarily many δ such that $A_\delta \subseteq A$. One can weaken this statement in various ways, for example requiring $|A_\delta \setminus A| < \aleph_0$ in place of $A_\delta \subseteq A$ above, and in general the weakened statements are not equivalent to ♣ (as opposed to the situation with $\diamondsuit$, see Džamonja-Shelah [DjSh 576] and Kunen's [Ku] respectively), and are not provable in ZFC. The question we consider here is if the corresponding situation holds at cardinals larger than $\aleph_1$. As an example of earlier results in this direction and connected to the statement of our main theorem, we mention a result of Erdös, Dushnik and Miller, and a much more recent one of Shelah. Erdös, Dushnik and Miller prove in [DuMi], that if $\alpha < \lambda^+$, then α can be written as $\bigcup_{n<\omega} A_n$, where for each n we have $\text{otp}(A_n) < \lambda^n$ (compare this with Note 1.6 below). Shelah proves in [Sh 572] that if $\lambda = \text{cf}(\lambda) > \kappa > \aleph_0$, then there is a sequence

$$\langle (C_\delta = \{\alpha_{\delta,\varepsilon} : \varepsilon < \lambda\}, h_\delta) : \delta \in S^{\lambda^+}_\lambda \rangle$$

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with each $\{\alpha_{\delta,\varepsilon} : \varepsilon < \lambda\}$ a continuous increasing sequence with $\sup_{\varepsilon<\lambda} \alpha_{\delta,\varepsilon} = \delta$, and $h_\delta : C_\delta \to \kappa$ an onto function, such that for every club E of λ^+, for stationarily many $\delta < \lambda^+$ we have, for every $i < \kappa$

$$\{\varepsilon < \lambda : \alpha_{\delta,\varepsilon}, \alpha_{\delta,\varepsilon+1} \in E \ \&\ h(\alpha_{\delta,\varepsilon}) = i\}$$

is stationary in λ (see Notation 0.4 below for $S^{\lambda^+}_\lambda$). Note that it is not known if the analogous result holds when "$\alpha_{\delta,\varepsilon}, \alpha_{\delta,\varepsilon+1} \in E$" is replaced by "$\alpha_{\delta,\varepsilon}, \alpha_{\delta,\varepsilon+1}, \alpha_{\delta,\varepsilon+2} \in E$". If it does, it would have interesting consequences regarding the generalized Suslin hypothesis, see [KjSh 449].

We prove that if λ is regular and at least equal to $\aleph_3$, then just in ZFC a version of $\clubsuit$ by which unbounded subsets of $S^{\lambda^+}_\lambda$ (i.e. the ordinals $< \lambda^+$ of cofinality λ) are guessed, holds. If $\lambda \geq \aleph_2, 2^{\aleph_0}$ we obtain a similar version of guessing. See Theorem 1.1.

Another result along these lines is one of Džamonja and Shelah from [DjSh 545]:

Definition 0.1. Suppose that λ is a cardinal. $\clubsuit^*_{-\lambda}(\lambda^+)$ is the statement saying that there is a sequence $\langle \mathcal{P}_\delta : \delta \text{ limit } < \lambda^+\rangle$ such that

(i) $\mathcal{P}_\delta$ is a family of $\leq |\delta|$ unbounded subsets of δ,

(ii) For $a \in \mathcal{P}_\delta$ we have $\mathrm{otp}(a) < \lambda$,

(iii) For all $X \in [\lambda^+]^{\lambda^+}$, there is a club C of λ^+ such that for all $\delta \in C$ limit, there is $a \in \mathcal{P}_\delta$ such that

$$\sup(a \cap X) = \delta.$$

Fact 0.2 (Džamonja-Shelah). [DjSh 545] If $\aleph_0 < \kappa = \mathrm{cf}(\lambda) < \lambda$, then $\clubsuit^*_{-\lambda}(\lambda^+)$.

See the discussion below for a related result of Shelah from [Sh 667].

Our Theorem 1.1 can be understood as an extension of the theorem from [DjSh 545] to the successor of a regular κ for some $\kappa \geq \aleph_3$, with the exception that our guessing has less guesses at each δ (just one), but the guessing is obtained stationarily often, as opposed to club often. Also note that the result in Theorem 1.1 is in some sense complementary to the "club guessing" results of Shelah ([Sh 365] for example) because here we are guessing unbounded subsets of λ^+ which are not necessarily clubs, but on the other hand, there are limitations on the cofinalities.

In §2, we investigate successors of singulars. On the one hand, we can hope to improve or at least modify in a non-trivial way the above result from [DjSh 545]. If λ is a strong limit singular, and $2^\lambda = \lambda^+$, it has already been done by Shelah in [Sh 667]:

Fact 0.3 (Shelah). [Sh 667] Suppose that λ is a strong limit singular with $2^\lambda = \lambda^+$, and S is a stationary subset of $S^{\lambda^+}_{\mathrm{cf}(\lambda)}$.

Then there is a sequence

$$\left\langle \langle \bar{\alpha}^\delta = \alpha_{\delta,i} : i < \mathrm{cf}(\lambda)\rangle : \delta \in S^{\lambda^+}_{\mathrm{cf}(\lambda)} \right\rangle$$

such that $\bar{\alpha}^\delta$ increases to δ, and for every $\theta < \lambda$ and $f \in {}^{\lambda^+}\theta$, there are stationarily many δ such that

$$(\forall^* i)\,[f(\alpha_{\delta,2i}) = f(\alpha_{\delta,2i+1})].$$

Here, the quantifier $\forall^* i$ means "for all but $< \mathrm{cf}(\lambda)$ many". For more on guessing of unbounded sets, see [Sh -e].

In §2, we show that to a large extent the assumption that $2^\lambda = \lambda^+$, is necessary above. See Theorem 2.1.

We finish this introduction by recalling some notation and facts which will be used in the following sections.

Notation 0.4. (1) Suppose that $\kappa = \mathrm{cf}(\kappa) < \delta$. We let

$$S^\delta_\kappa \stackrel{\text{def}}{=} \{\alpha < \delta : \mathrm{cf}(\alpha) = \kappa\}.$$

(2) Suppose that $C \subseteq \alpha$. We let

$$\mathrm{acc}(C) \stackrel{\text{def}}{=} \{\beta \in C : \beta = \sup(C \cap \beta)\},$$

and $\mathrm{nacc}(C) \stackrel{\text{def}}{=} C \setminus \mathrm{acc}(C)$, while $\lim(C) \stackrel{\text{def}}{=} \{\delta < \alpha : \delta = \sup(C \cap \delta)\}$.

Definition 0.5. Suppose that $\lambda \geq \aleph_1$ and γ is an ordinal, while $A \subseteq \lambda^+$. For $S \subseteq \lambda^+$, we say that S *has a square of type* $\leq \gamma$ *nonaccumulating in* A iff there is a sequence $\langle e_\alpha : \alpha \in S \rangle$ such that

(i) $\beta \in e_\alpha \Longrightarrow \beta \in S \ \& \ e_\beta = e_\alpha \cap \beta$,

(ii) e_α is a closed subset of α,

(iii) If $\alpha \in S \setminus A$, then $\alpha = \sup(e_\alpha)$ (so $\mathrm{nacc}(e_\alpha) \subseteq A$ for all $\alpha \in S$),

(iv) $\mathrm{otp}(e_\alpha) \leq \gamma$.

Fact 0.6 (Shelah). [[Sh 351]§4, [Sh -g]III§2] Suppose that

$$\lambda = \mathrm{cf}(\lambda) = \theta^+ > \theta = \mathrm{cf}(\theta) > \kappa = \mathrm{cf}(\kappa).$$

Further suppose that $S \subseteq S^\lambda_\kappa$ is stationary. <u>Then</u> there is $S_1 \subseteq \lambda$ on which there is a square of type $\leq \kappa$, nonaccumulating on A=the successor ordinals, and $S_1 \cap S$ is stationary.

Remark 0.7. In the proof of Fact 0.6 we can replace A=the successor ordinals with $A = S^\lambda_\sigma$ for any $\sigma = \mathrm{cf}(\sigma) < \kappa$.

Definition 0.8 (Shelah). [Sh -g] Suppose that $\delta < \lambda$ and $e \subseteq \delta$, while $E \subseteq \lambda$. We define

$$\mathrm{gl}(e, E) \stackrel{\mathrm{def}}{=} \{\sup(\alpha \cap E) : \alpha \in e \ \& \ \alpha > \min(E)\}.$$

Observation 0.9. Suppose that e and E are as in Definition 0.8, and both e and $E \cap \delta$ are clubs of δ. <u>Then</u>, we have that $\mathrm{gl}(e, E)$ is a club of δ with $\mathrm{otp}(\mathrm{gl}(e, E)) \leq \mathrm{otp}(e)$.

If e is just closed in δ, and $E \cap \delta$ is a club of δ <u>then</u> $\mathrm{gl}(e, E)$ is closed and $\mathrm{otp}(\mathrm{gl}(e, E)) \leq \mathrm{otp}(e)$.

Fact 0.10 (Shelah). [Sh 365] Suppose that $\mathrm{cf}(\kappa) = \kappa < \kappa^+ < \mathrm{cf}(\lambda) = \lambda$. Further suppose that $S \subseteq S^\lambda_\kappa$ is stationary and $\langle e_\delta : \delta \in S \rangle$ is a sequence such that each e_δ is a club of δ. Then there is a club E^* of λ such that the sequence

$$\bar{c} = \langle c_\delta \stackrel{\text{def}}{=} \text{gl}(e_\delta, E^*) : \delta \in S \cap E^* \rangle$$

has the property that for every club E of λ, there are stationarily many δ such that $c_\delta \subseteq E$.

Observation 0.11. Suppose that $\text{cf}(\kappa) = \kappa < \kappa^+ < \lambda$ and λ is a successor cardinal. Further assume that $S_1 \subseteq S^\lambda_\kappa$ is stationary, while $A = S^\lambda_\sigma$ for some $\sigma = \text{cf}(\sigma) < \kappa$, possibly $\sigma = 1$. Then there is stationary $S_2 \subseteq S_1$ and a square $\langle e_\delta : \delta \in S_2 \rangle$ of type $\leq \kappa$ nonaccumulating in A, such that each e_δ is a set of limit ordinals and $S_1 \cap S_2$ is stationary, while

$$E \text{ a club of } \lambda \Longrightarrow \{\delta \in S_1 \cap S_2 : e_\delta \subseteq E\} \text{ is stationary.}$$

[Why? The proof of this can be found in [[Sh -g], III, §2], but as it easily follows from the Facts we already quoted, we shall give a proof. By Fact 0.6 and Remark 0.7, there is $S_3 \subseteq \lambda$ with a square $\langle e_\alpha : \alpha \in S_3 \rangle$ of type $\leq \kappa$ nonaccumulating in A, and such that $S_1 \cap S_3$ is stationary. By Fact 0.10, there is club E^* of λ as in the conclusion of Fact 0.10, with $S_3 \cap S_1$ in place of S. Now, letting

$$S_2 \stackrel{\text{def}}{=} \{\sup(\alpha \cap E^*) : \alpha \in \bigcup_{\delta \in S_3} e_\delta \cup \{\delta\} \ \& \ \alpha > \min(E^*)\} \cap S_3,$$

and for $\delta \in S_2$, letting $c_\delta \stackrel{\text{def}}{=} \text{gl}(e_\delta, E^*)$, we observe that $S_2 \cap S_1$ is stationary (as $S_2 \cap S_1 \supseteq S_1 \cap S_3 \cap \text{acc}(E^*)$), and $\langle c_\delta : \delta \in S_2 \rangle$ is a square of type $\leq \kappa$ nonaccumulating in A, while

$$E \text{ a club of } \lambda \Longrightarrow \{\delta \in S_1 \cap S_2 : c_\delta \subseteq E\} \text{ is stationary.]}$$

Notation 0.12. Reg stands for the class of regular cardinals.

1 A ZFC version of ♣

Theorem 1.1. (1) Suppose that

(a) $\lambda = \mathrm{cf}(\lambda) > \kappa = \mathrm{cf}(\kappa) > \theta = \mathrm{cf}(\theta) \geq \aleph_1$.

(b) $S^* \subseteq S^{\lambda^+}_\theta$ is stationary, moreover

$$S_1 \stackrel{\text{def}}{=} \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \kappa \ \&\ S^* \cap \delta \text{ is stationary}\}$$

is stationary. (e.g. $S^* = S^{\lambda^+}_\theta$.)

Then there is a stationary $S' \subseteq S^*$ and $\langle E_\delta : \delta \in S' \rangle$ such that

(i) E_δ is a club of δ with $\mathrm{otp}(E_\delta) < \lambda^\omega \cdot \kappa$,

(ii) for every unbounded $A \subseteq S^{\lambda^+}_\lambda$, for stationarily many $\delta \in S'$, we have

$$\delta = \sup(A \cap \mathrm{nacc}(E_\delta)).$$

(2) If above we allow $\theta = \aleph_0$, but request $\lambda \geq 2^{\aleph_0}$, the conclusion of (1) remains true.

Remark 1.2. We explain why the assumption that S_1 is stationary in item (b) above implies that S^* is stationary. Notice that $\lambda^+ > \kappa^+$, hence club guessing holds between λ^+ and κ. In fact, by Fact 0.10, we can assume that this is exemplified by a sequence $\langle c_\delta : \delta \in S \subseteq S_1 \rangle$. Now suppose that C is a club of λ^+, and let $E \stackrel{\text{def}}{=} \mathrm{acc}(C)$. Let $\delta \in S$ be such that $c_\delta \subseteq E$. Hence, c_δ is a club of δ, so $c_\delta \cap S^* \neq \emptyset$, implying that $C \cap S^* \neq \emptyset$.

Obviously, S_1 being stationary is a necessary condition for our conclusion, as if S_1 were to be non-stationary, we could assume $S' \subseteq S^* \setminus S_1$, and $E_\delta \cap S^* = \emptyset$ for $\delta \in S'$. This would be a contradiction with (ii) above when $A = S^*$.

Proof. Let $S_0 \stackrel{\text{def}}{=} S^{\lambda^+}_\theta$ and let $A^* \stackrel{\text{def}}{=} S^{\lambda^+}_{\aleph_1}$.

By Observation 0.11 with λ^+ in place of λ and A^* in place of A, there is a $S_2 \subseteq S^{\lambda^+}_{\leq\kappa}$ such that there is a square $\bar{e} = \langle e_\delta : \delta \in S_2 \rangle$ of type $\leq \kappa$ nonaccumulating in A^*, the set $S_1 \cap S_2$ is stationary, and, moreover, for every E a club of λ^+, the set $\{\delta \in S_1 \cap S_2 : e_\delta \subseteq E\}$ is stationary. [Why can we assume that $S_2 \subseteq S^{\lambda^+}_{\leq\kappa}$? As if $\alpha \in S_2$ and $\alpha = \sup(e_\alpha)$, we have that

$\mathrm{cf}(\alpha) \le |e_\alpha| \le \kappa$ and if $\alpha \in S_2$ and $\alpha > \sup(e_\alpha)$ we have that $\alpha \in A^*$, hence $\mathrm{cf}(\alpha) = \aleph_1 \le \kappa$.

Let $S' \stackrel{\mathrm{def}}{=} S^* \cap S_2$, so be stationary. [Why? As otherwise there is a club C of λ^+ with $S' \cap C = \emptyset$. Let $E \stackrel{\mathrm{def}}{=} \mathrm{acc}(C)$ and let $\delta \in S_1 \cap S_2 \cap E$ be such that $e_\delta \subseteq E$. As $\mathrm{cf}(\delta) = \kappa \ne \aleph_1$, we have that $\delta \notin A^*$, and hence e_δ is a club of δ. On the other hand, $S^* \cap \delta$ is stationary in δ, hence $e_\delta \cap S^* \ne \emptyset$, a contradiction with $e_\delta \subseteq C$, as $e_\delta \subseteq S_2$ by the definition of a square (see Definition 0.5). So any point in $e_\delta \cap S^*$ is in $C \cap S'$, contrary to the choice of C.]

Claim 1.3. There is a function $g : S' \to \omega$ such that for every club E of λ^+, there are stationarily many $\delta \in S_1 \cap S_2$ such that $e_\delta \subseteq E$ and

$$(\forall n < \omega)[E \cap \delta \cap g^{-1}(\{n\}) \text{ is stationary in } \delta].$$

Proof of the Claim. For $\delta \in S'$, we choose a sequence $\bar{\xi}^\delta = \langle \xi_{\delta,i} : i < \theta \rangle$ increasing with limit δ, and such that $\xi_{\delta,i} \in e_\delta$ and $\mathrm{otp}(e_{\xi_{\delta,i}})$ depends only on i and $\mathrm{otp}(e_\delta)$, but not on δ. [The point is of course that $\mathrm{otp}(e_\delta)$ is in general larger than θ.] For each $i < \theta$, we define a function $h_i : S' \to \kappa$ by letting

$$h_i(\delta) \stackrel{\mathrm{def}}{=} \mathrm{otp}(e_{\xi_{\delta,i}}).$$

Subclaim 1.4. For each $\delta \in S_1 \cap S_2$ we can find $i(\delta) < \theta$ such that with $i = i(\delta)$,

$$A_i^\delta \stackrel{\mathrm{def}}{=} \{\beta \in e_\delta : \{\gamma \in e_\delta \cap S' : \xi_{\gamma,i} = \beta\} \text{ is stationary } \}$$

is unbounded in δ.

Proof of the Subclaim. If this fails for some $\delta \in S_1 \cap S_2$, then each A_i^δ for $i < \theta$ is bounded in δ. As $\theta < \kappa = \mathrm{cf}(\delta)$, we have $\beta^* \stackrel{\mathrm{def}}{=} \sup_{i<\theta} \sup(A_i^\delta) < \delta$. As $\delta \in S_1$, we have that $e_\delta \cap S^*$ is stationary in δ. For every $\gamma \in e_\delta \cap S^* \setminus \beta^*$, we in particular have that $\gamma \in S'$, so $\bar{\xi}^\gamma$ is defined. Hence, for such γ there is $i_\gamma < \theta$ such that $\gamma > \xi_{\gamma,i_\gamma} > \beta^*$. By Fodor's Lemma, there is ξ^* such that for stationarily many γ we have $\xi_{\gamma,i_\gamma} = \xi^*$, and applying the same lemma again, we can without loss of generality assume that for some $i^* < \theta$ we have

$i_\gamma = i^*$ for stationarily many γ for which $\xi_{\gamma,i_\gamma} = \xi^*$. But then $\xi^* > \beta^*$ and yet $\xi^* \in A^\delta_{i^*}$, a contradiction. $\bigstar_{1.4}$

Subclaim 1.5. For some $i(*) < \theta$, the set

$$S^{**} \stackrel{\text{def}}{=} \{\delta \in S_1 \cap S_2 : i(\delta) = i(*)\}$$

is stationary, and $\bar{e} \restriction S^{**}$ still guesses clubs of λ^+.

Proof of the Subclaim. Otherwise, for each $i < \theta$ such that

$$T_i \stackrel{\text{def}}{=} \{\delta \in S_1 \cap S_2 : i(\delta) = i\}$$

is stationary, there is a club C_i of λ^+ such that for no $\delta \in T_i$ do we have $e_\delta \subseteq C_i$. Let $C \stackrel{\text{def}}{=} \bigcap\{C_i : T_i \text{ stationary }\}$, hence a club of λ^+, and let E be a club of λ^+ such that

$$[i < \theta \ \&\ T_i \text{ not stationary}] \Longrightarrow T_i \cap E = \emptyset.$$

Let $\delta \in S_1 \cap S_2$ be such that $e_\delta \subseteq \operatorname{acc}(E \cap C)$. Hence $\delta \in E \cap T_{i(\delta)}$, so $T_{i(\delta)}$ is stationary. On the other hand, we have $e_\delta \subseteq C_{i(\delta)}$, a contradiction. $\bigstar_{1.5}$

Now notice that $\kappa > \aleph_1$, so club guessing holds between κ and $\aleph_0$, i.e. there is a sequence $\langle w_\zeta : \zeta \in S^\kappa_{\aleph_0}\rangle$ such that $w_\zeta \subseteq \zeta$ and $\operatorname{otp}(w_\zeta) = \omega$ for each ζ, while for every club C of κ, there are stationarily many ζ with $w_\zeta \subseteq C$. Let $W \stackrel{\text{def}}{=} \{w_\zeta : \zeta \in S^\kappa_{\aleph_0}\}$. For $\beta \in S'$, let

$$w^\zeta_\beta \stackrel{\text{def}}{=} \{\gamma \in e_\beta : \operatorname{otp}(e_\beta \cap \gamma) \in w_\zeta\}.$$

For each $\zeta \in S^\kappa_{\aleph_0}$, we define $g_\zeta : S' \to \omega$ by letting

$$g_\zeta(\gamma) \stackrel{\text{def}}{=} \operatorname{otp}(h_{i(*)}(\gamma) \cap w_\zeta).$$

If some g_ζ is as required in Claim 1.3, then we are done. Otherwise, for each ζ there is a club E_ζ of λ^+ with

$$[\delta \in S^{**} \cap E_\zeta \ \&\ e_\delta \subseteq E_\zeta] \Longrightarrow (\exists n \stackrel{\text{def}}{=} n_{\delta,\zeta})\, [E_\zeta \cap \delta \cap g_\zeta^{-1}(\{n\}) \text{ is non-stationary in } \delta].$$

Let $E \stackrel{\text{def}}{=} \bigcap_{\zeta<\kappa} E_\zeta$. Let $\delta^* \in S^{**} \cap E$ be such that $e_{\delta^*} \subseteq E$. As $\delta^* \in S^{**}$, we have that $A^{\delta^*}_{i(*)}$ is unbounded in δ^*. Let $C_{\delta^*} \stackrel{\text{def}}{=} \lim(A^{\delta^*}_{i(*)}) \cap e_{\delta^*}$, hence a club of δ^*. As $\text{cf}(\delta^*) = \kappa$, and so $\text{otp}(e_{\delta^*}) = \kappa$, we have that

$$C^-_{\delta^*} \stackrel{\text{def}}{=} \{\text{otp}(\gamma \cap e_{\delta^*}) : \gamma \in C_{\delta^*}\}$$

is a club of κ. Hence there is $\zeta^* \in S^\kappa_{\aleph_0}$ such that $w_{\zeta^*} \subseteq C^-_{\delta^*}$. Let $\{\xi_0, \xi_1, \xi_2, \ldots\}$ be the increasing enumeration of w_{ζ^*}, and stipulate $\xi_{-1} = 0$. For $l < \omega$, let γ_l be the unique $\gamma \in C_{\delta^*}$ such that $\text{otp}(\gamma_l \cap e_{\delta^*}) = \xi_l$, and let $\gamma_{-1} \stackrel{\text{def}}{=} \min(C_{\delta^*})$. Hence for every $l < \omega$ we have

$$A^{\delta^*}_{i(*)} \cap [\gamma_{l-1}, \gamma_l) \neq \emptyset.$$

Hence for some $\beta_l \in [\gamma_{l-1}, \gamma_l)$, we have that

$$\{\gamma \in e_{\delta^*} \cap S' : \xi_{\gamma, i(*)} = \beta_l\}$$

is stationary in δ^*. This means that for each $l < \omega$, the set $g^{-1}_{\zeta^*}(\{l\}) \cap e_{\delta^*} \cap S'$ is stationary in δ^*, a contradiction. $\bigstar_{1.3}$

Continuation of the Proof of Theorem 1.1. We now choose $\bar{c} = \langle c_\alpha : \alpha < \lambda^+ \rangle$, so that

(α) For every α, we have that c_α is a club of α with $\text{otp}(c_\alpha) \leq \lambda$, and for α a limit ordinal $\beta \in \text{acc}(c_\alpha) \Longrightarrow \text{cf}(\beta) < \lambda$, while if $\alpha = \beta + 1$, then $c_\alpha = \{\beta\}$.

(β) If $\delta \in S_2$, then $c_\delta \supseteq e_\delta$,

(γ) If $\delta \in S_2$ and $\sup(e_\delta) = \delta$, then $c_\delta = e_\delta$,

Now for any limit $\delta < \lambda^+$ we choose by induction on $n < \omega$ a club C^n_δ of δ of order type $\leq \lambda^{n+1}$, using the following algorithm:

Let $C^0_\delta \stackrel{\text{def}}{=} c_\delta$. Let

$$C^{n+1}_\delta \stackrel{\text{def}}{=} C^n_\delta \cup \{\alpha : (\exists \beta \in \text{nacc}(C^n_\delta)) [\sup(\beta \cap C^n_\delta) < \alpha < \beta \;\&\; \alpha \in c_\beta]\}.$$

Note 1.6. (1) The above algorithm really gives C^n_δ which is a club of δ with

$$\mathrm{otp}(C^n_\delta) \leq \lambda^{n+1}.$$

If $\delta \in S_2$ and $\sup(e_\delta) = \delta$, then $\mathrm{otp}(C^\delta_n) \leq \lambda^n \cdot \kappa$.

[Why? We prove this by induction on n. It is clearly true for $n = 0$. Assume its truth for n. Clearly C^{n+1}_δ is unbounded in δ, let us show that it is closed. Suppose $\alpha = \sup(C^{n+1}_\delta \cap \alpha) < \delta$. If $\alpha = \sup(C^n_\delta \cap \alpha)$, then $\alpha \in C^n_\delta \subseteq C^{n+1}_\delta$ by the induction hypothesis. So, assume

$$\alpha^* \stackrel{\mathrm{def}}{=} \sup(C^n_\delta \cap \alpha) < \alpha$$

and $\alpha \notin C^n_\delta$. Let $\langle \alpha_i : i < \mathrm{cf}(\alpha) \rangle$ be an increasing to α sequence in $(\alpha^*, \alpha) \cap C^{n+1}_\delta$. Hence for every i there is $\beta_i \in \mathrm{nacc}(C^n_\delta)$ such that $\alpha_i \in c_{\beta_i}$ and $\sup(C^n_\delta \cap \beta_i) < \alpha_i$. As $\sup(C^n_\delta \cap \alpha) = \alpha^* < \alpha_i$ and $\alpha \notin C^n_\delta$, we have $\beta_i > \alpha$, for every i, as $\beta_i \in \mathrm{nacc}(C^n_\delta)$. Suppose that $i \neq j$ and $\beta_i < \beta_j$. Hence $\sup(C^n_\delta \cap \beta_j) \geq \beta_i > \alpha_j$, a contradiction. So, there is β such that $\beta_i = \beta$ for all i, hence $\{\alpha_i : i < \mathrm{cf}(\alpha)\} \subseteq c_\beta$. As c_β is closed, and $\alpha < \beta$, we have $\alpha \in c_\beta$, and by the definition of C^{n+1}_δ we have $\alpha \in C^{n+1}_\delta$.

As for every β we have $\mathrm{otp}(c_\beta) \leq \lambda$, and by the induction hypothesis $\mathrm{otp}(C^n_\delta) \leq \lambda^{n+1}$, we have $\mathrm{otp}(C^{n+1}_\delta) \leq \lambda^{n+2}$.

Similarly, if $\delta \in S_2$ and $\sup(e_\delta) = \delta$, clearly $\mathrm{otp}(C^n_\delta) \leq \lambda^n \cdot \kappa$.]

(2) For every n, we have $\mathrm{acc}(C^n_\delta) \setminus \bigcup_{m<n} C^m_\delta \subseteq S^{\lambda^+}_{<\lambda}$.

[Why? Again by induction on n. For $n = 0$ it follows as $\mathrm{otp}(c_\delta) \leq \lambda$. Suppose this is true for C^n_δ. The analysis from the proof of (1) shows that for $\alpha \in \mathrm{acc}(C^{n+1}_\delta) \setminus C^n_\delta$, there is β such that $\alpha \in \mathrm{acc}(c_\beta)$, hence $\mathrm{cf}(\alpha) < \lambda$.]

(3) For every limit $\delta < \lambda^+$, we have $S^\delta_\lambda = \bigcup_{n<\omega} \mathrm{nacc}(C^n_\delta) \cap S^\delta_\lambda$.

[Why? Fix such δ and let $\alpha \in S^\delta_\lambda$. By item (2), it suffices to show that $\alpha \in C^n_\delta$ for some n. Suppose not, so let $\gamma_n \stackrel{\mathrm{def}}{=} \min(C^n_\delta \setminus \alpha)$ for $n < \omega$. Hence $\langle \gamma_n : n < \omega \rangle$ is a non-increasing sequence of ordinals $> \alpha$, and so there is n^* such that $n \geq n^* \Longrightarrow \gamma_n = \gamma_{n^*}$. In particular we have that $\gamma_{n^*} \in \mathrm{nacc}(C_\delta^{n^*})$. Let $\beta \in c_{\gamma_{n^*}} \setminus \alpha$. Hence $\sup(\gamma_{n^*} \cap C_\delta^{n^*}) < \alpha \leq \beta < \gamma_{n^*}$. By the definition of $C_\delta^{n^*+1}$, we have $\beta \in C_\delta^{n^*+1}$, a contradiction.]

Now for each $\delta \in S'$ we define

$$E_\delta \stackrel{\mathrm{def}}{=} e_\delta \cup \bigcup \{C_\alpha^{g(\delta)} \setminus \sup(e_\delta \cap \alpha) : \alpha \in \mathrm{nacc}(e_\delta)\}.$$

Note first that E_δ is a club of δ, for $\delta \in S'$.

[Why? Clearly, E_δ is unbounded. Suppose $\gamma = \sup(E_\delta \cap \gamma) < \delta$. Without loss of generality we can assume $\gamma \notin e_\delta$. Let $\gamma^* \stackrel{\mathrm{def}}{=} \sup(e_\delta \cap \gamma) < \gamma$. For every $\beta \in E_\delta \cap (\gamma^*, \gamma)$, there is $\alpha_\beta \in \mathrm{nacc}(e_\delta) \cap S'$ such that $\beta \in C_{\alpha_\beta}^{g(\delta)} \setminus \sup(e_\delta \cap \alpha_\beta)$. By the choice of γ^*, every such $\alpha_\beta > \gamma$. Suppose that $\beta_1 \neq \beta_2 \in E_\delta \cap (\gamma^*, \gamma)$ and $\alpha_{\beta_1} < \alpha_{\beta_2}$. Hence $\sup(e_\delta \cap \alpha_{\beta_2}) \geq \alpha_{\beta_1}$, a contradiction. So all α_β are a fixed α. Hence $\gamma < \alpha$ is a limit point of $C_\alpha^{g(\delta)}$, and we are done, as $C_\alpha^{g(\delta)}$ is closed.]

Also note that $\mathrm{otp}(E_\delta) < \lambda^\omega \cdot \kappa$.

Suppose that $A \subseteq S_\lambda^{\lambda^+}$ is unbounded and it exemplifies that $\langle E_\delta : \delta \in S' \rangle$ fails to satisfy the requirements of Theorem 1.1. Hence there is a club E of λ^+ such that

$$\delta \in E \cap S' \Longrightarrow \sup(A \cap \mathrm{nacc}(E_\delta)) < \delta.$$

Let $E^* \stackrel{\mathrm{def}}{=} \mathrm{acc}(E) \cap \{\delta : \delta = \sup(A \cap \delta)\}$, hence a club of λ^+. Let $\delta^* \in S_1 \cap S_2 \cap E^*$ be such that $e_{\delta^*} \subseteq E^*$ and for all $n < \omega$, the set $\delta^* \cap g^{-1}(\{n\})$ is stationary in δ^*.

For $\alpha \in \mathrm{nacc}(e_{\delta^*})$ we have that $A \cap \alpha$ is unbounded in α. Now we use Note 1.6(3). As $A \subseteq S_\lambda^{\lambda^+}$ we have $A \cap \alpha \subseteq S_\lambda^\alpha$. So $A \cap \alpha = \bigcup_{n<\omega} \mathrm{nacc}(C_\alpha^n) \cap A \cap \alpha$, by the above mentioned Note. As $\alpha \in \mathrm{nacc}(e_{\delta^*})$ and $\mathrm{nacc}(e_{\delta^*}) \subseteq S_{\aleph_1}^{\lambda^+}$, there is $n < \omega$ such that $A \cap \mathrm{nacc}(C_\alpha^n)$ is unbounded in α. Let $n^*(\alpha)$ be the smallest such n. There is n^* such that

$$\sup\{\alpha \in \mathrm{nacc}(e_{\delta^*}) : n^*(\alpha) = n^*\} = \delta^*,$$

as $\mathrm{cf}(\delta^*) > \aleph_0$. Let

$$e \stackrel{\mathrm{def}}{=} \{\beta \in \mathrm{acc}(e_{\delta^*}) : \beta = \sup\{\alpha \in \beta \cap \mathrm{nacc}(e_{\delta^*}) : n^*(\alpha) = n^*\}\},$$

hence e is a club of δ^*. By the choice of δ^*, the set $g^{-1}(\{n^*\}) \cap \delta^*$ is stationary in δ^*. So, there is $\beta \in e$ such that $g(\beta) = n^*$. In particular, $\beta \in S'$. For every $\alpha \in \mathrm{nacc}(e_\beta)$ such that $n^*(\alpha) = n^*$, we have that $A \cap \mathrm{nacc}(C_\alpha^{n^*})$ is unbounded in α. However,

$$C_\alpha^{n^*} \setminus \sup(\alpha \cap e_\beta) = E_\beta \cap [\sup(\alpha \cap e_\beta), \beta),$$

hence $\mathrm{nacc}(C_\alpha^{n^*}) \setminus \sup(\alpha \cap e_\beta) \subseteq \mathrm{nacc}(E_\beta)$. Now, on the one hand $\beta \in E \cap S'$, so $\alpha^* \stackrel{\mathrm{def}}{=} \sup(A \cap \mathrm{nacc}(E_\beta)) < \beta$, but on the other hand, the set of $\alpha \in \mathrm{nacc}(e_\beta)$ with $n^*(\alpha) = n^*$ is unbounded in β, hence there is $\gamma \in A$ with $\gamma \in E_\beta \setminus \alpha^*$. As $\gamma \in A$, we have $\mathrm{cf}(\gamma) = \lambda$, hence $\gamma \in \mathrm{nacc}(E_\beta)$, a contradiction.
(2) The statement of Claim 1.3 is true even when $\kappa = \aleph_1$, but if we assume that $\lambda \geq 2^{\aleph_0}$. Namely, under these assumptions, we have that $\theta = \aleph_0$, so there is $W \subseteq \{w \subseteq \omega_1 : \mathrm{otp}(w) = \omega\}$ such that $|W| \leq \lambda$, and for every club C of ω_1, for some $w \in W$, we have $w \subseteq C$. Now we can just repeat the proof of Claim 1.3, using W we have just defined. $\bigstar_{1.1}$

2 A negation of guessing

Theorem 2.1. Assume that there is a supercompact cardinal. Then

(1) It is consistent that there is λ a strong limit singular of cofinality $\aleph_0$, such that $2^\lambda > \lambda^+$ and

  (∗) There is a function $f : \lambda^+ \to \omega$ such that for every $\mathcal{P} \subseteq [\lambda^+]^{\aleph_0}$ of cardinality $< 2^\lambda$, for some $X \in [\lambda^+]^{\lambda^+}$ we have

    (i) $(\forall \zeta < \omega)[|X \cap f^{-1}(\{\zeta\})| = \lambda^+]$,

    (ii) If $a \in \mathcal{P}$, then $\sup(\mathrm{Rang}(f \restriction (a \cap X)) < \omega$.

(2) Moreover, in (1) we can replace $\aleph_0$ by any regular $\kappa < \lambda$.

Remark 2.2. So the theorem basically states that no $\mathcal{P}$ as above provides a guessing.

Proof. (1) We start with a universe in which λ is a supercompact cardinal and $2^\lambda = \lambda^+$ holds. We extend the universe by Laver's forcing ([La]), which makes the supercompactness of λ indestructible by any extension by a $(<\lambda)$-directed-closed forcing. This forcing will preserve the fact that $2^\lambda = \lambda^+$. Let us call the so obtained universe V.

Now choose μ such that $\mu = \mu^\lambda > \lambda^+$. By [Ba], there is a $(<\lambda)$-directed-closed λ^{++}-cc forcing notion P, not collapsing λ^+ of size μ adding μ unbounded subsets A_α $(\alpha < \mu)$ to λ^+ such that

$(**)$ $\alpha \neq \beta < \mu \Longrightarrow |A_\alpha \cap A_\beta| < \lambda$.

In particular, in V^P we have $\lambda^+ < 2^\lambda = \mu$ ([Ba], 6.1.), while λ is supercompact. In V^P, let Q be Prikry's forcing which does not collapse cardinals and makes λ singular with $\mathrm{cf}(\lambda) = \aleph_0$, [Pr]. As this forcing does not add bounded subsets to λ, in the extension λ is a strong limit singular and clearly satisfies $2^\lambda = \mu$. In $V^{P * \underset{\sim}{Q}}$ we have $(**)$. We now work in $V^{P * \underset{\sim}{Q}}$.

Let $\lambda = \sum_{\zeta<\omega} \lambda_\zeta$ where each $\lambda_\zeta < \lambda$ is regular. Let χ be large enough regular and $M \prec (\mathcal{H}(\chi), \in)$ with $||M|| = \lambda^+$ such that $\lambda^+ \subseteq M$ and $\langle A_\alpha : \alpha < \mu \rangle, \langle \lambda_\zeta : \zeta < \omega \rangle \in M$. We list $\bigcup_{\zeta<\omega}([\lambda^+]^{\lambda_\zeta} \cap M)$ as $\{b_i : i < \lambda^+\}$.

We define $f : \lambda^+ \to \omega$ by $f(i) = \zeta$ iff $|b_i| = \lambda_\zeta$. For $\alpha < \mu$, let $X_\alpha \stackrel{\text{def}}{=} \{i : b_i \subseteq A_\alpha\}$.

Now suppose that $\mathcal{P} \subseteq [\lambda^+]^{\aleph_0}$ is of cardinality $< 2^\lambda \leq \mu$, we shall look for X as required in $(*)$.

If $\alpha < \mu$ is such that X_α fails to serve as X, then at least one of the following two cases must hold:

Case 1. For some $\zeta < \omega$ we have $|\{i : b_i \subseteq A_\alpha \ \& \ |b_i| = \lambda_\zeta\}| < \lambda^+$, or

Case 2. For some $a \in \mathcal{P}$ we have $\sup(\mathrm{Rang}(f \restriction (a \cap X_\alpha))) = \omega$.

Considering the second case, we shall show that for any $a \in \mathcal{P}$, there are $< \lambda$ ordinals α such that the second case holds for X_α, a. Fix an $a \in \mathcal{P}$. If $\alpha < \mu$ is such that Case 2 holds for X_α, a, then

$$\sup(\{\zeta : (\exists i \in a)[b_i \subseteq A_\alpha \ \& \ |b_i| = \lambda_\zeta]\} = \omega.$$

For $\zeta < \omega$ and $\alpha < \mu$ let $B^\alpha_\zeta \stackrel{\text{def}}{=} \{i \in a : b_i \subseteq A_\alpha \ \& \ |b_i| = \lambda_\zeta\}$. Notice that if $\alpha \neq \beta < \mu$ we have that for some $\zeta_{\alpha,\beta}$ the intersection $A_\alpha \cap A_\beta$ has size $< \lambda_{\zeta_{\alpha,\beta}}$, hence for all $\zeta \geq \zeta_{\alpha,\beta}$ we have $B^\alpha_\zeta \cap B^\beta_\zeta = \emptyset$.

Let $A \stackrel{\rm def}{=} \{\alpha : \text{ Case 2 holds for } a, \alpha\}$. For every $\alpha \in A$, let

$$\bar{s_\alpha} \stackrel{\rm def}{=} \langle\, B^\alpha_\zeta : \zeta < \omega\rangle,$$

hence $\alpha \neq \beta \Longrightarrow \bar{s_\alpha} \neq \bar{s_\beta}$. Hence $|A| \leq 2^{\aleph_0} < \lambda$.

Now note that if $\alpha < \mu$, then $A_\alpha \in [\lambda^+]^{\lambda^+}$. For $\gamma_0 < \lambda^+$ for some $\gamma_1 < \lambda^+$ we have $|A_\alpha \cap \gamma_1 \setminus \gamma_0| = \lambda$. In M we have a sequence $\langle c_\xi : \xi < \omega\rangle$ such that $\cup_\xi c_\xi = \gamma_1 \setminus \gamma_0$ and $|c_\xi| = \lambda_\xi$. For every $\zeta, \varepsilon < \omega$, for some large enough $\xi < \omega$ we have $|A_\alpha \cap c_\xi| \geq \lambda_\varepsilon$. But $[A_\alpha \cap c_\xi]^{\lambda_\zeta} \subseteq \mathcal{P}(c_\xi) \subseteq M$ (as $\lambda^{<\lambda} < \lambda$), so for some i we have $b_i \subseteq A_\alpha \cap [\gamma_0, \gamma_1)$ and $|b_i| = \lambda_\zeta$. Hence

$$|\{i : \, b_i \subseteq A_\alpha \ \& \ |b_i| = \lambda_\zeta\}| = \lambda^+,$$

so Case 1 does not happen for this (any) α.

As we can find $\alpha < \mu$ such that Case 2 does not happen, we are finished.

(2) Use Magidor's forcing from [Ma] in place of Prikry's forcing in (1). $\bigstar_{2.1}$

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The address of the author for correspondence:
Mirna Džamonja
School of Mathematics
University of East Anglia
Norwich, NR4 7TJ
UK
Telephone: +44-1603-592981
Fax:+44-1603-593868
e-mail: M.Dzamonja@uea.ac.uk